Unfolding Low-Degree Orthotrees with Constant Refinement

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Abstract

We show that every orthotree of degree 3 or less can be unfolded with a 4×4 refinement of the grid faces. This is the first constant refinement unfolding result for orthotrees that are not required to be well-separated. Our approach shows promise of extending to arbitrary degree orthotrees.

1 Introduction

An *unfolding* of a polyhedron is obtained by cutting its surface in such a way that it can be flattened in the plane as a simple non-overlapping polygon called a *net*. An *edge unfolding* allows only cuts along the polyhedron's edges, while a *general unfolding* allows cuts anywhere on the surface. Edge cuts alone are not sufficient to guarantee an unfolding for non-convex polyhedra [BDE+03, BDD+98], however it is unknown whether all non-convex polyhedra have a general unfolding. In contrast, all convex polyhedra have a general unfolding [DO07, Sec. 24.1.1], but it is unknown whether they all have an edge unfolding [DO07, Ch. 22].

Prior work on unfolding algorithms for non-convex objects has focused on orthogonal polyhedra. This class consists of polyhedra whose edges and faces all meet at right angles. Because not all orthogonal polyhedra have edge unfoldings [BDD+98], the unfolding algorithms typically use additional non-edge cuts that follow one of two models. In the *grid unfolding model*, the surface is subdivided into rectangular *grid faces* by adding edges where axis-perpendicular planes through each vertex intersect the surface, and cuts along these added edges are also allowed. In the *grid refinement model*, each grid face under the grid unfolding model is further subdivided by an $(a \times b)$ orthogonal grid, for some positive integers $a, b \geq 1$, and cuts are also allowed along any of these grid lines.

A series of algorithms have been developed for unfolding arbitrary genus-0 orthogonal polyhedra, with each successive algorithm requiring less grid refinement. The first such algorithm [DFO07] required an exponential amount of grid refinement. This was reduced to quadratic refinement in [DDF14], and then to linear in [CY15]. These ideas were further extended in [DDFO17] to unfold arbitrary genus-2 orthogonal polyhedra with linear refinement.

The only unfolding algorithms for orthogonal polyhedra that use sublinear refinement are for specialized orthogonal shape classes. For example, there exist algorithms for unfolding orthostacks using 1×2 refinement [BDD+98] and Manhattan Towers using 4×5 refinement [DFO05]. There also exist unfolding algorithms for several classes of polyhedra composed of unit cubes. For example, orthotubes [BDD+98] and one layer block structures [LPW14] with an arbitrary number of unit holes can both be unfolded with cuts restricted to the cube edges.

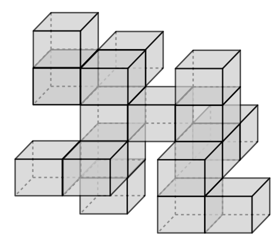

Figure 1: Orthotree of maximum degree three.

Our focus here is on the class of orthogonal polyhedra known as orthotrees. An *orthotree* $\mathcal{O}$ is composed of axis-aligned unit cubes (boxes) glued face to face, whose surface is a 2-manifold and whose dual graph $\mathcal{T}$ is a tree. (See Figure 1 for an example.) In the grid unfolding model, cuts are allowed along any of the cube edges. Each node in $\mathcal{T}$ is a box in $\mathcal{O}$ and two nodes are connected by an edge if the corresponding boxes are adjacent in $\mathcal{O}$ (i.e., if they share a face). In this paper we will use the terms *box* and *node* interchangeably. The *degree* of a box $b \in \mathcal{O}$ is defined as the degree of its corresponding node in the dual tree $\mathcal{T}$. We select any node of degree one to be the *root* of $\mathcal{T}$.

In an orthotree, each box can be classified as either a *leaf*, a *connector*, or a *junction*. A leaf is a box of degree one; a connector is a box of degree two whose two adjacent boxes are attached on opposite faces; all other boxes are junctions.

Because orthotrees are orthogonal polyhedra, they can be unfolded using the general algorithm in [CY15] with linear refinement. It is unknown whether orthotrees can be unfolded using sublinear refinement.

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Prior algorithms specialized for unfolding orthotrees have been limited to orthotrees that are *well-separated*, meaning that no two junction boxes are adjacent. In [DFMO05], the authors provide an algorithm for grid unfolding well-separated orthotrees. Recent work in [HCY17] shows that the related class of well-separated orthographs (which allow arbitrary genus) can be unfolded with a 2×1 refinement.

In this paper we provide an algorithm for unfolding orthotrees of degree up to three using a 4×4 refinement of the cube faces. For each box b in $\mathcal{T}$, the algorithm unfolds b and the boxes in the subtree rooted at b recursively. Intuitively, the algorithm unfolds surface pieces of b along a carefully constructed path. When the path reaches a child box of b, the child is recursively unfolded and then the path continues on b again to the next child (if there is one). The unfolding of b and its subtree is contained within a rectangular region having two staircase-like bites taken out of it.

This is the first sublinear refinement unfolding result for orthotrees that are not required to be well-separated. Our algorithm can handle trees with adjacent junction boxes of degrees two or three, which other constant refinement algorithms are unable to do. In addition, the ideas used here show promise of extending to arbitrary degree orthotrees.

2 Terminology

For any box $b \in \mathcal{O}$, R_b and L_b are the *right* and *left* faces of b (orthogonal to the x-axis); F_b and K_b are the *front* and *back* faces of b (orthogonal to the z-axis); and T_b and B_b are the *top* and *bottom* faces of b (orthogonal to the y-axis). We use a different notation for boxes adjacent to b, to clearly distinguish them from faces: E_b and W_b are the *east* and *west* neighbors of b (adjacent to R_b and L_b, resp.); N_b and S_b are the *north* and *south* neighbors of b (adjacent to T_b and B_b, resp.); and I_b and J_b are the *front* and *back* neighbors of b (adjacent to F_b and K_b, resp.). We omit the subscript whenever the box b is clear from the context. We use combined notations to refer to the east neighbor of N as NE, the back neighbor of NE as NEJ, and so on.

If a face of a box $b \in \mathcal{O}$ is also a face of $\mathcal{O}$, we call it an *open face*; otherwise, we call it a *closed face*. On the closed face shared by b with its parent box in $\mathcal{T}$, we identify a pair of opposite edges, one called the *entry port* and the other called the *exit port* (shown in red and labeled in Figure 2). The unfolding of b is determined by an *unfolding path* that starts on b's entry port, recursively visits all boxes in the subtree $\mathcal{T}_b \subseteq \mathcal{T}$ rooted at b, and ends on b's exit port.

To make it easier to visualize the unfolding path, we use an L-shaped guide (or simply L-guide) with two orthogonal pointers, namely a Hand *pointer* and a Head *pointer*, as shown in Figure 2, where the circle is the Head and the arrow is the Hand. With very few exceptions, the unfolding path extends in the direction of one of the two pointers. Whenever the unfolding path follows the direction of the Hand, we say that it extends Hand-*first*; otherwise, it extends Head-*first*. Surface pieces traversed in the direction of the Hand(Head) will flatten out horizontally (vertically) in the plane. We denote by $\mathcal{N}_b$ the unfolding net produced by a recursive unfolding of b.

We refer the reader to Figure 2a which shows the unfolding path for the simple case of a leaf box A. The L-guide is shown positioned on top of A's parent box I at the entry port. The unfolding path extends Head-first around the top, back, and bottom faces of A, and ends on the bottom of I at the exit port. The resulting unfolding net $\mathcal{N}_A$ consisting of A's open faces T_A, K_A, B_A, L_A, and R_A is shown. In all unfolding illustrations, the outer surface of $\mathcal{O}$ is shown. When describing and illustrating the unfolding of a box A, we will assume without loss of generality that the box is in *standard position* (as in Figure 2a), with its parent I_A attached to its front face F_A and its entry (exit) port on the top (bottom) edge of F_A.

The *ring* r of a box b includes all the points on the surface of b (not necessarily on the surface of $\mathcal{O}$) that are within distance $0 < \delta \leq 1/4$ of the closed face shared with b's parent. Thus, r consists of four $1/4 \times 1$ rectangular pieces (which we call *ring faces*) connected in a cycle. The *entry box* b_e of b is the box containing the open face in $\mathcal{T} \setminus \mathcal{T}_b$ adjacent to b's entry port. Note that b_e may be b's parent (as in Figure 4a), but this is not necessary (see Figure 4b where b_e is the box on top of the parent I). The *entry ring* r_e of b includes all points of b_e that are within distance $1/4$ of the closed face of b_e adjacent to b's entry port. (See Figure 4.) The face e of r_e adjacent to b's entry port is the *entry ring face*. Similarly, the *exit box* b_x of b is the box containing the open face in $\mathcal{T} \setminus \mathcal{T}_b$ adjacent to b's exit port. Note that b_x is not necessarily b's parent (see Figure 4b, where b_x is the box south of the parent I). The *exit ring* r_x of b includes all points of b_x that are within distance $1/4$ of the closed face of b_x adjacent to b's exit port. The face x of r_x adjacent to b's exit port is the *exit ring face*. Note that both e and x are *open* ring faces (by definition). When unclear from context, we will use subscripts (i.e., e_b and x_b) to specify box b's entry and exit faces.

In a Head-first unfolding of a box b, the L-guide begins on the entry ring face with the Head pointing toward the entry port, and it ends on the exit ring face with the Head pointing away from the exit port; the Hand has the same orientation at the start and end of the unfolding. (See Figure 2a.) Similarly, in a Hand-first unfolding, the L-guide begins on the entry ring face with the Hand pointing toward the entry port, and it

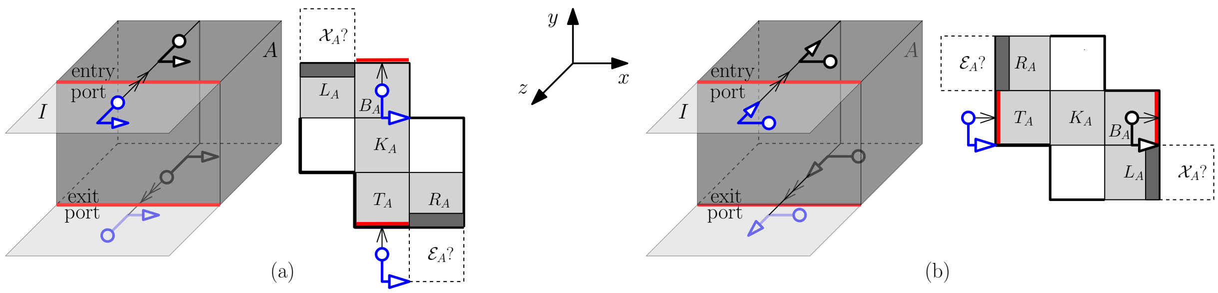

Figure 2: (a) HEAD-first and (b) HAND-first unfolding of leaf box.

ends on the exit ring face with the HAND pointing away from the exit port; the HEAD has the same orientation at the start and end of the unfolding. (See Figure 2b.) In standard position, the HAND in a HEAD-first unfolding will point either east or west. If it points east (west) we say that the unfolding is a HAND-east (west), HEAD-first unfolding. Similarly, in a HAND-first unfolding, the HEAD will either point east or west. If it points east (west), we say the unfolding is a HEAD-east (west), HAND-first unfolding.

In a HEAD-first (HAND-first) unfolding of b with entry ring face e, $\xrightarrow{e}$ is the ring face of r_e encountered immediately after e when cycling around r_e in the direction pointed to by the HAND(HEAD) of the L-guide as positioned on e at the start of b's unfolding. Similarly, in a HEAD-first (HAND-first) unfolding of b with exit ring face x, $\xleftarrow{x}$ is the ring face of r_x encountered just before x when cycling around r_x in the direction pointed to by the HAND(HEAD) of the L-guide as positioned on x at the end of b's unfolding path. Figure 4 shows $\xrightarrow{e}$ and $\xleftarrow{x}$ labeled. Note that although e and x are open ring faces by definition, $\xrightarrow{e}$ and $\xleftarrow{x}$ may not be open, as illustrated in Figure 4c.

3 Net Connections and Inductive Regions

Let $b \in \mathcal{T}$ be a box to be unfolded recursively. A HEAD-*first inductive region* for b is an orthogonally convex polygon shaped as in Figure 3a. Its bounding box is at least three units wide and at least three units tall. The lower (upper) convex vertex that lies strictly inside the bounding box is at unit vertical and horizontal distance from the lower left (upper right) corner of the bounding box, one unit away from the clockwise adjacent (reflex) vertex, and two units away from the counterclockwise adjacent (reflex) vertex. If the successor $\xrightarrow{e}$ of the entry ring face e is open, then the unit cell labeled $\mathcal{E}_b$ in Figure 3a is not part of the inductive region; otherwise, $\mathcal{E}_b$ is included as part of the inductive region. Similarly, if the predecessor $\xleftarrow{x}$ of the exit ring face x is open, then the unit cell labeled $\mathcal{X}_b$ in Figure 3a is not part of the inductive region; otherwise, $\mathcal{X}_b$ is included as part of the inductive region. (See Figure 4 for examples.) The *entry* (*exit*) port of the inductive region is the horizontal unit segment incident to the lower (upper) convex corner that lies strictly inside the bounding box of the region. A HEAD-first unfolding of b produces a net $\mathcal{N}_b$ that fits within the HEAD-first inductive region and whose entry and exit ports coincide with the entry and exit ports of the inductive region.

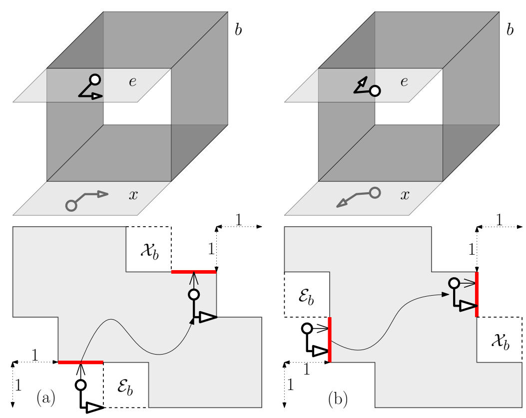

Figure 3: Inductive region for (a) HEAD-first unfolding (b) HAND-first unfolding

A HAND-*first inductive region* for b is an orthogonally convex polygon shaped as in Figure 3b. It is isometric to a HEAD-first inductive region, and one can be obtained from the other through a clockwise 90°-rotation, followed by a vertical reflection.

Lemma 1 *Let $\mathcal{N}_b$ be the unfolding net produced by a recursive* HAND-*east (west),* HEAD-*first recursive unfolding of b. If $\mathcal{N}_b$ is rotated clockwise by 90° and then reflected vertically, then the result is a* HEAD-*east (west),* HAND-*first recursive unfolding of b.*

Lemma 1 (whose proof is deferred to the appendix) enables us to focus the rest of our discussion on HEAD-first unfoldings only, and assume that the same results apply to the HAND-first unfoldings. Next we discuss the type of connections that each net must provide to

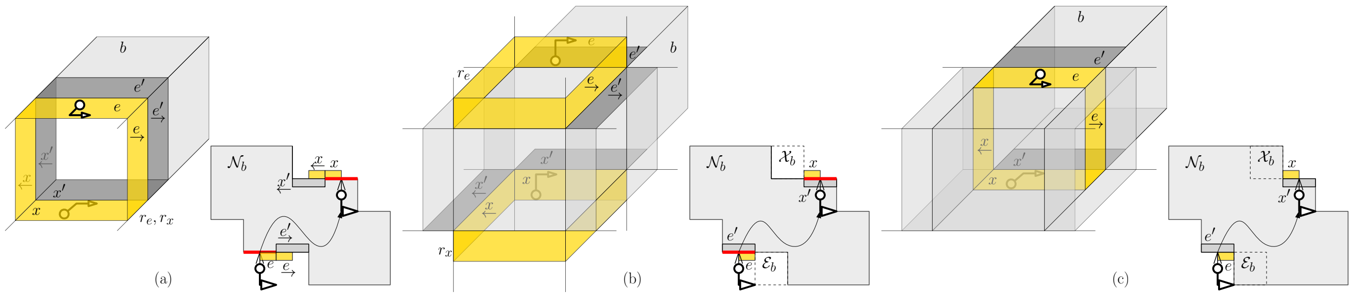

Figure 4: Entry/exit rings (in yellow) and entry/exit connections. (a) $\xrightarrow{e}$ and $\xleftarrow{x}$ open and adjacent to $\mathcal{T}_b$, type-2 entry and exit connections (type-1 entry connections would also be allowed here); (b) $\xrightarrow{e}$ and $\xleftarrow{x}$ non-adjacent to $\mathcal{T}_b$, type-1 entry and exit connections; (c) $\xrightarrow{e}$ and $\xleftarrow{x}$ closed: type-1 entry and exit connections; the squares labeled $\mathcal{E}_b$ and $\mathcal{X}_b$ belong to b's inductive region.

ensure it connects to the rest of $\mathcal{T}$'s unfolding. To do so, we need a few more definitions.

Let e' (x') be the open ring face of $\mathcal{T}_b$ that is adjacent to e (x) along the entry (exit) port. If $\xrightarrow{e}$ ($\xleftarrow{x}$) is open, let $\xrightarrow{e'}$ ($\xleftarrow{x'}$) be the open ring face adjacent to it along its side of unit length (see Figure 4). Note that, although e and $\xrightarrow{e}$ are ring faces from the same box by definition, ring faces e' and $\xrightarrow{e'}$ may be from different boxes (as in Figure 4b), and similarly for x' and $\xleftarrow{x'}$.

If b is not the root of $\mathcal{T}$, to ensure that b's net connects to the rest of $\mathcal{T}$'s unfolding, it must provide type-1 or type-2 connection pieces placed along the boundary inside its inductive region. These connections are defined as follows:

- A *type*-1 *entry connection* consists of the ring face e' placed alongside the entry port. (See Figure 4(b,c) for examples.)
- A *type*-1 *exit connection* consists of the ring face x' placed alongside the exit port. (See Figure 4(b,c) for examples.)
- A *type*-2 *entry connection* is used when the ring face $\xrightarrow{e}$ is open and adjacent to T_b. It consists of the ring face $\xrightarrow{e'}$ placed right of the entry port. (See Figure 4a for an example.)
- A *type*-2 *exit connection* is used when the ring face $\xleftarrow{x}$ is open and adjacent to $\mathcal{T}_b$. It consists of the ring face $\xleftarrow{x'}$ placed left of the exit port. (See Figure 4a for an example.)

Note that the unfolding of b begins on e (by definition) and is therefore adjacent to the entry port. The entry box b_e will provide a piece of e alongside the entry port of b's inductive region which connects to e' in b's net in a type-1 entry connection; for a type-2 entry connection, it places a piece of $\xrightarrow{e}$ next to e, which connects to $\xrightarrow{e'}$ in b's net. Examples of these pieces are shown in yellow along the boundary of the inductive regions in Figure 4. Similarly, the unfolding of b ends on x (by definition) and is therefore adjacent to the exit port. The exit box b_x will provide a piece of x alongside the exit port of b's inductive region which connects to x' in b's net in a type-1 exit connection; for a type-2 exit connection, it places a piece of $\xleftarrow{x}$ next to x, which connects to $\xleftarrow{x'}$ in b's net.

4 Inductive Hypothesis

We will make use of the following inductive hypothesis for the recursive unfolding of a box $b \in \mathcal{T}$ other than the root box:

(I1) The recursive HEAD-first (HAND-first) unfolding of b produces an unfolding net $\mathcal{N}_b$ that fits within a HEAD-first (HAND-first) inductive region and includes all open faces of $\mathcal{T}_b$, with cuts restricted to a 4×4 refinement of the box faces.

(I2) The unfolding net $\mathcal{N}_b$ provides the following entry and exit connections (see Figure 4):

(a) If $\xrightarrow{e}$ is open and adjacent to a face in $\mathcal{T}_b$, then $\mathcal{N}_b$ provides either a type-1 or type-2 entry connection. Otherwise, $\mathcal{N}_b$ provides a type-1 entry connection.

(b) If $\xleftarrow{x}$ is open and adjacent to a face in $\mathcal{T}_b$, then $\mathcal{N}_b$ provides either a type-1 or type-2 exit connection. Otherwise, $\mathcal{N}_b$ provides a type-1 exit connection.

(I3) Open faces of b's ring that are not used in N_b's entry and exit connections can be removed from $\mathcal{N}_b$ without disconnecting $\mathcal{N}_b$.

5 Unfolding Algorithm

Our unfolding algorithm uses an unfolding path that begins on the top face of the root box of $\mathcal{T}$, recursively visits all nodes in the subtree rooted at the child of the root box, and ends on the bottom face of the root box. The following theorem shows how the inductive hypothesis can be used to derive our main result.

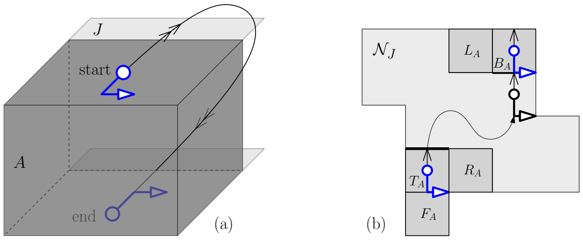

Figure 5: Head-first unfolding of root box A with back child J (a) unfolding path (b) unfolding net.

Theorem 2 *Let $\mathcal{O}$ be an orthotree of degree at most three. If the inductive hypothesis is met by all boxes in $\mathcal{T}$ other than the root box, then $\mathcal{O}$ can be unfolded into a net using a 4×4 refinement.*

Proof. Let $A \in T$ be the root of $\mathcal{T}$ (by definition, A is a node of degree one in $\mathcal{T}$). Assume A has a back child J (if this is not the case, reorient $\mathcal{O}$ to make this true). A recursive unfolding of $\mathcal{O}$ is depicted in Figure 5a: starting Head-first on the top face of A, the unfolding path recursively visits J and returns to the bottom face of A. The resulting net takes the shape depicted in Figure 5b.

Property (I2) applied to J tells us that $\mathcal{N}_J$ provides either type-1 or type-2 entry and exit connections. If of type-1, the entry (exit) connection attaches to T_A (B_A); otherwise, it attaches to R_A (L_A). In either case, the surface piece $\mathcal{N}_A$ depicted in Figure 5b is connected. Property (I1) applied to J tells us that $\mathcal{N}_J$ is a net that includes all open faces in the subtree $\mathcal{T}_J$ rooted at J and uses a 4×4 refinement. This along with the fact that the open faces of A attach to $\mathcal{N}_J$ without overlap shows that $\mathcal{N}_A$ is a net that uses a 4×4 refinement. □

The rest of the paper is devoted to proving that the inductive hypothesis holds for all boxes $A \in \mathcal{T}$ other than the root box. Lemma 1 allows us to restrict our attention to Head-first unfoldings only.

We discuss several cases depending on the node degree. The Head-first unfolding of a leaf node is depicted in Figure 2a, and it can be easily verified that this unfolding satisfies the inductive hypothesis. In the appendix we include a proof of this claim, and show that degree-2 nodes can be handled as degenerate cases of degree-3 nodes. Our analysis of degree-3 nodes is split into five different cases, depending on the position of A's children:

Case 3.1: E and J are children of A. The case where W and J are children of A is a vertical mirror plane reflection of this case.

Case 3.2: E and W are children of A.

Case 3.3: N and S are children of A.

Case 3.4: N and J are children of A. The case where S and J are children of A is a horizontal mirror plane reflection of this case, with the unfolding path traversed in the opposite direction.

Case 3.5: N and E are children of A. The case where N and W are children of A is a vertical mirror plane reflection of this case; the case where S and E are children of A is a horizontal mirror plane reflection of this case, with the unfolding path followed in the opposite direction; the case where S and W are children of A is a vertical mirror plane reflection of the case where S and E are children of A.

In this paper we discuss case 3.1 only, and defer the remaining cases to the appendix. Case 3.1 is handled by Lemma 5 and Theorem 6, which make use of the following two preliminary lemmas (whose proofs are deferred to the appendix).

Lemma 3 *Let $X \in \{E, W\}$ be a child of A. In a Head-first unfolding of A, if the Hand points in the direction of X (opposite to X), then the successor $\xrightarrow{e_A}$ (predecessor $\xleftarrow{x_A}$) of the entry (exit) ring face e_A (x_A) is open.*

Lemma 4 *Let ξ be the unfolding path and $\mathcal{N}$ the unfolding net produced by a recursive unfolding of a box in $\mathcal{T}$. Let $\overleftarrow{\xi}$ be the unfolding path traversed in reverse, starting at the exit port of $\mathcal{N}$ and ending at the entry port of $\mathcal{N}$, with the Head and Hand pointing in opposite direction. If $\mathcal{N}$ satisfies the inductive hypothesis, then the unfolding net induced by $\overleftarrow{\xi}$ also satisfies the inductive hypothesis.*

Lemma 5 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children E and J. There is a Hand-east, Head-first unfolding of A whose net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 6a: starting at A's entry port, the unfolding path moves Head-first on T_A, then proceeds Hand-first to recursively visit E; from E's exit face on B_A, it proceeds Head-first to recursively visit J; from J's exit face on T_A, it moves Hand-first to L_A and B_A and then to A's exit port. We now show that, when visited in this order and laid flat in the plane, the open faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by showing that $\mathcal{N}_A$ provides the appropriate entry and exit connection pieces. By Lemma 3, $\xrightarrow{e_A}$ is open, therefore $\mathcal{E}_A$ does not belong to A's inductive region (by definition). If $\xleftarrow{x_A}$ is closed (open), then $\mathcal{X}_A$ belongs (doesn't belong) to A's inductive region. This

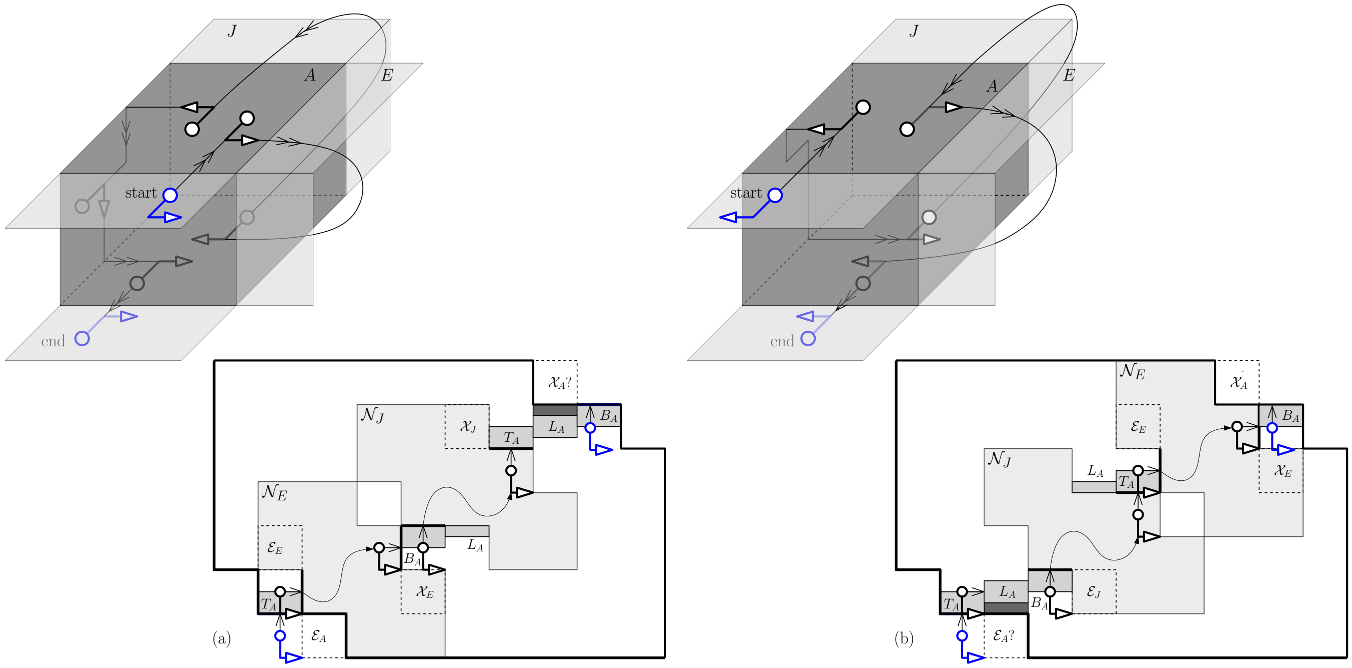

Figure 6: Box A with back and east children, HEAD-first unfolding (a) HAND pointing east (b) HAND pointing west.

dual case scenario is depicted by the cell labeled $\mathcal{X}_A$? in Figure 6a. Observe that $\mathcal{N}_A$ provides type-1 entry and exit connections since $e'_A \in T_A$ and $x'_A \in B_A$ are positioned alongside the entry and exit ports. Thus $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

Turning now to condition (I1) of the inductive hypothesis, we begin by showing that the unfolding $\mathcal{N}_A$ from Figure 6a is connected. Note that $\xrightarrow{e_E} \in K_A$ and $\xleftarrow{x_E} \in F_A$ are both closed, so $\mathcal{E}_E$ and $\mathcal{X}_E$ belong to E's inductive region (by definition). The inductive hypothesis applied to N_E tells us that $\mathcal{N}_E$ provides type-1 entry and exit connections, and thus it connects to the pieces of $e_E \in T_A$ and $x_E \in B_A$ placed along $\mathcal{N}_E$'s boundary at the entry and exit ports.

Next we show that the net $\mathcal{N}_J$ produced by a recursive unfolding of J connects to the pieces of B_A, L_A and T_A placed alongside its boundary. First note that $\xrightarrow{e_J} \in L_A$ is open and therefore $\mathcal{E}_J$ does not belong to J's inductive region. Because $\xrightarrow{e_J}$ is adjacent to $\mathcal{T}_J$, the inductive hypothesis applied to $\mathcal{N}_J$ tells us that $\mathcal{N}_J$ provides a type-1 or type-2 entry connection. If type-1, then it connects to the piece $e_J \in B_A$; if type-2, then it connects to $\xrightarrow{e_J} \in L_A$. Also note that $\xleftarrow{x_J} \in R_A$ is closed, so by definition $\mathcal{X}_J$ is inside J's inductive region. The inductive hypothesis applied to $\mathcal{N}_J$ tells us that $\mathcal{N}_J$ provides a type-1 exit connection, which connects to $x_J \in T_A$. It follows that the net $\mathcal{N}_A$ is connected.

By the inductive hypothesis, $\mathcal{N}_E$ covers all faces in $\mathcal{T}_E$ and $\mathcal{N}_J$ covers all faces in $\mathcal{T}_J$, both using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all points in T_A, L_A and B_A (which are A's open faces) using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ satisfies condition (I1) of the inductive hypothesis.

Turning to (I3) of the inductive hypothesis, observe that the only open ring face of A not used in A's entry or exit connections is part of L_A (this ring face is shown in dark gray in the unfolding in Figure 6a, left of the exit port). Its removal does not disconnect $\mathcal{N}_A$, and thus $\mathcal{N}_A$ satisfies condition (I3) of the inductive hypothesis. □

Theorem 6 *Any degree-3 node $A \in \mathcal{T}$ with children E and J can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. If the HAND points east, then by Lemma 5 there is an unfolding net $\mathcal{N}_A$ that satisfies the inductive hypothesis. If the HAND points west, the unfolding follows the same path but in reverse direction (compare Figure 6a and Figure 6b) This along with Lemma 4 implies that the unfolding net from Figure 6b satisfies the inductive hypothesis. □

A complete unfolding example is included in section 10 of the appendix.

6 Conclusion

This paper presents the first result on unfolding orthotrees of degree 3 or less with a 4×4 refinement. Our preliminary investigations show promise of this approach extending to arbitrary degree orthotrees. It is open whether all orthotrees can be grid-unfolded without any refinements.

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Appendix

7 Proofs of Preliminary Lemmas

Lemma 1 *Let $\mathcal{N}_b$ be the unfolding net produced by a recursive* Hand*-east (west),* Head*-first recursive unfolding of b. If $\mathcal{N}_b$ is rotated clockwise by $90°$ and then reflected vertically, then the result is a* Head*-east (west),* Hand*-first recursive unfolding of b.*

Proof. Note that, when applied to the L-guide, this combined ($90°$-rotation, vertical reflection) transformation switches the Head and the Hand positions, so a Head-first orientation at the beginning (end) of the unfolding becomes Hand-first. This implies that, when applied to the unfolded net, the same transformation turns a Hand-east (west), Head-first unfolding into a Head-east (west), Hand-first unfolding. □

Lemma 3 *Let $X \in \{E, W\}$ be a child of A. In a* Head*-first unfolding of A, if the* Hand *points in the direction of X (opposite to X), then the successor $\xrightarrow{e_A}$ (predecessor $\xleftarrow{x_A}$) of the entry (exit) ring face e_A (x_A) is open.*

Proof. Consider first the case where the Hand points east and E is a child of A. If T_I is open, then $e_A \in T_I$ and $\xrightarrow{e_A} \in R_I$; in this case $\xrightarrow{e_A}$ is necessarily open, otherwise $IE \in \mathcal{O}$ would close a cycle (I, A, E, IE), contradicting the fact that $\mathcal{O}$ is an orthotree. If T_I is not open, then $NI \in \mathcal{O}$, $e_A \in K_{NI}$ and $\xrightarrow{e_A} \in R_{NI}$. As noted earlier, $IE \notin O$. This along with the fact that $\mathcal{O}$ is a 2-manifold implies that R_{NI} is open (otherwise $NIE \in \mathcal{O}$ either meets E at an edge or closes a cycle (NIE,EN,E,A,I,NI)). It follows that $\xrightarrow{e_A}$ is open. Similar arguments hold for the other cases. □

Lemma 4 *Let ξ be the unfolding path and $\mathcal{N}$ the unfolding net produced by a recursive unfolding of a box in $\mathcal{T}$. Let $\overleftarrow{\xi}$ be the unfolding path traversed in reverse, starting at the exit port of $\mathcal{N}$ and ending at the entry port of $\mathcal{N}$, with the* Head *and* Hand *pointing in opposite direction. If $\mathcal{N}$ satisfies the inductive hypothesis, then the unfolding net induced by $\overleftarrow{\xi}$ also satisfies the inductive hypothesis.*

Proof. The unfolding net $\overleftarrow{\mathcal{N}}$ induced by $\overleftarrow{\xi}$ is a diagonal flip ($180°$-rotation) of $\mathcal{N}$. It can be verified that the inductive hypothesis is invariant under $180°$-rotations, therefore it holds for $\overleftarrow{\mathcal{N}}$ as well. □

8 Unfolding Degree-3 Nodes

In this section we discuss the unfoldings of cases 3.2 through 3.5 listed in Section 5.

Lemma 7 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children E and W. There is a* Hand*-east,* Head*-first unfolding of A whose net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 7a. The unfolding path is similar to the one from Figure 6a, with the only difference that the recursive unfolding of J in Figure 6a is replaced with a straight path across the back face of A in Figure 7a, and the straight path across the west face of A in Figure 6a is replaced with a recursive unfolding of W in Figure 7a. We now show that, when visited in this order and laid flat in the plane, the faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. By Lemma 3 $\xrightarrow{e_A}$ and $\xleftarrow{x_A}$ are both open, therefore $\mathcal{E}_A$ and $\mathcal{X}_A$ are outside A's inductive region. Observe that $\mathcal{N}_A$ provides type-1 entry and exit connections, since $e'_A \in T_A$ and $x'_A \in B_A$ are placed alongside the entry and exit ports. Thus $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

Next we turn to condition (I1) of the inductive hypothesis. We begin by showing that the unfolding $\mathcal{N}_A$ from Figure 7a is connected. First note that $\xrightarrow{e_E} \in K_A$ is open, therefore $\mathcal{E}_E$ is outside E's inductive region. Because $\xrightarrow{e_E}$ is open and adjacent to $\mathcal{T}_E$, the inductive hypothesis applied to $\mathcal{N}_E$ tells us that $\mathcal{N}_E$ provides a type-1 or type-2 entry connection: if a type-1 entry connection, then it connects to $e_E \in T_A$; if a type-2 entry connection, then it connects to $\xrightarrow{e_E} \in K_A$. Also note that $\xleftarrow{x_E} \in F_A$ is closed, therefore $\mathcal{X}_E$ is inside E's inductive region. The inductive hypothesis applied to $\mathcal{N}_E$ tells us that $\mathcal{N}_E$ provides a type-1 exit connection, which connects to $x_E \in B_A$.

Next we show that the net $\mathcal{N}_W$ produced by a recursive unfolding of W connects to the pieces T_A, B_A and K_A placed alongside its boundary. First note that $\xrightarrow{e_W} \in F_A$ is closed, therefore $\mathcal{E}_W$ is inside W's inductive region. The inductive hypothesis applied to $\mathcal{N}_W$ tells us that $\mathcal{N}_W$ provides a type-1 entry connection, which connects to $e_W \in T_A$. Also note that $\xleftarrow{x_W} \in K_A$ is open, therefore $\mathcal{X}_W$ is outside W's inductive region. Because $\xleftarrow{x_W}$ is open and adjacent to $\mathcal{T}_W$, the inductive hypothesis applied to $\mathcal{N}_W$ tells us that $\mathcal{N}_W$ provides a type-1 or type-2 exit connection: if a type-1 exit connection, then it connects to $x_W \in B_A$; if a type-2 exit connection, then it connects to $\xleftarrow{x_W} \in K_A$. It follows that the entire net $\mathcal{N}_A$ is connected.

By the inductive hypothesis $\mathcal{N}_E$ covers all faces in $\mathcal{T}_E$ and $\mathcal{N}_W$ covers all faces in $\mathcal{T}_W$, both using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all points in T_A, K_A and B_A (which are A's open faces) using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open

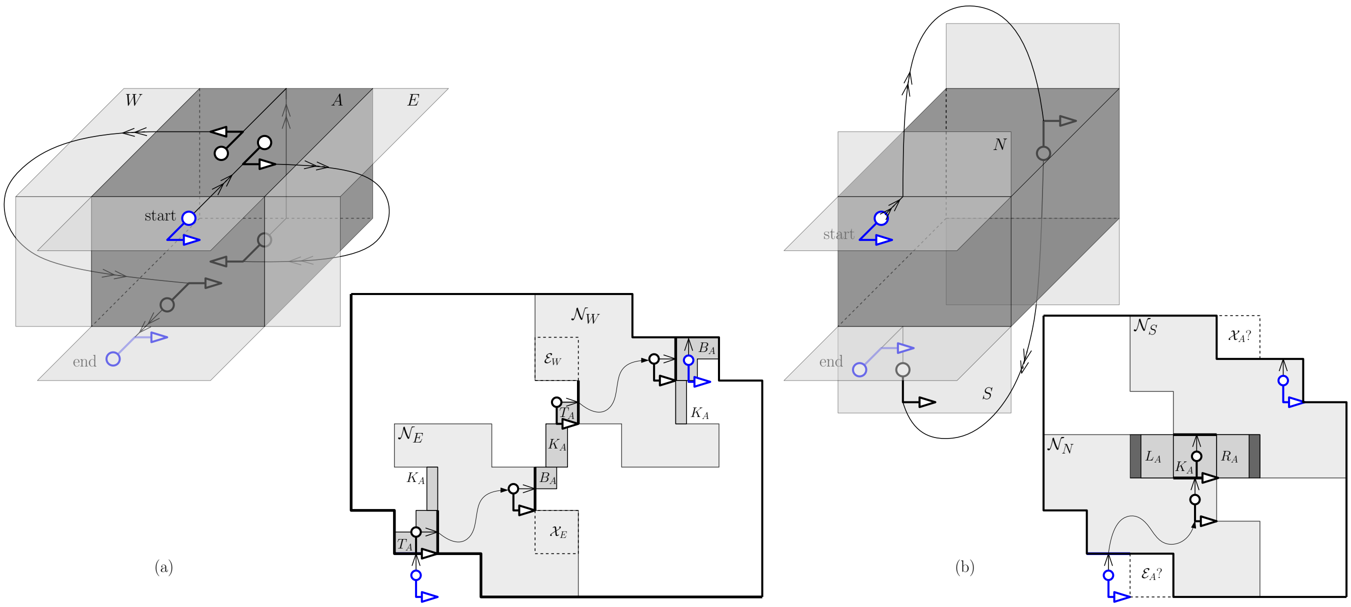

Figure 7: Box A of degree 3, Head-first unfolding (a) case when A has east and west children (b) case when A has north and south children.

faces of $\mathcal{T}_A$ and satisfies condition (I1) of the inductive hypothesis.

Condition (I3) of the inductive hypothesis is trivially satisfied, because only two ring faces of A are open, and they are both used in A's entry and exit connections. This concludes the proof. □

Theorem 8 *Any degree-3 box $A \in \mathcal{T}$ with children E and W can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to Head-first unfoldings of A. If the Hand points east, then by Lemma 7 there is an unfolding net $\mathcal{N}_A$ that satisfies the inductive hypothesis. Observe that the case when the Hand points west is symmetric, so arguments analogous to those in Lemma 7 show that its net also satisfies the inductive hypothesis. □

Lemma 9 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and S. There is a* Hand*-east,* Head*-first unfolding of A whose net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 7b: starting at the entry port on A, the unfolding path proceeds Head-first to recursively visit N, then crosses N's exit face K_A and proceeds Head-first to recursively visit S, ending at A's exit port. We now show that, when visited in this order and laid flat in the plane, the open faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. Note that the entry ports of A and N coincide, therefore the left boundaries of their inductive regions also coincide. If $\xrightarrow{e_A}=\xrightarrow{e_N}\in R_I$ is closed (open), then $\mathcal{E}_A = \mathcal{E}_N$ belongs (does not belong) to A and N's inductive region. This dual case scenario is depicted by the free cell labeled $\mathcal{E}_A$ in Figure 7b. Because $\xrightarrow{e_N}$ is not adjacent to $\mathcal{T}_N$, the inductive hypothesis tells us that N will provide a type-1 entry connection, which also serves as a type-1 entry connection for A since $e'_N = e'_A$. By analogous arguments, the right boundaries of A and S coincide, $\mathcal{X}_A = \mathcal{X}_S$ may or may not belong to their inductive regions, and S will provide a type-1 exit connection that also serves as a type-1 exit connection for A. Thus $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

Next we turn to condition (I1) of the inductive hypothesis. We begin by showing that the unfolding $\mathcal{N}_A$ from Figure 7b is connected. First note that $\xleftarrow{x_N}\in L_A$ is open, therefore $\mathcal{X}_N$ is outside N's inductive region. Because $\xleftarrow{x_N}$ is open and adjacent to $\mathcal{T}_N$, the inductive hypothesis applied to $\mathcal{N}_N$ tells us that $\mathcal{N}_N$ provides a type-1 or type-2 exit connection: if a type-1 exit connection, then it connects to $x_N \in K_A$; if a type-2 exit connection, then it connects to $\xleftarrow{x_N}\in L_A$.

Similarly, $\xrightarrow{e_S}\in R_A$ is open, therefore $\mathcal{E}_S$ is outside S's inductive region. Because $\xrightarrow{e_S}$ is open and adjacent to $\mathcal{T}_S$, the inductive hypothesis applied to $\mathcal{N}_S$ tells us that $\mathcal{N}_S$ provides a type-1 or type-2 entry connection: if a type-1 entry connection, then it connects to $e_S \in K_A$; if a type-2 entry connection, then it connects to $\xrightarrow{e_S}\in R_A$.

It follows that the entire net $\mathcal{N}_A$ is connected.

By the inductive hypothesis $\mathcal{N}_N$ covers all faces in $\mathcal{T}_N$ and $\mathcal{N}_S$ covers all faces in $\mathcal{T}_S$, both using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes L_A, K_A and R_A (which are A's open faces) using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies condition (I1) of the inductive hypothesis.

Finally we turn to condition (I3) of the inductive hypothesis. Note that the open ring faces of A not involved in A's entry and exit connections are the dark-shaded pieces of L_A and R_A from Figure 7b, whose removal does not disconnect $\mathcal{N}_A$. Thus $\mathcal{N}_A$ satisfies (I3) as well. $\square$

Theorem 10 *Any degree-3 node $A \in \mathcal{T}$ with children N and S can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. If the HAND points east, then by Lemma 9 there is an unfolding net $\mathcal{N}_A$ that satisfies the inductive hypothesis. Observe that the case when the HAND points west is symmetric, so arguments analogous to those in Lemma 9 show that its net also satisfies the inductive hypothesis. $\square$

Lemma 11 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and J. If R_I is open, there is a* HAND-*east,* HEAD-*first unfolding of A whose net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 8a: starting from the entry face T_I, the unfolding path moves HAND-first to R_I, HEAD-first to R_A, then it cycles HAND-first around A to L_A; from there it proceeds HEAD-first to recursively visit J and then HAND-first to recursively visit N; from N's exit face L_A it moves HAND-first to B_A and then to A's exit port. We now show that, when visited in this order and laid flat in the plane, the open faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. By the lemma statement $\xrightarrow{e_A} \in R_I$ is open, therefore $\mathcal{E}_A$ is outside the inductive region for A. Because $\xrightarrow{e_A} \in R_I$ is adjacent to T_A, the inductive hypothesis allows $\mathcal{N}_A$ to provide a type-2 entry connection, which it does in the form of $\xrightarrow{e'_A} \in R_A$ placed right of $\mathcal{N}_A$'s entry port. If $\xleftarrow{x_A} \in L_I$ is open (closed), then $\mathcal{X}_A$ is outside (inside) the inductive region for A. This dual case scenario is depicted by the free cell labeled $\mathcal{X}_A$ in Figure 8a. Note that $\mathcal{N}_A$ provides a type-1 exit connection, because $x'_A \in B_A$ is placed alongside its exit port. Thus $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

Next we turn to condition (I1) of the inductive hypothesis. We begin by showing that the unfolding $\mathcal{N}_A$ from Figure 8a is connected. First note that $\xrightarrow{e_J} \in T_A$ is closed, therefore $\mathcal{E}_J$ belongs to the inductive region for J. The inductive hypothesis applied to $\mathcal{N}_J$ tells us that $\mathcal{N}_J$ provides a type-1 entry connection, which connects to $e_J \in L_A$. Also note that $\xleftarrow{x_J} \in B_A$ is open and adjacent to T_J. In this case the inductive hypothesis applied to $\mathcal{N}_J$ tells us that $\mathcal{N}_J$ provides a type-1 or type-2 exit connection: if type-1, it connects to $x_J \in R_A$; if type-2, it connects to $\xleftarrow{x_J} \in B_A$.

Next we show that the net $\mathcal{N}_N$ produced by a recursive unfolding of N connects to the pieces of R_A and L_A placed alongside its boundary. Since $\xrightarrow{e_N} \in F_A$ and $\xleftarrow{x_N} \in K_A$ are closed, $\mathcal{E}_N$ and $\mathcal{X}_N$ are inside the inductive region for N. The inductive hypothesis applied to N tells us that $\mathcal{N}_N$ provides type-1 entry and exit connections, therefore it connects to the pieces of $e_N \in R_A$ and $x_N \in L_A$ placed alongside $\mathcal{N}_N$'s entry and exit ports. It follows that the entire net $\mathcal{N}_A$ is connected.

By the inductive hypothesis $\mathcal{N}_J$ covers all faces in $\mathcal{T}_J$ and $\mathcal{N}_S$ covers all faces in $\mathcal{T}_S$, both using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all points in L_A, B_A and R_A (which are A's open faces) using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies condition (I1) of the inductive hypothesis.

Finally we turn to condition (I3) of the inductive hypothesis. Note that the only open ring face of A not involved in A's entry and exit connections is the dark-shaded piece of L_A from Figure 8a, whose removal does not disconnect $\mathcal{N}_A$. Thus $\mathcal{N}_A$ satisfied (I3) as well. $\square$

Lemma 12 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and J. If R_I is closed, there is a* HAND-*east,* HEAD-*first unfolding of A whose net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 8b: starting at the entry port on A, the unfolding path moves HEAD-first to F_N and cycles HAND-first around N to K_N; it then proceeds HEAD-first to recursively visit NN, HAND-first to recursively visit NW, and then HEAD-first to recursively visit J; from J's exit face B_J it moves HEAD-first to A's exit port. We now show that, when visited in this order and laid flat in the plane, the open faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. By the lemma statement $\xrightarrow{e_A} \in R_I$ is closed, therefore $\mathcal{E}_A$ is inside the inductive region for A. If $\xleftarrow{x_A} \in L_I$ is closed (open), then $\mathcal{X}_A$ is inside (outside) the inductive region for A. Note that $\mathcal{N}_A$ provides a type-1 entry and exit connection, because $e'_A \in F_N$ and $x'_A \in B_A$ are placed alongside its entry and exit ports. Thus $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

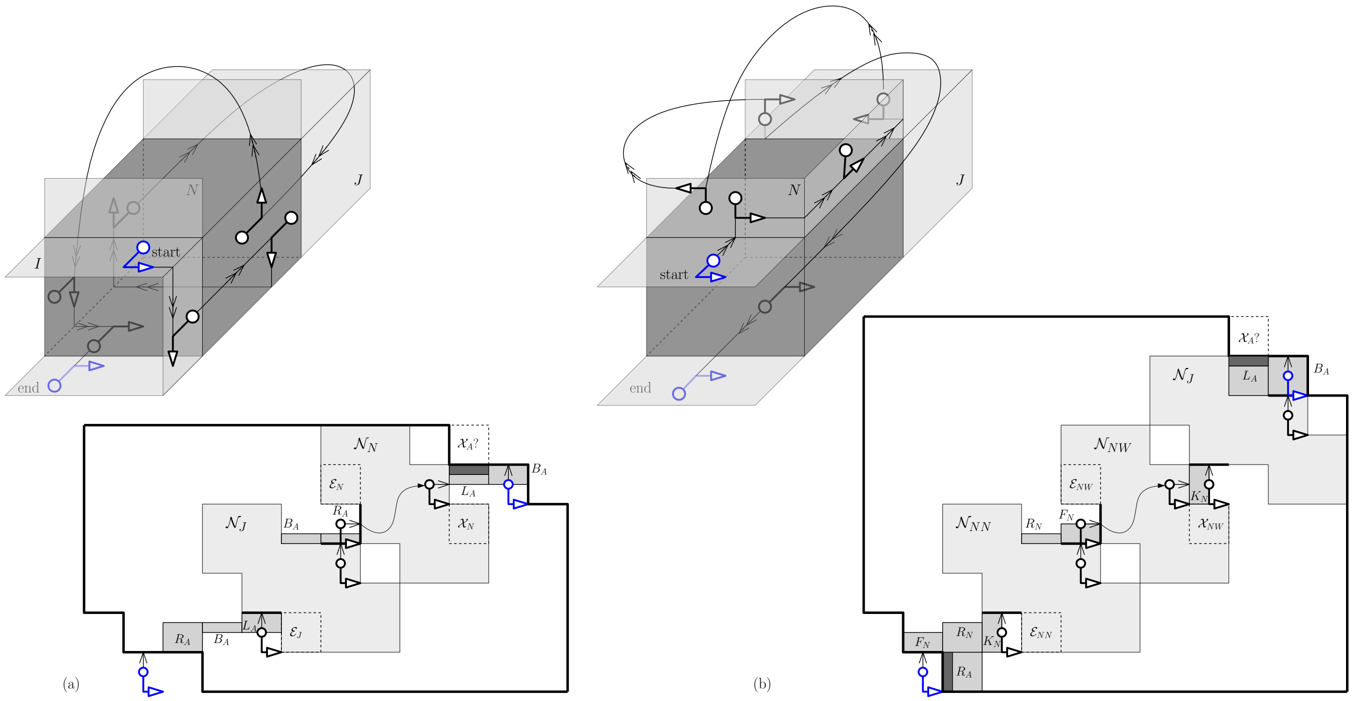

Figure 8: Box A of degree 3 with north and back children, HEAD-first unfolding (a) right face of I open (b) right face of I closed (so right face of N open).

Next we turn to condition (I1) of the inductive hypothesis. We begin by showing that the unfolding $\mathcal{N}_A$ from Figure 8b is connected. Here we assume the subtrees rooted at NN and NW are non-degenerate and thus consist of at least one box. (Handling cases in which one or both are degenerate requires only minor modifications.) By the lemma statement $\xrightarrow{e_{NN}}\in L_N$ is closed (and so $\mathcal{E}_{NN}$ is inside the inductive region for NN). The inductive hypothesis applied to $\mathcal{N}_{NN}$ tells us that $\mathcal{N}_{NN}$ provides a type-1 entry connection, which connects to $e_{NN} \in K_N$ placed alongside its entry port. Also note that $\xleftarrow{x_{NN}}\in R_N$ is open and adjacent to $\mathcal{T}_{NN}$ (and so $\mathcal{X}_{NN}$ is outside the inductive region for NN). In this case the inductive hypothesis applied to $\mathcal{N}_{NN}$ tells us that $\mathcal{N}_{NN}$ provides a type-1 or type-2 exit connection: if type-1, it connects to $x_{NN} \in F_N$; if type-2, it connects to $\xleftarrow{x_{NN}}\in R_N$.

Next we show that the net $\mathcal{N}_{NW}$ produced by a recursive unfolding of NW forms a connected component with the pieces of F_N and K_N placed alongside its boundary. Since $\xrightarrow{e_{NW}}\in B_N$ is closed, $\mathcal{E}_{NW}$ is inside the inductive region for NW. The inductive hypothesis applied to NW tells us that $\mathcal{N}_{NW}$ provides a type-1 entry connection, which connects to the piece of $e_{NW} \in F_N$ placed alongside its entry port. By the lemma statement $\xleftarrow{x_{NW}}\in T_N$ is closed (and so $\mathcal{X}_{NW}$ is inside the inductive region for NW). In this case the inductive hypothesis applied to $\mathcal{N}_{NW}$ tells us that $\mathcal{N}_{NW}$ provides a type-1 exit connection, which connects to $x_{NW} \in K_N$ placed alongside its exit port.

Next we show that the net $\mathcal{N}_J$ produced by a recursive unfolding of J connects to pieces of K_N, B_A, and L_A placed alongside its boundary. Note that $\xrightarrow{e_J}\in R_N$ is open (and so $\mathcal{E}_J$ is outside the inductive region for J) but not adjacent to T_J. The inductive hypothesis applied to J tells us that $\mathcal{N}_J$ provides a type-1 entry connection, which connects to $e_J \in K_N$ placed alongside its entry port. Also note that $\xleftarrow{x_J}\in L_A$ is open (and so $\mathcal{X}_J$ is outside J's inductive region) and adjacent to $\mathcal{T}_J$. The inductive hypothesis applied to $\mathcal{N}_J$ tells us that $\mathcal{N}_J$ provides a type-1 or type-2 exit connection: if type-1, it connects to $x_J \in B_A$; if type-2, it connects to $\xleftarrow{x_J}\in L_A$. It follows that the entire net $\mathcal{N}_A$ is connected.

By the inductive hypothesis $\mathcal{N}_{NN}$ covers all faces in $\mathcal{T}_{NN}$, $\mathcal{N}_{NW}$ covers all faces in $\mathcal{T}_{NW}$, and $\mathcal{N}_J$ covers all faces in $\mathcal{T}_J$, all using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all points in F_N, R_N and K_N (which are N's open faces) and in L_A, B_A and R_A (which are A's open faces), also using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies condition (I1) of the inductive hypothesis.

Finally we turn to condition (I3) of the inductive hypothesis. Note that the only open ring faces of A that are not involved in A's entry and exit connections are the dark-shaded piece of R_A and L_A from Figure 8b, whose removal does not disconnect $\mathcal{N}_A$. Thus $\mathcal{N}_A$ satisfied (I3) as well. □

Theorem 13 *Any degree-3 node $A \in \mathcal{T}$ with parent I*

and children N and J can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. If the HAND points east and R_I is open, then by Lemma 11 there is an unfolding net $\mathcal{N}_A$ that satisfies the inductive hypothesis; if R_I is closed, then by Lemma 12 there is an unfolding net $\mathcal{N}_A$ that satisfies the inductive hypothesis. Observe that the case when the HAND points west is symmetric, so arguments analogous to those in Lemma 11 (when L_I is open) and Lemma 12 (when L_I is closed) show that the theorem holds for this case as well. ☐

Lemma 14 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and E. If K_N is open, there is a* HAND*-east,* HEAD*-first unfolding of A whose output net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 9a: starting at the entry port on A, the unfolding path proceeds HEAD-first to recursively visit N; from N's exit port on K_N it moves HAND-first to R_N, then proceeds HEAD-first to recursively visit E; from E's exit face B_A it moves in the direction opposite the HAND to K_A, then HEAD-first to L_A, HAND-first B_A and then to A's exit port. We now show that, when visited in this order and laid flat in the plane, the open faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. Observe that the entry ports of A and N coincide, and thus the left boundaries of their inductive regions also coincide. By Lemma 3, $\xrightarrow{e_N} = \xrightarrow{e_A} \in R_I$ is open, therefore $\mathcal{E}_N = \mathcal{E}_A$ does not belong to the inductive region of N and A. Because $\xrightarrow{e_N}$ is not adjacent to T_N, the inductive hypothesis tells us that N will provide a type-1 entry connection, which is also a type-1 entry connection for A (since $e'_N = e'_A$). Now consider the right side of A's inductive region. Observe that $\xleftarrow{x_A}$ may or may not be open, therefore $\mathcal{X}_A$ may or may not belong to the inductive region of A, as indicated by the free cell in Figure 9a. As shown in Figure 9a, A places ring face $x'_A \in B_A$ adjacent to its exit port, and thus provides a type-1 exit connection. Therefore, $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

We now show that the net $\mathcal{N}_A$ is itself connected. First note that $\mathcal{X}_N$ is not part of N's inductive region because $\xleftarrow{x_N} \in L_A$ is open. Because $\xleftarrow{x_N}$ is adjacent to $\mathcal{T}_N$, it seems at first that N could provide a type-1 or type-2 exit connection. However, we can argue in this situation N must provide a type-1 exit connection. Specifically, if N is of degree 1 or 2, then it has a type-1 exit connection because all degree 1 and 2 unfoldings do [1]. If N is of degree 3, then it must be Case 3.1. To see this, orient N in standard position[2] and observe that it can only have a west and back child; its other faces are either open or attached to its parent. Observe that all Case 3.1 unfoldings have type-1 exit connections. This implies that N has a type-1 exit connection which tells us that $x'_N \in K_N$ is positioned as shown under N's exit port. This attaches to the piece of R_N (taken from N) on the right. By (I3) of the inductive hypothesis applied to N, it is safe to remove this piece of R_N from $\mathcal{N}_N$ without disconnecting $\mathcal{N}_N$.

Next we show that the net $\mathcal{N}_E$ connects to R_N, K_A, and B_A placed around its boundary. First note that $\xrightarrow{e_E} \in F_N$ and $\xleftarrow{x_E} \in K_A$ are open, so $\mathcal{E}_E$ and $\mathcal{X}_E$ are not part of E's inductive region. Because $\xrightarrow{e_E}$ is not adjacent to $\mathcal{T}_E$, E has a type-1 entry connection, and therefore $e_E \in R_N$ placed along E's entry port is sufficient to make the connection to $\mathcal{N}_E$. Now consider $\xleftarrow{x_E}$. It is adjacent to $\mathcal{T}_E$, so E may have a type-1 or a type-2 exit connection. Therefore to ensure $\mathcal{N}_E$ is connected to the rest of the unfolding, both $x_E \in B_A$ and $\xleftarrow{x_E} \in K_A$ are placed as shown alongside E's inductive region. Thus we have shown that N_A is connected.

By the inductive hypothesis, $\mathcal{N}_N$ covers all faces in $\mathcal{T}_N$ (except for the piece of R_N used by A) and $\mathcal{N}_E$ covers all faces in $\mathcal{T}_E$, both using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all pieces of L_A, K_A, and B_A (which are A's open faces), also using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies (I1) of the inductive hypothesis.

For I3, observe that the only open ring face of A not used in A's entry or exit connections is part of L_A, shown in dark gray in Figure 9a. Its removal does not disconnect $\mathcal{N}_A$ so (I3) is satisfied. ☐

Lemma 15 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and E. If K_N is closed, there is a* HAND*-east,* HEAD*-first unfolding of A whose output net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. First note that K_N closed implies that K_E is open, because otherwise box EJ exists and shares an edge with NJ and so either box AJ or EJN exists (because the orthotree is a manifold), which implies a cycle in the orthotree. The unfolding for this case is depicted in Figure 9b: from the entry ring face on T_I, the unfolding path moves HAND-first to R_I, HEAD-first to E, then proceeds HAND-first to recursively visit ES, then HEAD-first to recursively visit EE; from EE's exit face F_E it moves HAND-first to T_E, then proceeds HEAD-first to recursively visit N; from N's exit face L_A it moves HAND-first to K_A, then HEAD-first to B_A, and finally to A's exit port. We now show that, when visited in this

[1] check this

[2] defined in Section 1: entry and exit ports are the top and bottom edges of the front face.

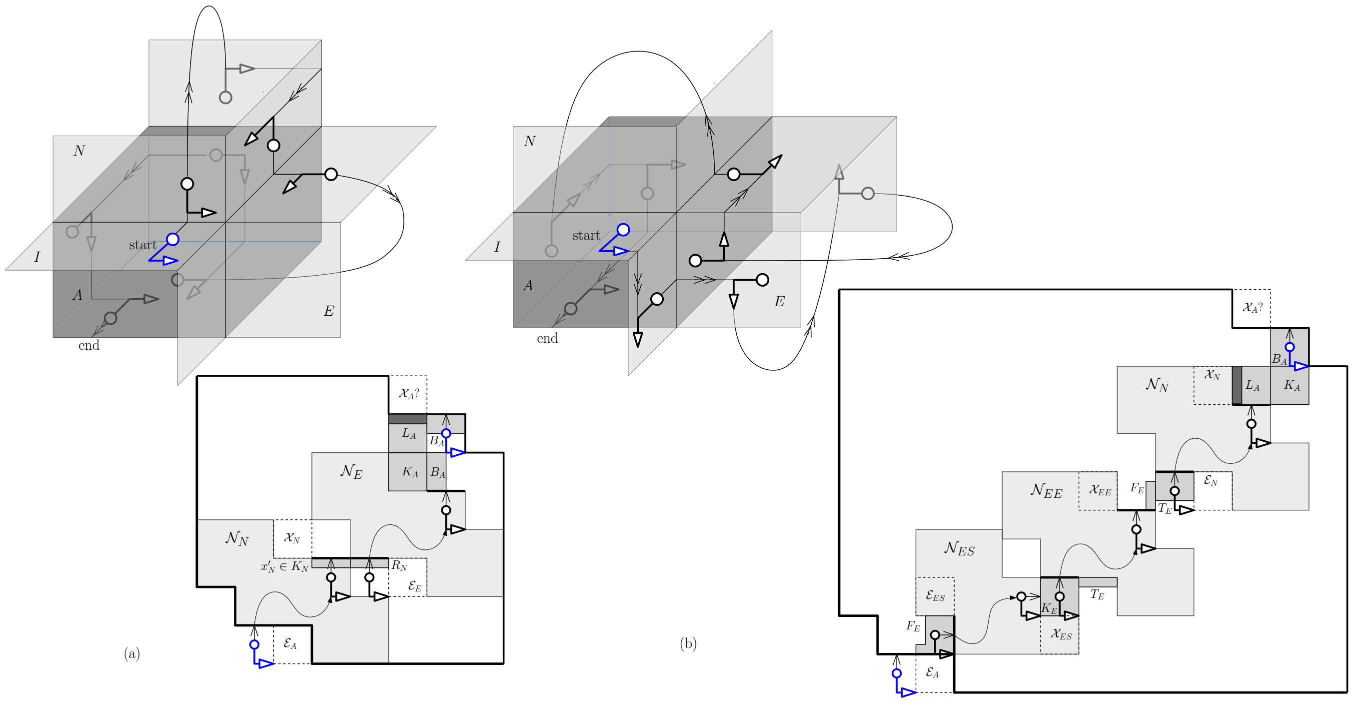

Figure 9: Box A of degree 3 with north and east children, HEAD-first unfolding, HAND pointing east (a) K_N open (b) K_N closed (and so K_E open).

order and laid flat in the plane, the open faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. By Lemma 3, $\xrightarrow{e_A}\in R_I$ is open, therefore $\mathcal{E}_A$ does not belong to A's inductive region. Because $\xrightarrow{e_A}$ is adjacent to T_A, the inductive hypothesis tells us that $\mathcal{N}_A$ can provide a type-2 entry connection, which it does in the form of $\xrightarrow{e'_A}\in F_E$ placed just to the right of the entry port. Now consider the right side of A's inductive region. Observe that $\xleftarrow{x_A}$ may or may not be open, therefore $\mathcal{X}_A$ may or may not belong to the inductive region of A, as indicated by the free cell in Figure 9b. Notice that A places the ring face $x'_A \in B_A$ adjacent to its exit port, and thus provides a type-1 exit connection. Therefore, $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

We now show that the net $\mathcal{N}_A$ is itself connected. We will assume that the subtrees rooted at ES and EE are non-degenerate and thus consist of at least one box. (Handling cases in which one or both are degenerate requires only minor modifications.) First note that $\mathcal{E}_{ES}$ and $\mathcal{X}_{ES}$ are both part of ES's inductive region because $\xrightarrow{e_{ES}}\in R_E$ and $\xleftarrow{x_{ES}}\in L_E$ are closed. The inductive hypothesis applied to ES tells us that $\mathcal{N}_{ES}$ provides type-1 entry and exit connections, which connect to pieces of $e_{ES} \in F_E$ and $x_{ES} \in K_E$.

Next we show that the net $\mathcal{N}_{EE}$ connects to K_E, T_E, and F_E placed around its boundary. First note that $\xrightarrow{e_{EE}}\in T_E$ is open, so $\mathcal{E}_{EE}$ is not part of EE's inductive region. Because $\xrightarrow{e_{EE}}$ is adjacent to $\mathcal{T}_{EE}$, the inductive hypothsis tells us that EE provides either a type-1 or type-2 entry connection, which attaches to $e_{EE} \in K_E$ (if type-1) or $\xrightarrow{e_{EE}}\in T_E$ (if type-2). Because $\xleftarrow{x_{EE}}\in B_E$ is closed, $\mathcal{X}_{EE}$ is part of EE's inductive region. The inductive hypothsis tells us that EE provides a type-1 exit connection, which attaches to the piece of $x_{EE} \in F_E$.

Next we show that the net $\mathcal{N}_N$ connects to T_E and L_A placed around its boundary. First note that $\xrightarrow{e_N}\in K_E$ is open, so $\mathcal{E}_N$ is not part of N's inductive region. Because $\xrightarrow{e_N}$ is not adjacent to $\mathcal{T}_N$, the inductive hypothsis tells us that $\mathcal{N}_N$ provides a type-1 entry connection, which attaches to $e_N \in T_E$. Because $\xleftarrow{x_N}\in F_A$ is closed, $\mathcal{X}_N$ is part of N's inductive region. The inductive hypothsis tells us that $\mathcal{N}_N$ provides a type-1 exit connection, which attaches to $x_N \in L_A$. Thus we have shown that N_A is connected.

By the inductive hypothesis, $\mathcal{N}_{ES}$, $\mathcal{N}_{EE}$, and $\mathcal{N}_N$ cover all faces in $\mathcal{T}_{ES}$, $\mathcal{T}_{EE}$ and $\mathcal{T}_N$, all using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all pieces of L_A, K_A, B_A, F_E, T_E, and K_E (which are A's and E's open faces), also using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies (I1) of the inductive hypothesis.

For condition (I3) of the inductive hypothesis, observe that the only open ring face of A not used in A's entry or exit connections is part of L_A, shown in dark gray in Figure 9b. Its removal does not disconnect $\mathcal{N}_A$ so (I3) is satisfied. □

Lemma 16 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and E. If B_I is open, then there is a* HAND*-west,* HEAD*-first unfolding of A whose output $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 10a: from the entry port on A, the unfolding path proceeds HEAD-first to recursively visit N; from N's exit face K_A it moves HAND-first to L_A, HEAD-first to B_A, in the direction opposite the HAND to K_A, then proceeds HEAD-first to recursively visit E; from E's exit port it moves HEAD-first to R_I and then HAND-first to the exit face B_I. We now show that, when visited in this order and laid flat in the plane, the open faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. Observe that the entry ports of A and N coincide, and thus the left boundaries of their inductive regions also coincide. Also note that $\xrightarrow{e_N} = \xrightarrow{e_A} \in L_I$ may be open or closed, therefore $\mathcal{E}_N = \mathcal{E}_A$ may or may not belong to the inductive region of N and A, as indicated by the free cell in Figure 10a. Because $\xrightarrow{e_N}$ is not adjacent to T_N, the inductive hypothesis applied to N tells us that N will provide a type-1 entry connection $e_N \in F_N$, which is also a type-1 entry connection for A (since $e'_N = e'_A$).

Now consider the right side of A's inductive region. Observe that $\xleftarrow{x_A} \in R_I$ is open, therefore $\mathcal{X}_A$ does not belong to the inductive region of A. Because $\xleftarrow{x_A}$ is adjacent to $\mathcal{T}_A$, A can provide either a type-1 or type-2 exit connection, and in this case A provides a type-2 connection. To see this, observe that the right boundaries of E's and A's inductive regions overlap along E's exit port, which is also where A would place a type-2 connection piece. Because $\xleftarrow{x_E} \in T_I$ is open but not adjacent to $\mathcal{T}_E$, the inductive hypothesis tells us that E provides a type-1 exit connection $x'_E \in F_E$ placed under its exit port. Because $x'_E = \xleftarrow{x'_A}$, this piece also serves as a type-2 connection for A. Therefore, $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

We now show that the net $\mathcal{N}_A$ is itself connected. First note that $\mathcal{X}_N$ is part of N's inductive region because $\xleftarrow{x_N} \in R_A$ is closed. The inductive hypothesis applied to N tells us that N provides a type-1 exit connection, which attaches to the piece $x_N \in K_A$.

Next we show that the net $\mathcal{N}_E$ connects to K_A and B_A placed along its left boundary. First note that $\xrightarrow{e_E} \in B_A$ is open, so $\mathcal{E}_E$ is not part of E's inductive region. Because $\xrightarrow{e_E}$ is adjacent to $\mathcal{T}_E$, $\mathcal{N}_E$ may have a type-1 or type-2 entry connection which will connect to $e_E \in K_A$ (if type-1) or $\xrightarrow{e_E} \in B_A$ (if type-2). Thus we have shown that N_A is connected.

By the inductive hypothesis, $\mathcal{N}_N$ covers all faces in $\mathcal{T}_N$ and $\mathcal{N}_E$ covers all faces in $\mathcal{T}_E$, both using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all pieces of L_A, K_A, and B_A (which are A's open faces), also using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies (I1) of the inductive hypothesis.

For (I3), observe that the only open ring face of A not used in A's entry or exit connections is part of L_A, shown in dark gray in Figure 10a. Its removal does not disconnect $\mathcal{N}_A$ so (I3) is satisfied. □

Lemma 17 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and E. If T_N is open, then there is a* HAND*-west,* HEAD*-first unfolding of A whose output $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 10b: from the entry port on A, the unfolding path moves HEAD-first across F_N to T_N and then proceeds HAND-first to recursively visit NW; from NW's exit face L_A it moves HAND-first to B_A and HEAD-first to K_A, then proceeds HEAD-first to recursively visit NJ; from NJ's exit face T_N it moves HAND-first to R_N and then proceeds HAND-first to recursively visit E; and from E's exit face it reaches directly A's exit port. We now show that, when visited in this order and laid flat in the plane, the faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

We start by considering the left/right boundaries of A's inductive region and show that A provides the appropriate entry and exit connection pieces. Because $\xrightarrow{e_A} \in L_I$ may be open or closed, $\mathcal{E}_A$ may or may not belong to A's inductive region, as indicated by the free cell in Figure 10b. Because $\xleftarrow{x_A} \in R_I$ is open, $\mathcal{E}_A$ does not belong to A's inductive region. Observing that A provides type-1 entry and exit connections $e'_A \in F_N$ and $x'_A \in B_A$, we conclude that $\mathcal{N}_A$ satisfies condition (I2) of the inductive hypothesis.

We now show that $\mathcal{N}_A$ is itself connected. We will assume that the subtrees rooted at NW and NJ are non-degenerate and thus consist of at least one box. (Handling cases in which one or both are degenerate requires only minor modifications.) First note that $\mathcal{E}_{NW}$ and $\mathcal{X}_{NW}$ are both part of NW's inductive region because $\xrightarrow{e_{NW}} \in K_N$ and $\xleftarrow{x_{NW}} \in F_A$ are closed. The inductive hypothesis applied to $\mathcal{N}_{NW}$ tells us that it provides type-1 entry and exit connections, which connect to pieces $e_{NW} \in T_N$ and $x_{NW} \in L_A$ placed alongside its entry and exit ports.

Next we show that $\mathcal{N}_{NJ}$ connects to the pieces of K_A and T_N placed along its boundary. First note that $\xrightarrow{e_{NJ}} \in R_A$ and $\xleftarrow{x_{NJ}} \in L_N$ are closed, so $\mathcal{E}_{NJ}$ and $\mathcal{X}_{NJ}$ are part of NJ's inductive region. By the inductive

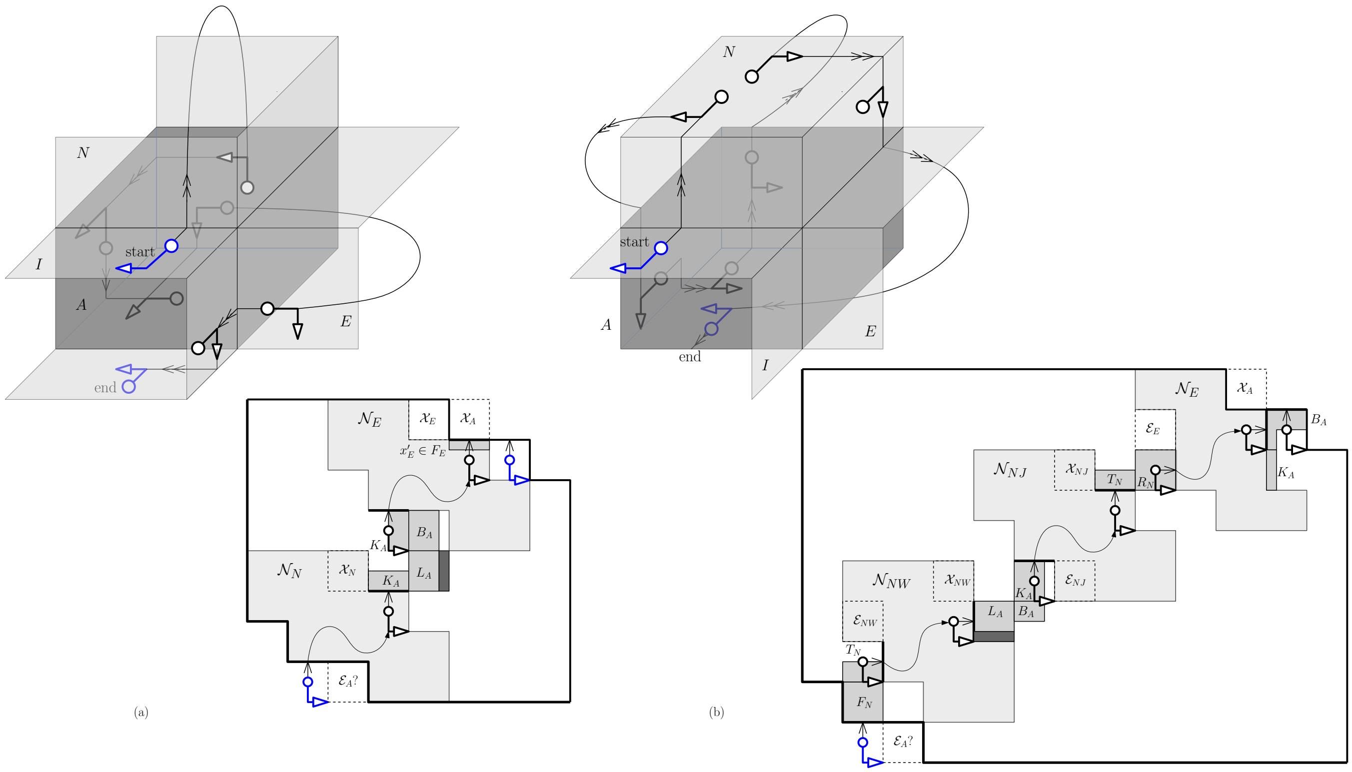

Figure 10: Box A of degree 3 with north and east children, HEAD-first unfolding, HAND pointing west. (a) B_I open (b) T_N open.

hypothesis, $\mathcal{N}_{NJ}$ provides type-1 entry and exit connections, which connect to $e_{NJ} \in K_A$ and $x_{NJ} \in T_N$.

Next we show that $\mathcal{N}_E$ connects to the pieces of R_N, B_A, and K_A placed along its boundary. First note that $\xrightarrow{e_E}\in F_N$ and $\xleftarrow{x_N}\in K_A$ are open, so $\mathcal{E}_E$ and $\mathcal{X}_E$ are not part of E's inductive region. Because $\xrightarrow{e_E}$ is not adjacent to $\mathcal{T}_E$, the inductive hypothesis tells us that E provides a type-1 entry connection, which connects to $e_E \in R_N$. Because $\xleftarrow{x_E}$ is adjacent to $\mathcal{T}_E$, the inductive hypothesis tells us that E provides a type-1 or type 2 exit connection: if type-1 it connects to $x_E \in B_A$ and if type-2 it connects to $\xleftarrow{x_E}\in K_A$. Thus we have shown that N_A is connected.

By the inductive hypothesis, $\mathcal{N}_{NW}$, $\mathcal{N}_{NJ}$, and $\mathcal{N}_E$ cover all faces in $\mathcal{T}_{NW}$, $\mathcal{T}_{NJ}$ and $\mathcal{T}_E$, all using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all pieces of L_A, K_A, B_A, F_N, T_N, and R_N (which are A's and N's open faces), also using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies (I1) of the inductive hypothesis.

For condition (I3) of the inductive hypothesis, observe that the only open ring face of A not used in A's entry or exit connections is part of L_A, shown in dark gray in Figure 10b. Its removal does not disconnect $\mathcal{N}_A$ so (I3) is satisfied. □

Lemma 18 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and E. If T_N and B_I are both closed and K_N is open, there is a* HAND*-west,* HEAD*-first unfolding of A whose output $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. One such unfolding is depicted in Figure 11a: from the entry port on A, the unfolding path moves HEAD-first to F_N, then proceeds HAND-first to recursively visit NW, then HEAD-first to recursively visit NN; from NN's exit face F_N it moves HAND-first across R_N to K_N, HEAD-first to K_A, HAND-first to L_A, HEAD-first to B_A, in the direction opposite the HAND across K_A to K_N, HEAD-first to R_N, HAND-first to T_E, then HEAD-first across F_E to B_E; it then proceeds HAND-first to recursively visit EE, then HEAD-first to recursively visit EJ; finally, from EJ's exit face B_E, it moves across B_A to the exit port. We now show that, when visited in this order and laid flat in the plane, the faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

Arguments identical to the ones used in the proof of Lemma 17 show that A provides the appropriate entry and exit connection pieces. We now show that the net $\mathcal{N}_A$ is itself connected. We will assume that NW is not degenerate and therefore it consists of at least one box. (Handling the case where NW is degenerate requires only minor modifications.) First note that $\xrightarrow{e_{NW}}\in T_N$

and $\xleftarrow{x_{NW}} \in B_N$ are closed, so $\mathcal{E}_{NW}$ and $\mathcal{X}_{NW}$ are part of NW's inductive region. The inductive hypothesis tells us that NW provides type-1 entry and exit connections, which connect to pieces of $e_{NW} \in F_N$ (along the entry port) and $x_{NW} \in K_N$ (along the exit port).

Next we show that the net $\mathcal{N}_{NN}$ connects to the pieces of K_E, R_N, and F_N placed along its boundary. First note that $\xrightarrow{e_{NN}} \in R_N$ is open and adjacent to $\mathcal{T}_{NN}$, so $\mathcal{E}_{NN}$ is not part of NN's inductive region. The inductive hypothesis applied to NN tells us that $\mathcal{N}_{NN}$ provides either a type-1 or type-2 entry connection. If type-1, it connects to $e_{NN} \in K_N$ and if type-2, it connects to $\xrightarrow{e_{NN}} \in R_N$. Because $\xleftarrow{x_{NN}} \in L_N$ is closed, $\mathcal{X}_{NN}$ is part of NN's inductive region. The inductive hypothesis tells us that NN provides a type-1 exit connection, which attaches to $x_{NN} \in F_N$.

We now consider $\mathcal{N}_{EE}$ and $\mathcal{N}_{EJ}$ and show they connect to the pieces of B_E, T_E, and F_E placed along their boundaries. We will assume that their subtrees are non-degenerate and thus consist of at least one box. (Handling cases in which one or both are degenerate require only minor modifications.) First note that $\mathcal{E}_{EE}$ and $\mathcal{E}_{EJ}$ are part of their respective inductive regions because $\xrightarrow{e_{EE}} \in K_E$ and $\xrightarrow{e_{EJ}} \in L_E$ are closed. Similarly, $\mathcal{X}_{EJ}$ is part of EJ's inductive region because $\xleftarrow{x_{EJ}} \in R_E$ is closed. The inductive hypothesis applied to these two nets tells us that $\mathcal{N}_{EE}$ has a type-1 entry connection (which attaches to $e_{EE} \in B_E$), and $\mathcal{N}_{EJ}$ has a type-1 entry and exit connection (which attach to $e_{EJ} \in T_E$ and $x_{EJ} \in B_E$). Also note that $\mathcal{X}_{EE}$ is not part of EE's inductive region because $\xleftarrow{x_{EE}} \in F_E$ is open. The inductive hypothesis applied to $\mathcal{N}_{EE}$ tells us that it provides a type-1 or type-2 exit connection, which connects $x_{EE} \in T_E$ (if type-1) or $\xleftarrow{x_{EE}} \in F_E$ (if type-2). Thus we have shown that N_A is connected.

By the inductive hypothesis, $\mathcal{N}_{NW}$, $\mathcal{N}_{NN}$, $\mathcal{N}_{EE}$, and $\mathcal{N}_{EJ}$ cover all faces in $\mathcal{T}_{NW}$, $\mathcal{T}_{NN}$, $\mathcal{T}_{EE}$ and $\mathcal{T}_{EJ}$, all using a 4×4 refinement. Observe that $\mathcal{N}_A$ includes all open faces of A, N, and E, also using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies (I1) of the inductive hypothesis.

For condition (I3) of the inductive hypothesis, observe that the only open ring face of A not used in A's entry or exit connections is part of L_A, shown in dark gray in Figure 11a. Its removal does not disconnect $\mathcal{N}_A$ so (I3) is satisfied. □

Lemma 19 *Let $A \in \mathcal{T}$ be a degree-3 node with parent I and children N and E. If T_N, B_I and K_N are all closed, then there is a* Hand*-west,* Head*-first unfolding of A whose output $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. First note that T_{NJ} is open, because otherwise boxes $\{NJN, NN, N, NJ\}$ form a cycle. Also note that R_{NJ} is open, because otherwise box NJE exists and is edge-adjacent to E; this implies that box EJ must also exist (to ensure that the orthotree is a manifold), but this creates the cycle $\{NJ, NJE, EJ, E, A, N\}$.

An unfolding that satisfies the conditions of the lemma is depicted in Figure 11b: from A's entry port the unfolding path moves to F_N and proceeds Hand-first to recursively visit NW, then Hand-first to recursively visit NJ; from R_{NJ} it moves Head-first to T_{NJ} and proceeds Hand-first to recursively visit NN; from NN's exit face F_N it moves in the direction opposite the Head to R_N and then proceeds Hand-first to recursively visit E; and from E's exit face B_A it reaches A's exit port. We now show that, when visited in this order and laid flat in the plane, the faces in $\mathcal{T}_A$ form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

Arguments identical to the ones used in the proof of Lemma 17 show that A provides the appropriate entry and exit connection pieces. We now show that the net $\mathcal{N}_A$ is itself connected. We will assume that NW is not degenerate and therefore consists of at least one box. (Handling the case where $\mathcal{T}_{NW}$ is empty requires only minor modifications.) First note that $\xrightarrow{e_{NW}} \in T_N$ is closed, so $\mathcal{E}_{NW}$ is part of NW's inductive region. The inductive hypothesis tells us that $\mathcal{N}_{NW}$ provides a type-1 entry connection, which connects to a piece of $e_{NW} \in F_N$ placed adjacent to its entry port. Also note that $\xleftarrow{x_{NW}} \in B_{NJ}$ is open but not adjacent to NW, so $\mathcal{X}_{NW}$ is not part of the inductive region. The inductive hypothesis applied to NW tells us that $\mathcal{N}_{NW}$ provides a type-1 exit connection.

Now we show that the net $\mathcal{N}_{NJ}$ connects to $\mathcal{N}_{NW}$'s type-1 exit connection and to the piece of T_{NJ} placed along its boundary. First note that $\xrightarrow{e_{NJ}} \in T_{NW}$ is open (so $\mathcal{E}_{NJ}$ is not part of NJ's inductive region) but not adjacent to $\mathcal{T}_{NJ}$. The inductive hypothesis applied to NJ tells us that $\mathcal{N}_{NJ}$ provides a type-1 entry connection $e'_{NJ} \in L_{NJ}$, which is adjacent to $\mathcal{N}_{NW}$'s type-1 exit connection $x'_{NW} \in K_{NW}$, so the two nets are connected to each other. Because $\xleftarrow{x_{NJ}} \in B_N$ is closed, $\mathcal{X}_{NJ}$ is part of NJ's inductive region. The inductive hypothesis applied to NJ tells us that $\mathcal{N}_{NJ}$ provides a type-1 exit connection $x'_{NJ} \in R_{NJ}$, which attaches along the bottom of the ring face piece of T_{NJ} extracted from $\mathcal{N}_{NJ}$. Because this ring face piece is not used in the entry or exit connections of $\mathcal{N}_{NJ}$, removing it from $\mathcal{N}_{NJ}$ does not disconnect $\mathcal{N}_{NJ}$, by the inductive hypothesis (I3) applied to NJ.

We now consider $\mathcal{N}_{NN}$ and $\mathcal{N}_E$ and show they connect to the pieces of T_{NJ}, F_N, R_N, B_A and K_A placed along their boundaries. First note that $\mathcal{E}_{NN}$ and $\mathcal{E}_E$ are not part of their respective inductive regions because $\xrightarrow{e_{NN}} \in L_{NJ}$ and $\xrightarrow{e_E} \in F_N$ are open. In addition, $\xrightarrow{e_{NN}}$ is not adjacent to $\mathcal{T}_{NN}$ and $\xrightarrow{e_E}$ is not adjacent to $\mathcal{T}_E$. By the inductive hypothesis, $\mathcal{N}_{NN}$ and $\mathcal{N}_E$ both provide type-1 entry connections, which connect to the piece of

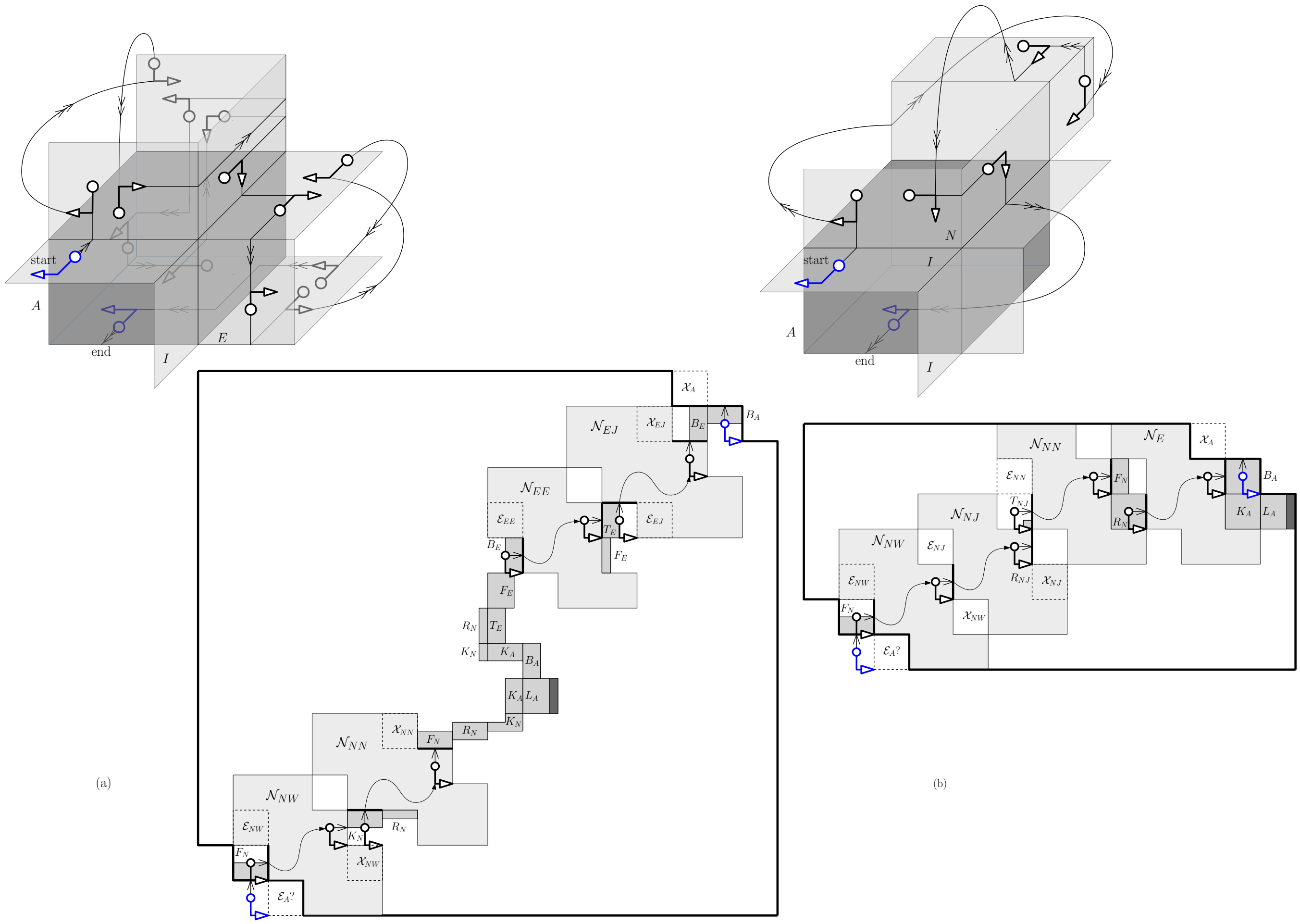

Figure 11: Box A of degree 3 with north and east children, HEAD-first unfolding, HAND pointing west, B_I closed (so B_E open) and T_N closed. (a) K_N open (b) K_N closed (and so T_{NJ} and E_{NJ} open).

$e_{NN} \in T_{NJ}$ and $e_E \in R_N$, respectively. Similarly, $\mathcal{X}_{NN}$ and $\mathcal{X}_E$ are also not part of their respective inductive regions because $\xleftarrow{x_{NN}} \in R_N$ and $\xleftarrow{x_E} \in K_A$ are both open. In addition, $\xleftarrow{x_{NN}}$ is adjacent to $\mathcal{T}_{NN}$ and $\xleftarrow{x_E}$ is adjacent to $\mathcal{T}_E$. By the inductive hypothesis, $\mathcal{N}_{NN}$ and $\mathcal{N}_E$ provide either type-1 or type-2 exit connections, therefore $\mathcal{N}_{NN}$ connects to F_N (if type-1) and R_N (if type-2), and $\mathcal{N}_E$ connects to B_A (if type-1) and K_A (if type-2). Thus we have shown that N_A is connected.

By the inductive hypothesis, $\mathcal{N}_{NW}$, $\mathcal{N}_{NJ}$, $\mathcal{N}_{NN}$, and $\mathcal{N}_E$ cover all faces in $\mathcal{T}_{NW}$, $\mathcal{T}_{NJ}$, $\mathcal{T}_{NN}$ and $\mathcal{T}_E$ using a 4×4 refinement, except for the piece of T_{NJ} that A uses. Observe that $\mathcal{N}_A$ includes all open faces of A and N, also using a 4×4 refinement. Thus we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies (I1) of the inductive hypothesis.

For condition (I3) of the inductive hypothesis, observe that the only open ring face of A not used in A's entry or exit connections is part of L_A, shown in dark gray in Figure 11b. Its removal does not disconnect $\mathcal{N}_A$ so (I3) is satisfied. □

Theorem 20 *Any degree-3 box $A \in \mathcal{T}$ with children N and E can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. Consider first the case with the HAND pointing east. We discuss two cases, depending on whether K_N is open or closed. If K_N is open, by Lemma 14 there is an unfolding net $\mathcal{N}_A$ that satisfies the inductive hypothesis, so the theorem holds. This unfolding is depicted in Figure 9a. If K_N is closed, by Lemma 15 there is an unfolding net $\mathcal{N}_A$ that satisfies the inductive hypothesis, so the theorem holds as well. This unfolding is depicted in Figure 9b.

Consider now the case with the HAND pointing west. We discuss four cases, depending on whether B_I, T_N and K_N are open or closed. If B_I is open, the theorem holds by Lemma 16; the unfolding for this case is depicted in Figure 10a. If T_N is open, the theorem holds by Lemma 17; the unfolding for this case is depicted in Figure 10b. If B_I and T_N are both closed and K_N

is open, the theorem holds by Lemma 18; the unfolding for this case is depicted in Figure 11a. Finally, if B_I and T_N are both closed and K_N is closed, the theorem holds by Lemma 19; the unfolding for this case is depicted in Figure 11b. ☐

8.1 Unfolding Degree-2 Nodes

In this section we turn our attention to degree-2 nodes in $\mathcal{T}$ and show that they can be unfolded into nets that satisfy the inductive hypothesis. We discuss three cases, depending on whether the degree-2 box has a back child, a north child or an east child. The cases where A has a south child or a west child are symmetric.

Theorem 21 *Any degree-2 box $A \in \mathcal{T}$ with back child J can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. One such unfolding is depicted in Figure 12a. Note that this unfolding is very similar to the unfolding of the root box from Figure 5. Because $\xrightarrow{e_A}$ ($\xleftarrow{x_A}$) may be open or closed, $\mathcal{E}_A$ ($\mathcal{X}_A$) may or may not belong to A's inductive region, as indicated by the free cells from Figure 12a. Since $\xrightarrow{e_J} \in R_A$ is open, the inductive hypothesis applied to $\mathcal{N}_J$ tells us that $\mathcal{N}_J$ provides a type-1 or type-2 entry (exit) connection, which attaches to either T_A or R_A (B_A or L_A). Thus the net $\mathcal{N}_A$ is connected. By the inductive hypothesis, $\mathcal{N}_J$ covers all open faces in $\mathcal{T}_J$ using a 4×4 refinement. Noting that $\mathcal{N}_A$ includes the open faces of A, we conclude that $\mathcal{N}_A$ includes all open faces of $\mathcal{T}_A$ and satisfies (I1) of the inductive hypothesis. Note that $\mathcal{N}_A$ provides type-1 entry and exit connections, therefore it satisfies (I2) of the inductive hypothesis. Finally, the open ring faces of A not used in its entry and exit connections (dark-shaded in Figure 12a) can be removed from $\mathcal{N}_A$ without disconnecting $\mathcal{N}_A$, therefore $\mathcal{N}_A$ satisfies (I3) of the inductive hypothesis. ☐

Theorem 22 *Any degree-2 box $A \in \mathcal{T}$ with north child N can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. One such unfolding is depicted in Figure 12b. Note that this is a degenerate case of the unfolding of the box with north and south children from Figure 7b. The only difference is that the net $\mathcal{N}_S$ for the south child from Figure 7b has been replaced by the south face B_A in Figure 12b. Arguments similar to the ones used in the proof of Theorem 10 show that $\mathcal{N}_A$ satisfies the inductive hypothesis. ☐

Theorem 23 *Any degree-2 box $A \in \mathcal{T}$ with east child E can be unfolded into a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. These unfoldings are depicted in Figure 13a for the case with the HAND pointing east, and in Figure 13b for the case with the HAND pointing west. Note that these unfoldings are degenerate cases of the unfoldings of the box with east and back children from Figure 6. One difference is that the unfolding net $\mathcal{N}_J$ for the back child from Figure 6 is replaced by the back face K_A in Figure 13. A few more minor modifications are necessary to accommodate for the fact that, in the HAND-east unfolding, $\xrightarrow{e_E} \in K_A$ is open (and so $\mathcal{E}_E$ does not belong to the inductive region for E) and therefore $\mathcal{N}_E$ may provide a type-1 or a type-2 entry connection. Similarly, in the HAND-west unfolding, $\xleftarrow{x_E} \in K_A$ is open (and so $\mathcal{X}_E$ does not belong to the inductive region for E) and thus $\mathcal{N}_E$ may provide a type-1 or a type-2 exit connection. These accommodations are reflected in Figure 13. Arguments similar to the ones used in the proof of Theorem 6 show that the nets $\mathcal{N}_A$ from Figure 6 satisfy the inductive hypothesis. ☐

9 Unfolding Leaf Nodes

Lemma 24 *Let $A \in \mathcal{T}$ be leaf box with parent I. There is an unfolding of A whose net $\mathcal{N}_A$ satisfies the inductive hypothesis.*

Proof. Lemma 1 enables us to restrict our attention to HEAD-first unfoldings of A. Consider the unfolding depicted in Figure 2a: starting at A's entry port, the unfolding path simply moves HEAD-first until it reaches A's exit port. We now show that, when laid flat in the plane, the open faces of A form a net $\mathcal{N}_A$ that satisfies the inductive hypothesis.

If $\xrightarrow{e}$ is closed (open), then $\mathcal{E}_A$ belongs (doesn't belong) to the inductive region for A. This dual case scenario is depicted by the free cell labeled $\mathcal{E}_A$ in Figure 2a. Similarly, if $\xleftarrow{x}$ is closed (open), then $\mathcal{X}_A$ belongs (doesn't belong) to the inductive region for A. Thus condition (I1) of the inductive hypothesis is trivially satisfied.

To check that (I2) is satisfied, note that $\mathcal{N}_A$ provides type-1 entry and exit connections since $e' \in T_A$ and $x' \in B_A$ are positioned alongside the entry and exit ports.

Turning to (I3) of the inductive hypothesis, observe that the open ring faces of A not used in A's entry or exit connections are the dark-shaded pieces from Figure 2a, whose removal does not disconnect $\mathcal{N}_A$. Thus $\mathcal{N}_A$ also satisfies condition (I3) of the inductive hypothesis. ☐

10 A Complete Example

Figure 14 illustrates a complete unfolding example for an orthotree composed of 9 boxes. The root A of the

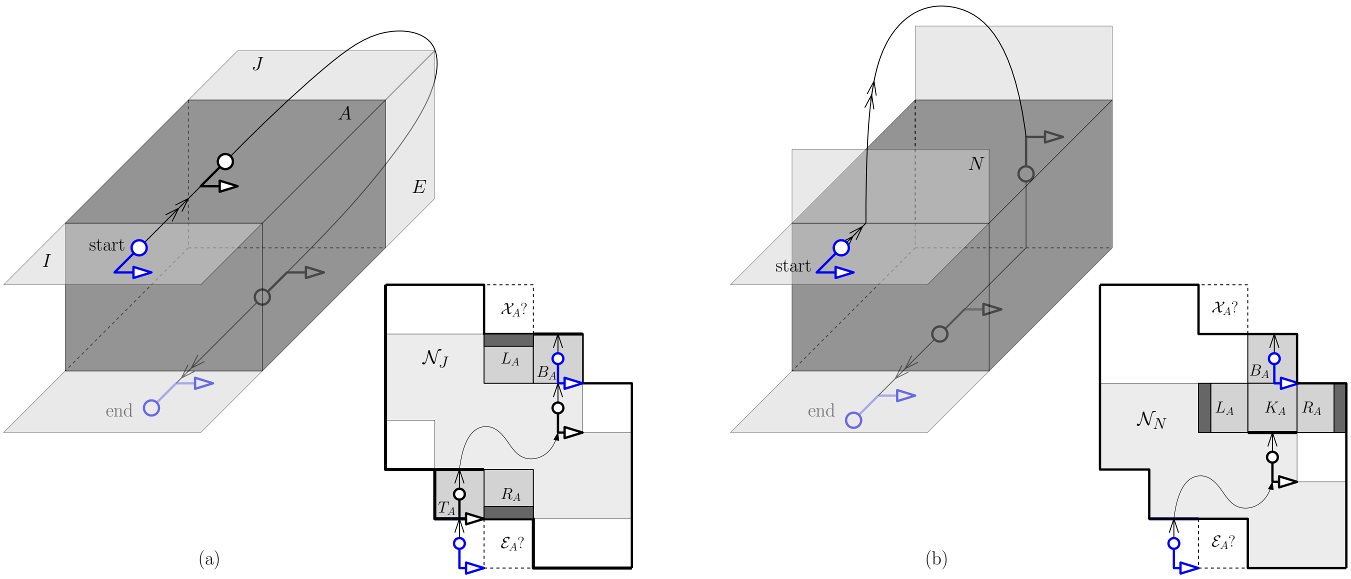

Figure 12: Box A of degree 2 with (a) back child (b) north child.

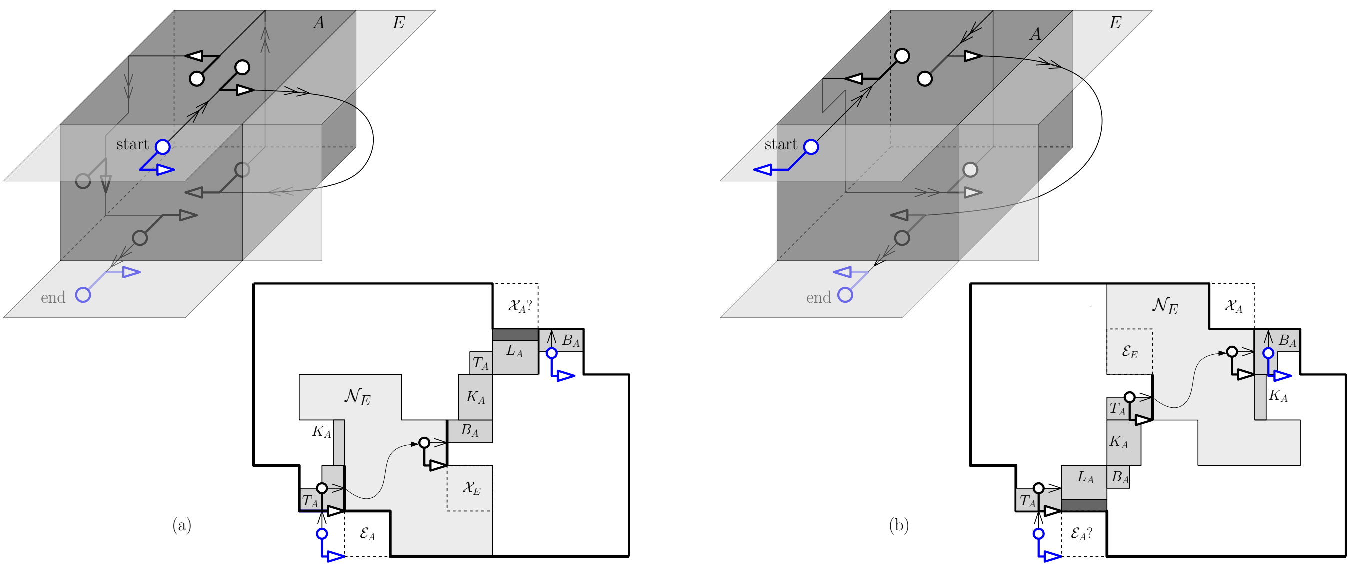

Figure 13: Box A of degree 2 with east child (a) HAND pointing east (b) HAND pointing west.

the unfolding tree is a degree-1 box with back child J, which is unfolded recursively. The unfolding of J follows the pattern depicted in Figure 9b, slightly adjusted to accommodate for the fact that J does not have a south-east child. The east-east child of J (labeled C in Figure 14) follows the unfolding pattern depicted in Figure 13a. The north child of J (labeled F in Figure 14) follows the unfolding pattern from Figure 10b, traversed on reverse (note that the subtree rooted at F is a horizontal mirror plane reflection of the case depicted in Figure 10b, after a clockwise 90°-rotation about a vertical axis followed by a clockwise 90°-rotation about a horizontal axis, to bring it in standard position). Finally, the leaves are unfolded as in Figure 2.

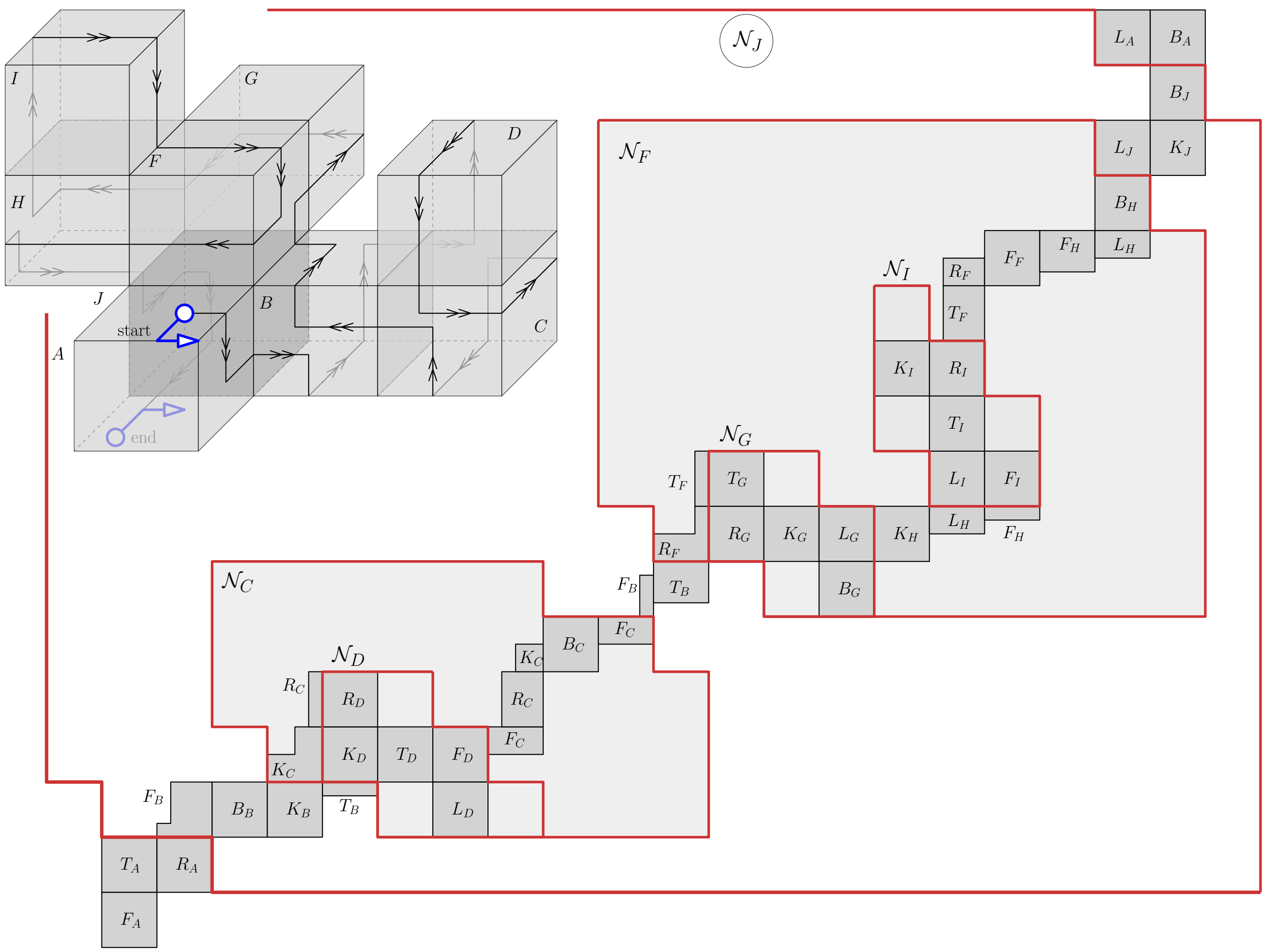

Figure 14: A complete unfolding example.