Intuitionistic Metric Temporal Logic

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ABSTRACT

We develop Intuitionistic Metric Temporal Logic (IMTL) that extends prior work on intuitionistic temporal logics in two ways: (1) it generalizes discrete time to dense time with intervals so it can, for example, express the duration of signals, and (2) every proof corresponds to a temporal computation.

Our main technical result is a syntactic proof of cut elimination for IMTL, which entails logical consistency and ensures that every proof executes while respecting the flow of time. Cut reductions in IMTL correspond to temporal interactions, although we do not fully develop a programming language in this paper.

Beyond the metatheory of IMTL, we illustrate the computational meaning of IMTL proofs by developing examples and a small case study where we apply IMTL to well-timed digital circuit design.

CCS CONCEPTS

• **Theory of computation** → **Proof theory**; **Modal and temporal logics**; **Constructive mathematics**; • **Hardware** → Static timing analysis.

1 INTRODUCTION

Temporal logic extends the standard logical connectives with time operations, allowing logic to describe temporal properties. Dating back to the seminal work of Pnueli [29] on *Linear Temporal Logic* (LTL), *temporal modalities* enable propositions to state that a property A holds at *all* points in the future, written as $\Box A$, at *some* point in the future, $\Diamond A$, or in the *next instant*, $\bigcirc A$.

Computer scientists have successfully applied LTL to a wide variety of computer science topics, from hardware specification [5, 26] to proving properties about sequential programs [24].

The semantics of LTL usually comes from an interpretation of its propositions relative to a model of a temporal system. In this setting, the notion of time is generally abstract in the sense that properties concern states that result from *discrete* (computational) steps taken by the system under study.

However, in real-world systems, the properties of interest often require reasoning about *real time*. To this end, (classical) *Metric Temporal Logic* (MTL) [23, 27, 28] is an extension of LTL where we constrain the usual LTL modalities by temporal intervals. In MTL, the proposition $\Box^I A$ denotes that A holds during the *entire* interval I instead of over all points in the future and $\Diamond^I A$ denotes that A holds *somewhere* within I. In this dense-time setting, the $\bigcirc$ modality does not play a central role, sometimes defined as $\Diamond$ or $\Box$ (either one works) indexed by a point (a singleton interval) such as $[\delta, \delta]$ for a small real δ considered the duration of a discrete step.

So far we described *classical* temporal logic, where the semantics is an interpretation of propositions over models. In this framework, we have two components: a *model* that captures the system of interest and *logical propositions* that codify the properties satisfied by the system.

In this paper, we propose an *intuitionistic* formulation of metric temporal logic (IMTL), studied from the perspective of structural *proof theory* [16, 30, 31] rather than *model theory*. Our semantics lies in the (syntactic) discipline of how propositions are *proved* rather than interpreted over an external model.

A characteristic of the intuitionistic approach is that the logic and the computational model coincide, in the spirit of propositions-as-types [10, 20]. From a practical standpoint, while the classical approach is suitable for *model checking* (a procedure that checks whether a model acts in accordance to a LTL proposition [3]), the intuitionistic approach is suitable for *temporal computation* (actions through time that realize a proposition while respecting the flow of time) and, thus, *temporal programming*. Thus, our main goal for the design of IMTL is for every proof to correspond to a temporal computation in this sense.

We introduce a sequent calculus for IMTL and show, as our main result, a syntactic proof of *cut elimination* which entails *temporal causality* — informally, "future events cannot affect the present"; and *temporal monotonicity*, the logical counterpart requirement for temporal computation, among other metatheorems. Furthermore, it is possible to extract a temporal *computational model* from the proof of cut elimination, although we do not fully develop a programming language in this paper.

To the best of the authors knowledge, this is the first attempt to develop an intuitionistic version of MTL, so we compare it with classical MTLs and other (non-metric) intuitionistic temporal and modal logics.

Our work shares similarities with intuitionistic versions of LTL and programming languages with temporal dynamics. Arguably the work most closely related to ours is that of Kojima and Igarashi [22], who present an LTL sequent calculus that respects temporal causality and establishes cut elimination syntactically. Their calculus, however, does not tackle the meaning of $\Box$ and $\Diamond$, and furthermore proofs do not correlate directly to temporal computations. For further comparisons with related work such as Simpson's intuitionistic modal logic [32], Davies' intuitionistic temporal logic [13, 14], and other proposals for intuitionistic temporal logics [1, 6] see Section 5.

Concretely, our principal contributions are:

- A judgmental account of IMTL based on Martin-Löf's [25] approach of distinguishing judgments from propositions (Section 2);
- A sequent calculus for IMTL whose proofs correspond to temporal computations (Section 2);

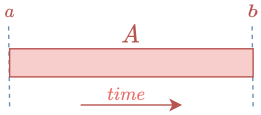
- Formal definitions (and proofs) for *temporal causality* — requirement for intuitionistic temporal logics — and *temporal monotonicity* — requirement for proofs to be temporally computable (Section 3);
- A syntactical result of cut elimination that entails several metatheorems including temporal *causality* and *monotonicity* (Section 3.3);
- A case study showcasing how IMTL can model well-timed digital circuits (Section 4);
- A comparison with other temporal and modal logics (Section 5);

2 INTUITIONISTIC METRIC TEMPORAL LOGIC

In this section, we develop IMTL. We start by introducing the basic judgments and then present the logical connectives individually with small examples. We then show what kinds of formulas are, and are not, valid in IMTL. Finally, we define an intuitionistic linear temporal logic within IMTL for comparison purposes with other temporal and modal logics.

2.1 Interval Judgment

IMTL builds on the methodology of Martin-Löf [25], in which *judgments* differ from propositions. We define our logic on top of a basic *interval judgment*:



A holds during interval $[a, b]$, *denoted* $A^{[a,b]}$

for a given proposition A and interval $[a, b]$, starting at a and ending at b (both possibly negative real numbers), measured relative to the same reference point. We also write A^{I}, for a given interval I, when referring to the interval limits is unnecessary.

2.2 Arithmetic Constraints

A key feature of IMTL is that our basic temporal judgment $A^{[a,b]}$ refers explicitly to real numbers a and b. It is therefore critical to clarify how we reason about them. We use standard arithmetic *constraints* and real number *expressions*

$$\begin{array}{lll} \textit{constraints} & \mathbb{C} & ::= x \text{ real} \mid e_1 < e_2 \mid e_1 \leqslant e_2 \mid e_1 = e_2 \\ & & \mid \top \mid \mathbb{C}_1 \wedge \mathbb{C}_2 \mid \bot \mid \mathbb{C}_1 \vee \mathbb{C}_2 \\ \textit{expressions} & e & ::= e_1 + e_2 \mid e_1 - e_2 \mid x \mid \textit{(real constants)} \end{array}$$

and a semantic constraint entailment

$$\Omega \vDash \mathbb{C} \text{ , with } \Omega \triangleq \mathbb{C}_1, \mathbb{C}_2, \cdots, \mathbb{C}_n$$

meaning Ω entails $\mathbb{C}$ in a standard theory of real numbers with variables (denoted by x in our case). Note that constraints can be inconsistent, in which case $\Omega \vDash \bot$, and can also branch into cases, $\Omega \vDash \mathbb{C}_1 \vee \mathbb{C}_2$.

2.3 Temporal Hypothetical Judgment

We develop a *temporal sequent* where both antecedents and succedent are interval judgments and depend on constraints Ω. By the sequent

$$\Omega \,; \Gamma \overset{s}{\vDash} A^{[a,b]}$$

$$\textit{where } \Gamma = A_1^{[a_1,b_1]}, A_2^{[a_1,b_1]}, \cdots, A_n^{[a_n,b_n]},$$

we mean that if propositions $A_1, A_2, \cdots, A_n$ are true during their respective intervals $[a_1, b_1]$, $[a_1, b_2]$, $\cdots$, $[a_n, b_n]$ then the proof, which is at time s, realizes A over interval $[a, b]$. It is useful to think of the proof as a process at time s that constructs evidence for the succedent $A^{[a,b]}$ from evidence for the antecedents. In many rules will abbreviate the succedent $A^{[a,b]}$ as γ.

Crucially, the collection of rules applicable to a sequent depends on the present time s, which is how the calculus enforces the application of rules only on propositions that are currently available.

A sequent is well-formed when satisfying the condition

$$\Omega \vDash (s \leqslant a) \wedge (a \leqslant b) \wedge (a_1 \leqslant b_1) \wedge (a_2 \leqslant b_2) \wedge \cdots \wedge (a_n \leqslant b_n)$$

where the $a_i \leqslant b_i$ clauses capture the notion that a *temporal interval* has to start before it ends and the clause $s \leqslant a$ enforces the succedents' interval to always start *after* the present time, otherwise, the proof cannot possibly provide evidence for its succedent without breaking temporal causality.

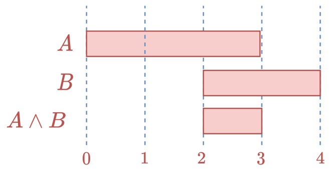
Having a well-formedness condition for sequents means we consider an inference rule well-formed if all premises of the rule are well-formed, assuming that the conclusion is well-formed. This "bottom-up" reading of rules is characteristic for many sequent calculi. We shall maintain this condition implicitly throughout the paper, only invoking it when necessary.

When the constraints Ω are inconsistent we are in an unreachable branch of the proof and succeed, echoing the intuitionistic proof of $\bot \supset A$. As we see in the proof of cut elimination, we are sometimes in a situation where we have two abstract times u and v, and Ω does not uniquely determine which of these comes first (i.e., is smaller). In this case we must split the proof into two branches, considering the cases $u \leq v$ and $v \leq u$. This is generalized in the split rule below.

$$\frac{\Omega \vDash \bot}{\Omega \,; \Gamma \overset{s}{\vDash} \gamma}\,\text{imposs}$$

$$\frac{\Omega \vDash \mathbb{C}_1 \vee \mathbb{C}_2 \quad \Omega, \mathbb{C}_1 \,; \Gamma \overset{s}{\vDash} \gamma \quad \Omega, \mathbb{C}_2 \,; \Gamma \overset{s}{\vDash} \gamma}{\Omega \,; \Gamma \overset{s}{\vDash} \gamma}\,\text{split}$$

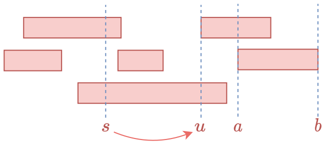
Example 2.1. The interval judgments in a sequent are interpreted relative to the temporal baseline s that annotates the turnstile. For example, in the following sequent:



$$A^{[0,3]}, B^{[2,4]} \overset{0}{\vDash} A \wedge B^{[2,3]}$$

the proof is at time 0, meaning that A is at the beginning of its truth interval, while B and $A \wedge B$ still lie in the future. Computationally, this means that the proof, viewed as a process, can interact at type A, but not yet at types B or $A \wedge B$.

However, proofs as in Example 2.1 would get stuck if there is no way to interact with future intervals. IMTL does not provide any *direct* way to interact with future intervals, but it provides the structural rule delay that forwards a proof through time.



$$\frac{\Omega \vDash (s \leqslant u \leqslant a) \qquad \Omega\ ; \Gamma \vDash^{u} A^{[a,b]}}{\Omega\ ; \Gamma \vDash^{s} A^{[a,b]}}\,\text{delay}$$

This rule is applicable given two conditions. Firstly, $\Omega \vDash s \leqslant u$ makes it impossible for a proof to go back in time, as it would break temporal causality. Secondly, $\Omega \vDash u \leqslant a$ ensures that proofs do not delay too much and end up ignoring their objective, namely proving A over the interval $[a, b]$.

Example 2.2 (Use of delay rule). The sequent

$$A^{[0,3]}, B^{[2,4]} \vDash^{0} A \wedge B^{[2,3]}$$

from Example 2.1, cannot advance to instant 2.5 because that would prevent it from producing evidence for $A \wedge B$ in the interval from 2 to 2.5. However, it can advance to time 2, and then interact with B, and $A \wedge B$.

2.4 Rules of Inference

We present the complete set of IMTL rules in Figure 1. We proceed by explaining the system of rules in detail. We separate the rules of *weakening* and *contraction* to unclutter the left and right rules that define the meaning of propositions.

$$\frac{\Omega\ ; \Gamma \vDash^{s} \gamma}{\Omega\ ; \Gamma, A^{[a,b]} \vDash^{s} \gamma}\,\text{weak} \qquad \frac{\Omega\ ; \Gamma, A^{[a,b]}, A^{[a,b]} \vDash^{s} \gamma}{\Omega\ ; \Gamma, A^{[a,b]} \vDash^{s} \gamma}\,\text{cntr}$$

Our *cut* and *identity* rules are straightforward. We usually present identity as the id rule, with the same interval $[a, b]$ on both sides, but the more precise definition of identity is id*, which checks whether both intervals match given constraints.

$$\frac{}{\Omega\ ; A^{[a,b]} \vDash^{s} A^{[a,b]}}\,\text{id} \qquad \frac{\Omega \vDash (a = a') \wedge (b = b')}{\Omega\ ; A^{[a,b]} \vDash^{s} A^{[a',b']}}\,\text{id*}$$

$$\frac{\Omega\ ; \Gamma_1 \vDash^{s} A^{[a,b]} \qquad \Omega\ ; \Gamma_2, A^{[a,b]} \vDash^{s} \gamma}{\Omega\ ; \Gamma_1\Gamma_2 \vDash^{s} \gamma}\,\text{cut}$$

Since every arithmetic expression in a sequent is interpreted with respect to the constraints Ω, we allow ourselves shorthands like the one on the left. In other words, if $\Omega \vDash e_1 = e_2$ we will take the liberty to silently replace e_1 by e_2 or vice versa.

We justify propositions following the proof-theoretic tradition [16, 30, 31] where the meaning of each proposition (over an interval) comes directly from its right and left rules. The syntax for IMTL propositions follows the grammar

$$\begin{aligned} A, B ::= \quad & P \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \\ \mid\ & \bigcirc^{\langle a,b \rangle} A \mid \Box^{\langle a,b \rangle} A \mid \Diamond^{\langle a,b \rangle} A \end{aligned}$$

with propositional variables P and (real) numbers a and b.

Proof-theoretically, we establish that all right and left rules cancel themselves out — a property called *harmony*. From harmony, we extract computational meaning and justify the connectives semantically. For now, we postulate harmony through informal discourse, but our proofs of *cut elimination* and *identity elimination* (Section 3) are formal evidence that the rules are harmonious.

We proceed by separating operations into logical and temporal and explain them separately.

2.5 Logical Operations as Synchronous Events

Let us start with implication: for $A \supset B$ to hold over an interval $[a, b]$, B must hold over the interval $[a, b]$ *given that* A holds over the same interval $[a, b]$. Using an implication amounts to presenting evidence of A over the interval $[a, b]$ and receiving a B over the same interval as a result.

$$\frac{\Omega\ ; \Gamma, A^{[a,b]} \vDash^{s} B^{[a,b]}}{\Omega\ ; \Gamma \vDash^{s} A \supset B^{[a,b]}}\,{\supset}\text{R(?)}$$

$$\frac{\Omega\ ; \Gamma_1 \vDash^{s} A^{[a,b]} \qquad \Omega\ ; \Gamma_2, B^{[a,b]} \vDash^{s} \gamma}{\Omega\ ; \Gamma_1\Gamma_2, A \supset B^{[a,b]} \vDash^{s} \gamma}\,{\supset}\text{L(?)}$$

However, these rules are too permissive when considering temporal computability. We want both the left and right rules to be at the same time when applied, enforcing *synchrononicity* between both rules. We shall restrict them to be applicable only if the sequent is at time a.

$$\frac{\Omega\ ; \Gamma, A^{[a,b]} \vDash^{a} B^{[a,b]}}{\Omega\ ; \Gamma \vDash^{a} A \supset B^{[a,b]}}\,{\supset}\text{R}$$

$$\frac{\Omega\ ; \Gamma_1 \vDash^{a} A^{[a,b]} \qquad \Omega\ ; \Gamma_2, B^{[a,b]} \vDash^{a} \gamma}{\Omega\ ; \Gamma_1\Gamma_2, A \supset B^{[a,b]} \vDash^{a} \gamma}\,{\supset}\text{L}$$

Here we present a concise version of the rules, similar to the identity rule case. By the sequent $\Omega\ ; \Gamma_1\Gamma_2 \vDash^{a} A \supset B^{[a,b]}$ we denote $\Omega\ ; \Gamma_1\Gamma_2 \vDash^{s} A \supset B^{[a,b]}$ with $\Omega \vDash s = a$, reasoning up to equality as derivable via Ω.

All other logical connectives ($\vee, \supset, \top, \bot$) share the same pattern of requiring the sequent to be at the start of the interval ($\Omega \vDash$ *present time* $= a$). This is one of the factors that allows us to relate proofs to temporal computations (more in Section 3).

We emphasize disjunction $\vee$ because it requires either choice to be stable during the entire duration of the interval, a different semantics from the one found in classical MTL:

$$\frac{\Omega\ ; \Gamma \vDash^{a} A^{[a,b]}}{\Omega\ ; \Gamma \vDash^{a} A \vee B^{[a,b]}}\,{\vee}\text{R}_1 \qquad \frac{\Omega\ ; \Gamma \vDash^{a} B^{[a,b]}}{\Omega\ ; \Gamma \vDash^{a} A \vee B^{[a,b]}}\,{\vee}\text{R}_2$$

$$\frac{\Omega\ ; \Gamma, A^{[a,b]} \vDash^{a} \gamma \qquad \Omega\ ; \Gamma, B^{[a,b]} \vDash^{a} \gamma}{\Omega\ ; \Gamma, A \vee B^{[a,b]} \vDash^{a} \gamma}\,{\vee}\text{L}$$

Computationally, harmony of all connectives (except $\bigcirc$) entails that some specific information is sent and received during cut reduction (which represents a single step of communication). The carrier of this information (say, a wire, a channel, or a memory location—let's call it an address generically) remains entirely abstract, but the information contents itself is specific. For example, a proof of $A \supset B[a, b]$ expects to receive the address of an A. A proof of $A \vee B$ will sent a (stable) bit of information over the interval [a,b], representing whether it consists of a proof of A or B. $\top[a, b]$ has only one proof, so it represents a point of synchronization without any further information being communicated.

Logical connectives represent *synchronous events* because they require both sides of the communication (the one providing the event and the one using it) to agree on a temporal interval $[a, b]$ and act/react during its entirety (see Figure 1).

Example 2.3 (Uncurrying). Implication works similar to $\wedge$, by requiring the present time to be the start of the interval (refer to Figure 1). Like in this example, we often omit constraints Ω when they are trivial or easily determined by context.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A^{[a,b]} \vdash^{a} A^{[a,b]}}\,\text{id} \qquad \dfrac{\dfrac{}{B^{[a,b]} \vdash^{a} B^{[a,b]}}\,\text{id} \quad \dfrac{}{C^{[a,b]} \vdash^{a} C^{[a,b]}}\,\text{id}}{B \supset C^{[a,b]}, B^{[a,b]} \vdash^{a} C^{[a,b]}}\supset \text{L}}{A \supset (B \supset C)^{[a,b]}, A^{[a,b]}, B^{[a,b]} \vdash^{a} C^{[a,b]}}\supset \text{L}}{A \supset (B \supset C)^{[a,b]}, A \wedge B^{[a,b]} \vdash^{a} C^{[a,b]}}\wedge\text{L}}{A \supset (B \supset C)^{[a,b]} \vdash^{a} (A \wedge B) \supset C^{[a,b]}}\supset \text{R}}{\cdot \vdash^{a} (A \supset (B \supset C)) \supset (A \wedge B) \supset C^{[a,b]}}\supset \text{R}}{\cdot \vdash^{0} (A \supset (B \supset C)) \supset (A \wedge B) \supset C^{[a,b]}}\,\text{delay}$$

2.6 Temporal Connectives

Our first temporal modality, $\bigcirc^{\langle a,b\rangle} A$ is the *internalization* of the *interval judgment* as a *proposition*, in the sense that $\bigcirc^{\langle a,b\rangle} A$ and $A^{[a,b]}$ both mean "A is true during interval $[a, b]$".

As an internalization of the judgment, it is clear that by $\bigcirc^{\langle \partial_1,\partial_2\rangle} A^{[0,0]}$ we mean $A^{[\partial_1,\partial_2]}$ and, stretching the concept to future points in time, by $\bigcirc^{\langle \partial_1,\partial_2\rangle} A^{[s,s]}$ we mean $A^{[s+\partial_1,s+\partial_2]}$.

Generally, by $\bigcirc^{\langle \partial_1,\partial_2\rangle} A^{[a,b]}$ we mean $A^{[a+\partial_1,b+\partial_2]}$, which subsumes the intuitions above by, formally, adding the endpoints to the interval $[a, b]$. The rules need to check that the resulting interval is well-formed.

$$\dfrac{\Omega \vDash a + \partial_1 \leqslant b + \partial_2 \qquad \Omega\,; \Gamma, A^{[a+\partial_1,b+\partial_2]} \vdash^{s} \gamma}{\Omega\,; \Gamma, \bigcirc^{\langle \partial_1,\partial_2\rangle} A^{[a,b]} \vdash^{s} \gamma}\bigcirc \text{L} \qquad \dfrac{\Omega \vDash a + \partial_1 \leqslant b + \partial_2 \qquad \Omega\,; \Gamma \vdash^{s} A^{[a+\partial_1,b+\partial_2]}}{\Omega\,; \Gamma \vdash^{s} \bigcirc^{\langle \partial_1,\partial_2\rangle} A^{[a,b]}}\bigcirc \text{R}$$

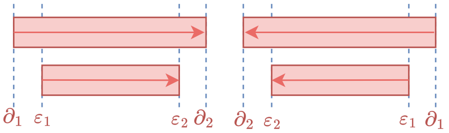
We use a different notation for $\langle \partial_1, \partial_2\rangle$ because it is a *differential* (of an interval) rather than an interval, in the sense that $\partial_1 \leqslant \partial_2$ might not hold. While a differential is not an interval, we add a differential to an interval to obtain the *next interval.*

The definition of *differential* enforces (and we will keep it implicit in our presentation), that both of its components are positive, (i.e., $\partial_1 \geqslant 0 \wedge \partial_2 \geqslant 0$), otherwise $\bigcirc$ would allow reasoning about past events and break causality of time. Note that every interval is a differential although not every differential is an interval. The use of differentials enables modeling interesting temporal phenomena as we will see in Section 4.

Computationally, $\bigcirc$ does not require any communication, we just change the interval at which a particular proposition may interact (which always lies in the future).

Also, because $\langle \partial_1, \partial_2\rangle$ is not an interval, it is not trivial from the conclusion that $[a + \partial_1, b + \partial_2]$ is an interval, which is why we need to require that $\Omega \vDash a + \partial_1 \leqslant b + \partial_2$.

While $\bigcirc^{\langle \partial_1,\partial_2\rangle}$ represents communication over a certain interval, $\square^{\langle \partial_1,\partial_2\rangle}$ and $\diamond^{\langle \partial_1,\partial_2\rangle}$ represent communications that are *uncertain* about their interval of interaction apart from the fact it must lie *within* $\langle \partial_1, \partial_2\rangle$. Here we use the notion of "within" as a generalization of the notion of *subinterval* $\subseteq$, extended to account for differentials.



$$\begin{aligned} \langle \varepsilon_1, \varepsilon_2\rangle \subseteq \langle \partial_1, \partial_2\rangle \quad &\triangleq \quad (\partial_1 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \leqslant \partial_2) \\ &\vee \quad (\partial_2 \leqslant \varepsilon_2 \leqslant \varepsilon_1 \leqslant \partial_1) \end{aligned}$$

where the first disjunct accounts for the usual subinterval relation while the second accounts for when a differential is not an interval, flipping the relations according to the figure.

$\square^{\langle \partial_1,\partial_2\rangle} A$ means that A must be true for all differentials within $\langle \partial_1, \partial_2\rangle$. As a succedent, we therefore must *prove* it for all subdifferentials. As an antecedent, we conversely *assume* that it is true for all subdifferentials. We capture this interplay by introducing fresh time parameters α_1 and α_2 in $\square$ R, and allowing arbitrary times ℓ_1 and ℓ_2 in $\square$ L. In both rules a requisite interval constraint must be satisfied to reflect the intuitive meaning of $\square^{\langle \partial_1,\partial_2\rangle}$.

$$\dfrac{\Omega, \langle \alpha_1, \alpha_2\rangle \subseteq \langle \partial_1, \partial_2\rangle \vDash a + \alpha_1 \leqslant b + \alpha_2 \qquad \Omega, \langle \alpha_1, \alpha_2\rangle \subseteq \langle \partial_1, \partial_2\rangle\,; \Gamma \vdash^{a} A^{[a+\alpha_1,b+\alpha_2]}}{\Omega\,; \Gamma \vdash^{a} \square^{\langle \partial_1,\partial_2\rangle} A^{[a,b]}}\square\, \text{R}^{\alpha_1,\alpha_2}$$

$$\dfrac{\Omega \vDash a + \ell_1 \leqslant b + \ell_2 \qquad \Omega \vDash \langle \ell_1, \ell_2\rangle \subseteq \langle \partial_1, \partial_2\rangle \quad \Omega\,; \Gamma, A^{[a+\ell_1,b+\ell_2]} \vdash^{a} \gamma}{\Omega\,; \Gamma, \square^{\langle \partial_1,\partial_2\rangle} A^{[a,b]} \vdash^{a} \gamma}\square\, \text{L}$$

Symmetrically, $\diamond^{\langle \partial_1,\partial_2\rangle} A$ means that A must be true over *some* subdifferential within $\langle \partial_1, \partial_2\rangle$. Therefore the $\diamond$ R can choose an arbitrary subdifferential $\langle \ell_1, \ell_2\rangle$, while $\diamond$ L must work for arbitrary subdifferentials and introduces fresh parameters α_1 and α_2.

$$\dfrac{\Omega \vDash a + \ell_1 \leqslant b + \ell_2 \qquad \Omega \vDash \langle \ell_1, \ell_2\rangle \subseteq \langle \partial_1, \partial_2\rangle \quad \Omega\,; \Gamma \vdash^{a} A^{[a+\ell_1,b+\ell_2]}}{\Omega\,; \Gamma \vdash^{a} \diamond^{\langle \partial_1,\partial_2\rangle} A^{[a,b]}}\diamond\, \text{R}$$

$$\dfrac{\Omega, \langle \alpha_1, \alpha_2\rangle \subseteq \langle \partial_1, \partial_2\rangle \vDash a + \alpha_1 \leqslant b + \alpha_2 \qquad \Omega, \langle \alpha_1, \alpha_2\rangle \subseteq \langle \partial_1, \partial_2\rangle\,; \Gamma, A^{[a+\alpha_1,b+\alpha_2]} \vdash^{a} \gamma}{\Omega\,; \Gamma, \diamond^{\langle \partial_1,\partial_2\rangle} A^{[a,b]} \vdash^{a} \gamma}\diamond\, \text{L}^{\alpha_1,\alpha_2}$$

In both sets of rules, $\langle \partial_1, \partial_2\rangle$, $\langle \ell_1, \ell_2\rangle$ and $\langle \alpha_1, \alpha_2\rangle$ are *differentials*, but α_i are variables while ∂_i and ℓ_i are expressions. Note that $[a, b]$, $[a + \alpha_1, b + \alpha_2]$ and $[a + \ell_1, b + \ell_2]$ must be *intervals*, according to sequent well-formedness (Section 2.3), which is why we have to add preconditions $\Omega \vDash a + \alpha_1 \leqslant b + \alpha_2$ and $\Omega \vDash a + \ell_1 \leqslant b + \ell_2$.

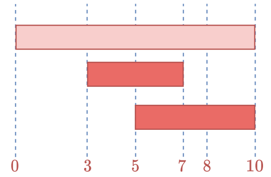
We can instantiate $\langle \alpha_1, \alpha_2 \rangle$ via an admissible *substitution* principle

$$\dfrac{\Omega \vDash \langle \ell_1, \ell_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle \quad \Omega, \langle \alpha_1, \alpha_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle \,; \Gamma \vdash^{s} A^{[a,b]}}{\Omega \,; [\ell_1, \ell_2 / \alpha_1, \alpha_2](\Gamma \vdash^{s} A^{[a,b]})}\;\text{subst}$$

where $[\ell_1, \ell_2/\alpha_1, \alpha_2](\Gamma \vdash^{s} A^{[a,b]})$ substitutes ℓ_i for α_i throughout the sequent. The harmony of both $\Box$ and $\Diamond$ is as follows: one side communicates the interval of interaction $\langle \ell_1, \ell_2 \rangle$ and the other replaces $\langle \alpha_1, \alpha_2 \rangle$ by it. Both sides agree, before the communication, that the base interval of interaction is $[a, b]$ and that the chosen differential must be within $\langle \partial_1, \partial_2 \rangle$.

The modalities $\Box$ and $\Diamond$ model events whose timing is constrained, but not fully determined. Combined with logical connectives and $\bigcirc$, which are synchronous, our logic is expressive enough to model a wide range of temporal phenomena while maintaining computational relevance as we will show during our case study in Section 4.

Example 2.4 (Derivation with $\Box$,$\Diamond$ and $\bigcirc$). The situation, depicted and described by the derivation below, involves all three temporal modalities to prove the formula $\Box^{\langle 0,10 \rangle}(A \supset \bigcirc^{\langle 2,3 \rangle} B) \supset \Diamond^{\langle 3,7 \rangle} A \supset \Diamond^{\langle 3,7 \rangle} \bigcirc^{\langle 2,3 \rangle} B$ at $[0, 0]$. We use $\Omega = \langle \alpha_1, \alpha_2 \rangle \subseteq \langle 3, 7 \rangle$.



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Omega \,; A^{[\alpha_1,\alpha_2]} \vdash^{\alpha_1} A^{[\alpha_1,\alpha_2]}}\,\text{id} \qquad \dfrac{}{\Omega \,; \bigcirc^{\langle 2,3\rangle} B^{[\alpha_1,\alpha_2]} \vdash^{\alpha_1} \bigcirc^{\langle 2,3\rangle} B^{[\alpha_1,\alpha_2]}}\,\text{id}}{\Omega \,; A \supset \bigcirc^{\langle 2,3\rangle} B^{[\alpha_1,\alpha_2]}, A^{[\alpha_1,\alpha_2]} \vdash^{\alpha_1} \bigcirc^{\langle 2,3\rangle} B^{[\alpha_1,\alpha_2]}}\,\supset \text{L}}{\Omega \,; A \supset \bigcirc^{\langle 2,3\rangle} B^{[\alpha_1,\alpha_2]}, A^{[\alpha_1,\alpha_2]} \vdash^{0} \bigcirc^{\langle 2,3\rangle} B^{[\alpha_1,\alpha_2]}}\,\text{delay}}{\Omega \,; A \supset \bigcirc^{\langle 2,3\rangle} B^{[\alpha_1,\alpha_2]}, A^{[\alpha_1,\alpha_2]} \vdash^{0} \Diamond^{\langle 3,7\rangle} \bigcirc^{\langle 2,3\rangle} B^{[0,0]}}\,\Diamond \text{R}}{\Omega \,; \Box^{\langle 0,10\rangle}(A \supset \bigcirc^{\langle 2,3\rangle} B)^{[0,0]}, A^{[\alpha_1,\alpha_2]} \vdash^{0} \Diamond^{\langle 3,7\rangle} \bigcirc^{\langle 2,3\rangle} B^{[0,0]}}\,\Box \text{L}}{\cdot \,; \Box^{\langle 0,10\rangle}(A \supset \bigcirc^{\langle 2,3\rangle} B)^{[0,0]}, \Diamond^{\langle 3,7\rangle} A^{[0,0]} \vdash^{0} \Diamond^{\langle 3,7\rangle} \bigcirc^{\langle 2,3\rangle} B^{[0,0]}}\,\Diamond \text{L}}{\cdot \,; \Box^{\langle 0,10\rangle}(A \supset \bigcirc^{\langle 2,3\rangle} B)^{[0,0]} \vdash^{0} \Diamond^{\langle 3,7\rangle} A \supset \Diamond^{\langle 3,7\rangle} \bigcirc^{\langle 2,3\rangle} B^{[0,0]}}\,\supset \text{R}}{\cdot \,; \vdash^{0} \Box^{\langle 0,10\rangle}(A \supset \bigcirc^{\langle 2,3\rangle} B) \supset \Diamond^{\langle 3,7\rangle} A \supset \Diamond^{\langle 3,7\rangle} \bigcirc^{\langle 2,3\rangle} B^{[0,0]}}\,\supset \text{R}$$

2.7 Examples of IMTL derivations

We say a formula A is *valid* if it holds at any time and for any interval:

<u>def:</u> $\quad A$ is valid $\quad$ *if* $\quad \cdot \,; s$ real, a real, b real $\vdash^{s} A^{[a,b]}$

where A does not mention s, a and b. In other words, A is valid if it holds for any (reasonable) instant of time and interval (remember $s \leqslant a \leqslant b$ holds implicitly by well-formedness.) However, an equivalent way to prove validity is to check whether A holds during $[0, 0]$ starting from point 0 (Lemma 2.5)

Lemma 2.5 (Alternative validity).

$$A \text{ is valid} \quad \textit{if and only if} \quad \cdot \,; \cdot \vdash^{0} A^{[0,0]}$$

Proof (sketch). From left to right by substitution. From right to left, we can construct a derivation that delays from s to a and then acts as the derivation on the right since IMTL restricts rule application by checking only whether the *present time* matches the *start of the interval.* □

We now proceed by analyzing IMTL valid and not valid propositions. We also present additional examples of IMTL derivations.

Lemma 2.6 (IMTL theorems). *For generic $\boldsymbol{\partial}$ and $\boldsymbol{\partial}'$, unless specified, the following formulas are* valid.

(1) $\bigcirc^{\boldsymbol{\partial}}(A \wedge B) \supset (\bigcirc^{\boldsymbol{\partial}} A \wedge \bigcirc^{\boldsymbol{\partial}} B)$
(2) $\Box^{\boldsymbol{\partial}}(A \wedge B) \supset (\Box^{\boldsymbol{\partial}} A \wedge \Box^{\boldsymbol{\partial}} B)$
(3) $\Diamond^{\boldsymbol{\partial}}(A \wedge B) \supset (\Diamond^{\boldsymbol{\partial}} A \wedge \Diamond^{\boldsymbol{\partial}} B)$
(4) $\bigcirc^{\boldsymbol{\partial}}(A \supset B) \supset \bigcirc^{\boldsymbol{\partial}} A \supset \bigcirc^{\boldsymbol{\partial}} B$
(5) $\Box^{\boldsymbol{\partial}}(A \supset B) \supset \Box^{\boldsymbol{\partial}} A \supset \Box^{\boldsymbol{\partial}} B$
(6) $\Diamond^{\boldsymbol{\partial}}(A \supset B) \supset \Box^{\boldsymbol{\partial}} A \supset \Diamond^{\boldsymbol{\partial}} B$
(7) $\Box^{\boldsymbol{\partial}}(A \supset B) \supset \Diamond^{\boldsymbol{\partial}} A \supset \Diamond^{\boldsymbol{\partial}} B$
(8) $\Box^{\boldsymbol{\partial}} A \supset \bigcirc^{\boldsymbol{\partial}'} A \quad$ *for* $\vDash \boldsymbol{\partial}' \subseteq \boldsymbol{\partial}$
(9) $\bigcirc^{\boldsymbol{\partial}'} A \supset \Diamond^{\boldsymbol{\partial}} A \quad$ *for* $\vDash \boldsymbol{\partial}' \subseteq \boldsymbol{\partial}$
(10) $\bigcirc^{\boldsymbol{\partial}} \bigcirc^{\boldsymbol{\partial}'} A \supset \bigcirc^{\boldsymbol{\partial}+\boldsymbol{\partial}'} A$
(11) $\bigcirc^{\boldsymbol{\partial}+\boldsymbol{\partial}'} A \supset \bigcirc^{\boldsymbol{\partial}} \bigcirc^{\boldsymbol{\partial}'} A$

Formulas (1), (2) and (3) show that all temporal modalities distribute over $\wedge$. Propositions (4), (5), (6) and (7) show how the modalities distribute over $\supset$, given the duality between $\Box$ and $\Diamond$. Formulas (8) and (9) clarify the order of strength between the modalities: $\Box$ entails $\bigcirc$, $\bigcirc$ entails $\Diamond$, and the reverse directions do not hold.

Example 2.7. We show the derivation of proposition (6). We use the shorthand $\boldsymbol{\alpha} = \langle \alpha_1, \alpha_2 \rangle$ for variables α_1 and α_2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boldsymbol{\alpha} \subseteq \boldsymbol{\partial} \,; A^{\boldsymbol{\alpha}} \vdash^{\alpha_1} A^{\boldsymbol{\alpha}}}\,\text{id} \qquad \dfrac{}{\boldsymbol{\alpha} \subseteq \boldsymbol{\partial} \,; B^{\boldsymbol{\alpha}} \vdash^{\alpha_1} B^{\boldsymbol{\alpha}}}\,\text{id}}{\boldsymbol{\alpha} \subseteq \boldsymbol{\partial} \,; A \supset B^{\boldsymbol{\alpha}}, A^{\boldsymbol{\alpha}} \vdash^{\alpha_1} B^{\boldsymbol{\alpha}}}\,\supset \text{L}}{\boldsymbol{\alpha} \subseteq \boldsymbol{\partial} \,; A \supset B^{\boldsymbol{\alpha}}, A^{\boldsymbol{\alpha}} \vdash^{0} B^{\boldsymbol{\alpha}}}\,\text{delay}}{\boldsymbol{\alpha} \subseteq \boldsymbol{\partial} \,; (A \supset B)^{\boldsymbol{\alpha}}, A^{\boldsymbol{\alpha}} \vdash^{0} \Diamond^{\boldsymbol{\partial}} B^{[0,0]}}\,\Diamond \text{R}}{\boldsymbol{\alpha} \subseteq \boldsymbol{\partial} \,; (A \supset B)^{\boldsymbol{\alpha}}, \Box^{\boldsymbol{\partial}} A^{[0,0]} \vdash^{0} \Diamond^{\boldsymbol{\partial}} B^{[0,0]}}\,\Box \text{L}}{\cdot \,; \Diamond^{\boldsymbol{\partial}}(A \supset B)^{[0,0]}, \Box^{\boldsymbol{\partial}} A^{[0,0]} \vdash^{0} \Diamond^{\boldsymbol{\partial}} B^{[0,0]}}\,\Diamond \text{L}}{\cdot \,; \Diamond^{\boldsymbol{\partial}}(A \supset B)^{[0,0]} \vdash^{0} \Box^{\boldsymbol{\partial}} A \supset \Diamond^{\boldsymbol{\partial}} B^{[0,0]}}\,\supset \text{R}}{\cdot \,; \cdot \vdash^{0} \Diamond^{\boldsymbol{\partial}}(A \supset B) \supset \Box^{\boldsymbol{\partial}} A \supset \Diamond^{\boldsymbol{\partial}} B^{[0,0]}}\,\supset \text{R}$$

The guideline for deriving the sequent is to apply $\Diamond L$ first, introducing the variables in $\boldsymbol{\alpha}$, then choosing the same $\boldsymbol{\alpha}$ when applying $\Box L$ and $\Diamond R$. Formulas 5 and 6 are similar.

IMTL proofs respect the flow of time, meaning they cannot prove propositions that break temporal causality.

Lemma 2.8 (Some IMTL counterexamples). *The following formulas are* not *valid for general A, B, $\boldsymbol{\partial}$, and $\boldsymbol{\partial}'$. Here we use $☆ \in \{\Box, \Diamond, \bigcirc\}$ to mean any temporal modality.*

(1) $☆^{\boldsymbol{\partial}}(A \vee B) \supset (☆^{\boldsymbol{\partial}} A \vee ☆^{\boldsymbol{\partial}} B)$
(2) $☆^{\boldsymbol{\partial}} \perp \supset \perp$
(3) $☆^{\boldsymbol{\partial}} A \supset A$
(4) $\Diamond^{[\partial_1,\partial_2]} \Diamond^{[\partial_1',\partial_2']} A \supset \Diamond^{[\partial_1+\partial_1',\partial_2+\partial_2']} A$
(5) $\Box^{[\partial_1,\partial_2]} \Box^{[\partial_1',\partial_2']} A \supset \Box^{[\partial_1+\partial_1',\partial_2+\partial_2']} A$

Example 2.9. We show that it is not possible to derive proposition (1) with $☆ = \Box$ as valid. Here we try every possible rule applicable to the sequent except *cut* which is always applicable. If all of the attempts lead to an ill-formed derivation, we can say the sequent is not provable. This works because IMTL has a cut elimination result

(Section 3) and, thus, if a proof has no *cut-free* derivation, then it does not have a derivation at all.

We start with the only option, the $\supset R$ rule, and then note we can either apply $\Box$ L or $\vee$R. If we apply $\vee$R first, we commit to showing either A or B from $A \vee B$ which is not possible. The other option is to apply $\Box\, L$ first:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \cdot\,; A \vee B^{\,[\ell_1,\ell_2]} \vdash^{0} \Box^{\partial} A \vee \Box^{\partial} B^{\,[0,0]}\end{array} \quad \begin{array}{c}(\cdot \vDash \ell_1 \leqslant \ell_2)\\ (\cdot \vDash [\ell_1, \ell_2] \subseteq \partial)\end{array}}{\cdot\,; \Box^{\partial}(A \vee B)^{\,[0,0]} \vdash^{0} \Box^{\partial} A \vee \Box^{\partial} B^{\,[0,0]}}\,\Box\,\mathsf{L}}{\cdot\,; \cdot \vdash^{0} \Box^{\partial}(A \vee B) \supset (\Box^{\partial} A \vee \Box^{\partial} B)^{\,[0,0]}}\supset \mathsf{R}$$

Note that now we have only one option, which is to use $\vee R$. This is because interacting with the antecedent would require delaying until the start of interval $[\ell_1, \ell_2]$ (which is generally not 0), which is not allowed since we cannot ignore the succedent at $[0, 0]$. Thus, we get stuck and the initial sequent is not derivable. A similar pattern arises if we replace $\Box$ by $\Diamond$ or $\bigcirc$.

Kojima and Igarashi [22] point out the importance of developing (constructive) temporal logics where $\vee$ does not distribute over temporal modalities, as it would break temporal causality. Intuitively it would mean we know information about an event before it happens. Their calculus, as is the case with IMTL, does not break causality in this sense, albeit using different technical devices (more about that in Section 5).

2.8 Linear-Time Temporal Logic in IMTL

For the purposes of comparing IMTL with other *intuitionistic linear temporal logics* (ILTLs) and other modal logics it is convenient to define an ILTL within IMTL.

Here we define ILTL as the logic with propositional variables and the standard propositional connectives $P, \wedge, \vee, \supset, \top$ and $\bot$ in addition to the modalities $\bullet\, A$, *next* (from now); $\blacksquare\, A$, *always* (starting from now); and $\blacklozenge\, A$, *eventually* (starting from now). Although our ILTL shares the same grammar as other intuitionistic linear temporal logics, such as, for example, $\mathtt{ITL}_{\Diamond\,\Box}$ in [7] or $\mathtt{ITL}^{e}$ in [1, 6], our insistence regarding *causality* and *computation* results in a different set of valid formulas.

We define ILTL modalities in terms of those of IMTL:

$$\blacksquare\, A \triangleq \Box^{\langle 0,\infty\rangle} A, \quad \blacklozenge\, A \triangleq \Diamond^{\langle 0,\infty\rangle} A, \quad \text{and} \quad \bullet\, A \triangleq \bigcirc^{\langle 1,1\rangle} A$$

where a "discrete" timestep has length 1 in real units. We also use ∞ by extending our real number domain with the symbol ∞ such that $\infty + x = \infty$ for any x. The use of ∞ does not invalidate any of our relevant metatheorems.

Furthermore, we restrict the selectable intervals to be either a singleton $[n, n]$ or an infinite duration interval such as $[n, \infty]$ for an integer n. As far as we know, this restriction means cut elimination may not hold anymore, but it makes it correspond to ILTL with *always* and *eventually* modalities.

In addition to the theorems in Lemma 2.6, the theorems in Lemma 2.10 also hold.

Lemma 2.10 (ILTL theorems). *For* $\bigstar \in \{\blacksquare, \blacklozenge, \bullet\}$*:*

(1) $\bullet\, \bot \supset \bot$ *is not* valid
(2) $\blacksquare\, \bot \supset \bot$ *is* valid
(3) $\blacklozenge\, \bot \supset \bot$ *is not* valid
(4) $\bigstar(A \vee B) \supset (\bigstar\, A \vee \bigstar\, B)$ *is not* valid
(5) $\bullet(A \supset B) \supset \bullet\, A \supset \bullet\, B$ *is* valid
(6) $\bullet\, \blacklozenge\, A \supset \blacklozenge\, \bullet\, A$ *is not* valid
(7) $\blacklozenge\, \bullet\, A \supset \bullet\, \blacklozenge\, A$ *is* valid
(8) $\bullet\, \blacksquare\, A \supset \blacksquare\, \bullet\, A$ *is* valid
(9) $\blacksquare\, \bullet\, A \supset \bullet\, \blacksquare\, A$ *is not* valid
(10) $\blacksquare\, A \supset A \wedge \bullet\, \blacksquare\, A$ *is not* valid
(11) $A \vee \bullet\, \blacklozenge\, A \supset \blacklozenge\, A$ *is not* valid
(12) $\blacksquare\, A \supset A$ *is* valid
(13) $A \supset \blacklozenge\, A$ *is* valid
(14) $\blacksquare\, \blacksquare\, A \supset \blacksquare\, A$ *is* valid
(15) $\blacksquare\, A \supset \blacksquare\, \blacksquare\, A$ *is not* valid
(16) $\blacklozenge\, \blacklozenge\, A \supset \blacklozenge\, A$ *is not* valid
(17) $\blacklozenge\, A \supset \blacklozenge\, \blacklozenge\, A$ *is* valid

In $\mathtt{ITL}^{\mathtt{e}}$ all of the formulas shown above are *valid*, which showcases differences between our ILTL (and by a stretch IMTL) to previous accounts of intuitionistic temporal logic whose semantics is not based on temporal execution.

Propositions (1), (3) and (4) not being valid is closely tied to *temporal causality*, as they would amount to knowing the contents of an event before it happened (same happens in [22] for $\bullet$ only). The rest of the propositions that are invalid in our formulation of ILTL (i.e., (6), (9), (10), (11), (15) and (16)) are thus because of *temporal monotonicity*. As each modality amounts to a computation that takes place along a temporal interval, the order of the modalities in a sequence of modalities matters.

Example 2.11. We show a derivation of validity of proposition (8).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle \alpha_1, \alpha_2\rangle \subseteq [0,\infty]\,; A^{\,[\alpha_1+1,\alpha_2+1]} \vdash^{1} A^{\,[\alpha_1+1,\alpha_2+1]}}\,\mathsf{id}}{\langle \alpha_1, \alpha_2\rangle \subseteq [0,\infty]\,; \Box^{\langle 0,\infty\rangle} A^{\,[1,1]} \vdash^{1} A^{\,[\alpha_1+1,\alpha_2+1]}}\,\Box\,\mathsf{L}}{\langle \alpha_1, \alpha_2\rangle \subseteq [0,\infty]\,; \Box^{\langle 0,\infty\rangle} A^{\,[1,1]} \vdash^{0} A^{\,[\alpha_1+1_1,\alpha_2+1_2]}}\,\mathsf{delay}}{\langle \alpha_1, \alpha_2\rangle \subseteq [0,\infty]\,; \Box^{\langle 0,\infty\rangle} A^{\,[1,1]} \vdash^{0} \bigcirc^{\langle 1,1\rangle} A^{\,[\alpha_1,\alpha_2]}}\,\bigcirc\,\mathsf{R}}{\cdot\,; \Box^{\langle 0,\infty\rangle} A^{\,[1,1]} \vdash^{0} \Box^{\langle 0,\infty\rangle} \bigcirc^{\langle 1,1\rangle} A^{\,[0,0]}}\,\Box\,\mathsf{R}}{\cdot\,; \bigcirc^{\langle 1,1\rangle} \Box^{\langle 0,\infty\rangle} A^{\,[0,0]} \vdash^{0} \Box^{\langle 0,\infty\rangle} \bigcirc^{\langle 1,1\rangle} A^{\,[0,0]}}\,\bigcirc\,\mathsf{R}}{\cdot\,; \vdash^{0} \bigcirc^{\langle 1,1\rangle} \Box^{\langle 0,\infty\rangle} A \supset \Box^{\langle 0,\infty\rangle} \bigcirc^{\langle 1,1\rangle} A^{\,[0,0]}}\supset \mathsf{R}$$

3 METATHEORY

In proof-theoretic terms, we want to show that our inference rules are both *sound* and *complete* (i.e., in harmony). In purely logical terms, we want to show that the *subformula property*, the *disjunction property* and that *consistency* hold. In temporal terms, we want to prove *temporal erasure*, *causality* and *monotonicity*. Fortunately, all of these properties, except completeness, are a consequence of *cut elimination* and induction on the structure of cut-free IMTL proofs. Completeness holds because of *identity elimination*.

We focus the discussion on the properties and proofs that differ the most from the standard literature, which are the proof-theoretic and the temporal ones.

$$\frac{}{\Omega\,;A^{[a,b]} \vdash^{s} A^{[a,b]}}\,\text{id} \qquad \frac{\Omega\,;\Gamma_1 \vdash^{s} A^{[a,b]} \quad \Omega\,;\Gamma_2, A^{[a,b]} \vdash^{s} \gamma}{\Omega\,;\Gamma_1\Gamma_2 \vdash^{s} \gamma}\,\text{cut} \qquad \frac{\Omega \vDash \bot}{\Omega\,;\Gamma \vdash^{s} \gamma}\,\text{imposs} \qquad \frac{\Omega \vDash \mathbb{C}_1 \vee \mathbb{C}_2 \quad \Omega, \mathbb{C}_1\,;\Gamma \vdash^{s} \gamma \quad \Omega, \mathbb{C}_2\,;\Gamma \vdash^{s} \gamma}{\Omega\,;\Gamma \vdash^{s} \gamma}\,\text{split}$$

$$\frac{\Omega\,;\Gamma \vdash^{s} \gamma}{\Omega\,;\Gamma, A^{[a,b]} \vdash^{s} \gamma}\,\text{weak} \qquad \frac{\Omega\,;\Gamma, A^{[a,b]}, A^{[a,b]} \vdash^{s} \gamma}{\Omega\,;\Gamma, A^{[a,b]} \vdash^{s} \gamma}\,\text{cntr} \qquad \frac{\begin{array}{c}\Omega \vDash (s \leqslant u \leqslant a)\\ \Omega\,;\Gamma \vdash^{u} A^{[a,b]}\end{array}}{\Omega\,;\Gamma \vdash^{s} A^{[a,b]}}\,\text{delay} \qquad \frac{}{\Omega\,;\cdot \vdash^{a} \top^{[a,b]}}\,\top\text{R} \qquad \frac{\Omega\,;\Gamma \vdash^{a} \gamma}{\Omega\,;\Gamma, \top^{[a,b]} \vdash^{a} \gamma}\,\top\text{L} \qquad \frac{}{\Omega\,;\bot^{[a,b]} \vdash^{a} \gamma}\,\bot\text{L}$$

$$\frac{\Omega\,;\Gamma, A^{[a,b]}, B^{[a,b]} \vdash^{a} \gamma}{\Omega\,;\Gamma, A \wedge B^{[a,b]} \vdash^{a} \gamma}\,\wedge\text{L} \qquad \frac{\Omega\,;\Gamma_1 \vdash^{a} A^{[a,b]} \quad \Omega\,;\Gamma_2 \vdash^{a} B^{[a,b]}}{\Omega\,;\Gamma_1\Gamma_2 \vdash^{a} A \wedge B^{[a,b]}}\,\wedge\text{R} \qquad \frac{\Omega\,;\Gamma, A^{[a,b]} \vdash^{a} B^{[a,b]}}{\Omega\,;\Gamma \vdash^{a} A \supset B^{[a,b]}}\supset\text{R} \qquad \frac{\Omega\,;\Gamma_1 \vdash^{a} A^{[a,b]} \quad \Omega\,;\Gamma_2, B^{[a,b]} \vdash^{a} \gamma}{\Omega\,;\Gamma_1\Gamma_2, A \supset B^{[a,b]} \vdash^{a} \gamma}\supset\text{L}$$

$$\frac{\Omega\,;\Gamma \vdash^{a} A^{[a,b]}}{\Omega\,;\Gamma \vdash^{a} A \vee B^{[a,b]}}\,\vee\text{R}_1 \qquad \frac{\Omega\,;\Gamma \vdash^{a} B^{[a,b]}}{\Omega\,;\Gamma \vdash^{a} A \vee B^{[a,b]}}\,\vee\text{R}_2 \qquad \frac{\Omega\,;\Gamma, A^{[a,b]} \vdash^{a} \gamma \quad \Omega\,;\Gamma, B^{[a,b]} \vdash^{a} \gamma}{\Omega\,;\Gamma, A \vee B^{[a,b]} \vdash^{a} \gamma}\,\vee\text{L}$$

$$\frac{\begin{array}{c}\Omega, \langle \alpha_1, \alpha_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle \vDash a + \alpha_1 \leqslant b + \alpha_2\\ \Omega, \langle \alpha_1, \alpha_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle\,;\Gamma \vdash^{a} A^{[a+\alpha_1, b+\alpha_2]}\end{array}}{\Omega\,;\Gamma \vdash^{a} \Box^{\langle \partial_1, \partial_2 \rangle} A^{[a,b]}}\,\Box\,\text{R}^{\alpha_1,\alpha_2} \qquad \frac{\begin{array}{c}\Omega \vDash a + \ell_1 \leqslant b + \ell_2\\ \Omega \vDash \langle \ell_1, \ell_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle \quad \Omega\,;\Gamma, A^{[a+\ell_1, b+\ell_2]} \vdash^{a} \gamma\end{array}}{\Omega\,;\Gamma, \Box^{\langle \partial_1, \partial_2 \rangle} A^{[a,b]} \vdash^{a} \gamma}\,\Box\,\text{L} \qquad \frac{\begin{array}{c}\Omega \vDash a + \partial_1 \leqslant b + \partial_2\\ \Omega\,;\Gamma, A^{[a+\partial_1, b+\partial_2]} \vdash^{s} \gamma\end{array}}{\Omega\,;\Gamma, \bigcirc^{\langle \partial_1, \partial_2 \rangle} A^{[a,b]} \vdash^{s} \gamma}\bigcirc\text{L}$$

$$\frac{\begin{array}{c}\Omega \vDash a + \partial_1 \leqslant b + \partial_2\\ \Omega\,;\Gamma \vdash^{s} A^{[a+\partial_1, b+\partial_2]}\end{array}}{\Omega\,;\Gamma \vdash^{s} \bigcirc^{\langle \partial_1, \partial_2 \rangle} A^{[a,b]}}\bigcirc\text{R} \qquad \frac{\begin{array}{c}\Omega \vDash a + \ell_1 \leqslant b + \ell_2\\ \Omega \vDash \langle \ell_1, \ell_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle \quad \Omega\,;\Gamma \vdash^{a} A^{[a+\ell_1, b+\ell_2]}\end{array}}{\Omega\,;\Gamma \vdash^{a} \Diamond^{\langle \partial_1, \partial_2 \rangle} A^{[a,b]}}\,\Diamond\,\text{R} \qquad \frac{\begin{array}{c}\Omega, \langle \alpha_1, \alpha_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle \vDash a + \alpha_1 \leqslant b + \alpha_2\\ \Omega, \langle \alpha_1, \alpha_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle\,;\Gamma, A^{[a+\alpha_1, b+\alpha_2]} \vdash^{a} \gamma\end{array}}{\Omega\,;\Gamma, \Diamond^{\langle \partial_1, \partial_2 \rangle} A^{[a,b]} \vdash^{a} \gamma}\,\Diamond\,\text{L}^{\alpha_1,\alpha_2}$$

Figure 1: IMTL rules

3.1 Identity elimination

Theorem 3.1 (Identity Elimination). *If $\Omega\,;\Gamma \vdash^{s} A^{[a,b]}$ then there is a proof of the same sequent that uses the id rule only for propositional variables.*

We prove identity elimination by proving the *admissibility of identity* first, then using it to replace all id rules by the results of the admissibility theorem.

Theorem 3.2 (Identity admissibility). *The following rule is admissible for any A in the system where identity is restricted to atomic propositions P:*

$$\frac{}{\Omega\,;A^{[a,b]} \vdash^{s} A^{[a,b]}}\,\text{id}$$

Proof. By induction on A. We show the case of $A = \Diamond^{\langle \partial_1, \partial_2 \rangle} A_1$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Omega, \langle \alpha_1, \alpha_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle\,;A_1{}^{[a+\alpha_1, b+\alpha_2]} \vdash^{a} A_1{}^{[a+\alpha_1, b+\alpha_2]}}\,\text{IH}}{\Omega, \langle \alpha_1, \alpha_2 \rangle \subseteq \langle \partial_1, \partial_2 \rangle\,;A_1{}^{[a+\alpha_1, b+\alpha_2]} \vdash^{a} \Diamond^{\langle \partial_1, \partial_2 \rangle} A_1{}^{[a,b]}}\,\Diamond\,\text{R}}{\Omega\,;\Diamond^{\langle \partial_1, \partial_2 \rangle} A_1[a,b] \vdash^{a} \Diamond^{\langle \partial_1, \partial_2 \rangle} A_1{}^{[a,b]}}\,\Diamond\,\text{L}}{\Omega\,;\Diamond^{\langle \partial_1, \partial_2 \rangle} A_1[a,b] \vdash^{s} \Diamond^{\langle \partial_1, \partial_2 \rangle} A_1{}^{[a,b]}}\,\text{delay}$$

□

3.2 Temporal properties

Temporal erasure is the idea that if we erase everything related to time from our calculus the result is a standard intuitionistic logic calculus.

Theorem 3.3 (Temporal erasure). *If $\Omega\,;\Gamma \vdash^{s} A^{[a,b]}$ for a consistent Ω in IMTL then $\Gamma* \vdash A*$ in* intuitionistic logic*, where $A*$ is the result of erasing all temporal modalities from the IMTL proposition A and $\Gamma*$ is the result of doing that to all elements in Γ.*

Proof. By induction on the IMTL derivation. Note most rules are standard intuitionistic logic rules without temporal information. The modality rules and delay collapse, while imposs is impossible and at least one premise of split must have consistent constraints.

□

Causality is informally described as ensuring that "future events cannot affect present decisions". Technically, it has two sides to it: *feasibility* and *strengthening*. Feasibility tells us a process cannot conclude something in its past while strengthening tells us a past event is the same as no event. We designed IMTL's inference rules to naturally enforce causality, so the proof goes by induction on the (cut-free) derivation.

Theorem 3.4 (Temporal causality).

Feasibility:
If $\Omega\,;\Gamma \vdash^{s} A^{[a,b]}$ then $\Omega \vDash s \leqslant a$.
Strengthening:
If $\Omega\,;\Gamma, A^{[a,b]} \vdash^{s} \gamma$ with $\Omega \vDash a < s$ then $\Omega\,;\Gamma \vdash^{s} \gamma$

Feasibility is implied by well-formedness of the sequent. It therefore holds for every sequent in a proof if it holds for the final conclusion, because each rule ensures that all premises are well-formed if the conclusion is well-formed. Formally, this would be proved by induction over the structure of a proof.

Similarly, strengthening holds because all our left rules require the present time to match the start of the interval (because of $\Omega \vDash s = a$). Again, formally this could be expressed as an induction.

Temporal causality is a requirement for intuitionistic temporal logics but it is not sufficient to achieve temporal computability because a temporal logic can conclude only causal sequents without temporally computable derivations (this is the case in Kojima and Igarashi [22]'s sequent calculus).

The logical counterpart of temporal computability is *temporal monotonicity*. Informally, monotonicity means a proof either stays at the same point in time or moves forward (and never backwards). Technically, monotonicity relies on a *timestamps* function (Definition 3.5) defined *inductively* on the structure of IMTL derivations that are *ground* (Ω is empty).

Definition 3.5 (Timestamps). We define a function $\mathbb{T}$ from closed cut-free derivations $\mathcal{D} :: \cdot\,; \Gamma \vdash^{s} \gamma$ to sets of *sequences of numbers*, corresponding to *all possible timestamps* of derivation $\mathcal{D}$. $\mathbb{T}$ is inductively defined on *cut-free* $\mathcal{D}$ with propositional identity only.

If the last rule in $\mathcal{D}$ has one subderivation $\mathcal{D}_1$, with the exception of $\bigcirc$ L, $\bigcirc$ R, $\Box$ R, $\Diamond$ L and delay, as for example

$$\mathcal{D} = \frac{\mathcal{D}_1 :: \Omega\,; \Gamma \vdash^{a} A^{[a,b]}}{\Omega\,; \Gamma \vdash^{a} A \vee B^{[a,b]}}\,\vee\mathsf{R}_1$$

then we add a to all possible sequences,

$$\mathbb{T}(\mathcal{D}) = \{(a, S) \mid S \in \mathbb{T}(\mathcal{D}_1)\}$$

This is the case of weak, cntr, $\top$L, $\wedge$L, $\vee$R$_1$, $\vee$R$_2$, $\supset$ R, $\Box$ L and $\Diamond$ R.

If the last rule in $\mathcal{D}$ has two subderivations $\mathcal{D}_1$ and $\mathcal{D}_2$, with the exception of split, as for example

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1 :: \cdot\,; \Gamma_1 \vdash^{a} A^{[a,b]} \\ \mathcal{D}_2 :: \cdot\,; \Gamma_2, B^{[a,b]} \vdash^{a} \gamma\end{array}}{\cdot\,; \Gamma_1\Gamma_2, A \supset B^{[a,b]} \vdash^{a} \gamma}\supset\mathsf{L}$$

then we add a to the union of all possible timestamps of both branches

$$\mathbb{T}(\mathcal{D}) = \{(a, S) \mid S \in \mathbb{T}(\mathcal{D}_1) \cup \mathbb{T}(\mathcal{D}_2)\}$$

This is the case of $\wedge$R, $\vee$L and $\supset$ L.

If the last rule in $\mathcal{D}$ is $\bigcirc$ L, $\bigcirc$ R or delay , as for example

$$\mathcal{D} = \frac{\begin{array}{c}\cdot \vDash (s \leqslant u \leqslant a) \\ \mathcal{D}_1 :: \cdot\,; \Gamma \vdash^{u} A^{[a,b]}\end{array}}{\cdot\,; \Gamma \vdash^{s} A^{[a,b]}}\,\text{delay}$$

then we do not add any timestamp

$$\mathbb{T}(\mathcal{D}) = \mathbb{T}(\mathcal{D}_1)$$

If the last rule in $\mathcal{D}$ is an axiom, like

$$\mathcal{D} = \frac{}{\cdot\,; P^{[a,b]} \vdash^{s} P^{[a,b]}}\,\text{id} \quad \text{or} \quad \mathcal{D} = \frac{}{\cdot\,; \bot^{[a,b]} \vdash^{a} \gamma}\,\bot\mathsf{L}$$

with the exception of imposs, then the only possible sequence is the one with a on it, (even for the id case),

$$\mathbb{T}(\mathcal{D}) = \{(a)\}$$

where (a) is a one-element sequence. This is the case of id, $\top$R, $\bot$L.

If the rule introduces variables, such as

$$\mathcal{D} = \frac{\mathcal{D}_1 :: [\alpha_1, \alpha_2] \subseteq [\partial_1, \partial_2]\,; \Gamma, A^{[a+\alpha_1, b+\alpha_2]} \vdash^{a} \gamma}{\cdot\,; \Gamma, \Diamond^{[\partial_1,\partial_2]} A^{[a,b]} \vdash^{a} \gamma}\,\Diamond\mathsf{L}$$

$\Diamond$ *L*, then we consider all the possible instantiations $[\ell_1, \ell_2]$ of $[\alpha_1, \alpha_2]$ that satisfy $[\ell_1, \ell_2] \subseteq [\partial_1, \partial_2]$:

$$\mathbb{T}(\mathcal{D}) = \{(a, S) \mid S \in \mathbb{S}\}, \text{ where}$$

$$\mathbb{S} = \bigcup\{\mathbb{T}([\ell_1, \ell_2/\alpha_1, \alpha_2]\mathcal{D}_1) \mid [\ell_1, \ell_2] \subseteq [\partial_1, \partial_2]\}$$

If the last rule in $\mathcal{D}$ is

$$\mathcal{D} = \frac{\cdot \vDash \bot}{\cdot\,; \Gamma \vdash^{s} \gamma}\,\text{imposs}$$

then we have that $\cdot \vDash \bot$, which is unsatisfiable, meaning timestamps will not realize this branch,

$$\mathbb{T}(\mathcal{D}) = \{\}$$

If the last rule in $\mathcal{D}$ is

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1 :: \mathbb{C}_1\,; \Gamma \vdash^{s} \gamma \\ \cdot \vDash \mathbb{C}_1 \vee \mathbb{C}_2 \quad \mathcal{D}_2 :: \mathbb{C}_2\,; \Gamma \vdash^{s} \gamma\end{array}}{\cdot\,; \Gamma \vdash^{s} \gamma}\,\text{split}$$

then by the meaning of $\cdot \vDash \mathbb{C}_1 \vee \mathbb{C}_2$, we know that $\cdot \vDash \mathbb{C}_1$ or $\cdot \vDash \mathbb{C}_2$, meaning we have either $\mathcal{D}_1$ or $\mathcal{D}_2$ closed. Then we define

$$\mathbb{T}(\mathcal{D}) = \mathbb{T}'(\mathcal{D}_1) \cup \mathbb{T}'(\mathcal{D}_2), \text{ where}$$

$$\begin{array}{ll} \mathbb{T}'(\mathcal{F}) = \{\} & \text{if constraints in } \mathcal{F} \text{ are inconsistent} \\ \mathbb{T}'(\mathcal{F}) = \mathbb{T}(\mathcal{F}) & \text{otherwise} \end{array}$$

Theorem 3.6 (Temporal monotonicity). *For any closed derivation $\mathcal{D} :: \cdot\,; \Gamma \vdash^{s} A^{[a,b]}$, all of its possible timestamps $\mathbb{T}(\mathcal{D})$ are monotonically non-decreasing.*

Proof. By induction on the derivation $\mathcal{D}$. □

Temporal monotonicity means that any execution of $\mathcal{D}$ will respect the flow of time granularly, given that all of its actions are monotonically ordered in time.

Note that temporal monotonicity subsumes causality since it forces proofs to obey the flow of time granularly, at every step, while temporal causality only requires every *sequent* to be temporally coherent in isolation.

3.3 Cut elimination

Theorem 3.7 (Cut Elimination). *If $\Omega\,; \Gamma \vdash^{s} A^{[a,b]}$ then there is a proof of the same sequent that does not use the cut rule.*

We prove cut elimination syntactically through *cut admissibility* as in Gentzen [18] and Dragalin [15]. Usually, we prove *cut admissibility* by strengthening the induction hypothesis with structural rules, making sure it covers all cases, including the commuting conversions, which are often the most intricate cases. Since our calculus is *local* and has *intervals* it is *a priori* unclear whether cut premises would always eventually interact despite the commuting conversions.

Proving cut admissibility for a sequent calculus with explicit structural rules was already solved by Gentzen for *weakening* and *contraction*. We do the same in IMTL, but we have to strengthen our cut with *delay* as well. The result is the *strong cut* principle, which connects two derivations at different times u and v and cuts multiple assumptions of $A^{[a,b]}$ at once (using the notation $\{A^{[a,b]}\}^*$).

Theorem 3.8 (Strong Cut Admissibility). *If there are cut-free derivations of $\Omega\,; \Gamma_1 \vdash^{u} A^{[a,b]}$ and $\Omega\,; \Gamma_2, \{A^{[a,b]}\}^* \vdash^{v} \gamma$ with $\Omega \vDash (s \leqslant u, v)$ then there is a cut-free derivation of $\Omega\,; \Gamma_1\Gamma_2 \vdash^{s} \gamma$.*

Proof. We express the theorem as the construction of derivation $\mathcal{F}$ from $\mathcal{D}$ and $\mathcal{E}$ in the form:

$$\dfrac{\begin{array}{c}\mathcal{D} :: \Omega\,;\Gamma_1 \vdash^{u} A^{[a,b]} \\ \Omega \vDash (s \leqslant u, v) \quad \mathcal{E} :: \Omega\,;\Gamma_2, \{A^{[a,b]}\}^{*} \vdash^{v} \gamma\end{array}}{\Omega\,;\Gamma_1\Gamma_2 \vdash^{s} \gamma}\ \mathsf{cut}^{*}$$

$$\rightsquigarrow \quad \mathcal{F} :: \Omega\,;\Gamma_1\Gamma_2 \vdash^{s} \gamma$$

By nested induction on A, then on $\mathcal{D}$ and $\mathcal{E}$. We divide the proof into (overlapping) cases depending on the last rules of $\mathcal{D}$ and $\mathcal{E}$.

The induction hypothesis is enough to solve most cases. We show the *principal case* for $\Box$ — where both $\mathcal{D}$ and $\mathcal{E}$ are interacting with the judgment $A^{[a,b]} = \Box^{[\partial_1,\partial_2]} A_1{}^{[a,b]}$ — and a *commuting case* — when either $\mathcal{D}$ or $\mathcal{E}$ are not interacting with $A^{[a,b]}$.

$\Box$ *principal case.* This is the case where $\mathcal{D}$ is $\Box$ R and $\mathcal{E}$ is $\Box$ L, corresponding to *cut reduction* for $\Box$.

$$\mathcal{D} = \dfrac{\mathcal{D}_1 :: \Omega'\,;\Gamma_1 \vdash^{a} A_1{}^{[a+\alpha_1,b+\alpha_2]}}{\Omega\,;\Gamma_1 \vdash^{a} \Box^{[\partial_1,\partial_2]} A_1{}^{[a,b]}}\ \wedge\mathsf{L}$$

$$\mathcal{E} = \dfrac{\mathcal{E}_1 :: \Omega\,;\Gamma_2, A_1{}^{[a+\ell_1,b+\ell_2]}, \{\Box^{[\partial_1,\partial_2]} A_1{}^{[a,b]}\}^{*} \vdash^{a} \gamma}{\Omega\,;\Gamma_2, \{\Box^{[\partial_1,\partial_2]} A^{[a,b]}\}^{*} \vdash^{a} \gamma}\ \supset\mathsf{R}$$

to construct $\mathcal{F} :: \Omega, \Gamma_1\Gamma_2 \vdash^{s} \gamma$ where $\Omega \vDash [\ell_1, \ell_2] \subseteq [\partial_1, \partial_2]$ and $\Omega' = \Omega, [\alpha_1, \alpha_2] \subseteq [\partial_1, \partial_2]$.

We first remove the copies of $\Box^{[\partial_1,\partial_2]} A_1{}^{[a,b]}$ cutting $\mathcal{D}$ and $\mathcal{E}_1$.

$$\mathcal{F}_1 = \dfrac{\begin{array}{c}\mathcal{D} :: \Omega\,;\Gamma_1 \vdash^{a} \Box^{[\partial_1,\partial_2]} A_1{}^{[a,b]} \\ \mathcal{E}_1 :: \Omega\,;\Gamma_2, A_1{}^{[a+\ell_1,b+\ell_2]}, \{\Box^{[\partial_1,\partial_2]} A_1{}^{[a,b]}\}^{*} \vdash^{a} \gamma\end{array}}{\Omega\,;\Gamma_1\Gamma_2, A_1{}^{[a+\ell_1,b+\ell_2]} \vdash^{s} \gamma}\ \mathsf{IH}$$

And cut $A_1{}^{[a+\ell_1,b+\ell_2]}$ using substitution on $\mathcal{D}_1$ and cutting with the result above.

$$\mathcal{F} = \dfrac{\begin{array}{c}[\ell_1, \ell_2/\alpha_1, \alpha_2]\mathcal{D}_1 :: \Omega'\,;\Gamma_1 \vdash^{a} A_1{}^{[a+\ell_1,b+\ell_2]} \\ \mathcal{F}_1 :: \Omega\,;\Gamma_1\Gamma_2, A_1{}^{[a+\ell_1,b+\ell_2]} \vdash^{s} \gamma\end{array}}{\Omega, \Gamma_1\Gamma_2 \vdash^{s} \gamma}\ \mathsf{IH}$$

Commuting case. Solving commuting cases while following temporal monotonicity impose new challenges, requiring the application of imposs and split rules.

We show an instance of commuting case where *both* $\mathcal{D}$ and $\mathcal{E}$ are not interacting with the principal judgment $A^{[a,b]}$). In this subcase, the last rules in $\mathcal{D}$ and in $\mathcal{E}$ are $\wedge L$ and $\supset R$ respectively.

$$\mathcal{D} = \dfrac{\mathcal{D}_1 :: \Omega\,;\Gamma_1, B_1{}^{[u,u']}, B_2{}^{[u,u']} \vdash^{u} A^{[a,b]}}{\Omega\,;\Gamma_1, B_1 \wedge B_2{}^{[u,u']} \vdash^{u} A^{I}}\ \wedge\mathsf{L}$$

$$\mathcal{E} = \dfrac{\mathcal{E}_1 :: \Omega\,;\Gamma_2, \{A^{[a,b]}\}^{*}, C_1{}^{[v,v']} \vdash^{v} C_2{}^{[v,v']}}{\Omega\,;\Gamma_2, \{A^{I}\}^{*} \vdash^{v} C_1 \supset C_2{}^{[v,v']}}\ \supset\mathsf{R}$$

to construct $\mathcal{F} :: \Omega\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{s} C_1 \supset C_2{}^{[v,v']}$ with $\Omega \vDash (s \leqslant u, v)$.

Note that it is unclear, *a priori*, whether we should progress through $\mathcal{D}$ or $\mathcal{E}$. Since we want to respect monotonicity (Theorem 3.6) we have to progress through the derivation that indexed by *smaller time first*, otherwise the cut elimination procedure could get stuck.

Since we cannot always know the relationship between u and v statically, we solve this case by *splitting* cases on $\Omega \vDash (u \leqslant v) \vee (v \leqslant u)$ and tackling the subcases separately. This is valid because at execution time, when Ω is concretely instantiated, at least one of the branches will have a constraint (equivalent to) $\top$. The cut-free derivation is

$$\mathcal{F} = \dfrac{\begin{array}{c}\Omega \vDash (u \leqslant v) \vee (v \leqslant u) \\ \mathcal{F}_{u\leqslant v} :: \Omega, u \leqslant v\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{s} C_1 \supset C_2{}^{[v,v']} \\ \mathcal{F}_{v\leqslant u} :: \Omega, v \leqslant u\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{s} C_1 \supset C_2{}^{[v,v']}\end{array}}{\Omega\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{s} C_1 \supset C_2{}^{[v,v']}}\ \mathsf{split}$$

where

$$\mathcal{F}_{u\leqslant v} = \dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1 :: \Omega, u \leqslant v\,;\Gamma_1, B_1{}^{[u,u']}, B_2{}^{[u,u']} \vdash^{u} A^{[a,b]} \\ \mathcal{E} :: \Omega, u \leqslant v\,;\Gamma_2, \{A^{[a,b]}\}^{*} \vdash^{u} C_1 \supset C_2{}^{[v,v']}\end{array}}{\Omega, u \leqslant v\,;\Gamma_1\Gamma_2, B_1{}^{[u,u']}, B_2{}^{[u,u']} \vdash^{u} C_1 \supset C_2{}^{[v,v']}}\ \mathsf{IH}}{\Omega, u \leqslant v\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{u} C_1 \supset C_2{}^{[v,v']}}\ \wedge\mathsf{L}}{\Omega, u \leqslant v\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{s} C_1 \supset C_2{}^{[v,v']}}\ \mathsf{delay}$$

and

$$\mathcal{F}_{v\leqslant u} = \dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D} :: \Omega, u \leqslant v\,;\Gamma_1, B_1 \wedge B_2{}^{[u,u']} \vdash^{u} A^{[a,b]} \\ \mathcal{E}_1 :: \Omega, u \leqslant v\,;\Gamma_2, \{A^{[a,b]}\}^{*}, C_1{}^{[v,v']} \vdash^{u} C_2{}^{[v,v']}\end{array}}{\Omega, u \leqslant v\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']}, C_1{}^{[v,v']} \vdash^{v} \supset C_2{}^{[v,v']}}\ \mathsf{IH}}{\Omega, u \leqslant v\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{v} C_1 \supset C_2{}^{[v,v']}}\ \supset\mathsf{R}}{\Omega, u \leqslant v\,;\Gamma_1\Gamma_2, B_1 \wedge B_2{}^{[u,u']} \vdash^{s} C_1 \supset C_2{}^{[v,v']}}\ \mathsf{delay}$$

We solve other subcases similarly. □

4 CASE STUDY: DIGITAL CIRCUITS AS IMTL DERIVATIONS

Digital circuits are an excellent way to explore the expressiveness of IMTL because they describe computations whose (functional) correctness depends on timing. By interpreting IMTL derivations as circuits, we develop a way to design circuits that are well-timed by construction.

We interpret circuit components as implication formulas inside a context Γ and the $\supset$ L rule as sending a (bit) signal to the circuit.

We start by modeling an *inverter*, taking into consideration its temporal behavior. Our model of digital gates captures timing to a reasonably realistic degree, but we abstract away several low-level phenomena related to the physics of gates.

4.1 Model of an inverter (NOT gate)

We model the temporal behavior of an *inverter*, with a *propagation delay* ∂_p — the (maximum) time between the start of the input and the start of the output — and a *contamination delay* ∂_c — the (minimum) time between the end of the input until the end of the output as in Figure 2 (details on the temporal behavior of digital gates can be found, for example, in [4]).

If an input bit, defined as

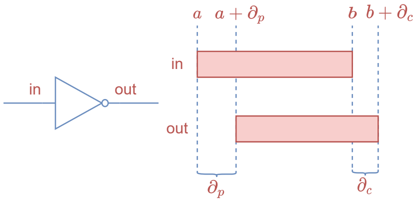


Figure 2: Inverter temporal behavior with propagation and contamination delays

$$\mathsf{Bit} \triangleq \mathsf{lo} \vee \mathsf{hi} \quad \text{where} \quad \mathsf{lo} \triangleq \top \quad \text{and} \quad \mathsf{hi} \triangleq \top,$$

is stable during interval $[a, b]$, the inverter's behavior would be to output at $[a+\partial_p, b+\partial_c]$. In this case study we will assume that $\partial_c < \partial_p$, meaning the output's duration is shorter than the input's by $\partial_p - \partial_c$ time units. We model this behavior using $\bigcirc$ and $\supset$

$$\bigcirc^{[a,b]}(\mathsf{Bit} \supset \bigcirc^{\boldsymbol{\partial}} \mathsf{Bit})$$

where $\boldsymbol{\partial} \triangleq [\partial_p, \partial_c]$.

However, note that our inverter currently works only for a concrete interval I. Instead, we would like it to be parametric over any interval, which hints at using $\Box$. An insufficient attempt would be to replace $\bigcirc^{[a,b]}$ for $\Box^{[a,b]}$, since we would allow for *durationless* interval inputs, which realistically do not work since electronic gates require a minimum duration $d \triangleq \partial_p - \partial_c$ to process the input. A solution is to use a $\Box$ followed by a $\bigcirc^{[0,d]}$, resulting in a gate that is parametric over an input interval but needs a minimum duration d. We are assuming here that the inverter is available for use at any time, so we use $\Box^{[0,\infty]}$.

$$\Box^{[0,\infty]}\,\bigcirc^{[0,d]}(\mathsf{Bit} \supset \bigcirc^{\boldsymbol{\partial}} \mathsf{Bit})$$

This is already a temporally detailed account of a gate. Additionally, we might want to add uncertainty in the output by using $\Diamond$. We then have a *certain delay* $\boldsymbol{\partial}^c = [\partial_1^c, \partial_2^c]$ and an *uncertain delay* $\boldsymbol{\partial}^u = [\partial_1^u, \partial_2^u]$ and d must be the worst case scenario given the uncertainty: $d \triangleq (\partial_1^c + \partial_1^u) - \partial_2^c$ (see Figure 3)

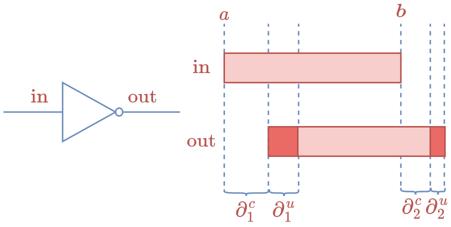


Figure 3: Inverter with uncertain delays

$$\Box^{[0,\infty]}\,\bigcirc^{[0,d]}(\mathsf{Bit} \supset \Diamond^{\boldsymbol{\partial}^u}\,\bigcirc^{\boldsymbol{\partial}^c}\,\mathsf{Bit})$$

Even if we treat components as primitives, notice the formula above is valid, and thus has a derivation describing its inner workings, temporally. Here we use $\Omega \triangleq [\alpha_1, \alpha_2] \subseteq [0, \infty]$.

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \Omega \vDash [\ell_1,\ell_2] \subseteq \boldsymbol{\partial}^u \qquad
        \cfrac{\begin{array}{c}\vdots\\ \Omega\,;\mathsf{Bit}^{[\alpha_1,\alpha_2+d]} \vdash^0 \mathsf{Bit}^{[\alpha_1+(\ell_1+\partial_1^c),\alpha_2+d+(\ell_2+\partial_2^c)]}\end{array}}{\Omega\,;\mathsf{Bit}^{[\alpha_1,\alpha_2+d]} \vdash^0 \bigcirc^{\boldsymbol{\partial}^c}\mathsf{Bit}^{[\alpha_1+\ell_1,\alpha_2+d+\ell_2]}}\;\bigcirc\mathsf{R}
      }{\Omega\,;\mathsf{Bit}^{[\alpha_1,\alpha_2+d]} \vdash^0 \Diamond^{\boldsymbol{\partial}^u}\bigcirc^{\boldsymbol{\partial}^c}\mathsf{Bit}^{[\alpha_1,\alpha_2+d]}}\;\Diamond\mathsf{R}
    }{\Omega\,;\cdot \vdash^0 \mathsf{Bit} \supset \Diamond^{\boldsymbol{\partial}^u}\bigcirc^{\boldsymbol{\partial}^c}\mathsf{Bit}^{[\alpha_1,\alpha_2+d]}}\;\supset\mathsf{R}
  }{\Omega\,;\cdot \vdash^0 \bigcirc^{[0,d]}(\mathsf{Bit} \supset \Diamond^{\boldsymbol{\partial}^u}\bigcirc^{\boldsymbol{\partial}^c}\mathsf{Bit})^{[\alpha_1,\alpha_2]}}\;\bigcirc\mathsf{R}
}{\cdot\,;\cdot \vdash^0 \Box^{[0,\infty]}\bigcirc^{[0,d]}(\mathsf{Bit} \supset \Diamond^{\boldsymbol{\partial}^u}\bigcirc^{\boldsymbol{\partial}^c}\mathsf{Bit})^{[0,0]}}\;\Box\mathsf{R}
$$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{}{\Omega\,;\mathsf{hi}^{[\alpha_1,\alpha_2+d]} \vdash^{\alpha_1+(\ell_1+\partial_1^c)} \mathsf{lo}^{[\alpha_1+(\ell_1+\partial_1^c),\alpha_2+d+(\ell_2+\partial_2^c)]}}\;\top\mathsf{R}
    }{\Omega\,;\mathsf{hi}^{[\alpha_1,\alpha_2+d]} \vdash^{\alpha_1+(\ell_1+\partial_1^c)} \mathsf{Bit}^{[\alpha_1+(\ell_1+\partial_1^c),\alpha_2+d+(\ell_2+\partial_2^c)]}}\;\vee\mathsf{R}_2
  }{\Omega\,;\mathsf{hi}^{[\alpha_1,\alpha_2+d]} \vdash^{\alpha_1} \mathsf{Bit}^{[\alpha_1+(\ell_1+\partial_1^c),\alpha_2+d+(\ell_2+\partial_2^c)]}}\;\mathsf{delay} \qquad (\cdots)
}{\Omega\,;\mathsf{Bit}^{[\alpha_1,\alpha_2+d]} \vdash^{\alpha_1} \mathsf{Bit}^{[\alpha_1+(\ell_1+\partial_1^c),\alpha_2+d+(\ell_2+\partial_2^c)]}}\;\vee\mathsf{L}
$$

where $(\cdots)$ is the opposite case where the input is lo and the inverter chooses to output hi. Here, $\langle \ell_1, \ell_2 \rangle \subseteq \boldsymbol{\partial}^u$ is the "uncertain" part of the inverter delay that the consumer does not know (but the gate knows.)

In the derivation, after the modalities disappear we get to an input interval $[\alpha_1, \alpha_2 + d]$, which is *any* interval with minimum duration d, and an output interval

$$[\alpha_1 + (\partial_1^c + \ell_1), \alpha_2 + d + (\partial_2^c + \ell_2)]$$

which is the same interval shifted by $\boldsymbol{\partial}^u$ and $\boldsymbol{\partial}^c$. Since we are assuming realistic values of $\boldsymbol{\partial}^c$ and $\boldsymbol{\partial}^u$ might have their first component greater than their second component, d must be big enough to ensure that

$$\Omega \vDash \alpha_1 + (\partial_1^c + \ell_1) \leqslant \alpha_2 + d + (\partial_2^c + \ell_2)$$

holds in all cases.

As soon as the circuit finds out whether the value of the Bit is hi or lo by using the $\vee L$ rule, the state of the circuit changes, allowing it to *construct* the opposite Bit in the future regardless if the original Bit is still available or not. The only requirement is that the output interval starts *after* the output interval, which is the case since $\alpha_1 \leqslant \alpha_1 + (\ell_1 + \partial_1^c)$. Note that IMTL derivations model the natural monotonic effect of information through time which adequately represents digital gates.

4.2 Model of two-input gates

The expressiveness of digital gates relies partly on two-input gates since any binary circuit is definable in terms of only *NOR*s or *NAND*s. The challenge lies in modeling two-input gates when each input is stable during different intervals (Figure 5). In our model, a two-input gate starts to process only when both inputs are present.

In IMTL it suffices to model what happens when the two input intervals are the same. Additionally, the $\Box$ takes care of mismatching intervals, as long as they have an intersection. An IMTL two-input gate, available at any time, is the formula

$$\Box^{[0,\infty]}\,\bigcirc^{[0,d]}(\mathsf{Bit} \supset \mathsf{Bit} \supset \Diamond^{\partial_u}\,\bigcirc^{\partial_c}\,\mathsf{Bit})$$

Just like the inverter, we can also assign a IMTL derivation to a two-input gate, as, for instance, an *and* gate. Here we omit some of the details of the derivation since it is similar to the inverter except it has two inputs. We start by eliminating the modalities and implications until we get to the sequent

$$
\dfrac{\dfrac{\dfrac{}{\text{Bit}^{[30,70]} \vdash^{30} \text{Bit}^{[30,70]}}\,\text{id} \qquad \dfrac{\dfrac{}{\text{Bit}^{[30,70]} \vdash^{30} \text{Bit}^{[30,70]}}\,\text{id} \quad \dfrac{}{\Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[30,70]} \vdash^{30} \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[30,70]}}\,\text{id}}{\text{Bit} \supset \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[30,70]}, \text{Bit}^{[30,70]} \vdash^{30} \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[30,70]}}\,{\supset}\text{L}}{\dfrac{\text{Bit} \supset \text{Bit} \supset \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[30,70]}, \text{Bit}^{[30,70]}, \text{Bit}^{[30,70]} \vdash^{30} \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[30,70]}}{\dfrac{\text{Bit} \supset \text{Bit} \supset \bigcirc^{[20,10]}\Diamond^{[5,0]}\text{Bit}^{[30,70]}, \text{Bit}^{[30,70]}, \text{Bit}^{[30,70]} \vdash^{0} \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[30,70]}}{\bigcirc^{[0,20]}(\text{Bit} \supset \text{Bit} \supset \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit})^{[30,50]}, \text{Bit}^{[30,70]}, \text{Bit}^{[30,70]} \vdash^{0} \bigcirc^{[30,70]}\Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[0,0]}}\,\bigcirc\text{L}(\times 2)}\,\text{delay}}\,{\supset}\text{L}}{\Box^{[0,\infty]}\bigcirc^{[0,20]}(\text{Bit} \supset \text{Bit} \supset \Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit})^{[0,0]}, \Box^{[30,100]}\text{Bit}^{[0,0]}, \Box^{[0,70]}\text{Bit}^{[0,0]} \vdash^{0} \bigcirc^{[30,70]}\Diamond^{[5,0]}\bigcirc^{[20,10]}\text{Bit}^{[0,0]}}\,\Box\text{L}(\times 3)
$$

Figure 4: Two-input gate example derivation

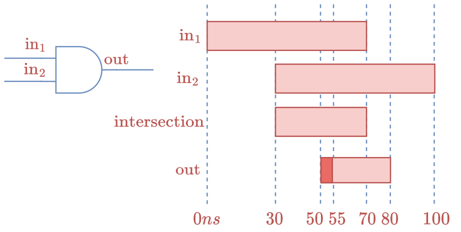


Figure 5: Temporal behavior of a two-input gate

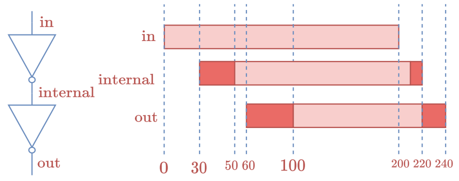


Figure 6: Connected inverters and example waveform

$$[\alpha_1, \alpha_2] \subseteq [0, \infty]\ ; \text{Bit}^{[\alpha_1,\alpha_2+d]}, \text{Bit}^{[\alpha_1,\alpha_2+d]} \vdash^{\alpha_1} \text{Bit}^{[\alpha_1+\ell_1+\partial_1^c, \alpha_2+d+\ell_2+\partial_2^c]}$$

again for a given hidden delay $[\ell_1, \ell_2] \subseteq \boldsymbol{\partial^u}$, in which case we proceed by covering all possible 4 input combinations using ∨Ls followed by ∨R.

The derivation in Example 4.1 represents the situation of applying a two-input gate to skewed inputs, as in Figure 5.

Example 4.1 (Using a two-input gate). We use $d = 20ns$, $\boldsymbol{\partial^c} = [20ns, 10ns]$ and $\boldsymbol{\partial^u} = [5ns, 0ns]$ with ns standing for *nanoseconds*. See Figure 5 for a depiction of the example and Figure 4 for a derivation representing the situation. We omit Ω and merge consecutive rules (indicated).

The derivation uses □ L and ◇ R to select the intersection between the input intervals $[30ns, 100ns]$ and $[0ns, 70ns]$, which is $[30ns, 70ns]$. After that the derivation delays to the right moment and sends both signals to the gate by consecutive applications of the ⊃ L rule.

4.3 Modeling Combinational Circuits

Combinational circuits are built by plugging (sub)circuits together. Checking temporal correctness of these circuits can often be difficult even for simple circuits [4], but IMTL can assist with this issue. Now that we have an inverter IMTL derivation, we want to plug two of them, one after the other, as in Figure 6.

However, the formulas do not exactly match, so cutting them together is not enough. Instead, we manipulate the modalities in such a way that the signals match.

Example 4.2 (Connected inverters). The derivation in Figure 7 models the situation depicted in Figure 6.

The derivation combines signals by manipulating □s and ◇s into matching intervals. The result, unsurprisingly, shows that the uncertainty introduced by ◇ is infectious (i.e, it cannot be eliminated).

We can model other combinational circuits using the framework presented in this section. Note that if a wire splits in two, as is customary in circuit design, we can use contraction to model it.

To analyze a circuit solely from the perspective of its functionality (without timing) we can retrieve an *atemporal* well-formed *intuitionistic logic* derivation from an IMTL one by the *time erasure* property, Theorem 3.3. Standard intuitionistic logic serves as a logical foundation for atemporal executable circuits, further enabling a circuit designer to prove functional properties of the circuit such as knowing whether it implements a given boolean function.

Modeling circuits with loops is a more challenging, albeit interesting, problem which seems to require a notion of proof-circularity. Furthermore, there are even more realistic ways to model circuits. These topics are not the locus of this paper but we plan to explore them in the future.

5 RELATED WORK

Classical Metric Temporal Logic (CMTL). CMTL [23] (see [3] for a survey) and IMTL are solutions to different problems: classical semantics solve model checking while our intuitionistic semantics define a temporally feasible computational interpretation of proof reductions.

Kojima and Igarashi [22] mention the main differences between intuitionistic and classical approaches to temporal logics in their concluding remarks and it seems like the conclusions they achieve somewhat generalize to MTL as well — in summary, CMTL is an IMTL without concerns for temporal causality, monotonicity and proof-relevance. Informally, removing (somehow) these three aspects from IMTL would yield a CMTL, but the details are unclear and there is little justification for exploring this direction.

$$\Omega \triangleq [x, y] \subseteq [20, 10]$$

$$
\dfrac{
\dfrac{}{\cdot\,; \mathsf{Bit}^{[0,200]} \vdash^{0} \mathsf{Bit}^{[0,200]}}\;\mathsf{id}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\Omega\,; \mathsf{Bit}^{[x+30,y+210]} \vdash^{x+3} \mathsf{Bit}^{[x+30,y+210]}}\;\mathsf{id}
\quad
\dfrac{}{\Omega\,; \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[x+30,y+210]} \vdash^{x+30} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[x+30,y+210]}}\;\mathsf{id}
}{\Omega\,; \mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[x+30,y+210]}, \mathsf{Bit}^{[x+30,y+210]} \vdash^{x+30} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[x+30,y+210]}}\;\supset\mathsf{L}
}{\Omega\,; \mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[x+30,y+210]}, \mathsf{Bit}^{[x+30,y+210]} \vdash^{0} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[x+30,y+210]}}\;\mathsf{delay}
}{\Omega\,; \mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[x+30,y+210]}, \mathsf{Bit}^{[x+30,y+210]} \vdash^{0} \Diamond^{[20,10]} \bigcirc^{[30,10]} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[0,200]}}\;\Diamond\mathsf{R}, \bigcirc\mathsf{R}
}{\Omega\,; \Box^{[0,\infty]} \bigcirc^{[0,200]} (\mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit})^{[0,0]}, \mathsf{Bit}^{[x+30,y+210]} \vdash^{0} \Diamond^{[20,10]} \bigcirc^{[30,10]} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[0,200]}}\;\Box\mathsf{L}, \bigcirc\mathsf{L}
}{\cdot\,; \Box^{[0,\infty]} \bigcirc^{[0,200]} (\mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit})^{[0,0]}, \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[0,200]} \vdash^{0} \Diamond^{[20,10]} \bigcirc^{[30,10]} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[0,200]}}\;\Diamond\mathsf{L}, \bigcirc\mathsf{L}
}{
\dfrac{\cdot\,; \Box^{[0,\infty]} \bigcirc^{[0,200]} (\mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit})^{[0,0]}, \mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[0,200]}, \mathsf{Bit}^{[0,200]} \vdash^{0} \Diamond^{[20,10]} \bigcirc^{[30,10]} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[0,200]}}{\begin{array}{c}\cdot\,; \Box^{[0,\infty]} \bigcirc^{[0,200]} (\mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit})^{[0,0]}, \Box^{[0,\infty]} \bigcirc^{[0,200]} (\mathsf{Bit} \supset \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit})^{[0,0]}, \bigcirc^{[0,200]} \mathsf{Bit}^{[0,0]} \\ \vdash^{0} \bigcirc^{[0,200]} \Diamond^{[20,10]} \bigcirc^{[30,10]} \Diamond^{[20,10]} \bigcirc^{[30,10]} \mathsf{Bit}^{[0,0]}\end{array}}\;\Box\mathsf{L}, \bigcirc\mathsf{L}(\times 2), \bigcirc\mathsf{R}
}\;\supset\mathsf{L}
$$

Figure 7: Derivation of two connected inverters

In CMTL the modalities □ and ◇ are definable in terms of one another, which is not possible intuitionistically, but usually, there is no metric version of ○ corresponding to IMTL's ○ modality.

IMTL seems to provide a more symmetric version of the temporal modalities by having symmetric □ and ◇ and a ○ in the middle, forming two adjunctions. The symmetry comes from defining our logic on top of *interval judgments*, rather than *instants in time*, as is usual from CMTL semantics. One result of this approach is that the classical and intuitionistic modalities (as well as logical connectives, such as ∨) do not trivially relate to each other.

Note that a definition of the *until operator* $\mathcal{U}$, common in CMTL, is not as natural in IMTL precisely because of interval judgments. We do not deny the possibility of an intuitionistic until operator, but it does not seem as foundational as its classical counterpart.

A few papers applied CMTL to prove properties of programs [8, 21], but inferring an execution model from the logic itself is a novel contribution. Some works show semantic proofs of cut elimination for CMTL (see, for example, [2, 17]). However, as far as we are aware, no syntactic proofs that might give rise to a concrete computational interpretation exist prior to this work.

Intuitionistic Temporal and Modal Logics. While we are not aware of prior work on intuitionistic versions of MTL, there are multiple accounts of intuitionistic *linear temporal logic* (LTL) and other *modal logics* (for a survey on the former see [7]). Despite their discrete-step semantics, they confront some of the issues we addressed.

Several accounts of LTL such as the ones described in [1, 6], do not respect *temporal causality*, in the sense that, for example, ● distributes over ∨. Kojima and Igarashi [22] being the first (purely logical) account of trying to incorporate causality into the calculus, resulting in temporal logic where ● does *not* distribute over ∨ and $\bullet \bot \supset \bot$ is *not* valid. We saw in Section 2.8 that IMTL's aim for causality and computation causes multiple properties valid in $\mathtt{ITL}^{\mathtt{e}}$ to be invalid in IMTL-based ILTL.

Simpson's elegant account of intuitionistic modal logic [32] is based on Kripke-style world structures where proofs can reason with propositions true at any worlds. A proposition such as $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$ is then valid regardless of the properties of the accessibility relation between worlds, but will not be valid under any natural mapping into IMTL.

Davies [13] provides a different take on *intuitionistic temporal logic* including only the discrete-time ○ to capture binding times and support partial evaluation. His system is temporally monotonic, which is enforced in an entirely different manner using natural deduction and discrete time transformation from a program at time n to a residual program at time $n + 1$. His *logical* system does not include positive types like disjunction and it is not clear if such an extension is easily possible so that normalization/cut elimination would hold. He also does not have □ and ◇ which are a priori semantically incompatible with ○. We recommend [14, Section 6] for an extended discussion of the related complexities.

The work of Kojima and Igarashi [22] seems to be the closest to ours in the sense they provided a syntactic cut elimination result as well as cared about a notion of *temporal causality*, represented by the fact their calculus cannot prove $\bullet(A \vee B) \supset \bullet A \vee \bullet B$ (IMTL also cannot prove it; see Lemma 2.8) because of rule restrictions. Although their goal is for their calculus to respect temporal causality, their proofs do not correspond to temporal computations, at least not directly, since a notion of temporal monotonicity would not hold. Our work extends theirs by replacing points by intervals, adding the □ and ◇ modalities, and by making sure proofs are computational while retaining the notion of causality.

Accounts similar to ours can be found in the combinations of intuitionistic LTL and linear logic (in the sense of Girard [19]) [11, 12]. Proofs in linear logic correspond to communicating processes adhering to session-typed protocols [9]. The types are then augmented with temporal modalities that capture the number of discrete steps taken by a flexible cost model. While the programming language satisfies preservation and progress (incorporating cost), it does not appear that a corresponding logic would satisfy cut elimination.

6 CONCLUDING REMARKS

We defined and studied IMTL, an intuitionistic account of metric temporal logic where proofs respect temporal causality and monotonicity, entailing a *proofs as temporal programs* interpretation.

The concrete description of temporal computation comes from cut reductions, derived from our syntactic proof of cut elimination, but we have not yet developed a programming notation and extracted an operational semantics.

We plan on (1) extending the current logical foundations with recursion (both at the level of types and the level of programs) and explore IMTL's modeling limitations, and (2) developing the modeling of digital circuits with IMTL further than this paper did, tackling interesting issues, such as feedback loops and lenient gates, that seem to interest logicians as well as hardware designers.

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