Optimal Control and Estimation, Chapters 1 and 2

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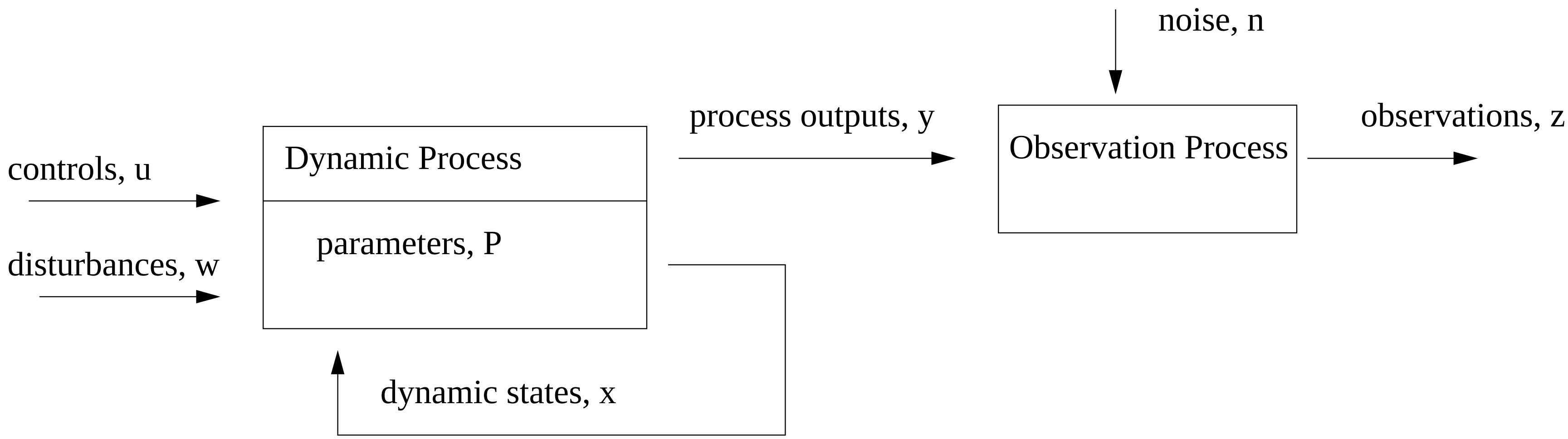
Outline

1. The optimal control problem: Modeling dynamic Systems
2. Introduction to Laplace Transforms for systems of differential equations
3. Solving linear systems of first order ODE's
4. Stability criteria

Modeling Dynamic Systems: Schematic



Modeling Dynamic Systems: Equations

Dynamics Equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{p}(t), t]$$

Output Equation:

$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{p}(t), t]$$

Observation Equation:

$$\mathbf{z}(t) = \mathbf{j}[\mathbf{y}(t), \mathbf{n}(t), t]$$

Sanity Check: Roadmap for Generating a Trajectory

This is a straightforward framework, but we do have 10 different vector quantities floating around!

To Generate a Trajectory in State Space:

1. Fix a control function $\mathbf{u}(t)$, a disturbance function $\mathbf{w}(t)$, and a parameter function $\mathbf{p}(t)$.

2. Fix initial conditions $\mathbf{x}(t_0)$.

3. Plug these functions into the differential equation for $\dot{\mathbf{x}}(t)$ and solve (somehow) for $\mathbf{x}(t)$.

Note that while we write $\mathbf{u}(t)$ as a function of time, it might actually be a function of the current state.

Higher Order ODEs $\Rightarrow$ Systems of First Order ODEs

An example will be the easiest way to demonstrate this. Consider the differential equation

$$\dddot{x} + c_1\ddot{x} + c_2\dot{x} = bu$$

and perform the variables substitution

$$\mathbf{x} = [x_1, x_2, x_3]^T = [x, \dot{x}, \ddot{x}]$$

and now we can write the differential equation as a system of first order equations:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ (-c_3 x_3 - c_2 x_2 - c_1 x_1 + bu) \end{bmatrix}$$

Or

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_3 & -c_2 & -c_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} u$$

Or

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$$

The Laplace Transform

Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ in the time domain, the Laplace transform $\mathfrak{L}[\cdot]$ produces a function $F : \mathbb{C} \rightarrow \mathbb{R}$ in the frequency domain:

$$\mathfrak{L}[f(t)] = f(s) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

We will be able to use this to reduce the solution of certain differential equations to algebraic manipulations by first transforming the problem into the frequency domain, manipulating the problem, and then applying the inverse Laplace transform to get a time-domain solution.

It's Like Multiplication with Logs

To find $a \cdot b$ we can convert to logarithms:

$$t = \ln a + \ln b = \ln(a \cdot b)$$

and now "solve" in the log domain by performing the addition, and then convert back to a solution by performing the inverse operation:

$$a \cdot b = e^t = e^{\ln a + \ln b}$$

Laplace Transform on Derivatives

Using integration by parts with $u = e^{-st}$ and $dv = f'(t)dt$, we see that

$$\begin{aligned}
\mathfrak{L}[f'(t)] &= \int_0^\infty f(t)e^{-st}dt \\
&= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})dt \\
&= -f(0) + s\int_0^\infty f(t)(e^{-st})dt \\
&= -f(0) + s\mathfrak{L}[f(t)]
\end{aligned}$$

We are assuming for some s, $e^{-st}f(t) \to 0$ as $t \to \infty$.

More about the Laplace Transform

We can express the Laplace transform of a $f'(t)$ as a simple function of $f(t)$. This is the main point.

Another important fact: the Laplace transform is a linear operator

$$\begin{aligned} \mathfrak{L}[af(x)+bg(x)] &= a\mathfrak{L}[f(x)]+b\mathfrak{L}[g(x)] \\ &= af(s)+bg(s) \end{aligned}$$

Important Laplace Transform

$$\mathfrak{L}[1] = \frac{1}{s} \qquad L[e^{at}] = \frac{1}{s-a}$$

$$\mathfrak{L}[t] = \frac{1}{s^2} \quad L[\sin(at)] = \frac{a}{s^2+a^2}$$

$$\mathfrak{L}[f^{(n)}(t)] = s^n \mathfrak{L}[f(t)] - \sum_{i=1}^{n} s^{n-1} f^{(n-i)}(0)$$

Laplace Transform of Vector Equations

Note that when we apply the Laplace transform to a vector or matrix, we are just applying it element-wise. For example, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Then,

$$\mathfrak{L}[A\mathbf{x}(t)] = \mathfrak{L}\left[\begin{bmatrix} ax_1(t) + bx_2(t) \\ cx_1(t) + dx_2(t) \end{bmatrix}\right] = \begin{bmatrix} a\mathfrak{L}[x_1(t)] + b\mathfrak{L}[x_2(t)] \\ c\mathfrak{L}[x_1(t)] + d\mathfrak{L}[x_2(t)] \end{bmatrix} = A\mathfrak{L}[\mathbf{x}(t)]$$

where we have used the fact that $\mathfrak{L}$ is a linear operator.

Laplace Transform of Linear Dynamics

If we have linear dynamics, we have the system of differential equations:

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)$$

Taking the Laplace transform of both sides gives:

$$s\mathbf{x}(s) - \mathbf{x}(0) = \mathbf{F}\mathbf{x}(s) + \mathbf{G}\mathbf{u}(s) + \mathbf{L}\mathbf{w}(s)$$

and we can solve for $\mathbf{x}(s)$ as

$$\mathbf{x}(s) = (s\mathbf{I}_n - \mathbf{F})^{-1}[\mathbf{x}(0) + \mathbf{G}\mathbf{u}(s) + \mathbf{L}\mathbf{w}(s)]$$

The Inverse Laplace Transform

We have now found $\mathbf{x}(s)$, but we want a time-domain solution. This requires applying the inverse Laplace operator,

$$\mathfrak{L}^{-1}[f(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} x(s)e^{st}ds$$

where we assume $s = \sigma + j\omega$ (In control literature, $j = \sqrt{-1}$).

Rather than doing this integration directly, usually one can expand $\mathbf{x}(s)$ via partial fractions and then use lookup-tables for the necessary inverse transforms.

Definition: Static Equilibrium and Quasistatic Equilibrium

For fixed controls $\mathbf{u}^*$ and disturbances $\mathbf{w}^*$, a point $\mathbf{x}^*$ is a static equilibrium if

$$\mathbf{0} = \mathbf{f}[\mathbf{x}^*, \mathbf{u}^*, \mathbf{w}^*],$$

that is, if the state is not changing.

The point is a quasistatic equilibrium point if some state variables are fixed at zero, and other variables are not.

System Dynamics

Recall that we had determined that

$$\mathbf{x}(s) = (s\mathbf{I}_n - \mathbf{F})^{-1}[\mathbf{x}(0) + \mathbf{G}\mathbf{u}(s) + \mathbf{L}\mathbf{w}(s)]$$

and so if the initial-conditions response (no disturbances and no controls) is given by

$$\mathbf{x}(s) = (s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{x}(0) = \mathbf{A}\mathbf{x}(0)$$

The inverse Laplace transform, $\mathfrak{L}^{-1}[\cdot]$, will be applied element-wise to the vector $\mathbf{A}\mathbf{x}(0)$ to determine our time-domain initial-conditions response $\mathbf{x}(t)$. What form will this function take?

Modes of Motion

$$(s\mathbf{I}_n - \mathbf{F})^{-1} = \frac{\text{Adj}(s\mathbf{I}_n - \mathbf{F})}{|s\mathbf{I}_n - \mathbf{F}|}$$

Thus, the entries

$$a_{ij} = \frac{k(s-\beta_1)(s-\beta_2)\cdots(s-\beta_n)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}$$

and so

$$x_j(s) = a_{j1}x_1(0) + \cdots + a_{jn}(s)x_n(0)$$

will be a ratio of two polynomials.

What does this give us?

1. We can apply partial fractions and express $x_j(s)$ as the sum of ratios of low degree polynomials. These polynomials have inverse Laplace transforms that are either exponentials, sine or cosine. These define the "modes of motion" of the system.

2. If we examine the role the λ_i's play in the resulting equation, it becomes clear that if all $\lambda_i < 0$ the system will be stable.

3. But the λ_i's are really just the Eigenvalues of F

What's Next?

1. Bode diagrams

2. Transfer Functions

3. Root Locus techniques