Subtyping on Nested Polymorphic Session Types

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Abstract. The importance of subtyping to enable a wider range of well-typed programs is undeniable. However, the interaction between subtyping, recursion, and polymorphism is not completely understood yet. In this work, we explore subtyping in a system of nested, recursive, and polymorphic types with a coinductive interpretation, and we prove that this problem is undecidable. Our results will be broadly applicable, but to keep our study grounded in a concrete setting, we work with an extension of session types with explicit polymorphism, parametric type constructors, and nested types. We prove that subtyping is undecidable even for the fragment with only internal choices and nested unary recursive type constructors. Despite this negative result, we present a subtyping algorithm for our system and prove its soundness. We minimize the impact of the inescapable incompleteness by enabling the programmer to seed the algorithm with subtyping declarations (that are validated by the algorithm). We have implemented the proposed algorithm in Rast and it showed to be efficient in various example programs.

1 Introduction

Subtyping is a standard feature in modern programming languages, whether functional, imperative, or object-oriented. The two principal approaches to understanding the meaning of subtyping is via coercions or as subsets. Either way, subtyping allows more programs as well-typed, programs can be more concise, and types can express more informative properties of programs.

When it comes to the interaction between advanced features of type systems, specifically recursive and polymorphic types, there are still some gaps in our understanding of subtyping. In this paper we analyze a particular interaction of features, but as we explain below, we contend that the lessons are broadly applicable.

– Recursive types and type constructors are *equirecursive* (rather than isorecursive) and *structural* (rather than generative).
– Recursively defined type constructors may be *nested* rather than being restricted to be regular (in the terminology of [45]).
– Types are interpreted *coinductively* rather than inductively.
– Polymorphism is *explicit* rather than implicit.
– Types are *linear*.

We prove that subtyping for a language with even a small subset of these features is *undecidable*, including only internal choice and nested unary recursive type constructors, but no function or product types, type quantification, or type constructors with multiple parameters. This may be disheartening, but we also present an incomplete algorithm that has performed well over a range of examples. One of its features is that it is inherently extensible through natural programmer declarations. We provide a simple prototypical example of how nested recursive types and subtyping can cooperate to allow some interesting program properties to be checked and expressed. Other examples of nested polymorphic types and applications in functional programming can be found in the literature [4,41,29,33]. Closely related issues arise in object-oriented programs with path-dependent types [3,40]. Recent semantic investigations have demonstrated that nested types exhibit many of the properties we come to expect from regular types, including induction [36] and parametricity principles [34].

This paper continues an investigation of nested equirecursive types [11]. This prior work did not treat quantified types and considered only *type equality* rather than subtyping. Remarkably, type equality is *decidable*, although no practical and complete algorithm is known. In [11], we also proposed a practical, but incomplete algorithm, which provided some inspiration for the algorithm here. In experiments with nested recursive types restricted to type equality we found that some programs became prohibitively complex, since we needed to explicitly implement (often recursive) coercions between subtypes. This provided some motivation for the generalizations present in this paper. We found that the additional complexities of subtyping are considerable, regarding both pragmatics and theory.

Which of our language features above are essential for our undecidability result and practical algorithm? The nature of the subtyping changes drastically if recursive types are *generative* (or *nominal*), essentially blocking many applications of subtyping. However, the issues addressed in this paper arise again if datasort refinements [22,20,17] are considered. Specifically, the interaction between polymorphism and datasort refinements was identified as problematic [44] and somewhat drastically restricted in SML Cidre [18]. Our results therefore apply to refinement types in the presence of polymorphic constructors.

Regarding nesting, in the non-generative setting it seems difficult to justify the exclusion of nesting. If we disallow it and artificially restrict subtyping, then it becomes decidable by standard subtyping algorithms, whether types are interpreted inductively [38] or coinductively [2,5,25] with the standard notion of *constructor variance* [31,37,1]. We also expect that, once nesting is permitted, there is no essential difference between equi- and isorecursive subtyping. We expect that it will be easy to map isorecursive types to corresponding equirecursive types while preserving subtyping, in which case our algorithm could be applied to isorecursive types. But we leave the details to future work.

Our system of subtyping remains sound when certain types (for example, purely positive recursive types) are considered inductively. However, it then suffers from an additional source of incompleteness by not identifying all empty

types [38]. Since in particular function types are almost always considered nonempty (in essence, coinductively) this does not change our undecidability result. For a purely inductive fragment of the types this can also be obtained directly by a reduction from the undecidability of the language inclusion problem for deterministic pushdown automata [23] following Solmon's construction [45].

Linearity actually plays no role whatsoever in subtyping, and we use it here only because coinductively defined linear types naturally model binary session types [30,26,7,47]. Session typed concurrent programs form the basis for our experimental implementation in Rast [14].

However, because our type simulation is based on the observable communication behavior of processes (like [25]), the choice of Rast as an underlying programming language is of secondary interest. We chose Rast because its coinductive interpretation of types and bidirectional type checker allow for easy experimentation. Lastly, type safety for Rast with explicit polymorphism can be proved in a straightforward way following our previous work [11].

In summary, our main contributions are:

- A definition of subtyping for session types with parametric type constructors, explicit polymorphism, and nested types.
- A proof of undecidability of subtyping for even a small fragment of the proposed nested polymorphic session types.
- A practical (but incomplete) algorithm to check subtyping and its soundness.
- A sound extension of the subtyping algorithm through programmer declarations, to minimize the impact of incompleteness.
- An implementation of the subtyping algorithm.

The remainder of the paper is organized as follows. We provide an overview and simple examples in Section 2. Our language of types and criteria for their validity are presented in Section 3. Section 4 gives a semantic definition of subtyping followed by an undecidability proof by reduction from language inclusion for Basic Process Algebras. In Section 5, we present our algorithm for subtyping and its soundness proof. Section 6 briefly sketches the implementation, followed by further examples in Section 7. We make some comparisons with related work in Section 8 and then conclude with Section 9.

2   Overview

Understanding the interaction between polymorphism and subtyping is not a new concern. In this paper, we study the interaction of explicit polymorphism, recursion, type nesting, and subtyping on (linear) session types [30,26,7]. However, nothing in our analysis depends on linearity, so our key results apply more broadly to any system of equirecursive, structural, coinductive types with nesting and explicit polymorphism.

Consider a type of unary representations of natural numbers given by

$$\mathsf{nat} = \oplus\{\mathbf{z} : \mathbf{1}, \mathbf{s} : \mathsf{nat}\}\,.$$

Each nat is either **z** or **s** followed by a nat. As a session type, it is represented as an *internal choice* between two possible labels: **z** terminates the communication (represented by **1**), and **s** recursively calls nat. The types of even and odd natural numbers can now be defined as

$$\mathsf{even} = \oplus\{\mathbf{z} : \mathbf{1}, \mathbf{s} : \mathsf{odd}\} \quad \text{and} \quad \mathsf{odd} = \oplus\{\mathbf{s} : \mathsf{even}\} \,.$$

As mentioned in the previous section, our recursive types and type constructors are equirecursive and structural, not generative. As such, it becomes unavoidable that we ought to have even and odd as subtypes of type nat, based on a subset interpretation of subtyping.

Moreover, type nesting is also unavoidable in the structural setting. Consider two type definitions for lists that differ only in their type constructors' names:

$$\mathsf{List}[\alpha] = \oplus\{\mathbf{nil} : \mathbf{1}, \mathbf{cons} : \alpha \otimes \mathsf{List}[\alpha]\} \text{ and } \mathsf{List}'[\alpha] = \oplus\{\mathbf{nil} : \mathbf{1}, \mathbf{cons} : \alpha \otimes \mathsf{List}'[\alpha]\} \,.$$

Each $\mathsf{List}[\alpha]$ is either **nil** or a **cons** followed by an α and another $\mathsf{List}[\alpha]$. As a session type, it is an internal choice between sending either the **nil** or **cons** labels. If **cons** is chosen, then the a channel of type α is sent, after which the session continues at type $\mathsf{List}[\alpha]$. In a generative setting, the types $\mathsf{List}[A]$ and $\mathsf{List}'[B]$ would necessarily be unrelated by subtyping – generativity would ensure that $\mathsf{List}[-]$ would remain distinct from $\mathsf{List}'[-]$. However, in the structural setting, $\mathsf{List}[A]$ ought to be a subtype of $\mathsf{List}'[B]$ whenever A is a subtype of B, again based on a subset interpretation of subtyping – the two types differ only in the type constructor name, after all. Moreover, $\mathsf{List}[\mathsf{List}[\mathsf{even}]]$ ought to be a subtype of $\mathsf{List}'[\mathsf{List}'[\mathsf{nat}]]$, and so deciding subtyping relationships between such nested types is inescapable.

Using nested types, we can define type constructors

$$\mathsf{Nil} = \oplus\{\mathbf{nil} : \mathbf{1}\} \quad \text{and} \quad \mathsf{Cons}[\alpha, \kappa] = \oplus\{\mathbf{cons} : \alpha \otimes \kappa\} \,.$$

Both Nil and $\mathsf{Cons}[A, L]$ are subtypes of $\mathsf{List}[A]$ if L is a subtype of $\mathsf{List}[A]$. The subtype Nil characterizes the empty list, whereas $\mathsf{Cons}[A, \mathsf{Cons}[A, \mathsf{Nil}]]$ characterizes lists of length 2, for example. (Roughly speaking, $\mathsf{Cons}^n[A, \mathsf{Nil}]$ characterizes lists of length n.)

Variances will be inferred for all parameters used in type constructors. For example, in the type constructor $\mathsf{List}[\alpha]$ above, the type parameter α is covariant because it occurs in only postive positions. On the other hand, the type $\mathsf{List}[\beta] \multimap \mathsf{List}[\alpha]$ is covariant in α but contravariant in β because α and β appear in positive and negative positions, respectively. This means that, for example, $\mathsf{List}[\mathsf{nat}] \multimap \mathsf{List}[\mathsf{nat}]$ is a subtype of $\mathsf{List}[\mathsf{even}] \multimap \mathsf{List}[\mathsf{nat}]$ – the former type is sufficient anywhere that the latter is required.

As a further example of variances, the type constructor $\mathsf{Seg}[\alpha]$ for list segments given by

$$\mathsf{Seg}[\alpha] = \mathsf{List}[\alpha] \multimap \mathsf{List}[\alpha]$$

has both positive and negative occurrences of α, owing to the presence of $\multimap$. This makes α bivariant in $\mathsf{Seg}[\alpha]$, and it means that, for example, there is no subtyping relationship between $\mathsf{Seg}[\mathsf{nat}]$ and $\mathsf{Seg}[\mathsf{even}]$ despite even being a subtype of nat.

Lastly, our language of types includes explicit polymorphic quantifiers $\exists x.\, A$ and $\forall x.\, A$. We can use the existential quantifier to describe heterogeneous lists:

$$\mathsf{HList} = \oplus\{\mathbf{nil} : \mathbf{1}, \mathbf{cons} : \exists x.\, (x \otimes \mathsf{HList})\}\,.$$

Like each $\mathsf{List}[\alpha]$, each HList is either **nil** or a **cons**; the difference from the type $\mathsf{List}[\alpha]$ is in the existential quantification over the type of data in each **cons**. As a session type, HList types a process that sends either **nil** or **cons**. If **cons** is sent, then some type T and a channel of type T are sent, after which the session continues at type HList (for the list's tail). We can even define type constructors

$$\mathsf{HNil} = \oplus\{\mathbf{nil} : \mathbf{1}\} \quad \text{and} \quad \mathsf{HCons}[\kappa] = \oplus\{\mathbf{cons} : \exists x.\, (x \otimes \kappa)\}\,.$$

Our notion of subtyping is such that both HNil and $\mathsf{HCons}[L]$ are subtypes of HList if L is a subtype of HList. However, the explicit polymorphic quantifiers induce communication, so we do not have $\mathsf{Cons}[A, L]$ as a subtype of HList for any type A, even if L is a subtype of HList.

3   Description of Types

The underlying base system of session types is derived from a Curry-Howard interpretation [7,8] of intuitionistic linear logic [27]. Das et al. extended this type system with parametric type constructors [11]. The *nested polymorphic session types* we propose are also endowed with explicit polymorphism, variances, and subtyping.

3.1   Syntax

We describe *nested polymorphic session types*, their operational interpretation and continuation types in Figure 1.

A, B, C	::=	$\oplus\{\ell : A_\ell\}_{\ell \in L}$	send label $k \in L$	continue at type A_k
	|	$\&\{\ell : A_\ell\}_{\ell \in L}$	receive label $k \in L$	continue at type A_k
	|	$A \otimes B$	send channel $a : A$	continue at type B
	|	$A \multimap B$	receive channel $a : A$	continue at type B
	|	$\mathbf{1}$	send **close** message	no continuation
	|	$\exists x.\, A$	send type B	continue at type $A[B/x]$
	|	$\forall x.\, A$	receive type B	continue at type $A[B/x]$
	|	x	quantified variable	
	|	α	type parameter	
	|	$V[\theta]$	defined type name	

Fig. 1. Description of nested polymorphic session types, their operational semantics and continuation types.

The basic type operators have the usual interpretation: the *internal choice* operator $\oplus\{\ell : A_\ell\}_{\ell \in L}$ selects a branch with label $k \in L$ with corresponding

continuation type A_k; the *external choice* operator $\&\{\ell \colon A_\ell\}_{\ell \in L}$ offers a choice with labels $\ell \in L$ with corresponding continuation types A_ℓ; the *tensor* operator $A \otimes B$ represents the channel passing type that consists of sending a channel of type A and proceeding with type B; dually, the *lolli* operator $A \multimap B$ consists of receiving a channel of type A and continuing with type B; the *terminated session* $\mathbf{1}$ is the operator that closes the session.

We also have two explicit *type quantifier operators*. The existential type $\exists x.\, A$ is interpreted as sending an arbitrary well-formed type B and continuing as type $A[B/x]$. Dually, the universal type $\forall x.\, A$ receives a type B and continues at type $A[B/x]$.

Nested polymorphic session types support *parametrized type definitions* to define new type names. In a type definition, each *type parameter* is assigned a variance. Substitutions θ for these parameters need to comply with the prescribed variances. Below we describe variances, substitutions, and signatures.

$$
\begin{array}{lrl}
\text{Variables} & \mathcal{V} &::= \cdot \mid \mathcal{V}, \alpha \mid \mathcal{V}, x \\
\text{Variance} & \xi &::= + \mid - \mid \top \mid \bot \\
\text{Variances} & \Xi &::= \cdot \mid \Xi, \alpha \,\#\, \xi \\
\text{Substitution } \theta &::= \cdot \mid \theta, A/\alpha \\
& \sigma &::= \cdot \mid \sigma, A/x \\
\text{Signature} & \Sigma &::= \cdot \mid \Sigma, V[\Xi] = A
\end{array}
$$

We distinguish *quantified variables* x from *type parameters* α and, in general, refer to them as *variables*. A *type name* V is defined according to a *type definition* $V[\Xi] = A$ in *signature* Σ, that is parametrized by a sequence of distinct *type parameters* $\overline{\alpha}$ that the type A can refer to. Each type parameter α in Ξ has variance ξ, establishing the position of the occurrences of α in A: covariant $(+)$, contravariant $(-)$, bivariant $(\top)$, or nonvariant $(\bot)$. We instantiate a definition $V[\Xi] = A$ by writing $V[\theta]$, where θ is a substitution for the type parameters in Ξ. We distinguish substitutions for type parameters, θ, from substitutions for quantified variables, σ, because the former require the validation of variances. The set of *free variables* in type A refer to type variables that occur freely in A. Types without any free variables are called *closed types*. Any type not of the form $V[\theta]$ is called *structural*.

3.2   Variances

We define an implication relation on variances. The implication relation constitutes a partial ordering and is defined by the following rules:

$$
\frac{}{\xi \leq \xi} \; \text{refl} \qquad \frac{}{\bot \leq \xi} \; \bot \qquad \frac{}{\xi \leq \top} \; \top
$$

The least upper bound of this lattice is $\top$—if a type name V is covariant, contravariant, or nonvariant in a type parameter α, then it also bivariant—whereas the greatest lower bound is $\bot$. The relation $\leq$ on variances can be shown to be transitive.

The *nesting* of types is echoed in a *nesting operator* on variances, as defined in the adjacent table. Observe that: nesting with $+$ preserves variance; nesting with $-$ converts covariance to contravariance and vice versa; nesting with $\top$ converts covariance and contravariance to bivariance; and $\bot$ is the absorbing element of this operator. The nesting operator can be naturally extended to a set of variances as follows:

$\xi \mid \xi'$	$\bot$	$+$	$-$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$+$	$\bot$	$+$	$-$	$\top$
$-$	$\bot$	$-$	$+$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$

$$(\cdot) \mid \xi = (\cdot) \qquad (\varXi, \alpha \mathbin{\#} \xi) \mid \xi' = (\varXi \mid \xi'), \alpha \mathbin{\#} (\xi \mid \xi')$$

To simplify notation, we sometimes use $\neg\xi$ as an abbreviation for $- \mid \xi$.

Lemma 1 (Properties of nesting).

- *Commutativity:* $\xi \mid \xi' = \xi' \mid \xi$
- *Associativity:* $(\xi_1 \mid \xi_2) \mid \xi_3 = \xi_1 \mid (\xi_2 \mid \xi_3)$
- *Monotonicity: If $\xi \leq \xi'$ then $\xi \mid \zeta \leq \xi' \mid \zeta$ and if $\zeta \leq \zeta'$ then $\xi \mid \zeta \leq \xi \mid \zeta'$*

Proof. Proofs by case analysis on the variances, using the definition of the nesting operator. Monotonicity also uses the definition of the partial order $\leq$.

3.3 Signatures, types, and substitutions

Now we present the criteria for valid signatures, types, and substitutions.

Valid signatures The judgment for valid signatures is written as $\vdash \varSigma \ valid$ and is defined by the following rules:

$$\frac{\vdash_{\varSigma_0} \varSigma_0 \ valid}{\vdash \varSigma_0 \ valid} \ \mathsf{sig} \qquad \frac{}{\vdash_{\varSigma_0} (\cdot) \ valid} \ (\cdot)$$

$$\frac{\vdash_{\varSigma_0} \varSigma \ valid \quad \cdot\,; \varXi \vdash_{\varSigma_0} A \mathbin{\#} + \quad A \neq V'[\theta]}{\vdash_{\varSigma_0} (\varSigma, V[\varXi] = A) \ valid} \ \mathsf{def}$$

Type definitions may be mutually recursive and each parameter of each defined type constructor is assigned a variance that is stored in $\varXi$. In a *valid signature*, all definitions $V[\varXi] = A$ are defined over *structural* types that are *valid* in a context of variance $+$. (The former restriction is usually called *contractivity* [25].)

We take an *equirecursive* view of type definitions, which means that unfolding a type definition does not require communication. More concretely, the type $V[\theta]$ is considered equal to its unfolding $A[\theta]$. (We expect that we can easily adapt our definitions to an *isorecursive* view [39,19] with explicit unfold messages, but we leave the details as future work.) All type names V occurring in a valid signature must be defined, and all type parameters defined in a valid type definition must be distinct. Furthermore, for a valid definition $V[\varXi] = A$, the free variables occurring in A must be contained in $\varXi$.

Valid types The judgment for valid types is written as $\mathcal{V} \; ; \; \Xi \vdash_{\Sigma_0} A \mathbin{\#} \xi$ and is defined over a valid signature Σ_0. $\mathcal{V}$ stores the free variables in A and Ξ stores the assignments of type parameters to variances. This judgment expresses that if A appears in a context of variance ξ then all the type variables $\beta \mathbin{\#} \zeta$ in Ξ will occur only as prescribed by ζ. For example, if $\xi = -$ and $\beta \mathbin{\#} -$ then all occurrences of β in A must be in covariant or nonvariant positions. If $\xi = \bot$ then no requirement is imposed on any occurrences. We elide the subscript Σ_0 in the rules presented below.

Internal and external choices are covariant in the continuation types, *lolli* is contravariant in the first component and covariant in the second component, and tensor is covariant in both components.

$$\frac{\mathcal{V} \; ; \; \Xi \vdash A_\ell \mathbin{\#} \xi \qquad (\forall \ell \in L)}{\mathcal{V} \; ; \; \Xi \vdash \oplus\{\ell : A_\ell\}_{\ell \in L} \mathbin{\#} \xi} \; \oplus \qquad \frac{\mathcal{V} \; ; \; \Xi \vdash A_\ell \mathbin{\#} \xi \qquad (\forall \ell \in L)}{\mathcal{V} \; ; \; \Xi \vdash \&\{\ell : A_\ell\}_{\ell \in L} \mathbin{\#} \xi} \; \&$$

$$\frac{\mathcal{V} \; ; \; \Xi \vdash A_1 \mathbin{\#} \neg\xi \quad \mathcal{V} \; ; \; \Xi \vdash A_2 \mathbin{\#} \xi}{\mathcal{V} \; ; \; \Xi \vdash A_1 \multimap A_2 \mathbin{\#} \xi} \; \multimap \qquad \frac{\mathcal{V} \; ; \; \Xi \vdash A_1 \mathbin{\#} \xi \quad \mathcal{V} \; ; \; \Xi \vdash A_2 \mathbin{\#} \xi}{\mathcal{V} \; ; \; \Xi \vdash A_1 \otimes A_2 \mathbin{\#} \xi} \; \otimes$$

Explicitly quantified types are covariant in the continuation type. Explicitly quantified variables do not carry any variance information, so they are kept in the set of variables $\mathcal{V}$.

$$\frac{\mathcal{V}, x \; ; \; \Xi \vdash A \mathbin{\#} \xi}{\mathcal{V} \; ; \; \Xi \vdash \exists x.\, A \mathbin{\#} \xi} \; \exists x \qquad \frac{\mathcal{V}, x \; ; \; \Xi \vdash A \mathbin{\#} \xi}{\mathcal{V} \; ; \; \Xi \vdash \forall x.\, A \mathbin{\#} \xi} \; \forall x$$

The terminated session $\mathbf{1}$ and quantified variables x are valid in a context with any variance, whereas type parameters $\alpha \mathbin{\#} \xi' \in \Xi$ are valid in contexts of variance at most ξ'. For instance, $\cdot \; ; \; \alpha \mathbin{\#} \top \vdash \alpha \mathbin{\#} +$.

$$\frac{}{\mathcal{V} \; ; \; \Xi \vdash \mathbf{1} \mathbin{\#} \xi} \; \mathbf{1} \qquad \frac{x \in \mathcal{V}}{\mathcal{V} \; ; \; \Xi \vdash x \mathbin{\#} \xi} \; \mathsf{var} \qquad \frac{\alpha \mathbin{\#} \xi' \in \Xi \quad \xi \leq \xi'}{\mathcal{V} \; ; \; \Xi \vdash \alpha \mathbin{\#} \xi} \; \mathsf{par}$$

We instantiate a definition $V[\Xi_V] = A_V$ by writing $V[\theta]$, where θ is a *valid substitution* for type parameters in Ξ_V. Each type in θ must be valid according to the variance in the context of $V[\theta]$ and by Ξ_V. We apply the nesting operator to combine these variances.

$$\frac{V[\Xi_V] = A_V \in \Sigma_0 \quad \mathcal{V} \; ; \; \Xi \vdash \theta \mathbin{\#} (\Xi_V \mid \xi)}{\mathcal{V} \; ; \; \Xi \vdash V[\theta] \mathbin{\#} \xi} \; \mathsf{def}$$

Valid substitutions The judgment for valid substitutions is written as $\mathcal{V} \; ; \; \Xi \vdash_{\Sigma_0} \theta \mathbin{\#} \Xi$ and is defined over a valid signature Σ_0, whose reference we omit. Again, $\mathcal{V}$ is a set of variables and Ξ is a set of type parameters and their corresponding variances. This judgment expresses that a substitution θ is valid on a set of variances Ξ if any type variable α in Ξ is substituted for a type A with (at least) the same variance.

$$\frac{}{\mathcal{V} \; ; \; \Xi \vdash (\cdot) \mathbin{\#} (\cdot)} \; (\cdot) \qquad \frac{\mathcal{V} \; ; \; \Xi \vdash \theta \mathbin{\#} \Xi_\theta \quad \mathcal{V} \; ; \; \Xi \vdash A \mathbin{\#} \xi}{\mathcal{V} \; ; \; \Xi \vdash (\theta, A/\alpha) \mathbin{\#} (\Xi_\theta, \alpha \mathbin{\#} \xi)} \; \mathsf{subs}$$

Properties of variance We now identify some properties of variance on nested polymorphic session types and revisit some examples. Additional properties can be found in the supplementary material.

Lemma 2. *The following properties hold.*

1. *If $\mathcal{V}$; $\Xi \vdash A$ # ξ and $\xi' \leq \xi$ then $\mathcal{V}$; $\Xi \vdash A$ # ξ'.*
2. *If $\mathcal{V}$; $\Xi \vdash A$ # ξ and $\mathcal{V}$; Ξ, α # $\xi \vdash C$ # ζ then $\mathcal{V}$; $\Xi \vdash C[A/\alpha]$ # ζ.*
3. *If $\mathcal{V}$; $\Xi \vdash \theta$ # Ξ' and $\mathcal{V}$; $\Xi' \vdash C$ # ζ then $\mathcal{V}$; $\Xi \vdash C[\theta]$ # ζ.*
4. *If $\mathcal{V}$; $\Xi \vdash A$ # ξ then $\mathcal{V}$; $\Xi \mid \xi' \vdash A$ # $\xi \mid \xi'$.*
5. *If $\mathcal{V}$; $\Xi \vdash V[\theta']$ # ξ and $V[\Xi'] = A' \in \Sigma_0$ then $\mathcal{V}$; $\Xi \vdash A'[\theta']$ # ξ.*

Proof. (1) Proof by induction, using transitivity of $\leq$ and monotonicity of nesting. (2) Proof by induction, using (1). (3) Proof by induction, using (2). (4) Proof by induction, using associativity of nesting. (5) Proof by inversion, using (4), (3), and $+ \mid \xi = \xi$.

Example 1. Consider a signature composed of definitions for the type of *list segments* presented in section 2:

$$\Sigma_0 = \{\ \mathsf{List}[\alpha \,\#\, +] = \oplus\{\mathbf{nil} : \mathbf{1}, \mathbf{cons} : \alpha \otimes \mathsf{List}[\alpha]\},$$
$$\mathsf{Seg}[\alpha \,\#\, \top] = \mathsf{List}[\alpha] \multimap \mathsf{List}[\alpha]\ \}$$

To prove that Σ_0 is valid, we initiate the validity check of $\mathsf{List}[\alpha$ # $+]$ with $\cdot$; α # $+ \vdash_{\Sigma_0} \oplus\{\mathbf{nil} : \mathbf{1}, \mathbf{cons} : \alpha \otimes \mathsf{List}[\alpha]\}$ # $+$. We then explore each branch using the $\oplus$ rule. Since the **nil** branch is trivial, we show the **cons** branch. Here, we check $\cdot$; α # $+ \vdash_{\Sigma_0} \alpha \otimes \mathsf{List}[\alpha]$ # $+$, which, in turn, checks $\cdot$; α # $+ \vdash_{\Sigma_0} \alpha$ # $+$ and $\cdot$; α # $+ \vdash_{\Sigma_0} \mathsf{List}[\alpha]$ # $+$. The former follows from par rule, while the latter reduces to checking $\cdot$; α # $+ \vdash_{\Sigma_0} \alpha$ # $(+ \mid +)$, and then par rule applies.

To verify validity of $\mathsf{Seg}[\alpha$ # $\top]$, we check $\cdot$; α # $\top \vdash_{\Sigma_0} \mathsf{List}[\alpha] \multimap \mathsf{List}[\alpha]$ # $+$. Intuitively, this should follow since α has variance $\top$ in the context, and therefore can occur covariantly and contravariantly. Formally, this reduces (by the $\multimap$ rule) to checking $\cdot$; α # $\top \vdash_{\Sigma_0} \mathsf{List}[\alpha]$ # $+$ and $\cdot$; α # $\top \vdash_{\Sigma_0} \mathsf{List}[\alpha]$ # $-$, which follows by the def rule and by property (1) of Lemma 2.

4 Subtyping

The ability to verify subtyping is paramount to enable the specification of more expressive (well-typed) programs. However, the combination of recursion, polymorphism, and subtyping has proven to be challenging. The subtyping relation for session types was proposed by Gay and Hole [25] to enhance the flexibility of the type system. We say that A is a subtype of B, written $A \leq B$, if every behavior permitted by A is also permitted by B. However, also for session types, the combination of subtyping and explicit polymorphic quantifiers has proven difficult [24]. In this section, we provide the definition of subtyping for nested polymorphic session types and build on the legacy of the undecidability

of the inclusion problem for simple languages [23] to prove that subtyping is also undecidable.

Our results show that even for a very simple fragment of nested polymorphic session types, subtyping is undecidable. This fragment is composed of types defined *mutual recursively* through *unary type definitions* involving non-deterministic (labelled) choices. In our session-typed setting, that means *nested* session types defined through *unary* type definitions whose type parameter is always assigned positive variance; these definitions only provide *internal choices*.

4.1   Subtyping definition

We start by defining the notion of *unfolding* on nested polymorphic session types. We define $\mathsf{unfold}_{\Sigma_0}(A)$ recursively on the structure of A, according to the following rules:

$$\frac{V[\Xi'] = B \in \Sigma_0 \quad \mathcal{V}\,;\, \Xi \vdash \theta \mathbin{\#} \Xi'}{\mathsf{unfold}_{\Sigma_0}(V[\theta]) = B[\theta]}\ \mathsf{def} \qquad \frac{}{\mathsf{unfold}_{\Sigma_0}(A) = A}\ \mathsf{str}$$

Simulation and variance-based type relations. The subtyping relation for closed types is defined based on the notion of *type simulation*. Let *Type* denote the set of closed types.

Definition 1. *A relation $\mathcal{R} \subseteq \mathit{Type} \times \mathit{Type}$ is a type simulation if $(A, B) \in \mathcal{R}$ implies the following conditions:*

- *If $\mathsf{unfold}_{\Sigma_0}(A) = \oplus\{\ell : A_\ell\}_{\ell \in L}$, then $\mathsf{unfold}_{\Sigma_0}(B) = \oplus\{m : B_m\}_{m \in M}$ where $L \subseteq M$ and $(A_\ell, B_\ell) \in \mathcal{R}$ for all $\ell \in L$.*
- *If $\mathsf{unfold}_{\Sigma_0}(A) = \&\{\ell : A_\ell\}_{\ell \in L}$, then $\mathsf{unfold}_{\Sigma_0}(B) = \&\{m : B_m\}_{m \in M}$ where $L \supseteq M$ and $(A_m, B_m) \in \mathcal{R}$ for all $m \in M$.*
- *If $\mathsf{unfold}_{\Sigma_0}(A) = A_1 \multimap A_2$, then $\mathsf{unfold}_{\Sigma_0}(B) = B_1 \multimap B_2$ and $(B_1, A_1) \in \mathcal{R}$ and $(A_2, B_2) \in \mathcal{R}$.*
- *If $\mathsf{unfold}_{\Sigma_0}(A) = A_1 \otimes A_2$, then $\mathsf{unfold}_{\Sigma_0}(B) = B_1 \otimes B_2$ and $(A_1, B_1) \in \mathcal{R}$ and $(A_2, B_2) \in \mathcal{R}$.*
- *If $\mathsf{unfold}_{\Sigma_0}(A) = \mathbf{1}$, then $\mathsf{unfold}_{\Sigma_0}(B) = \mathbf{1}$.*
- *If $\mathsf{unfold}_{\Sigma_0}(A) = \exists x.\, A'$, then $\mathsf{unfold}_{\Sigma_0}(B) = \exists y.\, B'$ and for all $C \in \mathit{Type}$, $(A'[C/x], B'[C/y]) \in \mathcal{R}$.*
- *If $\mathsf{unfold}_{\Sigma_0}(A) = \forall x.\, A'$, then $\mathsf{unfold}_{\Sigma_0}(B) = \forall y.\, B'$ and for all $C \in \mathit{Type}$, $(A'[C/x], B'[C/y]) \in \mathcal{R}$.*

Relying on the idea that covariance is *the variance*, we now define relations based on each kind of variance.

Definition 2. *Given a relation $\mathcal{R} \subseteq \mathit{Type} \times \mathit{Type}$, we define variance-based relations as follows:*

- *The covariant-relation of $\mathcal{R}$ is $\mathcal{R}^+ = \mathcal{R}$.*
- *The contravariant-relation of $\mathcal{R}$ is $\mathcal{R}^- = \{(A, B) \mid (B, A) \in \mathcal{R}\}$.*
- *The bivariant-relation of $\mathcal{R}$ is $\mathcal{R}^\top = \{(A, B) \mid (A, B) \in \mathcal{R} \text{ and } (B, A) \in \mathcal{R}\}$.*
- *The nonvariant-relation of $\mathcal{R}$ is $\mathcal{R}^\perp = \{(A, B) \mid A, B \in \mathit{Type}\}$.*

The subtyping relation. The definition of subtyping is based on the notion of type simulation and takes advantage of variance-based relations to ensure the required sensitivity to variances.

Definition 3 (Subtyping). *Given closed types A and B s.t. $\cdot \,;\, \cdot \vdash A \,\#\, \xi$ and $\cdot \,;\, \cdot \vdash B \,\#\, \xi$, we say that A is a subtype of B at variance ξ, written $A \leq B \,\#\, \xi$, if there exists a type simulation $\mathcal{R}$ such that $(A, B) \in \mathcal{R}^\xi$.*

Since our algorithms have to deal with open types, we can lift this definition by considering suitably valid substitution instances as follows.

Definition 4 (Subtyping of open types). *Given types A and B s.t. $\mathcal{V} \,;\, \cdot \vdash A \,\#\, \xi$ and $\mathcal{V} \,;\, \cdot \vdash B \,\#\, \xi$, we say that A is a subtype of B at variance ξ, written $\forall \mathcal{V}.\, A \leq B \,\#\, \xi$, if there exists a type simulation $\mathcal{R}$ such that $(A[\sigma], B[\sigma]) \in \mathcal{R}^\xi$ for all closed substitutions σ over $\mathcal{V}$.*

The definition of subtyping could be extended to open types defined over a non-empty set $\varXi$, just by additionally closing the types with all substitutions θ and θ' such that $\vdash \theta \,\#\, \varXi$ and $\vdash \theta' \,\#\, \varXi$, and $(C, D) \in \mathcal{R}^\xi$ for each $C/\alpha \in \theta$ and $D/\alpha \in \theta'$ and $\alpha \,\#\, \xi \in \varXi$. However, at present, we only need to consider an empty $\varXi$ because (a) syntactic subtyping is only invoked from the type checker for the program, and the program only has the quantified variables in $\mathcal{V}$, (b) definitions are unfolded during the algorithm so we never need to consider a type with free type parameters.

4.2   Undecidability of subtyping

In this section, we prove that the subtyping of nested polymorphic session types is *undecidable*. Subtyping is undecidable even for a small fragment of these types.

In previous work we observed that the type system for nested session types (without explicit quantifiers) has many similarities with deterministic pushdown automata [11]. Even though we have shown that the type equality problem is decidable, it is thus perhaps not surprising that the subtyping problem inherits the famous undecidability of the language inclusion problem for simple languages [23]. We now prove that the subtyping problem for nested polymorphic session types is undecidable by reducing the language inclusion problem for Basic Process Algebra (BPA) processes [28] to our problem.

Basic Process Algebras. BPA *expressions* are defined by the grammar:

$$\text{BPA Expressions } p, q ::= a \mid X \mid p + q \mid p \cdot q$$

where a ranges over a set of *atomic actions*, X is a *variable*, $+$ represents *nondeterministic choice*, and $\cdot$ represents *sequential composition* [9]. Recursive *BPA processes* are defined by means of *process equations* $\varDelta = \{X_i \triangleq p_i\}_{i \in I}$, where X_i are distinct variables, one of which is identified as the *root*. A BPA expression is *guarded* if any variable occurs in the scope of an atomic action. A system

$\Delta = \{X_i \triangleq p_i\}_{i \in I}$ of process equations is guarded if p_i is a guarded expression, for all $i \in I$. We consider that *all* process equations are guarded. The *empty process* ε is the neutral element of sequential composition, but does not occur in a process definition—it is only considered for the operational semantics (for more details, see [9]).

The operational semantics of a BPA process is a labelled transition relation, defined over a set Δ of guarded equations as follows:

$$\frac{}{a \xrightarrow{a} \varepsilon} \qquad \frac{p \xrightarrow{a} p'}{p + q \xrightarrow{a} p'} \qquad \frac{q \xrightarrow{a} q'}{p + q \xrightarrow{a} q'} \qquad \frac{p \xrightarrow{a} p'}{p \cdot q \xrightarrow{a} p' \cdot q} \qquad \frac{p \xrightarrow{a} \varepsilon}{p \cdot q \xrightarrow{a} q}$$

$$\frac{p \xrightarrow{a} p' \ (X \triangleq p \in \Delta)}{X \xrightarrow{a} p'}$$

The language accepted by a BPA process p is defined as $L(p) = \{\overline{a} \mid p \xrightarrow{\overline{a}} \varepsilon\}$. A process p is *deterministic* if whenever $p \xrightarrow{a} p'$ and $p \xrightarrow{a} p''$, then $p' = p''$. A set of process equations Δ is *normed* if for all variables X in Δ, there is a *trace* $\overline{a}$ s.t. $X \xrightarrow{\overline{a}} \varepsilon$. A BPA process p is normed if it is defined through a normed set of equations.

Translation of BPA to Nested Polymorphic Session Types In the remainder of this section, we focus on normed and deterministic processes, defined over a set of guarded process equations. Let $\mathcal{P}$ denote the set of such BPA processes. We present the translation of a BPA process $p_0 \in \mathcal{P}$, defined over a set of BPA equations Δ_0 with root X_0, in three steps: (1) we propose a general translation of BPA guarded expressions to nested session types, (2) we convert Δ_0 into a type signature Σ_0, and (3) we propose a translation for p_0.

A *guarded* BPA expression is translated to a nested session type without explicit quantifiers and parametrized by a type variable α. This translation is denoted by $(\!|\cdot|\!)_\alpha$ and is defined as follows:

$$(\!|a|\!)_\alpha = \oplus\{a : \alpha\} \qquad (\!|X|\!)_\alpha = X[\alpha/\alpha] \qquad (\!|a \cdot p + b \cdot q|\!)_\alpha = \oplus\{a : (\!|p|\!)_\alpha, b : (\!|q|\!)_\alpha\}$$

$$(\!|p \cdot q|\!)_\alpha = (\!|p|\!)_\alpha[(\!|q|\!)_\alpha/\alpha] \qquad\qquad (\!|\varepsilon|\!)_\alpha = \alpha$$

Through this translation, BPA expressions are converted into session types characterized by providing internal choices and nesting. Atomic actions are translated into an internal (single) choice with label a. Process variables lead to type names that are further defined through type definitions (detailed below). Non-deterministic choices occurring in *guarded* process expressions are characterized by providing atomic actions in the scope of any type variable; for this reason, any non-deterministic choice can be written in the form $a \cdot p + b \cdot q$, where a and b are atomic actions and p and q are processes. These choices are converted into internal choices where each branch is labelled by the corresponding atomic action. Sequential composition is translated into nested types. Empty continuations are captured by the type parameter α. Since we are only considering deterministic processes, the translated types are well-defined.

The type signature Σ_0 is composed by the translation of all process equations in Δ_0.

$$\Sigma_0 = (\!|\Delta_0|\!)_\alpha = \{X[\alpha \ \# \ +] = (\!|p|\!)_\alpha \mid X \triangleq p \in \Delta\}$$

All type definitions are parametrized by α. The parameter α is assigned variance $+$ because it only occurs in covariant positions in the type constructors used in the translation.

Finally, the translation of process $p_0 \in \mathcal{P}$, defined over Δ_0 with root X_0, is the nested session type $X_p[\mathbf{1}/\alpha]$ defined over signature $\Sigma_0 = (\!|\Delta|\!)_\alpha$. We denote the translation of p_0 by $(\!|p_0|\!)$.

Example 2. The BPA process p_0 with root X_0, defined over $\Delta_0 = \{X_0 \triangleq a \ \cdot \ X_0 \cdot c + b \cdot X_1, \ \ X_1 \triangleq a\}$ is translated to type $(\!|p_0|\!) = X_0[\mathbf{1}/\alpha]$, defined over the signature composed by the definitions:

$$X_0[\alpha \ \# \ +] = \oplus\{a : (\!|X_0 \cdot c|\!)_\alpha, b : (\!|X_1|\!)_\alpha\} = \oplus\{a : X_0[\oplus\{c : \alpha\}/\alpha], b : X_1[\alpha/\alpha]\}$$
$$X_1[\alpha \ \# \ +] = \oplus\{a : (\!|\varepsilon|\!)_\alpha\} \qquad\qquad\qquad = \oplus\{a : \alpha\}.$$

The parallel between p_0's labelled transitions – a, b, c – and $(\!|p_0|\!)$'s simulation steps – $\oplus a, \oplus b, \oplus c$ – should now become clear.

Undecidability of Subtyping We now use the translation above to reduce the language inclusion problem for BPA processes to the subtyping problem of nested polymorphic session types. Groote and Huttel proved that the former is undecidable [28], thus our subtyping problem is also undecidable. Indeed, this result tells us more: the subtyping problem is undecidable for the smaller fragment of nested polymorphic session types that allows mutual (parametrized) type definitions only involving internal choices and type nesting.

We start with two auxiliary lemmas.

Lemma 3. *Given two deterministic and normed processes $p, q \in \mathcal{P}$, if $L(p) \subseteq L(q)$ and $p \xrightarrow{a} p'$ and $q \xrightarrow{a} q'$, then $L(p') \subseteq L(q')$.*

Proof. Assume that exists $w \in L(p')$ s.t. $w \notin L(q')$. Obviously, $a \cdot w \in L(p)$. However, $a \cdot w \notin L(q)$ because q is deterministic and there is no $q'' \neq q'$ s.t. $q \xrightarrow{a} q''$. Thus, we would have $L(p) \nsubseteq L(q)$.

Lemma 4. *Given a deterministic and normed process $p \in \mathcal{P}$, $\mathsf{unfold}_{\Sigma_0}((\!|p|\!)_\alpha) = \oplus\{a : (\!|p_a|\!)_\alpha\}_{a \in L}$ where $L = \{a \mid p \xrightarrow{a} p_a\}$.*

Proof. Proof by induction on the structure of p. The cases for atomic action and non-deterministic choice are immediate from the definition of $(\!|\cdot|\!)_\alpha$.

Case $p = X$ and $X \triangleq q \in \Delta$. In this case, for each $a \in L$, by the last labelled transition rule, we have $q \xrightarrow{a} p_a$. Since q is guarded, q is of the form $q = \sum_{a \in L} a \cdot p_a$. Thus, $(\!|q|\!)_\alpha = \oplus\{a : (\!|p_a|\!)_\alpha\}_{a \in L}$. On the other hand, the type signature contains the definition $X[\alpha \ \# \ +] = (\!|q|\!)_\alpha$ and $(\!|p|\!)_\alpha = X[\alpha/\alpha]$. Hence, $\mathsf{unfold}_{\Sigma_0}((\!|p|\!)_\alpha) = \oplus\{a : (\!|p_a|\!)_\alpha\}_{a \in L}$.

Case $p = p_1 \cdot p_2$. In this case, for each $a \in L$, either: $p_1 \xrightarrow{a} p'_a$ and $p_a = p'_a \cdot p_2$; or, $p_1 \xrightarrow{a} \varepsilon$ and $p_a = p_2$. In any of this subcases, by induction hypothesis, $\mathsf{unfold}_{\Sigma_0}(\llparenthesis p_1 \rrparenthesis_\alpha) = \oplus\{a : \llparenthesis p'_a \rrparenthesis_\alpha\}_{a \in L}$, with $p'_a = \varepsilon$ in the second subcase. Hence, $\mathsf{unfold}_{\Sigma_0}(\llparenthesis p \rrparenthesis_\alpha) = \oplus\{a : \llparenthesis p'_a \rrparenthesis_\alpha[\llparenthesis p_2 \rrparenthesis_\alpha/\alpha]\}_{a \in L} = \oplus\{a : \llparenthesis p'_a \cdot p_2 \rrparenthesis_\alpha\}_{a \in L}$.

Theorem 1. *Given two deterministic and normed BPA processes $p, q \in \mathcal{P}$, $L(p) \subseteq L(q)$ if and only if $\llparenthesis p \rrparenthesis \leq \llparenthesis q \rrparenthesis$ # $+$.*

Proof. For the direct implication, assume that $L(p) \subseteq L(q)$ and consider a (covariant-)relation over nested polymorphic session types defined by:

$$\mathcal{R} = \{(\llparenthesis p_0 \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q_0 \rrparenthesis_\alpha[\mathbf{1}/\alpha]) \mid L(p_0) \subseteq L(q_0)\} \cup \{(\mathbf{1}, \mathbf{1})\}.$$

Note that, by definition of process translation, we have $(\llparenthesis p \rrparenthesis, \llparenthesis q \rrparenthesis) \in \mathcal{R}$. To check that $\mathcal{R}$ is a simulation, we just need to verify that the conditions of Definition 1 are met. For that, let $(\llparenthesis p_0 \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q_0 \rrparenthesis_\alpha[\mathbf{1}/\alpha]) \in \mathcal{R}$. Since $\mathcal{R}$ is only composed by images of the translation, the only conditions of Definition 1 that we end up verifying are the closure conditions for $\oplus$ and $\mathbf{1}$. The closure condition for $\mathbf{1}$ is handled by $(\mathbf{1}, \mathbf{1}) \in \mathcal{R}$, provided that p_0 and q_0 do not represent the empty process. For $\oplus$, we proceed by case analysis on p_0. Throughout the proof we use $\mathsf{unfold}_{\Sigma_0}(\cdot)$ to handle type definitions that arise from the process equations and from the application of the last rule for the labelled transitions.

Case $p_0 = a$. In this case, $p_0 \xrightarrow{a} \varepsilon$. Since $L(p_0) \subseteq L(q_0)$, then $q_0 \xrightarrow{a} \varepsilon$. By definition, $\llparenthesis p_0 \rrparenthesis_\alpha = \oplus\{a : \alpha\}$. Using Lemma 4, we have $\mathsf{unfold}_{\Sigma_0}(\llparenthesis q_0 \rrparenthesis_\alpha) = \oplus\{a : \alpha\}$. To prove that $\mathcal{R}$ is a simulation, we need to conclude that $(\alpha[\mathbf{1}/\alpha], \alpha[\mathbf{1}/\alpha]) \in \mathcal{R}$, which is immediate from $\mathcal{R}$'s definition, noting that $\alpha[\mathbf{1}/\alpha] = \mathbf{1}$.

Case $p_0 = X$ and $X \triangleq p_1 \in \Delta$. Since $L(p_0) \subseteq L(q_0)$, for every a s.t. $p_1 \xrightarrow{a} p_a$ (i.e., $p_0 \xrightarrow{a} p_a$), we know that $q_0 \xrightarrow{a} q_a$ and, using Lemma 3, $L(p_a) \subseteq L(q_a)$. By Lemma 4, we know that $\mathsf{unfold}_{\Sigma_0}(\llparenthesis p_0 \rrparenthesis_\alpha) = \oplus\{a : \llparenthesis p_a \rrparenthesis_\alpha\}_{a \in L}$ where $L = \{a \mid p_1 \xrightarrow{a} p_a\}$ and $\mathsf{unfold}_{\Sigma_0}(\llparenthesis q_0 \rrparenthesis_\alpha) = \oplus\{b : \llparenthesis q_b \rrparenthesis_\alpha\}_{b \in M}$, with $L \subseteq M$. Since $L(p_a) \subseteq L(q_a)$, we also have $(\llparenthesis p_a \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q_a \rrparenthesis_\alpha[\mathbf{1}/\alpha]) \in \mathcal{R}$, for all $a \in L$.

Case $p_0 = a \cdot p_1 + b \cdot p_2$. Recall that this case is representative for non-deterministic choice because all process equations are guarded. In this case, $p_0 \xrightarrow{a} p_1$ and $p_0 \xrightarrow{b} p_2$. Hence, $q_0 \xrightarrow{a} q_1$ and $q_0 \xrightarrow{b} q_2$ and $L(p_1) \subseteq L(q_1)$ and $L(p_2) \subseteq L(q_2)$. Thus, $\llparenthesis p_0 \rrparenthesis_\alpha = \oplus\{a : \llparenthesis p_1 \rrparenthesis_\alpha, b : \llparenthesis p_2 \rrparenthesis_\alpha\}_{a \in L}$ and, by Lemma 4, $\mathsf{unfold}_{\Sigma_0}(\llparenthesis q_0 \rrparenthesis_\alpha) = \oplus\{\ell : \llparenthesis q_\ell \rrparenthesis_\alpha\}_{\ell \in L}$, where $a, b \in L$ and $q_a = q_1$ and $q_b = q_2$. Since $L(p_1) \subseteq L(q_1)$ and $L(p_2) \subseteq L(q_2)$, we know that $(\llparenthesis p_1 \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q_1 \rrparenthesis_\alpha[\mathbf{1}/\alpha])$, $(\llparenthesis p_2 \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q_2 \rrparenthesis_\alpha[\mathbf{1}/\alpha]) \in \mathcal{R}$.

Case $p_0 = p_1 \cdot p_2$. In this case, by Lemma 4, we have $\mathsf{unfold}_{\Sigma_0}(\llparenthesis p_0 \rrparenthesis_\alpha) = \oplus\{a : \llparenthesis p_a \rrparenthesis_\alpha\}_{a \in L}$ with $L = \{a \mid p \xrightarrow{a} p_a\}$. Since $L(p_0) \subseteq L(q_0)$, by Lemma 4, we know that $\mathsf{unfold}_{\Sigma_0}(\llparenthesis q_0 \rrparenthesis_\alpha) = \oplus\{m : \llparenthesis q_m \rrparenthesis_\alpha\}_{m \in M}$ with $L \subseteq M$. Furthermore, using Lemma 3, for each $a \in L$, $L(p_a) \subseteq L(q_a)$. Thus, $(\llparenthesis p_a \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q_a \rrparenthesis_\alpha[\mathbf{1}/\alpha]) \in \mathcal{R}$.

Reciprocally, assume that $L(p) \not\subseteq L(q)$ and let $w \in L(p)$ be such that $w \notin L(q)$. Let w_0 be its greatest prefix that occurs in $\mathsf{L}(q)$. We have $p \xrightarrow{w_0} p'$ and $q \xrightarrow{w_0} q'$. We can prove, by induction on the length of w_0 and using Lemma 4, that any simulation $\mathcal{R}$ for $(\llparenthesis p \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q \rrparenthesis_\alpha[\mathbf{1}/\alpha])$ is such that $(\llparenthesis p' \rrparenthesis_\alpha[\mathbf{1}/\alpha], \llparenthesis q' \rrparenthesis_\alpha[\mathbf{1}/\alpha]) \in \mathcal{R}$.

However, since w_0 is the greatest (proper) prefix of w for which $p \xrightarrow{w_0} p'$ and $q \xrightarrow{w_0} q'$, we know that there is a labelled transition a for p', $p' \xrightarrow{a} p''$, that is not applicable to q'. Hence, we would have a choice label for $(\!|p'|\!)_\alpha[\mathbf{1}/\alpha]$ distinct from those in $(\!|q'|\!)_\alpha[\mathbf{1}/\alpha]$. For that, note that all internal labels of $(\!|q'|\!)_\alpha[\mathbf{1}/\alpha]$ are derived from labelled transitions in q'. Thus, we have $((\!|p'|\!)_\alpha[\mathbf{1}/\alpha], (\!|q'|\!)_\alpha[\mathbf{1}/\alpha]) \notin \mathcal{R}$. We conclude that there would be no simulation for $((\!|p|\!)_\alpha[\mathbf{1}/\alpha], (\!|q|\!)_\alpha[\mathbf{1}/\alpha])$.

Theorem 2. *Checking $A \leq B$ # $+$ is undecidable.*

Proof. Theorem 1 reduces the language inclusion problem for deterministic and normed BPA processes to the subtyping problem of (closed) nested polymorphic session types. Groote and Huttel proved the former is undecidable [28], thus the subtyping problem for nested polymorphic session types is also undecidable.

These results show that subtyping is already undecidable even for a *small fragment* of nested polymorphic session types: the fragment composed of *nested* session types (mutually) defined through *unary* type definitions whose type parameter is always assigned positive variance; these definitions only provide *internal choices*.

5 Practical Algorithm for Subtyping

Although the subtyping problem is undecidable, we have designed a coinductive algorithm for approximating this problem. The algorithm is sound but incomplete. The undecidability of subtyping precludes us from achieving a complete algorithm. However, we propose a recourse that enables the programmer to provide a *seed* and help the algorithm *generalize the coinductive hypothesis*.

Taking inspiration from Gay and Hole [25], we attempt to construct a type simulation. Our algorithm can terminate in three states: *(i)* types are proved to have a subtyping relation by constructing a simulation, *(ii)* a counterexample is detected by identifying a position where the subtype and the supertype differ, or *(iii)* no conclusive answer is obtained due to algorithm's incompleteness. We interpret both *(ii)* and *(iii)* as a failure of subtyping verification, but the extension presented in subsection 5.4 comes to the rescue in the case of *(iii)*.

Our subtyping algorithm is deterministic (with no backtracking) and is presented in subsection 5.2. This algorithm is proved to be sound (subsection 5.3). Our algorithm assumes a preliminary pass over the given types to introduce *fresh internal names*.

5.1 Internal renaming

The fundamental operation in the subtyping algorithm of our recursive structural types is *loop detection*, where we determine if we have already added a subtyping relation $A \leq B$ to the type simulation. However, our simulation also contains open types with free variables and, therefore, determining if we have

already considered two types in the subtyping relation becomes a difficult operation. Following our previous approach [11], we have reduced this problem to the verification of loop detection on defined type names. For this purpose, we perform a renaming of the given types by introducing fresh internal type names and definitions.

Renaming A preliminary type transformation assigns a *fresh* name to each intermediate (structural) type expression in the given types. The new internal names are parametrized over their type variables, and their definitions are added to the signature. After the *internal renaming*, the type grammar becomes:

$$A ::= \oplus\{\ell : T\}_{\ell \in L} \mid \&\{\ell : T\}_{\ell \in I} \mid T \multimap T \mid T \otimes T \mid \mathbf{1} \mid \exists x.\, T \mid \forall x.\, T \mid x \mid \alpha$$
$$T ::= V[\theta]$$

Note that the substitutions θ are also *internally renamed*, implying that the continuation types are a nesting of type names. In the resulting signature, type names and structural types alternate, thus allowing loop detection to be entirely performed on defined type names.

Example 3. After creating internal names for a list of natural numbers, $\mathsf{List}[\mathsf{nat}]$ where $\mathsf{List}[\alpha] = \oplus\{\mathbf{nil} : \mathbf{1}, \mathbf{cons} : \alpha \otimes \mathsf{List}[\alpha]\}$ and $\mathsf{nat} = \oplus\{\mathbf{z} : \mathbf{1}, \mathbf{s} : \mathsf{nat}\}$, we obtain the following declarations:

$$\mathsf{nat} = \oplus\{\mathbf{z} : X_1, \mathbf{s} : \mathsf{nat}\} \qquad \mathsf{List}[\alpha] = \oplus\{\mathbf{nil} : X_2, \mathbf{cons} : X_3[\alpha]\}$$

$$X_1 = \mathbf{1} \qquad X_2 = \mathbf{1} \qquad X_3[\alpha] = X_4[\alpha] \otimes \mathsf{List}[\alpha] \qquad X_4[\alpha] = \alpha$$

(To ease the notation, when the type constructors are *unary*, we often omit the explicit substitution for the parameter. In List, for instance, a substitution by nat should be considered as $\theta_{\mathsf{nat}} = (\mathsf{nat}/\alpha)$.)

To illustrate the invariant that continuation types are a nesting of type names, notice that: a list $\mathsf{List}[\oplus\{\mathbf{s} : \oplus\{\mathbf{z} : \mathbf{1}\}\}]$ is renamed to $\mathsf{List}[X_5]$ under a signature extended with definitions

$$X_5 = \oplus\{\mathbf{s} : X_6\} \qquad X_6 = \oplus\{\mathbf{z} : X_7\} \qquad X_7 = \mathbf{1},$$

whereas a list of lists of natural numbers, $\mathsf{List}[\mathsf{List}[\mathsf{nat}]]$, is already renamed, assuming the previous renaming for lists of natural numbers. For the latter, the continuation types are now X_2 (for branch **nil**) and $X_3[\mathsf{List}[\mathsf{nat}]]$ (for branch **cons**) – both type names. By unfolding them, we get the structural types $\mathbf{1}$ and $X_4 \otimes \mathsf{List}[\mathsf{List}[\mathsf{nat}]]$. The former type does not have any continuation, but the latter has two continuations: X_4 and $\mathsf{List}[\mathsf{List}[\mathsf{nat}]]$ – both type names. By unfolding, we get structural types again, and so on. Our subtyping algorithm takes advantage of this alternation.

Variance assignment Variance assignment is done by inspecting the signature. We start with declarations $V[\overline{\alpha}] = A$ to elaborate into declarations $V[\Xi] = A$, assigning a variance ξ to each type parameter α in each declaration.

Variances are computed using the variants construction method of Altidor et al. [1]: we start with the approximation $\perp$ for each variance and calculate the least fixed point to get the most informative variances. Since we have a finite lattice of variances, this procedure terminates. If a type parameter occurs in both covariant and contravariant positions, it is assigned variance $\top$. In the degenerate case where α does not occur in A, we end up with $\alpha \mathbin{\#} \perp \in \varXi$.

Example 4. After variance assignment, the signature for our list of natural numbers becomes:

$$\mathsf{nat} = \oplus\{\mathbf{z} : X_1, \mathbf{s} : \mathsf{nat}\} \qquad \mathsf{List}[\alpha \mathbin{\#} +] = \oplus\{\mathbf{nil} : X_2, \mathbf{cons} : X_3[\alpha]\}$$

$$X_1 = \mathbf{1} \qquad X_2 = \mathbf{1} \qquad X_3[\alpha \mathbin{\#} +] = X_4[\alpha] \otimes \mathsf{List}[\alpha] \qquad X_4[\alpha \mathbin{\#} +] = \alpha$$

Subtyping preservation Our internal renaming preserves subtyping. To prove this, recall that our subtyping relation relies on the notion of simulation. Since type simulations are defined over *unfolded* types, the notion of simulation is naturally preserved by the internal renaming.

Lemma 5. *Given a type simulation $\mathcal{R} \subseteq \mathit{Type} \times \mathit{Type}$, if $(A, B) \in \mathcal{R}$ and A and B are internally renamed, then the following conditions hold:*

- *If $\mathsf{unfold}_{\varSigma_0}(A) = \oplus\{\ell : V_\ell[\theta_{A,\ell}]\}_{\ell \in L}$, then $\mathsf{unfold}_{\varSigma_0}(B) = \oplus\{m : U_m[\theta_{B,m}]\}_{m \in M}$ for $L \subseteq M$ and $(V_\ell[\theta_{A,\ell}], U_\ell[\theta_{B,\ell}]) \in \mathcal{R}$ for all $\ell \in L$.*
- *If $\mathsf{unfold}_{\varSigma_0}(A) = \&\{\ell : V_\ell[\theta_{A,\ell}]\}_{\ell \in L}$, then $\mathsf{unfold}_{\varSigma_0}(B) = \&\{m : U_m[\theta_{B,m}]\}_{m \in M}$ for $L \supseteq M$ and $(V_m[\theta_{A,m}], U_m[\theta_{B,m}]) \in \mathcal{R}$ for all $m \in M$.*
- *If $\mathsf{unfold}_{\varSigma_0}(A) = V_1[\theta_{A,1}] \multimap V_2[\theta_{A,2}]$, then $\mathsf{unfold}_{\varSigma_0}(B) = U_1[\theta_{B,1}] \multimap U_2[\theta_{B,2}]$ and we have $(U_1[\theta_{B,1}], V_1[\theta_{A,1}]) \in \mathcal{R}$ and $(V_2[\theta_{A,2}], U_2[\theta_{B,2}]) \in \mathcal{R}$.*
- *If $\mathsf{unfold}_{\varSigma_0}(A) = V_1[\theta_{A,1}] \otimes V_2[\theta_{A,2}]$, then $\mathsf{unfold}_{\varSigma_0}(B) = U_1[\theta_{B,1}] \otimes U_2[\theta_{B,2}]$ and we have $(V_1[\theta_{A,1}], U_1[\theta_{B,1}]) \in \mathcal{R}$ and $(V_2[\theta_{A,2}], U_2[\theta_{B,2}]) \in \mathcal{R}$.*
- *If $\mathsf{unfold}_{\varSigma_0}(A) = \mathbf{1}$, then $\mathsf{unfold}_{\varSigma_0}(B) = \mathbf{1}$.*
- *If $\mathsf{unfold}_{\varSigma_0}(A) = \exists x.\, V[\theta_A]$, then $\mathsf{unfold}_{\varSigma_0}(B) = \exists y.\, U[\theta_B]$ and for all $C \in \mathit{Type}$, we have $(V[\theta_A][C/x], U[\theta_B][C/y]) \in \mathcal{R}$.*
- *If $\mathsf{unfold}_{\varSigma_0}(A) = \forall x.\, V[\theta_A]$, then $\mathsf{unfold}_{\varSigma_0}(B) = \forall y.\, U[\theta_B]$ and for all $C \in \mathit{Type}$, we have $(V[\theta_A][C/x], U[\theta_B][C/y]) \in \mathcal{R}$.*

Proof. Proof by case analysis on the structure of types after internal renaming and applying Definition 1.

5.2 Subtyping algorithm

The subtyping algorithm capitalizes on the invariants established by the internal renaming and, thus, only needs to compare two structural types or two type names. The judgment is written as $\mathcal{V} \mathbin{;} \varGamma \vdash_{\varSigma_0} A \leq B \mathbin{\#} \delta$ and is defined over a valid signature $\varSigma_0$. $\mathcal{V}$ stores the free variables in A and B and $\varGamma$ stores the subtyping constraints that are collected while following the algorithm recursively.

Under constraints $\varGamma$, we attempt to construct a simulation. The subtyping constraints are of the form $\langle \mathcal{V} \mathbin{;} V_1[\theta_1] \leq V_2[\theta_2] \mathbin{\#} \delta \rangle$ and are called *closures*. If a

derivation can be constructed, all *closed instances* of all closures are included in the resulting simulation (see the proof of Theorem 3). A closed instance of closure $\langle \mathcal{V} \; ; \; V_1[\theta_1] \leq V_2[\theta_2] \; \# \; \delta \rangle$ is obtained by applying a closed substitution σ over variables in $\mathcal{V}$ in a way that $V_1[\theta_1][\sigma]$ and $V_2[\theta_2][\sigma]$ have no free type variables. Note that, since type names are not defined over free quantified variables (recall the rules for valid signatures, subsection 3.3), we can write $V_1[\theta_1][\sigma]$ and $V_2[\theta_2][\sigma]$ as $V_1[\theta_1[\sigma]]$ and $V_2[\theta_2[\sigma]]$.

We present the rules in the remainder of this subsection. Since the signature is fixed, we elide it from the rules. The algorithm is initiated with an empty Γ and follows the rules of the judgment. In the rules below, T and U denote arbitrary (internally renamed) types, while V denotes a type name.

Rules for $\oplus$ and $\&$ are specialized to variance. Although the continuation types preserve the variance of the context, not surprisingly, the inclusion of the choice labels is sensitive to the variance. Covariance preserves the usual inclusion for subtyping, contravariance reverts the inclusion, bivariance requires both inclusions and, thus, the sets of labels are equal.

$$\frac{L \subseteq M \quad \mathcal{V} \; ; \; \Gamma \vdash T_\ell \leq U_\ell \; \# \; + \quad (\forall \ell \in L)}{\mathcal{V} \; ; \; \Gamma \vdash \oplus\{l : T_\ell\}_{\ell \in L} \leq \oplus\{m : U_m\}_{m \in M} \; \# \; +} \oplus_+$$

$$\frac{L \supseteq M \quad \mathcal{V} \; ; \; \Gamma \vdash T_m \leq U_m \; \# \; + \quad (\forall m \in M)}{\mathcal{V} \; ; \; \Gamma \vdash \&\{\ell : T_\ell\}_{\ell \in L} \leq \&\{m : U_m\}_{m \in M} \; \# \; +} \&_+$$

$$\frac{L \supseteq M \quad \mathcal{V} \; ; \; \Gamma \vdash T_m \leq U_m \; \# \; - \quad (\forall m \in M)}{\mathcal{V} \; ; \; \Gamma \vdash \oplus\{\ell : T_\ell\}_{\ell \in L} \leq \oplus\{m : U_m\}_{m \in M} \; \# \; -} \oplus_-$$

$$\frac{L \subseteq M \quad \mathcal{V} \; ; \; \Gamma \vdash T_\ell \leq U_\ell \; \# \; - \quad (\forall \ell \in L)}{\mathcal{V} \; ; \; \Gamma \vdash \&\{\ell : T_\ell\}_{\ell \in L} \leq \&\{m : U_m\}_{m \in M} \; \# \; -} \&_-$$

$$\frac{L = M \quad \mathcal{V} \; ; \; \Gamma \vdash T_\ell \leq U_\ell \; \# \; \top \quad (\forall \ell \in L)}{\mathcal{V} \; ; \; \Gamma \vdash \oplus\{\ell : T_\ell\}_{\ell \in L} \leq \oplus\{m : U_m\}_{m \in M} \; \# \; \top} \oplus_\top$$

$$\frac{L = M \quad \mathcal{V} \; ; \; \Gamma \vdash T_\ell \leq U_\ell \; \# \; \top \quad (\forall \ell \in L)}{\mathcal{V} \; ; \; \Gamma \vdash \&\{\ell : T_\ell\}_{\ell \in L} \leq \&\{m : U_m\}_{m \in M} \; \# \; \top} \&_\top$$

Tensor preserves the context variances for both components, whereas *lolli* considers the contravariant position of the first component and *negates* the variance of its context, while maintaining the variance for the second component.

$$\frac{\mathcal{V} \; ; \; \Gamma \vdash T_1 \leq U_1 \; \# \; \delta \quad \mathcal{V} \; ; \; \Gamma \vdash T_2 \leq U_2 \; \# \; \delta}{\mathcal{V} \; ; \; \Gamma \vdash T_1 \otimes T_2 \leq U_1 \otimes U_2 \; \# \; \delta} \otimes$$

$$\frac{\mathcal{V} \; ; \; \Gamma \vdash T_1 \leq U_1 \; \# \; \neg\delta \quad \mathcal{V} \; ; \; \Gamma \vdash T_2 \leq U_2 \; \# \; \delta}{\mathcal{V} \; ; \; \Gamma \vdash T_1 \multimap T_2 \leq U_1 \multimap U_2 \; \# \; \delta} \multimap$$

The type **1** is only a subtype of itself.

$$\frac{}{\mathcal{V} \; ; \; \Gamma \vdash \mathbf{1} \leq \mathbf{1} \; \# \; \delta} \; \mathbf{1}$$

Explicit quantifiers preserve the variance for their continuations. Since explicitly quantified variables are only substituted in the program, they are stored in $\mathcal{V}$, renamed with a *fresh* (quantified) variable z.

$$\frac{\mathcal{V}, z \;;\; \Gamma \vdash T[z/x] \leq U[z/y] \;\#\; \delta}{\mathcal{V} \;;\; \Gamma \vdash \exists x.\, T \leq \exists y.\, U \;\#\; \delta} \; \exists^z \qquad \frac{\mathcal{V}, z \;;\; \Gamma \vdash T[z/x] \leq_\delta U[z/y]}{\mathcal{V} \;;\; \Gamma \vdash \forall x.\, T \leq_\delta \forall y.\, U} \; \forall^z$$

For quantified variables, we only relate a variable to itself, in a context of any variance.

$$\frac{}{\mathcal{V} \;;\; \Gamma \vdash x \leq x \;\#\; \delta} \; \mathsf{var}$$

So far, we have covered the subtyping of structural types. The rules for type operators compare the components according to the variances of their context. If the type constructors do not match, or the label sets do not respect the inclusions (for $\oplus$ and $\&$), the subtyping fails having constructed a counterexample to simulation. Similarly, two type variables are in a subtyping relation if and only if they have the same name, as exemplified by the var rule.

We have one special rule for non-variance: all types are related in a context with variance $\bot$.

$$\frac{}{\mathcal{V} \;;\; \Gamma \vdash T \leq U \;\#\; \bot} \; \bot$$

Taking advantage of the invariants established by the internal renaming, we can now focus on the subtyping of two type names. When comparing type names, we analyze the following cases: either we are comparing the same type name and, taking advantage of it, we focus on their substitutions (refl rule), or we already came across a subtyping relation involving the same type names and we instantiate the quantified variables, preserving the relation (def rule), or we expand their definitions (expd rule).

In the expd rule, we expand the definitions of $V_1[\Xi_1]$ and $V_2[\Xi_2]$, keeping the variances of the context and adding the closure $\langle \mathcal{V} \;;\; V_1[\theta_1] \leq V_2[\theta_2] \;\#\; \delta \rangle$ to Γ. Since the subtyping relation between $V_1[\theta_1]$ and $V_2[\theta_2]$, must hold for all its closed instances, the extension of Γ with the corresponding closure enables to remember that.

$$\frac{V_1[\Xi_1] = A \in \Sigma_0 \quad V_2[\Xi_2] = B \in \Sigma_0 \qquad \mathcal{V} \;;\; \Gamma, \langle \mathcal{V} \;;\; V_1[\theta_1] \leq V_2[\theta_2] \;\#\; \delta \rangle \vdash A[\theta_1] \leq B[\theta_2] \;\#\; \delta}{\mathcal{V} \;;\; \Gamma \vdash V_1[\theta_1] \leq V_2[\theta_2] \;\#\; \delta} \; \mathsf{expd}$$

The refl rule takes advantage of being comparing the same type name, and calls the subtyping algorithm on the substitutions. While doing so, the algorithm updates the variance of the context by *nesting* the current variance with the variances of the type parameters. This rule is where the nesting of types *harmonizes* with the nesting of variances. The judgment for the subtyping of substitutions is (mutual recursively) defined below.

$$\frac{V[\Xi] = A \in \Sigma_0 \qquad \mathcal{V} \;;\; \Gamma \vdash \theta_1 \leq \theta_2 \;\#\; \Xi \mid \delta}{\mathcal{V} \;;\; \Gamma \vdash V[\theta_1] \leq V[\theta_2] \;\#\; \delta} \; \mathsf{refl}$$

The def rule only applies when there already exists a closure in Γ with the same type names V_1 and V_2, over a set of quantified variables $\mathcal{V}'$. In that case, we try to find a substitution σ from $\mathcal{V}'$ to the type expressions defined over variables in $\mathcal{V}$ – that we denote by $\mathcal{T}(\mathcal{V})$ – and such that $V_1[\theta_1]$ and $V_1[\theta_1'[\sigma]]$ preserve the subtyping relation, and $V_2[\theta_2'[\sigma]]$ and $V_2[\theta_2]$ also preserve the subtyping relation, under the same variance. The substitution σ is computed by a standard match algorithm on first-order terms (which is linear-time), applied to the syntactic structure of the types. The existence of such substitution ensures that any closed instance of $\langle \mathcal{V} \ ; \ V_1[\theta_1] \leq V_2[\theta_2] \ \text{\#} \ \delta \rangle$ is also a closed instance of $\langle \mathcal{V}' \ ; \ V_1[\theta_1'] \leq V_2[\theta_2'] \ \text{\#} \ \delta \rangle$, and these are already present in the constructed type simulation. So, we can terminate our subtyping check, having successfully detected a loop.

$$\frac{\langle \mathcal{V}' \ ; \ V_1[\theta_1'] \leq V_2[\theta_2'] \ \text{\#} \ \delta \rangle \in \Gamma \qquad \exists \sigma : \mathcal{V}' \to \mathcal{T}(\mathcal{V}) \ \text{s.t.} \quad (\mathcal{V} \ ; \ \Gamma \vdash V_1[\theta_1] \leq V_1[\theta_1'[\sigma]] \ \text{\#} \ \delta \wedge \mathcal{V} \ ; \ \Gamma \vdash V_2[\theta_2'[\sigma]] \leq V_2[\theta_2] \ \text{\#} \ \delta)}{\mathcal{V} \ ; \ \Gamma \vdash V_1[\theta_1] \leq V_2[\theta_2] \ \text{\#} \ \delta} \ \text{def}$$

Subtyping of substitutions The subtyping relation is easily extended to substitutions. This judgement is used in the refl rule above and is written as $\mathcal{V} \ ; \ \Gamma \vdash \theta_1 \leq \theta_2 \ \text{\#} \ \Xi$. Again, the judgement is defined over a valid signature Σ_0 (that is elided), a set of free variables $\mathcal{V}$ with variables occurring in the types that compose θ_1 and θ_2, and a collection of subtyping closures Γ. The two rules for the subtyping of substitutions ensure that two substitutions are in a subtyping relation on a context of variance Ξ if the types substituting each type parameter $\alpha \ \text{\#} \ \delta \in \Xi$ preserve the subtyping relation under variance δ.

$$\frac{}{\mathcal{V} \ ; \ \Gamma \vdash (\cdot) \leq (\cdot) \ \text{\#} \ (\cdot)} \ (\cdot) \qquad \frac{\mathcal{V} \ ; \ \Gamma \vdash \theta_1 \leq \theta_2 \ \text{\#} \ \Xi \qquad \mathcal{V} \ ; \ \Gamma \vdash A \leq B \ \text{\#} \ \delta}{\mathcal{V} \ ; \ \Gamma \vdash (\theta_1, A/\alpha) \leq (\theta_2, B/\alpha) \ \text{\#} \ (\Xi, \alpha \ \text{\#} \ \delta)} \ \text{subs}$$

Lemma 6. *In a goal $\mathcal{V} \ ; \ \Gamma \vdash A \leq B \ \text{\#} \ \delta$, either A and B are both structural or both type names.*

Proof. By induction on the algorithmic subtyping rules, using the fact that type definitions are contractive and that after internal renaming every continuation becomes by a type name.

Example 5. To check that $\mathsf{List}[\mathsf{nat}] \multimap \mathsf{List}[\mathsf{nat}]$ is a subtype of $\mathsf{List}[\mathsf{even}] \multimap \mathsf{List}[\mathsf{nat}]$ in a context of variance $+$, formally written as

$$\cdot \ ; \ \cdot \vdash \mathsf{List}[\theta_\mathsf{nat}] \multimap \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \multimap \mathsf{List}[\theta_\mathsf{nat}] \ \text{\#} \ +$$

where $\theta_\mathsf{nat} = (\mathsf{nat}/\alpha)$ and $\theta_\mathsf{even} = (\mathsf{even}/\alpha)$, we use the signature of Example 4 extended with definitions for even and odd. Using the subtyping rule for $\multimap$ we need to check that $\cdot \ ; \ \cdot \vdash \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \ \text{\#} \ -$ and $\cdot \ ; \ \cdot \vdash \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{nat}] \ \text{\#} \ +$. Focusing on the former and using the expd rule we have

$$\cdot \ ; \ \langle \cdot \ ; \ \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \ \text{\#} \ - \rangle \vdash$$
$$\oplus\{\mathbf{nil} : X_2, \mathbf{cons} : X_3[\theta_\mathsf{nat}]\} \leq \oplus\{\mathbf{nil} : X_2, \mathbf{cons} : X_3[\theta_\mathsf{even}]\} \ \text{\#} \ -$$

which then compares the continuations. For branch **cons** we get

$$\cdot \; ; \; \langle \cdot \; ; \; \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \; \text{\#} \; - \rangle \vdash X_3[\theta_\mathsf{nat}] \leq X_3[\theta_\mathsf{even}] \; \text{\#} \; -$$

to which we now apply the refl rule to get

$$\cdot \; ; \; \langle \cdot \; ; \; \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \; \text{\#} \; - \rangle \vdash \theta_\mathsf{nat} \leq \theta_\mathsf{even} \; \text{\#} \; (\alpha \; \text{\#} \; +) \mid - \; .$$

Recalling that $(\alpha \; \text{\#} \; +) \mid - = (\alpha \; \text{\#} \; -)$, the rules for subtyping on substitutions reduce us to

$$\cdot \; ; \; \langle \cdot \; ; \; \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \; \text{\#} \; - \rangle \vdash \mathsf{nat} \leq \mathsf{even} \; \text{\#} \; - \; .$$

Recalling that $\mathsf{nat} = \oplus\{\mathbf{z} : X_1, \mathbf{s} : \mathsf{nat}\}$ and $\mathsf{even} = \oplus\{\mathbf{z} : X_5, \mathbf{s} : \mathsf{odd}\}$ and $\mathsf{odd} = \oplus\{\mathbf{s} : \mathsf{even}\}$, with $X_5 = \mathbf{1}$, we expand the definitions and enrich the context (expd rule). Then, the relations for branch **z**, eventually follow from the rule for **1**, and for the **s** branch we proceed with

$$\cdot \; ; \; \langle \cdot \; ; \; \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \; \text{\#} \; - \rangle, \langle \cdot \; ; \; \mathsf{nat} \leq \mathsf{even} \; \text{\#} \; - \rangle \vdash \mathsf{nat} \leq \mathsf{odd} \; \text{\#} \; - \; .$$

After expanding, note that, although we have an internal choice, we are in a context with negative variance and, thus, the subset relation in the labels is reverted (according to rule $\oplus_+$). We proceed with a comparison of the (single) branch **s**, which enables us to conclude.

$$\cdot \; ; \; \langle \cdot \; ; \; \mathsf{List}[\theta_\mathsf{nat}] \leq \mathsf{List}[\theta_\mathsf{even}] \; \text{\#} \; - \rangle, \langle \cdot \; ; \; \mathsf{nat} \leq \mathsf{even} \; \text{\#} \; - \rangle, \langle \cdot \; ; \; \mathsf{nat} \leq \mathsf{odd} \; \text{\#} \; - \rangle \vdash$$
$$\mathsf{nat} \leq \mathsf{even} \; \text{\#} \; - \; .$$

5.3   Soundness of subtyping

We prove the soundness of the subtyping algorithm by constructing a simulation from a derivation $\mathcal{V} \; ; \; \Gamma \vdash A \leq B \; \text{\#} \; \delta$ by *(i)* collecting the conclusions of all the sequents, and *(ii)* forming all closed instances from them.

Definition 5. *Given a derivation $\mathcal{D}_0$ of $\mathcal{V}_0 \; ; \; \Gamma_0 \vdash A_0 \leq B_0 \; \text{\#} \; \delta$, we define the set $\mathcal{S}(\mathcal{D}_0)$ of closures. For each sequent of the form $\mathcal{V} \; ; \; \Gamma \vdash A \leq B \; \text{\#} \; \delta$ in $\mathcal{D}_0$, we include the closure $\langle \mathcal{V} \; ; \; A \leq B \; \text{\#} \; \delta \rangle$ in $\mathcal{S}(\mathcal{D}_0)$.*

Lemma 7 (Closure Invariants). *For any valid derivation $\mathcal{D}$ with the set of closures $\mathcal{S}(\mathcal{D})$,*

- *If $\langle \mathcal{V} \; ; \; \oplus\{\ell : T_\ell\}_{\ell \in L} \leq \oplus\{m : U_m\}_{m \in M} \; \text{\#} \; + \rangle \in \mathcal{S}(\mathcal{D})$ from $\oplus_+$ rule, then $L \subseteq M$ and $\langle \mathcal{V} \; ; \; T_\ell \leq U_\ell \; \text{\#} \; + \rangle \in \mathcal{S}(\mathcal{D})$ for all $\ell \in L$.*
- *If $\langle \mathcal{V} \; ; \; \oplus\{\ell : T_\ell\}_{\ell \in L} \leq \oplus\{m : U_m\}_{m \in M} \; \text{\#} \; - \rangle \in \mathcal{S}(\mathcal{D})$ from $\oplus_-$ rule, then $L \supseteq M$ and $\langle \mathcal{V} \; ; \; T_m \leq U_m \; \text{\#} \; - \rangle \in \mathcal{S}(\mathcal{D})$ for all $m \in M$.*
- *If $\langle \mathcal{V} \; ; \; \oplus\{\ell : T_\ell\}_{\ell \in L} \leq \oplus\{m : U_m\}_{m \in M} \; \text{\#} \; \top \rangle \in \mathcal{S}(\mathcal{D})$ from $\oplus_\top$ rule, then $L = M$ and $\langle \mathcal{V} \; ; \; T_\ell \leq U_\ell \; \text{\#} \; \top \rangle \in \mathcal{S}(\mathcal{D})$ for all $\ell \in L$.*
- *If $\langle \mathcal{V} \; ; \; \&\{\ell : T_\ell\}_{\ell \in L} \leq \&\{m : U_m\}_{m \in M} \; \text{\#} \; + \rangle \in \mathcal{S}(\mathcal{D})$ from $\&_+$ rule, then $L \supseteq M$ and $\langle \mathcal{V} \; ; \; T_m \leq U_m \; \text{\#} \; + \rangle \in \mathcal{S}(\mathcal{D})$ for all $m \in M$.*

- If $\langle \mathcal{V} \ ; \ \&\{\ell : T_\ell\}_{\ell \in L} \leq \&\{m : U_m\}_{m \in M} \ \# \ -\rangle \in \mathcal{S}(\mathcal{D})$ *from* $\&_-$ *rule, then* $L \subseteq M$ *and* $\langle \mathcal{V} \ ; \ T_\ell \leq U_\ell \ \# \ -\rangle \in \mathcal{S}(\mathcal{D})$ *for all* $\ell \in L$.
- If $\langle \mathcal{V} \ ; \ \&\{\ell : T_\ell\}_{\ell \in L} \leq \&\{m : U_m\}_{m \in M} \ \# \ \top\rangle \in \mathcal{S}(\mathcal{D})$ *from* $\&_\top$ *rule, then* $L = M$ *and* $\langle \mathcal{V} \ ; \ T_\ell \leq U_\ell \ \# \ \top\rangle \in \mathcal{S}(\mathcal{D})$ *for all* $\ell \in L$.
- If $\langle \mathcal{V} \ ; \ T_1 \otimes T_2 \leq U_1 \otimes U_2 \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *from* $\otimes$ *rule, then* $\langle \mathcal{V} \ ; \ T_1 \leq U_1 \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *and* $\langle \mathcal{V} \ ; \ T_2 \leq U_2 \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$.
- If $\langle \mathcal{V} \ ; \ T_1 \multimap T_2 \leq U_1 \multimap U_2 \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *from* $\multimap$ *rule, then* $\langle \mathcal{V} \ ; \ T_1 \leq U_1 \ \# \ \neg\delta\rangle \in \mathcal{S}(\mathcal{D})$ *and* $\langle \mathcal{V} \ ; \ T_2 \leq U_2 \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$.
- If $\langle \mathcal{V} \ ; \ \exists x. T \ \leq \ \exists y. U \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *from* $\exists^z$ *rule, then* $\langle \mathcal{V}, z \ ; \ T[z/x] \leq U[z/y] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$.
- If $\langle \mathcal{V} \ ; \ \forall x. T \ \leq \ \forall y. U \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *from* $\forall^z$ *rule, then* $\langle \mathcal{V}, z \ ; \ T[z/x] \leq U[z/y] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$.
- If $\langle \mathcal{V} \ ; \ V_1[\theta_1] \leq V_2[\theta_2] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *and* $V_1[\Xi_1] = A \in \Sigma_0$ *and* $V_2[\Xi_2] = B \in \Sigma_0$ *from* expd *rule, then* $\langle \mathcal{V} \ ; \ A[\theta_1] \leq B[\theta_2] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$.
- If $\langle \mathcal{V} \ ; \ V[\theta_1] \leq V[\theta_2] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *and* $V[\Xi] = A \in \Sigma_0$ *from* refl *rule, then* $\langle \mathcal{V} \ ; \ A \leq B \ \# \ \xi \ | \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *for all* $A/\alpha \in \theta_1$ *and* $B/\alpha \in \theta_2$ *and* $\alpha \ \# \ \xi \in \Xi$.
- If $\langle \mathcal{V} \ ; \ V_1[\theta_1] \leq V_2[\theta_2] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *and* $\langle \mathcal{V}' \ ; \ V_1[\theta_1'] \leq V_2[\theta_2'] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *from* def *rule, then exists* $\sigma : \mathcal{V}' \to \mathcal{T}(\mathcal{V})$ *s.t.* $\langle \mathcal{V} \ ; \ V_1[\theta_1] \leq V_1[\theta_1'[\sigma]] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$ *and* $\langle \mathcal{V} \ ; \ V_2[\theta_2'[\sigma]] \leq V_2[\theta_2] \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D})$

Proof. By induction on the subtyping judgement.

Now, we prove that the subtyping algorithm that we propose is sound.

Theorem 3. *For valid types A and B s.t. $\mathcal{V} \ ; \ \cdot \vdash A \ \# \ \delta$ and $\mathcal{V} \ ; \ \cdot \vdash B \ \# \ \delta$, if the subtyping algorithm proves $\mathcal{V} \ ; \ \cdot \vdash A \leq B \ \# \ \delta$, then $\forall \mathcal{V}. A \leq B \ \# \ \delta$.*

Proof. Given A_0, B_0 s.t. $\mathcal{V}_0 \ ; \ \cdot \vdash A_0 \ \# \ \delta$ and $\mathcal{V}_0 \ ; \ \cdot \vdash B_0 \ \# \ \delta$ and a derivation $\mathcal{D}_0$ of $\mathcal{V}_0 \ ; \ \cdot \vdash A_0 \leq B_0 \ \# \ \delta$, construct the set of closures $\mathcal{S}(\mathcal{D}_0)$. Define a relation $\mathcal{R}_0$ such that

$$\mathcal{R}_0 = \{(A[\sigma], B[\sigma]) \mid \langle \mathcal{V} \ ; \ A \leq B \ \# \ +\rangle \in \mathcal{S}(\mathcal{D}_0), \ \sigma \text{ closed substitution over } \mathcal{V}\}$$
$$\cup \{(B[\sigma], A[\sigma]) \mid \langle \mathcal{V} \ ; \ A \leq B \ \# \ -\rangle \in \mathcal{S}(\mathcal{D}_0) \text{ and } \sigma \text{ closed substitution over } \mathcal{V}\}$$
$$\cup \{(A[\sigma], B[\sigma]), (B[\sigma], A[\sigma]) \mid \langle \mathcal{V} \ ; \ A \leq B \ \# \ \top\rangle \in \mathcal{S}(\mathcal{D}_0), \ \sigma \text{ closed subst. over } \mathcal{V}\}$$

Recall the variance-based relation defined in Definition 2 and note that $\langle \mathcal{V} \ ; \ A \leq B \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D}_0)$ iff $(A[\sigma], B[\sigma]) \in \mathcal{R}_0^\delta$, for any closed substitution σ over $\mathcal{V}$. Then, let $\mathcal{R}$ be the reflexive and transitive closure of $\bigcup_{i \geq 0} \mathcal{R}_i$ where, for $i \geq 1$:

$$\mathcal{R}_i = \{(V[\theta_1], V[\theta_2]) \mid V[\Xi] = C \in \Sigma_0 \text{ and } (\alpha \ \# \ \xi) \in \Xi \Rightarrow$$
$$(A/\alpha) \in \theta_1 \text{ and } (B/\alpha) \in \theta_2 \ \forall (A, B) \in R_{i-1}^\xi\}$$

We now prove that $\mathcal{R}$ is a type simulation. Then, our theorem follows because $\mathcal{V}_0 \ ; \ \cdot \vdash A_0 \leq B_0 \ \# \ \delta$ implies $\langle \mathcal{V}_0 \ ; \ A_0 \leq B_0 \ \# \ \delta\rangle \in \mathcal{S}(\mathcal{D}_0)$. This means that, for any closed substitution σ over $\mathcal{V}_0$, we have $(A_0[\sigma], B_0[\sigma]) \in \mathcal{R}^\delta$, proving to be in the conditions of Definition 4.

Note that extending a relation by its reflexive and transitive closure preserves the simulation properties. To prove that $\mathcal{R}$ is a type simulation, we consider

$(A[\sigma], B[\sigma]) \in \mathcal{R}^\delta$ where $\langle \mathcal{V} \; ; \; A \leq B \; \# \; \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$ and σ is a substitution over $\mathcal{V}$. We case analyze on the rule in the derivation which added the above closure to $\mathcal{R}$. We detail the most representative cases; some other cases can be found in the supplementary material.

Consider the case where $\oplus_-$ rule is applied to add $(B[\sigma], A[\sigma]) \in \mathcal{R}$, i.e., $(A[\sigma], B[\sigma]) \in \mathcal{R}^-$. The rule dictates that $A = \oplus\{\ell : A_\ell\}_{\ell \in L}$ and $B = \oplus\{m : B_m\}_{m \in M}$. Since $\langle \mathcal{V} \; ; \; A \leq B \; \# \; - \rangle \in \mathcal{S}(\mathcal{D}_0)$, by Lemma 7, we know that $L \supseteq M$ and $\langle \mathcal{V} \; ; \; A_m \leq B_m \; \# \; - \rangle \in \mathcal{S}(\mathcal{D}_0)$ for all $m \in M$. By definition of $\mathcal{R}$, we get $(B_m[\sigma], A_m[\sigma]) \in \mathcal{R}$. Also, $A[\sigma] = \oplus\{\ell : A_\ell[\sigma]\}_{\ell \in L}$ and $B[\sigma] = \oplus\{m : B_m[\sigma]\}_{m \in M}$. Hence, $\mathcal{R}$ satisfies the first condition of Definition 1. Cases for $\oplus_+$, $\oplus_\top$, $\&_+$, $\&_-$, $\&_\perp$ are analogous.

If the applied rule is $\multimap$, then $A = A_1 \multimap A_2$ and $B = B_1 \multimap B_2$. Lemma 7 implies that $\langle \mathcal{V} \; ; \; A_1 \leq B_1 \; \# \; \neg\delta \rangle \in \mathcal{S}(\mathcal{D}_0)$ and $\langle \mathcal{V} \; ; \; A_2 \leq B_2 \; \# \; \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$. We thus get $(A_1[\sigma], B_1[\sigma]) \in \mathcal{R}^{\neg\delta}$ and $(A_2[\sigma], B_2[\sigma]) \in \mathcal{R}^\delta$. From which we conclude that $(B_1[\sigma], A_1[\sigma]) \in \mathcal{R}^\delta$ and $\mathcal{R}$ satisfies the third closure condition from Definition 1. The case of $\otimes$ is analogous. The case for $\mathbf{1}$ is trivial.

If the applied rule is $\exists^z$, then $A = \exists x. A'$ and $B = \exists y. B'$ and $(A[\sigma], B[\sigma]) \in \mathcal{R}^\delta$. Lemma 7 implies that $\langle \mathcal{V}, z \; ; \; A'[z/x] \leq B'[z/y] \; \# \; \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$. Thus, the definition of $\mathcal{R}$ ensures that $(A'[z/x][\sigma'], B'[z/y][\sigma']) \in \mathcal{R}^\delta$ for any closed σ' over $\mathcal{V}, z$. Also, $A[\sigma] = \exists x. A'[\sigma]$ and $B[\sigma] = \exists y. B'[\sigma]$. To prove that we have $(A'[\sigma][C/x], B'[\sigma][C/y]) \in \mathcal{R}^\delta$ for any $C \in Type$, we set $\sigma' = \sigma, C/z$, and get $A'[z/x][\sigma'] = A'[z/x][\sigma, C/z] = A'[\sigma][C/x]$. Similarly, $B'[z/x][\sigma'] = B'[\sigma][C/y]$. Combining, we get that $(A'[\sigma][C/x], B'[\sigma][C/y]) \in \mathcal{R}^\delta$. The case for $\forall^z$ is analogous.

If the applied rule is var, then $A = x$ and $B = x$. In this case, the relation $\mathcal{R}$ contains any (C, C) for a closed session type C. Since the next applied rule has to be a structural rule, we can use Lemma 7 again to obtain the closure conditions of a type simulation. If the applied rule is $\perp$, nothing is added to $\mathcal{R}$; thus, the result holds vacuously.

When the expd rule is applied, $A = V_1[\theta_1]$ and $B = V_2[\theta_2]$ and $(A[\sigma], B[\sigma]) \in \mathcal{R}^\delta$ with definitions $V_1[\Xi_1] = C_1$ and $V_2[\Xi_2] = C_2$. From Lemma 7, we have $\langle \mathcal{V} \; ; \; C_1[\theta_1] \leq C_2[\theta_2] \; \# \; \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$. Again, the next applied rule is a structural rule, so we use Lemma 7 to obtain the closure conditions.

When the applied rule is def, we have $\langle \mathcal{V} \; ; \; A \leq B \; \# \; \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$, with $A = V_1[\theta_1]$ and $B = U_1[\theta_2]$, and definitions $V_1[\Xi_1] = C$ and $U_1[\Xi_2] = D$. We also have $\langle \mathcal{V}' \; ; \; V_1[\theta_1'] \leq U_1[\theta_2'] \; \# \; \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$. The types $\mathsf{unfold}_{\Sigma_0}(A)$ and $\mathsf{unfold}_{\Sigma_0}(B)$ are structural. Due to the internal renaming, the continuation types for C and D are type names, say $V_2[\theta_A]$ and $U_2[\theta_B]$, respect., occurring in a position of variance ξ. Recalling that $(A[\sigma], B[\sigma]) \in \mathcal{R}^\delta$, we want to prove that: $(V_2[\theta_A[\theta_1]][\sigma], U_2[\theta_B[\theta_2]][\sigma]) \in \mathcal{R}^{\delta|\xi}$. From Lemma 7, we know that exists $\sigma' : \mathcal{V}' \to \mathcal{T}(\mathcal{V})$ s.t. $\langle \mathcal{V} \; ; \; V_1[\theta_1] \leq V_1[\theta_1'[\sigma']] \; \# \; \delta \rangle, \langle \mathcal{V} \; ; \; U_1[\theta_2'[\sigma']] \leq U_1[\theta_2] \; \# \; \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$. Hence, we have $(V_1[\theta_1[\sigma]], V_1[\theta_1'[\sigma'[\sigma]]]), (U_1[\theta_2'[\sigma'[\sigma]]], U_1[\theta_2[\sigma]]) \in \mathcal{R}^\delta$. By applying successive simulation steps on these types, we eventually reach the

type variables occurring in C and D. Thus, we know that:

$$(A_1[\sigma], A_1'[\sigma'[\sigma]]) \in \mathcal{R}^{\delta|\xi_1} \text{ for all } A_1/\alpha \in \theta_1 \text{ and } A_1'/\alpha \in \theta_1' \text{ and } \alpha \mathbin{\#} \xi_1 \in \Xi_1$$
$$(A_2'[\sigma'[\sigma]], A_2[\sigma]) \in \mathcal{R}^{\delta|\xi_2} \text{ for all } A_2'/\beta \in \theta_2' \text{ and } A_2/\beta \in \theta_2 \text{ and } \beta \mathbin{\#} \xi_2 \in \Xi_2$$

$$(1)$$

The internal renaming ensures that θ_A and θ_B are a nesting of type names. Hence, from (1) and definition of $\mathcal{R}_i$, we conclude that: for some $i \geq 1$,

$$(V_2[\theta_A[\theta_1[\sigma]]], V_2[\theta_A[\theta_1'[\sigma'[\sigma]]]]), (U_2[\theta_B[\theta_2'[\sigma'[\sigma]]]], U_2[\theta_B[\theta_2[\sigma]]]) \in \mathcal{R}_i^{\delta|\xi}. \quad (2)$$

Since we also have $\langle \mathcal{V}' \; ; \; V_1[\theta_1'] \leq U_1[\theta_2'] \mathbin{\#} \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$, we know that the pair $(V_1[\theta_1'][\sigma'[\sigma]], U_1[\theta_2'][\sigma'[\sigma]]) \in \mathcal{R}^\delta$. Proceeding to the continuation types $V_2[\theta_A]$ and $U_2[\theta_B]$ (occurring at variance ξ) and using (2) and transitivity, we conclude that $(V_2[\theta_A[\theta_1]][\sigma], U_2[\theta_B[\theta_2]][\sigma]) \in \mathcal{R}^{\delta|\xi}$, as we wanted.

The last two cases concern reflexivity, one that comes directly from the closure obtained from applying the refl rule, and the other comes from the relation $\mathcal{R}_i$. We consider the representative case where closure $\langle \mathcal{V} \; ; \; V[\theta_1] \leq V[\theta_2] \mathbin{\#} \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$, i.e., $(V[\theta_1][\sigma], V[\theta_2][\sigma]) \in \mathcal{R}^\delta$. Assume that $V[\Xi_V] = C \in \Sigma_0$. Lemma 7 ensures that $\langle \mathcal{V} \; ; \; A \leq B \mathbin{\#} \xi \mid \delta \rangle \in \mathcal{S}(\mathcal{D}_0)$, for all $A/\alpha \in \theta_1$ and $B/\alpha \in \theta_2$ and $\alpha \mathbin{\#} \xi \in \Xi_V$. By definition of $\mathcal{R}_0$, we thus have

$$(A[\sigma], B[\sigma]) \in \mathcal{R}_0^{\xi|\delta}, \text{ for each } A/\alpha \in \theta_1 \text{ and } B/\alpha \in \theta_2 \text{ and } \alpha \mathbin{\#} \xi \in \Xi_V. \quad (3)$$

We proceed with a continuation of C. Let $V_0[\theta_0]$ denote a continuation occurring in a position of variance η on C, with definition $V_0[\Xi_0] = D$. We can easily show that $\Xi_0 \mid \eta \subseteq \Xi_V$. To conclude, recall that the intermediate renaming ensures that θ_0 is itself a nesting of type names $V_1 \cdots V_k$, for $k \geq 1$. Using (3) and the definition of $\mathcal{R}_{k+1}$, we have $(V_0[\theta_0[\theta_1[\sigma]]], V_0[\theta_0[\theta_2[\sigma]]]) \in \mathcal{R}^{\delta|\eta}$, as we wanted. In the subcase where $C = \alpha$ for $\alpha \mathbin{\#} \xi \in \Xi$, the validity of Σ_0 ensures that $\xi = +$. In this particular case, we have $V[\theta_1] = A$ and $V[\theta_2] = B$ and $A/\alpha \in \theta_1$ and $B/\alpha \in \theta_2$. From (3), we conclude that $(A[\sigma], B[\sigma]) \in \mathcal{R}_0^{+|\delta}$ and so $(A[\sigma], B[\sigma]) \in \mathcal{R}^\delta$. Note that $\mathsf{unfold}_{\Sigma_0}(V[\theta_1][\sigma]) = A[\sigma]$. The next applied rule has to be a structural rule, so we can use Lemma 7 again to obtain the closure conditions of a type simulation.

Thus, $\mathcal{R}$ is a type simulation.

Our subtyping algorithm makes a clear distinction between the two classes of type variables: *type parameters* are always substituted by unfolding definitions during algorithm's execution, so they do not arise in the subtyping algorithm, whereas *explicitly quantified variables* arise in the program (as parameters to process definitions) or even in the types, so they are kept in $\mathcal{V}$. Quantified variables are only substituted in the program and never in the subtyping algorithm. This is the reason why we handle a set of quantified variables $\mathcal{V}$, but not of type parameters Ξ in our subtyping algorithm.

5.4   Subtyping declarations

One source of incompleteness in our algorithm is its inability to *generalize the coinductive hypothesis*. As an illustration, consider the following two types D

and D'.

$$T[\alpha\ \#\ +] = \oplus\{\mathbf{L} : T[T[\alpha]], \mathbf{R} : \alpha\} \qquad D\ = \oplus\{\mathbf{L} : T[D], \$: \mathbf{1}\}$$
$$T'[\beta\ \#\ +] = \oplus\{\mathbf{L} : T'[T'[\beta]], \mathbf{R} : \beta\} \qquad D' = \oplus\{\mathbf{L} : T'[D'], \mathbf{R} : \mathbf{1}, \$: \mathbf{1}\}$$

To establish $D \leq D'\ \#\ +$, our algorithm explores the $\mathbf{L}$ branch and checks $T[D] \leq T'[D']\ \#\ +$. A corresponding closure $\langle \cdot \ ; \ T[D] \leq T'[D']\rangle$ is added to Γ, and our algorithm then checks $T[T[D]] \leq T'[T'[D']]\ \#\ +$. This process repeats until it exceeds the depth bound and terminates with an inconclusive answer. What the algorithm never realizes is that $T[\theta] \leq T'[\theta]\ \#\ +$ for all substitutions θ over α; it fails to generalize to this hypothesis and is always inserting closed subtyping constraints to Γ.

To allow a recourse, we permit the programmer to declare *subtyping declarations* easily verified by our algorithm (concrete syntax):

```
eqtype T[x]   <= T'[x]
```

Indeed, here we could even *declare* an equality constraint, meaning that, for T and T', subtyping holds in both directions:

```
eqtype T[x] = T'[x]
```

Then, we *seed* the Γ in the subtyping algorithm with the corresponding closure from the eqtype declaration which can then be used to establish $D \leq D'\ \#\ +$:

$$\cdot \ ; \ \langle x \ ; \ T[x/\alpha] \leq T'[x/\beta]\rangle, \langle x \ ; \ T'[x/\beta] \leq T[x/\alpha]\rangle \vdash D \leq D'\ \#\ + .$$

Upon exploring the $\mathbf{L}$ branch, it reduces to

$$\cdot \ ; \ \langle x \ ; \ T[x/\alpha] \leq T'[x/\beta]\rangle, \langle x \ ; \ T'[x/\beta] \leq T[x/\alpha]\rangle, \langle \cdot \ ; \ D \leq D'\rangle \vdash$$
$$T[D] \leq T'[D']\ \#\ + .$$

This last inequality holds under substitution $\sigma = D/x$ over $\langle x \ ; \ T[x/\alpha] \leq T'[x/\alpha]\rangle$, as required by the def rule.

In the implementation, we first collect all the eqtype declarations in the program into a global set of closures Γ_0. We then verify the *validity* of every eqtype declaration, by checking $\mathcal{V} \ ; \ \Gamma_0 \vdash A \leq B\ \#\ +$ for every pair (A, B) (with free variables $\mathcal{V}$) in the eqtype declarations. Essentially, this ensures that all subtyping declarations are valid with respect to each other. Finally, all subtyping checks are performed under this more general Γ_0. The soundness of this approach can be proved with the following theorem.

Theorem 4 (Seeded Soundness). *For a valid set of eqtype declarations Γ_0, if $\mathcal{V} \ ; \ \Gamma_0 \vdash A \leq B\ \#\ \delta$, then $\forall \mathcal{V}. A \leq B\ \#\ \delta$.*

Our soundness proof can easily be modified to accommodate this requirement. Intuitively, since Γ_0 is valid, all closed instances of Γ_0 are already proven to be valid subtyping relations and consistent with each other. Thus, all properties of a type simulation are preserved when all closed instances of Γ_0 are added to it.

One final note on the rule of reflexivity: if a type name does *not* depend on its parameter, at the variance assignment stage, the type parameter is assigned variance $\perp$. Hence, its type definition is of the form $V[\alpha \text{ \# } \perp] = A$. To prove that $V[T] \leq V[U] \text{ \# } \xi$ using the refl rule, we need to prove that $T \leq U \text{ \# } \perp$, which holds due to our special (permissive) rule for $\perp$ and regardless of T and U.

6   Implementation

We have implemented a prototype for nested polymorphic session types with subtyping and integrated it with the open-source Rast system [14]. Rast (Resource-Aware Session Types) is a programming language that implements the intuitionistic version of session types [7] with support for arithmetic refinements [15,16], ergometric [13] and temporal [12] types for complexity analysis, and nested session types [11].

In this work, however, we focus on the subtyping implementation and how that connects to the Rast language. The Rast implementation uses a bi-directional type checker [42] requiring programmers to specify the initial type of each channel of a process. The intermediate types are then reconstructed automatically using the syntax-directed typing system.

Syntax For clarity of the examples in section 7, we provide the syntax of our programs, omitting details on process definitions. A program contains a series of mutually recursive type and process declarations and definitions. It is also provided with auxiliary subtyping declarations.

```
type V[x1]...[xn] = A
eqtype V[x1]...[xm] <= U[y1]...[yk]
decl f[x1]...[xi] : (c1 : A1) ... (cj : Aj) |- (c : B)
```

The first line represents a *type definition*, where V is the type name parameterized by type parameters $x_1, \ldots, x_n$ and A is its definition. The second line stands for *subtyping declarations*, that seed the subtyping algorithm with additional constraints. The third line is a *process declaration*, where f is a process name parameterized by type variables $x_1, \ldots, x_i$ and $(c_1 : A_1) \ldots (c_j : A_j)$ are the used channels and corresponding types; the offered channel is c and has type B. A process declaration is followed by its definition, which we omit because it is not completely necessary for understanding the program's communication behavior.

Once the program is parsed and its abstract syntax tree is extracted, we perform a *validity check* on it. This includes checking that type definitions, process declarations, and process definitions are closed with respect to the type variables in scope. To simplify and improve the subtyping algorithm's efficiency, we also assign internal names to type subexpressions parameterized over their free variables. These internal names are not visible to the programmer. Once the internal names are collected, the most informative variances are assigned to all type parameters in all type definitions. For that purpose, we start by assigning variance $\perp$ to all type parameters and then we calculate the least fixed point [1].

Type Checking After the most informative variances are assigned to all type parameters, we check the validity of `eqtype` declarations. This validity is performed by checking each declaration against each other (recall subsection 5.4). These declarations are then incorporated in the set of closures Γ, where the algorithm will collect subtyping constraints during its execution.

The type checker explicitly calls the subtyping algorithm in 3 places: (i) forwarding, (ii) sending/receiving channels, and (iii) process spawning. Earlier, Rast used type equality for typing these 3 constructs. We relax type equality in Rast to subtyping in two ways. If a process P offers channel of type A, we also safely allow P to offer A' if $A \leq A'$. Dually, if a process P uses a channel of type A, we allow it to use A' if $A' \leq A$.

When a type equality needs to be verified, either in the provided declarations or during algorithm's execution, the type checker verifies subtyping in both directions.

7 Examples

In the following examples, we focus on types, rather than process code, because the types alone suffice to describe the essential communication behavior. The complete Rast code for these examples can be found in the supplementary material.

7.1 Stacks

As an example of nested types and subtyping, we will present an implementation of stacks of natural numbers, where the stack type tracks the stack's shape using the nested type parameter. (These stacks could also be made polymorphic in the type of data elements, but here we choose to use monomorphic data so that we may focus on the use of nested types, subtyping, and keep the syntactic overhead down. At the end of this section, we will give the polymorphic types.)

Stacks of natural numbers At a basic level, stacks of natural numbers can be described by the (lightly nested) type

```
type Stack' = &{ push: nat -o Stack' , pop: Option[Stack'] }
```

where

```
type Option[k] = +{ some: nat * k , none: 1 }
```

Each stack supports push and pop operations with an external choice between `push` and `pop` labels. If the stack's client chooses `push`, then the subsequent type, `nat -o Stack'`, requires that the client send a `nat`; then the structure recurs at type `Stack'` to continue serving push and pop requests. If the stack's client instead chooses `pop`, then the subsequent type, `Option[Stack']`, requires the client to branch on whether the stack is nonempty – whether there is `some` natural number or `none` at all at the top of the stack.

Type nesting enforces an invariant Implicit in this description of how a stack would offer type `Stack'` is a key invariant about the stack's shape: popping from a stack onto which a natural number was just pushed should always yield `some` natural number, never `none` at all. However, the type `Stack'` cannot enforce this pop-after-push invariant precisely because it does not track the stack's shape – `Stack'` can be used equally well to type empty stacks as to type stacks containing three natural numbers, for instance. But by taking advantage of the expressive power provided by nested types, we can enforce the invariant by defining a type `Stack[k]` of `k`-shaped stacks of natural numbers that can enforce the pop-after-push invariant.

We will start by defining two types that describe shapes.

```
type Some[k] = +{ some: nat * k }
type None = +{ none: 1 }
```

The shape `Some[k]` describes a `k` with `some` natural number added on top; the type `None` describes an empty shape. Notice that both `Some[k]` and `None` are subtypes of `Option[k]`, for all types `k`.

The idea is that `Stack[None]` will type empty stacks – because they have the shape `None` – and that `Stack[Some[Stack[None]]]` will type stacks containing one natural number – because they have `Some` natural number on top of an empty stack – and so on for stacks of larger shapes.

More generally, the type `Stack[k]` describes `k`-shaped stacks:

```
type Stack[k] = &{ push: nat -o Stack[Some[Stack[k]]] , pop: k }
```

Once again, each stack supports push and pop operations with an external choice between `push` and `pop` labels. This time, however, pushing a `nat` onto the stack leads to type `Stack[Some[Stack[k]]]`, i.e., a stack with `Some` natural number on top of a `k`-shaped stack. Equally importantly, popping from a `k`-shaped stack exposes the shape `k`.

Together these two aspects of the type `Stack[k]` serve to enforce the pop-after-push invariant. Suppose that a client pushes a natural number, say 0, onto a stack `s` of type `Stack[k]`. After the push, the stack `s` will have type `Stack[Some[Stack[k]]]`. If the client then pops from `s`, the type becomes `Some[Stack[k]]`, which is `+{ some: nat * Stack[k] }`. This means that the client will always receive `some` natural number, never `none` at all because `none` is not part of this type. And that is how the type `Stack[k]` enforces the pop-after-push invariant.

Given this type constructor, the empty stack can be expressed as a process of type `Stack[None]`:

```
decl empty : . |- (s : Stack[None])
```

And we can define a process `elem[k]` that constructs a stack with shape given by `Some[Stack[k]]`, from a natural number and a stack of shape `k`:

```
decl elem[k] : (x:nat) (t : Stack[k]) |- (s : Stack[Some[Stack[k]]])
```

Subtyping So far, we have taken advantage of the expressive power of nested types to guarantee adherence to an invariant about a stack's shape. The strength and precision of these types is a double-edged sword, however. Sometimes we will want to implement operations for which it is difficult or even impossible to maintain the stronger, more precise types across the operation. Subtyping will allow us to fall back on the weaker, less precise types in these cases.

For example, our more precise stack types `Stack[None]` and `Stack[Some[k]]` are both subtypes of the less precise `Stack'` type if k is a subtype of `Stack'`. So, as a specific example, from stacks of type `Stack[None]` or `Stack[Some[Stack']]`, we can always fall back on the less precise `Stack'`. Our Rast implementation is capable of verifying this instance using the following `eqtype` declarations.

```
eqtype Stack[None] <= Stack'
eqtype Stack[Some[Stack']] <= Stack'
eqtype Stack[Option[Stack']] <= Stack'
```

As an example of where this subtyping can be useful, consider an operation to reverse a stack. Because our system does not include intersection types – an opportunity for future work – it is difficult to express the invariant that reversing a stack preserves its shape, making it difficult to use the more precise `Stack[None]` and `Stack[Some[k]]` types. Instead, we use the type `Stack'` to describe the stack before and after reversal:

```
decl reverse : (t : Stack') |- (s : Stack')
```

This `reverse` process would be implemented by way of a helper process:

```
decl rev_append : (t : Stack') (a : Stack') |- (s : Stack')
```

Specifically, we would call `rev_append` with the accumulator `a` being the `empty` stack process mentioned previously. But `empty` has the type `Stack[None]`, whereas `rev_append` uses the less precise `Stack'` type. Subtyping is how we are able to bridge this gap and implement stack reversal.

Above, we showed that we can move from a more precise `Stack[None]` or `Stack[Some[Stack']]` type to a less precise `Stack'` type using subtyping, and showed how that can be useful. But by moving to the less precise `Stack'` type, we are not irreversibly losing information about the stack's shape. Given a `Stack'`, we can recover the more precise type `Stack[Option[Stack']]` for that stack by using the process

```
decl shaped : (t : Stack') |- (s : Stack[Option[Stack']])
```

This `shaped` process acts as a kind of coercion that analyzes the shape of `Stack'` and constructs a `Stack[Option[Stack']]`. This coercion is operationally not the identity function, so `Stack'` is not a *subtype* of `Stack[Option[Stack']]`, but we can still use it as an explicit coercion for recovering information about the stack's shape.

Polymorphic stacks Lastly, we close this example by generalizing the above types for stacks of natural numbers to types for stacks that are polymorphic in the type `a` of data that they hold.

The type for polymorphic stacks of unknown shape is:

```
type Stack'[a] = &{push: a -o Stack'[a], pop: Option[a][Stack'[a]]}
```

and the type for polymorphic stacks of shape `k` is:

```
type Stack[a][k] = &{push: a -o Stack[a][Some[a][Stack[a][k]]], pop: k}
```

where

```
type Option[a][k] = +{ some: a * k , none: 1 }
type Some[a][k] = +{ some: a * k }
type None = +{ none: 1 }
```

Notice that the type `Stack'[a]` is bivariant in `a`, with `a` occurring in both negative and positive positions. Thus there is no subtyping relationship at all between `Stack'[nat]` and `Stack'[even]`.

The type `Stack[a][k]` is also bivariant in `a`, although for a slightly more intricate reason: `Stack[a][k]` is covariant in `k` and so the occurrence of `a` in `Some[a][Stack[a][k]]` in the larger type `Stack[a][Some[a][Stack[a][k]]]` is a positive one, in contrast with the negative occurrence in the linear implication. Like for `Stack'[ ]`, there is no subtyping relationship between types `Stack[nat][k]` and `Stack[even][k]`.

7.2   Queues

We can similarly use a combination of nested types and subtyping to describe queues. The type `Queue'` describes a queue with an external choice between `enq` and `deq` labels:

```
type Queue' = &{ enq: nat -o Queue' , deq: Option[Queue'] }
```

As for stacks, there is a dequeue-after-enqueue invariant that is not enforced by this type. We can again use nested types to enforce this invariant:

```
type Queue[k] = &{ enq: nat -o Queue[Some[Queue[k]]] , deq: k }
```

At first glance, it seems obvious that this should be possible for queues – it worked for stacks, after all. However, the nature of parametric type constructors like `Stack[k]` and `Queue[k]` is that we can only build up types from the parameter `k`, never analyze `k` to break it down into smaller types. This means that the nested type operates something like a stack. But unlike stacks, queues insert new elements at the back, so it is, in fact, somewhat surpring that this same technique works to enforce the dequeue-after-enqueue invariant for queues.

Similar to the stacks, we can move from the more precise shaped types `Queue[None]` and `Queue[Some[Queue']]` to the less precise type `Queue'` by way of subtyping. But, like the stacks, we can also express a coercion from the less precise `Queue'` to the more precise `Queue[Option[Queue']]`:

```
decl shaped : (q' : Queue') |- (q : Queue[Option[Queue']])
```

7.3   Dyck language

As a simple example, consider the language of `$`-terminated strings of balanced parentheses (also known as the Dyck language) [21]. This language can be described with the type `D0` that uses a type constructor `D[k]` to track the number of unmatched `l`s:

```
type D0 = +{ l: D[D0] , $: 1 }
type D[k] = +{ l: D[D[k]] , r: k }
```

Each occurrence of `D[ ]` correspond to an unmatched `l`.

We can also describe the related language $l^n r^n \$$ (with $n \geq 0$) using nested types:

```
type E0 = +{ l: E[+{$:1}] , $: 1 }
type E[k] = +{ l: E[R[k]] , r: k }
type R[k] = +{ r: k }
```

Here the number of type constructors `R[ ]` corresponds to the number of unmatched `l`s.

The `$`-terminated language of balanced parentheses includes language $l^n r^n \$$, and this is reflected in the subtyping relationship between `E0` and `D0`: `E0` is a subtype of `D0`. Our Rast implementation will verify this, provided that we sufficiently generalize the coinductive hypothesis using `eqtype` declarations:

```
eqtype R[k] <= D[k]
eqtype E[R[k]] <= D[D[k]]
```

(Strictly speaking, our types are coinductively defined, so it is not quite right to think of this as language inclusion for finite strings, but instead a property of all of the finite prefixes of potentially infinite strings.)

8   Additional Related Work

The literature on subtyping in general is vast, so we cannot survey it; some related work is already mentioned in the introduction. Once recursive types and parameterized type constructors are considered together, there are significantly fewer results, as far as we are aware.

The seminal work in this area in general is that of Solomon [45] who showed that the problem of language equivalence for deterministic pushdown automata (DPDA) is interreducible with *type equality* for a language with purely positive parameterized recursive types under an inductive interpretation. Our construction for undecidability has been inspired by his technique, although we use a reduction from the related problem of language inclusion for BPAs, rather than DPDAs, due to the coinductive nature of our definition of subtyping.

Unfortunately, despite the eventual discovery that DPDA language equality is decidable [43] it does not yield a practical algorithm. Instead, our algorithm is based on Gay and Hole's construction of a simulation [25] for session types,

combined with careful considerations of constructor variance (following [1]) and instantiation of type variables when (in essence) constructing circular proofs [6] of subtyping.

Perhaps closest to our system in the functional world is that by Im et al. which features equirecursive nested types with a mixed inductive and coinductive definition of type equality and subtyping [32]. It also includes modules, but not first-class universal or existential quantification. The authors show their system to be sound for a call-by-value language under the assumption of contractiveness for type definitions (which we also make). They only briefly mention algorithmic issues with no resolution.

9   Conclusion

We presented an undecidability proof for a system of subtyping with nested recursive and polymorphic types under a coinductive interpretation of types. We also presented a practical algorithm for subtyping. When embedded in a language with bidirectional type checking and explicit polymorphism, type checking directly reduces to subtyping and we have implemented and explored our algorithm in the context of Rast [14].

Since undecidability of a small fragment is definitive, the most interesting items of future work concern extensions of the algorithm.

It seems plausible that we can hypothesize additional pairs to add to the partial simulation in situations when the subtyping algorithm exceeds its depth bound and consequently fails without a counterexample. Currently, this must be done by the programmer.

We conjecture it requires only a minor adaptation to incorporate a mixed inductive/coinductive definition of subtyping into our algorithm [10,5]. This would make it easily applicable in strict functional languages such as ML.

Much work has been done on subtyping polymorphic types [46] in implicit form so that $\forall \alpha.\alpha \to \alpha \leq \mathsf{int} \to \mathsf{int}$. We do not consider such subtyping here, since it introduces new sources of undecidability and significant additional algorithmic complexities. It is an interesting question to what extent our *algorithm* might still apply in the setting of implicit polymorphism, possibly restrict to the prefix case.

A final item for future work is to consider *bounded polymorphism* and extension from nested types to the related *generalized algebraic data types* (GADTs). A positive sign here is the uniform semantics by Johann and Polonsky [35], disturbed slightly by more recent results pointing out some obstacles to parametricity [34].

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