Engineering Letters, 14:2, EL_14_2_14 (Advance online publication: 16 May 2007)

Stabilization of LTI Switched Systems with Input Time Delay

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***Abstract*—This paper deals with stabilization of LTI switched systems with input time delay. A description of systems' stabilization is presented. Common Lyapunov function is introduced to construct switching law in order to stabilize this kind of system. Necessary and sufficient conditions are presented for both asynchronous switching and synchronous one. Precisely, after a proper change in state space the conditions can be expressed in terms of matrix inequalities. At the same time, definitions, theorems and corollaries as well as simulation result of one example are presented.**

***Index Terms*— Hybrid Systems, Switched Systems, Quadrat-
Ic Stabilization, Common Lyapunov function.**

I. Introduction

In real systems, input delays are often encountered because of transmission of the measurement information. The existence of these delays may be the source of instability or serious deterioration in the performance of the closed-loop system.

Recently, the control design problem of input delayed systems has attracted considerable attention. In [1,2] the memory less controllers were proposed and the stability criteria were independent of the size of the time delay. Moreover, these stability criteria are expressed in the form of Riccati matrix equations. Although the memory less controllers in [1] and [2] are easy to implement, it was pointed out in [3] that they tend to be more conservative when the time delay is small. Based on the reduction method [4], [3] proposed a robust controller for the uncertain input-delayed systems, which has a feedback of the current state and the past input history. It was shown by examples that the controller with delay compensation can have more robustness that the memory less controllers. However, the shortcoming of the method is that the exact value of the time delay must be known.

Manuscript received on 9th June, 2006

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In recent years, study of hybrid systems has achieved rich results [5,6,7,8]. Among various modelling frameworks of hybrid systems, switched systems, as the special class of hybrid modelling, are the most studied. However, switched systems with input-delay have got only less attention. In this paper we are concerned with the stabilization problem of this kind of switched systems. The switched systems are consisted of several linear time invariant systems with control input delays.

The existence of a common Lyapunov function for all subsystems has been found to be a necessary and sufficient condition for the stabilization of switched hybrid systems under arbitrary switching law [5]. Hence, a number of methods to construct such a Lyapunov function are presented [5,9,10]. A lot of systems, however, do not possess a common Lyapunov function, yet they still may be stable under some properly chosen switching law. Therefore, multiple Lyapunov function methods are effective tools for finding such a switching law [12,13,14]

In this paper, the common Lyapunov function method is still used to find a suitable switching law to stabilize the switched systems with time delay. We will first introduce a definition of stabilization of the switched systems with time-delay inputs. Then, we will derive a necessary and sufficient condition of the stabilization via constructing a quadratic Lyapunov function of the form

$$V(x) = x^{'}Px \tag{1}$$

The conditions consist of a collection of matrix inequalities. Through the paper, a positive definite and semi-definite matrix here is denoted by $P > 0$ and $P \geq 0$.

The structure of the paper is organized as follows. Section 2 introduces the definition of quadratic stabilization via asynchronous switching and presents a necessary and sufficient condition for this kind of stabilization. Then a sufficient condition to stabilize the switched systems with input-delay is obtained as a corollary. Section 3 addresses the problem of quadratic stabilization via synchronous switching. In section 4, we give a switched system with input delay and stabilize it based on the designed conditions in application.

II. Quadratic Stabilization Of Switched Systems With Time Delay Via Asynchronous Switching

Consider a switched system with time delay:

$$\dot{x}(t) = A_{i(t)}x(t) + B_{i(t)}u(t-h), i(t) \in S \quad (2)$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input. $S = \{1, 2, \cdots M\}$ is a finite symbols set. Let $i(t) : [0, \infty) \to S$ be a symbolic piecewise function that maps the time to the set of symbols. The matrices $A_{i(t)}, B_{i(t)}$ $(i = 1, 2, \cdots, M)$ are the given constant matrices. h is a positive constant, the delay.
Now our aim is to convert the switched systems with input delay into systems without delay.

Theorem 2.1 Consider a switched system with time delay as the form (2), now define

$$y(t) = x(t) + \int_{t-h}^{t} e^{(t-s-h)A_i} B_i u(s) ds \quad (3)$$

Then we can change the form (2) into

$$\dot{y}(t) = A_i y(t) + e^{-hA_i} B_i u(t) \quad (4)$$

The result is not difficult to obtain.

Furthermore, we consider a class of state feedback control laws generated by switching between the given state feedback controller for each subsystem:

$$u(t) = L_i \cdot y(t) \quad (5)$$

where L_i are given linear matrix functions.
So

$$\dot{y}(t) = [A_i + e^{-hA_i} \cdot B_i \cdot L_i] y(t) \quad (6)$$

Now, Lyapunov functions are introduced to construct switching function $i(t)$ in order to stabilize (2)
Here, we give a description of stabilizibility of switched systems with input delay.

Definition 2.2. System (2) is said to be stabilizable, if for any initial state $x_0 \in R^n$, there exits feedback controllers L_i for each subsystems, after a properly switching $i(t)$, the trajectory of $x(t)$ enters into a sphere with the origin centre and a finite radius R at a finite time t_0 then remains in it henceforth. Namely,

$$\|x(t)\| \le R, t \ge t_0 \quad (7)$$

Hence an asynchronous switching strategy is a rule for switching from one subsystem to another based on the measured value of the switched systems' state.

Definition2.3. Suppose that there exists a matrix $P = P' > 0$ and a class of state feedback controllers which satisfy (5) such that for the derivative $\dot{V}[x(t)]$ of the quadratic Lyapunov function (1) along trajectories of the switched system (2), the condition

$$\dot{V}[x(t)] \le 0 \quad (8)$$

holds for all $x \in R^n$. Then, the switched system (2) is said to be quadratic stabilizable via asynchronous switching.

Remark: As in the case of asynchronous switching, it can be easily shown that quadratic stabilization implies the stabilizability defined in Definition 2.2. [15].

We are now in a position to present a necessary and sufficient condition for quadratic stabilization via asynchronous switching.

Theorem 2.4. Consider the switched system (2). Then it is quadratic stabilizable via asynchronous switching if and only if there exits a square matrix $P = P'$ >0 and a class of state feedback controllers L_i such that the set of matrices

$$\begin{aligned} Q_i = (A_i' + L_i' \cdot B_i' \cdot e^{-hA_i'}) \cdot P + \\ P \cdot (A_i + e^{-hA_i} \cdot B_i \cdot L_i) \le 0 \end{aligned} \quad (9)$$

Then, after a state-space transformation

$$z(t) = H \cdot y(t) = \begin{bmatrix} z_p(t) \\ z_m(t) \end{bmatrix} \quad (10)$$

with a non-singular matrix T, for the switched signal at any time, there exists an index $i \in S$ such that:

$$\bar{Q}_{i,m\times m} < 0 \quad (11)$$

where

$$m = \text{rank} Q_i \;\; p = n - m, \;\; H^{-1} Q_i H = \begin{bmatrix} 0_{p\times p} & 0_{(n-m)\times(n-p)} \\ 0_{(n-p)\times(n-m)} & \bar{Q}_{i,m\times m} \end{bmatrix}$$

Furthermore, Suppose that inequalities (9) hold and introduce a symbolic function $i(t)$, where it is an index for which the minimum in

$$\min_{i=1,2,\cdots M} \dot{V}[x(t)] \quad (12)$$

is achieved. Then, asynchronous switching $i(t)$ can be constructed to stabilize switched systems (2).

Proof:

If the switched system (2) is quadratically stabilizable via asynchronous switching according to definition 2.3, we get

$$\frac{dV}{dt} = \dot{y}^{'}(t)Py(t) + y^{'}(t)P\dot{y}(t)$$
$$= \dot{y}^{'}(t)Py(t) + y^{'}(t)P\dot{y}(t)$$
$$= y^{'}(t)\cdot[(A_i^{'} + L_i^{'}\cdot B_i^{'}\cdot e^{-hA_i^{'}})\cdot P$$
$$+P\cdot(A_i + e^{-hA_i}\cdot B_i\cdot L_i)]\cdot y(t) \le 0$$

here assume that

$$Q_i = [A_i^{'} + L_i^{'}\cdot B_i^{'}\cdot e^{-hA_i^{'}}]\cdot P + P\cdot[A_i + e^{-hA_i}\cdot B_i\cdot L_i],$$

according to a state-space transformation

$$z(t) = H\cdot y(t) = \begin{bmatrix} z_p(t) \\ z_m(t) \end{bmatrix}$$

with a non-singular matrix T transfers the above ones into

$$\begin{bmatrix} z_p^{'}(t) & z_m^{'}(t) \end{bmatrix}\cdot\begin{bmatrix} 0_{p\times p} & 0_{(n-m)\times(n-p)} \\ 0_{(n-p)\times(n-m)} & \bar{Q}_{m\times m} \end{bmatrix}\cdot\begin{bmatrix} z_p(t) \\ z_m(t) \end{bmatrix}$$

So
$$\frac{dV}{dt} = y_m^{'}(t)\bar{Q}_{m\times m}y_m(t)$$

It is obvious that inequality (8) holds under(9),(11). In other words, we have proved that for any $x \in R^n$, there exists an index $i \in S$ satisfies inequality (8).
Now we introduce the function

$$\alpha[x(t)] = \min_{i=1,2,\cdots M} \dot{V}[x(t)]$$

Inequality (8) implies that $\alpha[x(t)] \le 0$. Furthermore, let

$$\alpha_0 = \max \alpha(x)$$

Because the function $\alpha(x)$ is continuous and $\alpha(x) \le 0$, the maximum is achieved and $\alpha_0 \le 0$. Finally, the inequalities (9), (11) hold with the asynchronous switching defined in the statement of the theorem. This completes the proof of the theorem 2.4.

The following corollary provides a simplified sufficient condition to check whether a switched system of the form (2) can be quadratically stabilized.

Corollary 2.5. Consider the switched system (2). Then for the switched signal at any time it is quadratic stabilizable via asynchronous switching if there exits a square matrix $P = P^{'} > 0$ and a class of state feedback controllers L_i, we can find an index $i \in S$ such that:

$$Q_i = [A_i^{'} + L_i^{'}\cdot B_i^{'}\cdot e^{-hA_i^{'}}]\cdot P + P\cdot[A_i + e^{-hA_i}\cdot B_i\cdot L_i] < 0 \quad (13)$$

Furthermore, suppose that (13) hold and introduce a symbolic function $i(t)$, where it is an index for which the minimum in (12) is achieved. Then, the asynchronous switching $i(t)$ quadratically stabilizes the switched systems (2).

The condition in the above corollary provides a simplified sufficient condition to stabilize the switched system via asynchronous switching, and it is not difficult to check.

III. Quadratic Stabilization Of Switched Systems With Time Delay Via Synchronous Switching

In this section, we address the case where switching can only occur at pre-specified times.

Suppose that now they can only be switched at the discrete times $lT, \{l = 0,1,\cdots\}$ and $T > 0$ is the switching interval. More precisely, let $T > 0$ be a given time, and let $i_l(\cdot)$ be a function that maps from the set of plant state measurements $\{x(\cdot)\big|_0^{lT}\}$ to the set of symbols S. Then a synchronous switching strategy is a rule for switching from one subsystem to another at the discrete times lT.

Now we consider the problem of quadratic stabilization of the system (2) via synchronous switching.

Define

$$G_i = \exp([A_i + e^{-hA_i}\cdot B_i\cdot L_i]\cdot T) \quad (14)$$

be the state transition matrix for the system (2) under the influence of the subsystem i between the time instants lT and $(l+1)T$. So we change the system (6) in to:

$$y((l+1)T) = G_i\cdot y(lT) \quad (15)$$

Hence an synchronous switching strategy is a rule for switching from one subsystem to another based on the measured value of the switched systems' state, and it can only be occurred at pre-specified switching times.

Definition3.1. Suppose that there exists a matrix $P = P' > 0$ such that the following condition holds for the quadratic Lyapunov function (1) along all the solutions of the systems (15)

$$V[x((l+1)T)] - V[x(lT)] \le 0 \qquad (16)$$

for all $l = 0,1,2,\cdots$ then, the system (15) is said to be quadratic stabilizable via synchronous switching.
Remark: As in the case of synchronous switching, it can be easily shown that quadratic stabilization implies the stabilizability defined in Definition 2.1.[15].

Following the similarity of theorem 2.4, we are now in a position to present a necessary and sufficient condition for quadratic stabilization via synchronous switching.

Theorem 3.2. Consider the system (15). Then it is quadratic stabilizable via synchronous switching if and only if there exits a square matrix $P = P' > 0$ and a class of state feedback controllers L_i such that the set of matrices

$$Q_i = G_i' P G_i - P \le 0 \qquad (17)$$

then, after a state-space transformation

$$z(lT) = H \cdot y(lT) = \begin{bmatrix} z_p(lT) \\ z_m(lT) \end{bmatrix} \qquad (18)$$

with a non-singular matrix T , for the switched signal at any time, there exists an index $i \in S$ such that:

$$\bar{Q}_{i,m\times m} < 0 \qquad (19)$$

where

$$m = \mathrm{rank} Q_i,\ p = n - m,\ H^{-1} Q_i H = \begin{bmatrix} 0_{p\times p} & 0_{(n-m)\times(n-p)} \\ 0_{(n-p)\times(n-m)} & \bar{Q}_{i,m\times m} \end{bmatrix}$$

Furthermore, Suppose that inequalities (17), (19) hold and introduce a symbolic function $i(t)$, where it is an index for which the minimum in

$$\min_{i=1,2,\cdots M} V[(l+1)T] - V[lT] \qquad (20)$$

is achieved. Then, synchronous switching $i(lT)$ can be constructed to stabilize systems (15).

The following corollary provides a simplified sufficient condition to check whether a system of the form (15) can be quadratically stabilized.

Corollary 3.3. Consider the system of the form (15). Then for the switched signal at any time it is quadratic stabilizable via synchronous switching if there exits a square matrix $P = P' > 0$ and a class of state feedback controllers L_i , we can find an index $i \in S$ such that:

$$Q_i = G_i' P G_i - P < 0 \qquad (21)$$

Furthermore, suppose that (21) hold and introduce a symbolic function $i(t)$, where it is an index for which the minimum in (20) is achieved. Then, synchronous switching quadratically stabilizes the systems (15)

The condition in the above corollary provides a simplified sufficient condition to stabilize the system via synchronous switching, and it is not difficult to check.

IV. EXAMPLE

In this section, we present a simple example to illustrate the theoretical results. Note that various switching laws will make the behaviour of a switched system quite different from the behaviours of its components. A switched system can be unstable even if all of its subsystems are stable [12], so we focus on how to construct the switching law to stabilize the switched system even if its subsystems are stable.

We consider the problem of quadratic stabilization of the LTI switched system with input delay via asynchronous controller switching with, a positive constant $h = 0.04$, the delay, and the initial state $x_0 = [1,1]$.

$subsystem1:$

$$\dot{x}(t) = \begin{bmatrix} -2.2698 & 6.5396 \\ -10.9636 & 0.9272 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t-h)$$

$subsystem2:$

$$\dot{x}(t) = \begin{bmatrix} -10.5806 & 9.1612 \\ 2.2592 & 3.3058 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot u(t-h)$$

We apply theorem 2.1 and change subsystem 1, subsystem 2 into

$$\dot{y}(t) = \begin{bmatrix} -1.5 & 5 \\ -10 & -1 \end{bmatrix} y(t); \qquad \dot{y}(t) = \begin{bmatrix} -1 & -10 \\ 5 & -1.5 \end{bmatrix} y(t).$$

Furthermore, we will consider the case of two linear basic state feedback controllers (5) as

$$L_1 = \begin{bmatrix} 1 & -2 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 3 & -6 \end{bmatrix}$$

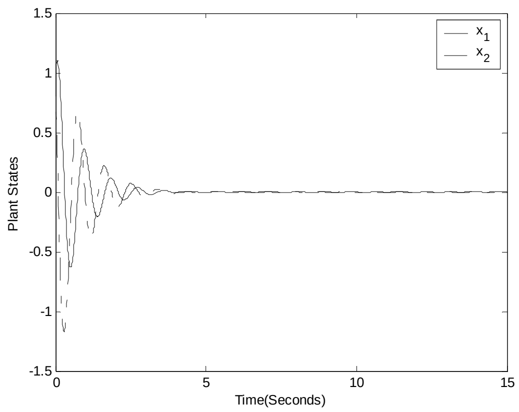
It can be easily seen that both matrices $A_i + e^{-hA_i} \cdot B_i \cdot L, i = 1,2$ are stable (i.e., they have eigenvalues in the left half complex plane). The state trajectory has been shown in figure 1.

We apply theorem 2.4 and get

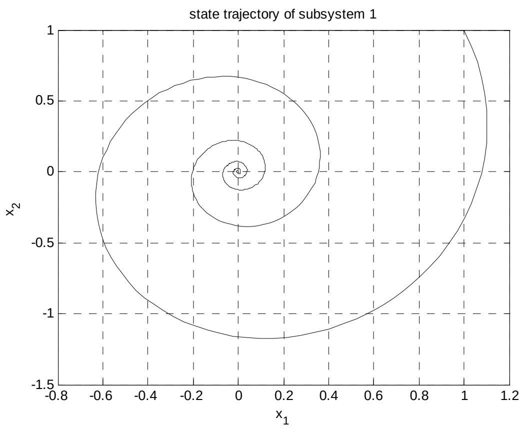
$$P = P^{'} = \begin{bmatrix} 0.1524 & 0.0393 \\ 0.0393 & 0.2499 \end{bmatrix} > 0$$

And then we can stabilize this kind of switched system with input time delay.

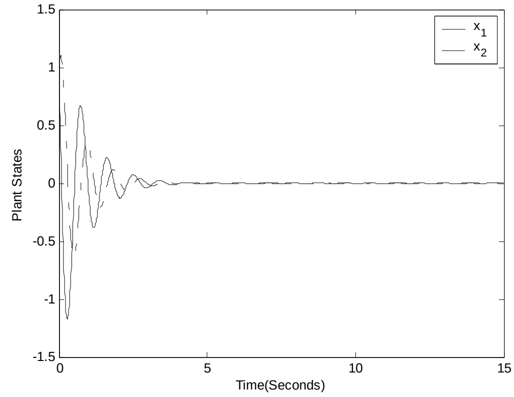
Figure 2 (a) shows the controller index $i(t)$ that describes which of the two basic controllers we must use at each point of the state space. In other words, if at switching time jT our trajectory is in the region U_i, then we use the controller i over the time interval $(jT,(j+1)T]$. The quadratic Lyapunov function via time and switching law are shown in figure2 (c) and the stabilized state trajectory of switched systems is shown in figure 2 (b).. figure 2 (d) shows the switching sequence of switched systems.



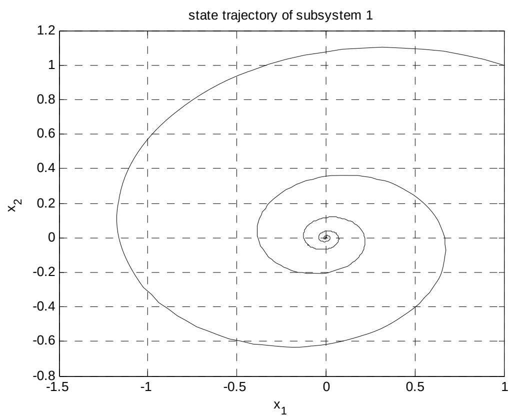
(a)



(b)

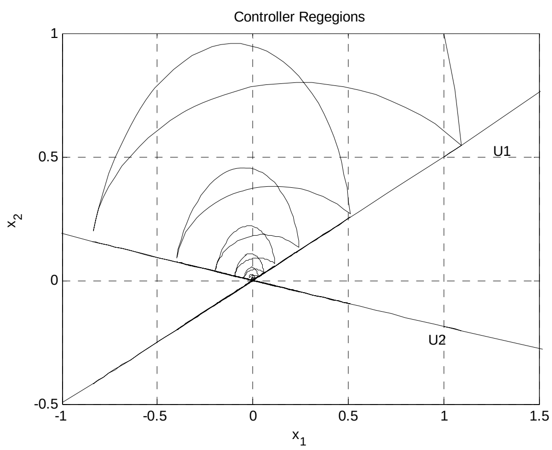


(c)

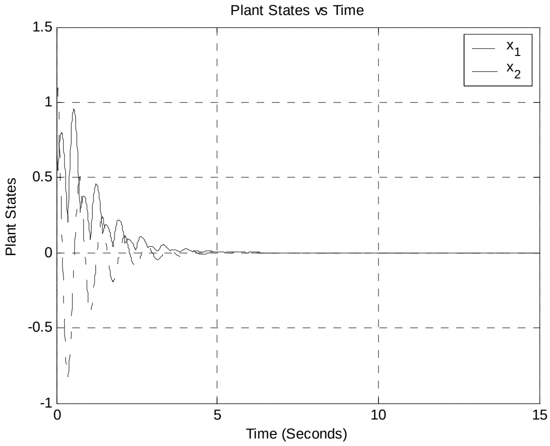


(d)

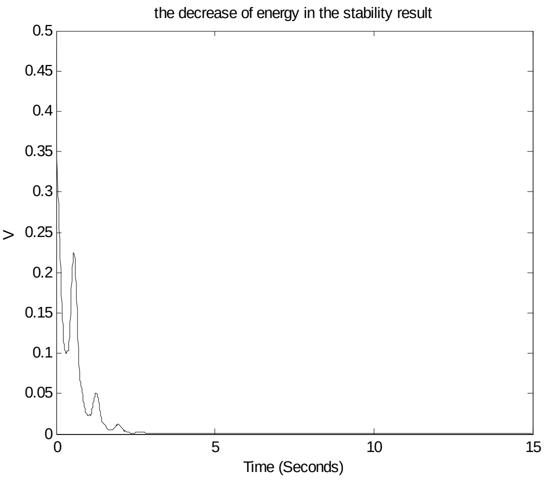
Fig. 1. (a) The state trajectory of subsystem 1 vs time. (b) The state trajectory of subsystem 1. (c) The state trajectory of subsystem 2 vs time. (d) The state trajectory of subsystem 2.



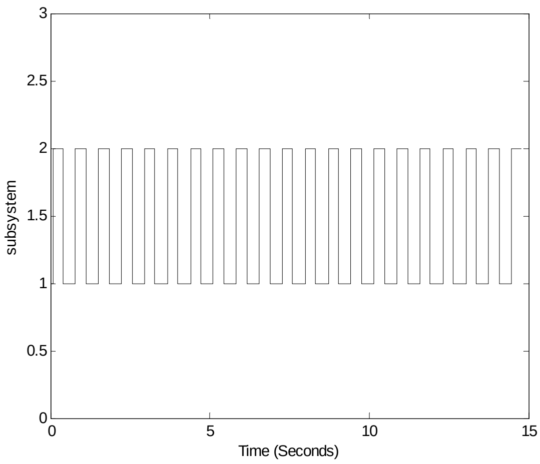
(a)



(b)



(c)



(d)

Fig. 2. (a) State trajectory of switched systems with time delay. (b)The state trajectory of switched system vs time..(c) Lyapunov function vs time (d) the switching law in the stabilization of switched system.

V. CONCLUSION

In this paper, stabilization of LTI switched systems with input delay has been solved. With a quadratic Lyapunov function (1) we design a suitable rule for switching from one subsystem to another such that the system is stabilizable. Precisely, necessary and sufficient conditions for both asynchronous switching and synchronous one have been obtained. At last, a numerical example has been used to show the effectiveness of the main results

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