Characterization of Shortest Paths on Directional Frictional Polyhedral Surfaces

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Abstract

In this paper, we address a shortest path problem where an autonomous vehicle moves on a polyhedral surface according to a distance function that depends on the direction of the movement (directional) and on the friction of the space (frictional). This shortest path problem generalizes a hierarchy of problems and finds geometric structure to solve several proximity problems. We perform the characterization of shortest paths for a directional frictional geodesic (DFG) distance function on polyhedral surfaces. We derive the local optimality criterion necessary to solve the corresponding shortest path problem using the continuous Dijkstra algorithm [11]. The derivation of this optimality criterion essentially involves demonstrating the strict convexity of the DFG distance function. This contribution is the most fundamental result that enables all constructions of the continuous Dijkstra algorithm to solve the corresponding DFG shortest path problem.

1 Introduction

Path planning still remains an active field with modern applications such as autonomous vehicles and surveillance systems [1, 5, 7]. Although some effort has been made on path planning in unknown and dynamic environments [4, 19], the shortest path planning problems in known static environments are a fundamental step towards these spaces.

In this paper, we address a shortest path problem where an autonomous vehicle moves on a polyhedral surface according to a distance function that depends on the direction of the movement (directional) and on the friction of the space (frictional). More specifically, the distance function considers the total work done by an external force applied to the vehicle to move it from a point to another. Therefore, we generalize the shortest path problem on polyhedral surfaces [11, 12] to consider the moving direction, friction, and slope.

The continuous Dijkstra paradigm [11] is an algorithm that solves several shortest path problems by simulating the propagation of a wave from a source point to all points in the space. The structure of the wave is updated at discrete events when the wave reaches vertices and edges. This structure consists of *intervals of optimality* subdividing each edge according to the sequences of vertices and edges that uniquely define a path from the source to any point in the interval.

The continuous Dijkstra algorithm finds shortest paths by exploring characteristic properties related to the local behavior of the shortest paths. The characterization of shortest paths consists of the determination of a local optimality criterion. In this paper, we perform the characterization of shortest paths for a directional frictional geodesic (DFG) distance function on polyhedral surfaces. We derive the local optimality criterion necessary to solve the corresponding shortest path problem using the continuous Dijkstra algorithm. The derivation of this optimality criterion essentially involves demonstrating the strict convexity of the DFG distance function.

Section 2 defines the shortest path problem addressed in this paper and the corresponding distance function considered in this problem. Section 3 reduces a hierarchy of shortest path problems to the DFG shortest path problem. In section 4, we present the characterization of geodesics and shortest paths on polyhedral surfaces according to this distance function. Section 5 has our concluding remarks.

2 Directional Frictional Shortest Path Problem

Let S be a polyhedral surface, possibly non-convex, specified by a set of faces, edges, and vertices. We assume that all faces of S are triangles since simple polygons may be triangulated in linear time on the number of vertices [2]. We consider *bounded polyhedral surfaces*, that is, a surface with a finite number of bounded faces.

Based on Newtonian mechanics, we address a shortest path problem related to a point particle moving on the surface S according to a path of minimum resistance. As a *distance function*, we consider the total amount of mechanical work that must be done to move the particle through the path. More specifically, the distance function considers the work done by an *external force* F to move the particle between points on the same face of S. The forces acting on the particle, besides the external

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force F, are the weight P ($|P| = m.g$), the normal N ($|N| = |P|\cos\alpha_f$), and the friction A ($|A| = \mu_f|N|$), where m is the mass of the particle, g is the acceleration of gravity, α_f is the acute angle corresponding to the slope between a face f and the horizontal plane, and μ_f is the kinetic friction coefficient of face f. We assume a constant kinetic friction coefficient μ_f and a constant slope α_f for all points on each face f of S. The static friction force is ignored. We also assume that μ_f is finite and positive. Thus, we do not consider obstacles or free regions.

The vector sum of the forces weight and normal ($P \oplus N$) results in a force R_{PN}. This force is contained on the face with direction of the gradient, that is, perpendicular to the intersection between the face and the horizontal plane (see Fig. 1). The forces weight-normal R_{PN} and friction A implies in a resulting force $R_{PNA} = R_{PN} \oplus A$ according to angle β (see Fig. 1). We assume the acceleration of the particle is zero (*i.e.*, a constant positive speed). Therefore, since the resulting force of all forces is null, we conclude that F has the same direction and size of R_{PNA}, but opposite orientation. Thus, the size of force F is:

$$|F| = mg\sqrt{\sin^2\alpha_f + 2\mu_f \sin\alpha_f \cos\alpha_f \cos\beta + \mu_f^2 \cos^2\alpha_f}.$$

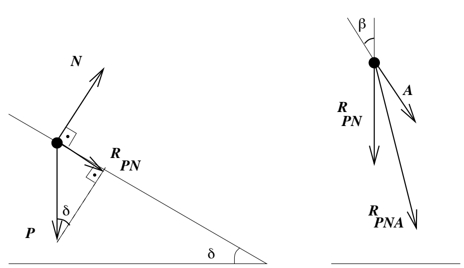

Figure 1: Resulting force acting on the body.

We define the *directional frictional geodesic* (DFG) distance function such that each face f is associated with a kinetic friction coefficient $\mu_f > 0$ that specifies the resistance to move on the interior of face f. Similarly, each edge e is associated with a kinetic friction coefficient $\mu_e > 0$. According to the DFG distance, the length of a line segment from a point s to a point t in the edge e is the size of the external force F for $\beta \in \{0, \pi\}$ times the euclidean distance between s and t:

$$mg|\mu_e \cos\alpha_e + \sin\alpha_e \cos\beta||st|,$$

where α_e is the slope angle between e and its orthogonal projection into the horizontal plane. The length of a line segment from a point s to a point t on the interior of face f is the size of force F times the euclidean distance between s and t:

$$mg\sqrt{\sin^2\alpha_f + 2\mu_f \sin\alpha_f \cos\alpha_f \cos\beta + \mu_f^2 \cos^2\alpha_f}|st|,$$

where β is the angle between the friction force and the weight component projected into the plane of the face.

Formally, the DFG shortest path problem is stated as follows: Given one source point s on a triangulated polyhedral surface S, an assignment of kinetic friction coefficients to edges and faces, and an error tolerance $\varepsilon > 0$; build a structure that allows the computation of an ε-optimal path (according to the DFG distance function) from s to any query point t, such that the path stays on the surface S.

3 A Hierarchy of Shortest Path Problems

Several shortest path problems are reduced to the directional frictional shortest path problem addressed in this paper. The most specific shortest path problem considers the interior of a simple polygon (ISP) [17]. The Euclidean shortest path problem with polygonal obstacles (EPO) [6] consists of finding shortest paths in a plane avoiding a set of disjoint simple polygonal obstacles. This problem generalizes the ISP problem when the complement of the polygon is considered an obstacle. A special case of the EPO problem considers parallel straight line segments (PLS) as obstacles [9]. In a simpler version of the PLS problem, the obstacles are parallel half-lines (PHL) [18]. Another special case of the EPO problem consider only polygonal obstacles with disjoint convex hulls (DCH) [16].

A generalization of the EPO problem consists of finding shortest paths according to the Euclidean metric on the surface of a (possibly non-convex) polyhedron [11]. This problem is called the discrete geodesic problem (DGP). To reduce the EPO problem to the DGP problem, we construct a surface where each obstacle becomes an infinite orthogonal prism whose base is on the plane. A shortest path on this surface between points in the plane is fully contained in the plane and avoids all prisms which corresponds to a shortest path avoiding obstacles. Another special case of the DGP problem, is the shortest path problem on the surface of a convex polyhedron (SCP) [13].

The weighted region problem (WRP) [12] consists of finding a path in a planar subdivision that minimizes the total cost according to a weighted Euclidean metric. The WRP problem generalizes the EPO problem. In this case, the weights associated with the free space and obstacles are 1 and $+\infty$, respectively. A special case of the WRP problem arises when the weights of regions are 0, 1, or $+\infty$ which has applications to the maximum concealment problem (MCP).

The DFG problem addressed in this paper generalizes the DGP problem when there is only one constant force applied to the particle for the whole polyhedral surface. In this case, the total work done to move the particle considers only the weight force. On the other hand, if the polyhedral surface is embedded into a single plane, the only force applied to the particle is the constant friction on each face, hence, the WRP problem is a special case of the DFG problem.

4 Geodesic and Optimal Paths

Assuming that each face has a constant kinetic friction coefficient and a constant slope, the following lemma states that geodesic paths are piecewise linear. A *piecewise linear* path is a path whose intersection with any face is the union of disjoint line segments.

Lemma 1 *Let f be a face with $\mu_f > 0$. Let s and t be points on the interior of f. A subpath from s to t fully contained on the interior of f is geodesic if and only if it is a straight line segment.*

Proof. If the subpath from s to t is geodesic, then it must be locally optimal with regards to the DFG distance function. We assume that the subpath from s to t is the arc of a differential *parametric curve* $\Phi : I \to \mathbb{R}^2$ from an open range $I \subset \mathbb{R}$ into $\mathbb{R}^2$ that represents the plane of face f, where Φ is a function that leads $i \in I$ to a point $\Phi(i) = (x(i), y(i)) \in \mathbb{R}^2$. The tangent vector $\Phi'(i)$ is the vector $(x'(i), y'(i))$, where $x'(i)$ is the first derivative of $x(i)$ in $i \in I$.

The length d_{i_0,i_1} of the arc of the parametric curve Φ according to the DFG distance function, from $\Phi(i_0) = s$ to $\Phi(i_1) = t$, where $i_0, i_1 \in I$ is [3]:

$$d_{i_0,i_1} = \left| \int_{i_0}^{i_1} F_{x_i} V_x \left|\Phi'(i)\right| di \oplus \int_{i_0}^{i_1} F_{y_i} V_y \left|\Phi'(i)\right| di \right|.$$

The forces $F_{x_i} V_x$ and $F_{y_i} V_y$ are horizontal and vertical components of the external force F_i, where V_x and V_y are the unit vectors in the direction of the axes. We denote by $|st|$ the Euclidean length of the straight line segment $\overline{st}$. Therefore, we have the following:

$$\left| \int_{i_0}^{i_1} F_{x_i} V_x \left|\Phi'(i)\right| di \right| = -\left|A\right| \cos\beta \left|st\right|,$$

$$\left| \int_{i_0}^{i_1} F_{y_i} V_y \left|\Phi'(i)\right| di \right| = -\left|R_{PN}\right| \int_{i_0}^{i_1} \left|\Phi'(i)\right| di - \left|A\right| \sin\beta \left|st\right|,$$

where $|\Phi'(i)| = \sqrt{(x'(i))^2 + (y'(i))^2}$ denotes the size of vector $\Phi'(i)$, A is the friction force, and R_{PN} is the weight and normal resulting force.

The length of the line segment $\overline{st}$ according to the DFG distance function is $d_{s,t} = |F_x V_x |st| \oplus F_y V_y |st||$, where

$$\begin{aligned} \left|F_x V_x \left|st\right|\right| &= -\left|A\right| \cos\beta \left|st\right|, \\ \left|F_y V_y \left|st\right|\right| &= -\left|R_{PN}\right| \left|st\right| - \left|A\right| \sin\beta \left|st\right|. \end{aligned}$$

Thus, we must show that $d_{s,t} \le d_{i_0,i_1}$. However, since

$$\left|F_x V_x \left|st\right|\right| = \left| \int_{i_0}^{i_1} F_{x_i} V_x \left|\Phi'(i)\right| di \right|,$$

we need only to demonstrate that $|F_y V_y |st|| \le \left|\int_{i_0}^{i_1} F_{y_i} V_y |\Phi'(i)| di\right|$. Since $|st| \le \int_{i_0}^{i_1} |\Phi'(i)| di$, we have

$$\left|\left(-\left|R_{PN}\right| \left|st\right|\right) V_y\right| \le \left| \left(-\left|R_{PN}\right| \int_{i_0}^{i_1} \left|\Phi'(i)\right| di\right) V_y \right|.$$

Therefore,

$$\left|\left(-\left|R_{PN}\right| \left|st\right| - \left|A\right| \sin\beta \left|st\right|\right) V_y\right| \le \left| \left(-\left|R_{PN}\right| \int_{i_0}^{i_1} \left|\Phi'(i)\right| di - \left|A\right| \sin\beta \left|st\right|\right) V_y \right|,$$

that is, the length of the line segment $\overline{st}$ is less than or equal to the length of the arc of any parametric curve according to the DFG distance function. Thus, the subpath represented by this arc is geodesic if and only if it is a straight line segment. Furthermore, since the geodesic subpath is a line segment, we can drop our assumption that the curve Φ is differential. $\square$

Corollary 4.1 *Geodesic paths on polyhedral surfaces according to the DFG distance function are piecewise linear.*

The locus of points t on a face f with constant distance δ from a source point s, according to the DFG distance function, consists of a curve defined by the following polar equation: $|st| =$

$$\frac{\delta}{mg\sqrt{\mu_f^2 \cos^2\alpha_f + 2\mu_f \cos\alpha_f \sin\alpha_f \cos\beta_t + \sin^2\alpha_f}},$$

where β_t is the angle between the vector of the force R_{PN} in the direction of the gradient and the vector of the friction force A in the direction of the line segment $\overline{st}$. This curve has an oval shape. The curve is a circle when $\alpha_f = 0$, but it has a degenerated shape when $mg\sqrt{\mu_f^2 \cos^2\alpha_f + 2\mu_f \cos\alpha_f \sin\alpha_f \cos\beta_t + \sin^2\alpha_f} = 0$. This only occurs when $\cos\beta_t = -1$. In this case, we have $(\mu_f \cos\alpha_f - \sin\alpha_f)^2 = 0$, that is, $\mu_f = \tan\alpha_f$. We assume that $\mu_f \neq \tan\alpha_f$ to avoid this degenerated case and to guarantee the strict convexity of the DFG distance function. This locus is a strong evidence of the convexity of the DFG function. However, the algebraic proof [1] of this fact is necessary to guarantee that there exists a local optimality criterion.

The *angle of incidence* θ is the acute angle between a segment of a geodesic path that crosses (incoming ray) the boundary of face f and a vector perpendicular to the boundary of f. The *angle of refraction* θ' is the acute angle between a segment of a geodesic path that crosses (outgoing ray) the boundary of face f' and a vector perpendicular to the boundary of f'.

A geodesic path must pass through the interior of an edge e according to a local optimality criterion for the directional frictional geodesic shortest path problem.

[1] This proof is presented in appendix B.

Lemma 2 *Let f and f' be two faces that share an edge e. Let s be a point on the interior of f and let t be a point on the interior of f'. Let p be a geodesic path between s and t that passes through only one point x^* in the interior of e, then x^* is uniquely defined.*

Proof. The proof consists of solving a minimization problem on the length of a path from a point $s(0, -y_0, z_0)$ on face f to a point $t(x_1, y_1, z_1)$ on face f' passing through only one point x^* in edge $e = f \cap f'$ according to the DFG distance function (see Fig. 2). The faces f and f' are defined by points s, t, and by the edge e. The points in the edge e are projected into the y axis and they have height equal to $(ax+b)$, where a and b are constants.

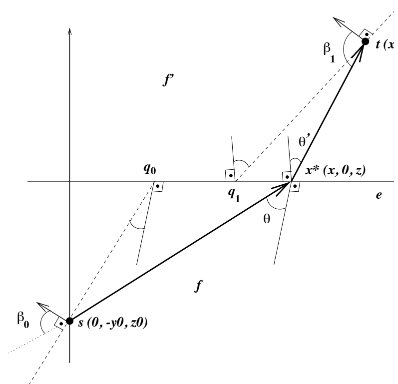

Figure 2: Local optimality criterion.

We must find the minimum point x^* in the following function [2] with a single real variable x:

$$\sqrt{\mu_f^2 \cos^2 \alpha_f + 2\mu_f \cos \alpha_f \sin \alpha_f \cos \beta_{f_x} + \sin^2 \alpha_f}\ |sx^*| +$$
$$\sqrt{\mu_{f'}^2 \cos^2 \alpha_{f'} + 2\mu_{f'} \cos \alpha_{f'} \sin \alpha_{f'} \cos \beta_{f'_x} + \sin^2 \alpha_{f'}}\ |x^*t|,$$

where $\mu_f, \mu_{f'}$ are friction coefficients; $\alpha_f, \alpha_{f'}$ are slope angles between faces and the horizontal plane; and $\beta_{f_x}, \beta_{f'_x}$ are the angles between the friction force A in the direction of the movement and the resulting force R_{PN} in the direction of the gradient for each face, respectively. Note that the angles [3] β_{f_x} and $\beta_{f'_x}$ change according to x.

The convexity of the DFG distance function between points on the same face guarantees that there exists a single point x_s (x_t) in e whose distance from s (to t) is minimum. Therefore, the point x^* must be in the range $[x_s, x_t]$. The determination of the minimum point x_s is achieved using the first derivative of the DFG distance function from s to x^*. Analogously, we find the point x_t. In appendix C, we get the following expression that specifies x_s in function of the angle of incidence θ_s in edge e, where $c_{s_0}, \ldots, c_{s_4}$ are constants:

$$c_{s_0} + c_{s_1} \sin \theta_s + c_{s_2} \sin^2 \theta_s + c_{s_3} \sin^3 \theta_s + c_{s_4} \sin^4 \theta_s = 0.$$

The point x^* is uniquely specified by the first derivative of the DFG function from s to x^* added to the DFG function from x^* to t (see Exp. 5):

$$\frac{c_5(\cos \beta_{f_{x^*}})'\ |sx^*| + (2c_5 \cos \beta_{f_{x^*}} + c_6)\ (|sx^*|)'}{\sqrt{2c_5 \cos \beta_{f_{x^*}} + c_6}} +$$
$$\frac{c_7(\cos \beta_{f'_{x^*}})'\ |x^*t| + (2c_7 \cos \beta_{f'_{x^*}} + c_8)\ (|x^*t|)'}{\sqrt{2c_7 \cos \beta_{f'_{x^*}} + c_8}} = 0,$$

where $c_5 = \mu_f \cos \alpha_f \sin \alpha_f$, $c_6 = \mu_f^2 \cos^2 \alpha_f + \sin^2 \alpha_f$, $c_7 = \mu_{f'} \cos \alpha_{f'} \sin \alpha_{f'}$, and $c_8 = \mu_{f'}^2 \cos^2 \alpha_{f'} + \sin^2 \alpha_{f'}$.

We simplify the equation above, seeking an equivalent expression in function of angles θ and θ' (see Exp. 6):

$$\frac{\sin \theta (c_5 \cos \beta_{f_{x^*}} + c_6) + c_9}{\sqrt{2c_5 \cos \beta_{f_{x^*}} + c_6}} + \frac{\sin \theta' (c_7 \cos \beta_{f'_{x^*}} + c_8) + c_{10}}{\sqrt{2c_7 \cos \beta_{f'_{x^*}} + c_8}} = 0, \tag{1}$$

where $c_9 = \frac{c_5 c_4}{\sqrt{1+a^2}|sq'|}$, q' is the intersection of the straight line passing by s in the same direction of the vector of the force R_{PN} with the edge e, c_{10} is a constant analogous to c_9 related to face f'. Note that $\cos \beta_{f_{x^*}} = \pm \cos \psi \cos \theta \pm \sin \psi \sin \theta$ and $\cos \beta_{f'_{x^*}} = \pm \cos \psi' \cos \theta' \pm \sin \psi' \sin \theta'$, where ψ and ψ' are constants. Therefore, the algebraic expression above uniquely specifies x^*. $\square$

The *critical angle* $\theta_{f,f'}$ for e consists of the angle of incidence when the angle of refraction $\theta' = \frac{\pi}{2}$. In this case, the local optimality criterion (see Eq. 1) has the following form:

$$\frac{\sin \theta_{f,f'} (c_5(\pm \cos \psi \cos \theta_{f,f'} \pm \sin \psi \sin \theta_{f,f'}) + c_6) + c_9}{\sqrt{2c_5(\pm \cos \psi \cos \theta_{f,f'} \pm \sin \psi \sin \theta_{f,f'}) + c_6}} +$$
$$\frac{\pm c_7 \sin \psi' + c_8 + c_{10}}{\sqrt{2c_7 \sin \psi' + c_8}} = 0.$$

Then, analogously to θ_s and θ_t, the critical angle $\theta_{f,f'}$ is given by the expression

$$c_{f,f'_0} + c_{f,f'_1} \sin \theta_{f,f'} + c_{f,f'_2} \sin^2 \theta_{f,f'} +$$
$$c_{f,f'_3} \sin^3 \theta_{f,f'} + c_{f,f'_4} \sin^4 \theta_{f,f'} = 0,$$

where c_{f,f'_i} are constants for $i = 0, \ldots, 4$ (see Eq. 7).

A geodesic path *critically uses* part of an edge e when it reaches the edge e at the critical angle $\theta_{f,e}$ at a point q interior to e, travels along edge e for some distance, and leaves edge e into the interior of f' at the critical angle $\theta_{e,f'}$ at a point r interior to e (see Fig. 3(a)).

A geodesic path is *critically reflected* by e from face f when it is incident to edge e at critical angle $\theta_{f,e}$ at a point q interior to e, travels along edge e for some distance, and exits edge e back into face f at a point r interior to e, leaving the edge at angle $\theta_{e,f}$ (see Fig. 3(b)).

[2] We prove the strict convexity of this function in appendix B.

[3] Details about the angles β_{f_x} and $\beta_{f'_x}$ are found in appendix A.

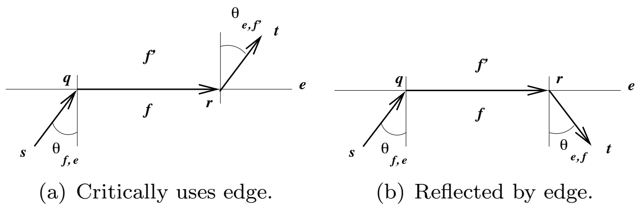

Figure 3: Geodesic paths on an edge.

We generalize Lemma 2 to consider critical angles, that is, paths that can either critically use part of an edge or be critically reflected. In this case, we have a two-variable minimization problem.

Lemma 3 *A geodesic path crosses edge $e = f \cap f'$ in one of two ways: either it intersects edge e at one crossing point and satisfies the local optimality criterion at that point, or it hits edge e at a critical angle $\theta_{f,e}$, travels along the edge for some distance, and exits the edge into the other face (into the same face) at a critical angle $\theta_{e,f'}$ ($\theta_{e,f}$).*

Proof. The proof consists of solving a convex (nonlinear) programming problem [10] in two real variables x and x', where x and x' are the coordinates of the points q and r at the x axis, respectively (see Fig. 3). Our goal is to minimize the function $d(x, x') =$

$$\sqrt{\mu_f^2 \cos^2\alpha_f + 2\mu_f \cos\alpha_f \sin\alpha_f \cos\beta_{f_x} + \sin^2\alpha_f}\ |sx^*| + (\mu_e \cos\alpha_e - \sin\alpha_e)\ |x^* x'^*| + \sqrt{\mu_{f'}^2 \cos^2\alpha_{f'} + 2\mu_{f'} \cos\alpha_{f'} \sin\alpha_{f'} \cos\beta_{f'_x} + \sin^2\alpha_{f'}}\ |x'^* t|$$

subject to $g(x, x') = x - x' \leq 0$. The Karush-Kuhn-Tucker conditions [8] imply the three relations $\nabla d(x, x') + l \nabla g(x, x') = 0$, $l\ g(x, x') = 0$, $l \geq 0$, where $\nabla d(x, x')$ and $\nabla g(x, x')$ are gradient vectors and l is the Lagrange multiplier. Therefore, if $l = 0$ we have $\nabla d(x, x') = \left(\frac{\partial d(x,x')}{\partial x}, \frac{\partial d(x,x')}{\partial x'}\right) = 0$, otherwise,

$$\frac{c_5 (\cos\beta_{f_x})'\ |sx|}{\sqrt{c_6 + 2c_5 \cos\beta_{f_x}}} + \sqrt{c_6 + 2c_5 \cos\beta_{f_x}}\ (|sx|)' + \sqrt{1 + a^2}(\mu_e \cos\alpha_e - \sin\alpha_e) = 0 \text{ and}$$

$$\frac{c_7 (\cos\beta_{f'_{x'}})'\ |x't|}{\sqrt{c_8 + 2c_7 \cos\beta_{f'_{x'}}}} + \sqrt{c_8 + 2c_7 \cos\beta_{f'_{x'}}}\ (|x't|)' - \sqrt{1 + a^2}(\mu_e \cos\alpha_e - \sin\alpha_e) = 0.$$

Simplifying in terms of critical angles $\theta_{f,e}$ and $\theta_{e,f'}$,

$$\frac{\sin\theta_{f,e}(c_6 + c_5 \cos\beta_{f_x}) + c_9}{\sqrt{c_6 + 2c_5 \cos\beta_{f_x}}} + (\mu_e \cos\alpha_e - \sin\alpha_e) = 0 \text{ and}$$

$$\frac{\sin\theta_{e,f'}(c_8 + c_7 \cos\beta_{f'_{x'}}) + c_{10}}{\sqrt{c_8 + 2c_7 \cos\beta_{f'_{x'}}}} - (\mu_e \cos\alpha_e - \sin\alpha_e) = 0.$$

However, the Lagrange multiplier is not zero if and only if $g(x, x') = 0$, that is, $x = x'$. In this case, the path crosses the edge at the single crossing point and satisfies the following local optimality criterion:

$$\frac{(c_6 + c_5 \cos\beta_{f_x}) \sin\theta + c_9}{\sqrt{c_6 + 2c_5 \cos\beta_{f_x}}} + \frac{(c_8 + c_7 \cos\beta_{f'_{x'}}) \sin\theta' + c_{10}}{\sqrt{c_8 + 2c_7 \cos\beta_{f'_{x'}}}} = 0.$$

Depending on the Lagrange multiplier, a geodesic path either intercepts the edge at a single point ($l \neq 0$), or travels along the edge and exits at critical angle $\theta_{e,f'}$ ($l = 0$) according to the local optimality criterion. There is a similar proof for critically reflected paths. □

The intersection of a geodesic path p with an edge e is a set, probably empty, of points and segments. These points are called *crossing points* of the edge e for path p and the segments, *shared segments* for e and p.

The convexity of the DFG function uniquely specifies a geodesic path that intercepts an edge sequence.

Lemma 4 *If p is a geodesic path from a point s to a point t that intercepts the edge sequence $E = (e_1, \ldots, e_k)$ with $e_i \neq e_{i+1}$ (so that there are no shared segments), then p is the unique geodesic path connecting s to t.*

Proof. We show that the function that gives the DFG length of the path from point s to point t intercepting the edge sequence E is a strictly convex function of the crossing points at each edge (see Fig. 4).

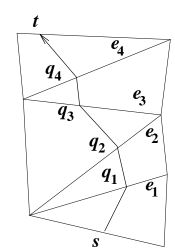

Figure 4: A path intercepting edge sequence E.

The DFG function of the length of the path from point s to point t intercepting the edge sequence E is given by $d(q_1, \ldots, q_k) = d_1 + \sum_{i=1}^{k-1} d_{i+1} + d_{k+1} =$

$$\sqrt{\mu_1^2 \cos^2\alpha_1 + 2\mu_1 \cos\alpha_1 \sin\alpha_1 \cos\beta_1 + \sin^2\alpha_1}\ |sq_1| + \sum_{i=1}^{k-1} \sqrt{\mu_{i+1}^2 \cos^2\alpha_{i+1} + 2\mu_{i+1} \cos\alpha_{i+1} \sin\alpha_{i+1} \cos\beta_{i+1} + \sin^2\alpha_{i+1}}\ |q_i q_{i+1}| + \sqrt{\mu_{k+1}^2 \cos^2\alpha_{k+1} + 2\mu_{k+1} \cos\alpha_{k+1} \sin\alpha_{k+1} \cos\beta_{k+1} + \sin^2\alpha_{k+1}}\ |q_k t|,$$

where q_i is the crossing point at edge e_i. Our goal is to show that $d(q_1, \ldots, q_k)$ is a strictly convex function of the crossing points at each edge. According to theorems in appendix B, the functions d_1 and d_{k+1} are strictly convex. The function d_{i+1} is strictly convex in two scalar variables that specify the points q_i and q_{i+1}. This follows from the strict convexity of this function when one of the points q_i or q_{i+1} is fixed. Thus, the intersection of an orthogonal plane to the horizontal plane (parallel to x or y axis) with this function implies a strictly convex curve. Generalizing the direction of the orthogonal plane, then the function in two variables is strictly convex.

Since function $d(q_1, \ldots, q_k)$ is a summation of strictly convex functions, then $d(q_1, \ldots, q_k)$ is strictly convex. Therefore, function $d(q_1, \ldots, q_k)$ has a unique global minimum, and any local minimum must be global. Since p is a local minimum, it is also the unique global minimum intercepting the edge sequence E. □

A *critical point of entry* of a geodesic path p into face f consists of a point q (the closer endpoint of a shared segment to the source s) interior to an edge $e = f \cap f'$ when p hits q from the side of f. Similarly, a *critical point of exit* of path p into face f is a point r interior to $(f \cap f')$ (the further endpoint of a shared segment from the source s) when p goes from r into face f.

Let v and v' be consecutive vertices encountered in the list of points describing a geodesic path p. The characterization of geodesic paths implies that the structure of the subpath of p between v and v' is an alternate list of crossing points and shared segments. A geodesic path p may be uniquely specified by a list of vertices, edges, and faces whose interiors contain a portion of p. Edges and faces may be repeated in this list.

Finally, we have the following characterization of geodesic and shortest paths on polyhedral surfaces according to the DFG function:

Theorem 5 *The general form of either a geodesic or a shortest path is a piecewise linear path that goes through an alternating sequence of vertices, (possibly empty) edge sequences, and shared segments, such that the path satisfies the local optimality criterion at each edge along any edge sequence and at the endpoints of each shared segment.*

Proof. Follows from Lemmas 2 and 3. □

5 Conclusions

We performed the characterization of shortest paths on polyhedral surfaces according to the DFG distance function. We derived the local optimality criterion by showing the strict convexity of this distance function. The DFG shortest path problem generalizes a hierarchy of shortest path problem [11, 12]. This implies in a single framework to address several shortest path problems and in the versatility necessary to consider several applications. Furthermore, this framework finds geometric structures embedded on polyhedral surfaces (i.e., a shortest path Voronoi diagram [14])that allows the solution of proximity problems (closest pair of points, all nearest neighbors, minimum expanding tree) on polyhedral surfaces according to the DFG distance.

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