Anisotropic Diagrams: Labelle Shewchuk approach revisited *

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Abstract

F. Labelle and J. Shewchuk [3] have proposed a discrete definition of anisotropic Voronoi diagrams. These diagrams are parametrized by a metric field. Under mild hypotheses on the metric field, such Voronoi diagrams can be refined so that their dual is a triangulation, with elements shaped according to the specified anisotropic metric field.

We propose an alternative view of the construction of these diagrams and a variant of Labelle and Shewchuk's algorithm. This variant computes the Voronoi vertices using a higher dimensional power diagram and refines the diagram as long as dual triangles overlap. We see this variant as a first step toward a 3-dimensional anisotropic meshing algorithm.

1 Introduction

Anisotropic meshes are triangulations of a given domain in the plane or in higher dimension, with elements elongated along prescribed directions. Anisotropic triangulations have been shown [5] to be particularly well suited for interpolation of functions or numerical modeling. They allow to minimize the number of triangles in the mesh while retaining a good accuracy in computations.

Various heuristic solutions for generating anisotropic meshes have been proposed. Li et al. [4] and Shimada et al. [6] use packing methods. Bossen and Heckbert [2] use a pliant method consisting in centroidal smoothing, retriangulating and inserting or removing sites. Borouchaki et al. [1] adapt the classical Delaunay refinement algorithm to the case of an anisotropic metric.

Recently, Labelle and Shewchuk [3] have settled the foundations for a rigorous approach based on the so-called anisotropic Voronoi diagrams. We propose an alternative view of the construction of these diagrams and a variant of the algorithm of Labelle and Shewchuk. This variant computes the Voronoi vertices using a higher dimensional power diagram and refines the diagram as long as dual triangles overlap.

In this paper, most of the proofs are omitted. The full paper is available online [1].

2 Anisotropic Meshing

2.1 Anisotropic Voronoi Diagrams

We expose in this section the definitions proposed by Labelle and Shewchuk [3]. Figures illustrating most of these definitions can be found in [3]. An anisotropic Voronoi diagram is defined over a domain $\Omega \subset \mathbb{R}^d$, and each point $p \in \Omega$ has an associated metric. More specifically, a point p is given a symmetric positive definite quadratic form represented by a $d \times d$ matrix M_p. The *distance* between two points x and y *as viewed by* p is defined as

$$d_p(x,y) = \sqrt{(x-y)^t M_p (x-y)} \ ,$$

and the distance between p and q is defined as $d(p,q) = \min(d_p(p,q), d_q(p,q))$. In a similar way, the *angle* $\angle xqy$ *as viewed by* p is defined as

$$\angle_p xqy = \arccos \frac{(x-q)^t M_p (y-q)}{d_p(q,x) d_p(q,y)} \ .$$

In order to compare the metric at points p and q, a transfer application is needed. Given the quadratic form M_p of a point p, we denote by F_p any matrix such that $\det(F_p) > 0$ and $F_p^t F_p = M_p$. Then, the transfer application from p to q is

$$F_{p,q} = F_q F_p^{-1} \ .$$

Application $F_{p,q}$ is in fact an isometry between the metric spaces $(\mathbb{R}^d, M_p)$ and $(\mathbb{R}^d, M_q)$. The *distortion* between p and q is then defined as $\gamma(p,q) = \gamma(q,p) = \max\{\|F_{p,q}\|_2, \|F_{q,p}\|_2\}$. For any two points x, y, we have $1/\gamma(p,q)\ d_q(x,y) \le d_p(x,y) \le \gamma(p,q)\ d_q(x,y)$.

Let S be a set of points, called *sites* in the sequel. The *Voronoi cell* of a site $p \in S$ is

$$\mathrm{Vor}(p) = \{x \in \mathbb{R}^d, d_p(p,x) \le d_q(q,x), \forall q \in S\} \ .$$

Any subset of sites $R \subset S$ defines a Voronoi face $\mathrm{Vor}(R) = \cap_{p \in R} \mathrm{Vor}(p)$ which is the set of points equally

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[1] ftp://ftp-sop.inria.fr/geometrica/cwormser/aniso.ps

close to the sites in R and not closer to any other. The set R is called the *label* of $\mathrm{Vor}(R)$. The *anisotropic Voronoi diagram* of S is the subdivision formed by the non-empty faces $\{\mathrm{Vor}(R), R \subset S, R \neq \emptyset, \mathrm{Vor}(R) \neq \emptyset\}$.

The *restriction* of the anisotropic Voronoi diagram to Ω is the diagram formed by the non-empty intersections $\mathrm{Vor}(R) \cap \Omega$.

It should be noted that each site is in the topological interior of its cell. The cells are not always connected, and the boundary between two adjacent cells may be composed of several patches. These patches are all contained in the same quadric, which is the bisector of the two sites. Moreover, there can be more than one Voronoi vertex with a given label.

2.2 Dual Complex

The *dual complex* of an anisotropic Voronoi diagram is the simplicial complex whose set of vertices is the set S, with a simplex associated to each subset $R \subset S$ such that $\mathrm{Vor}(R) \neq \emptyset$. We associate to each of those simplices a *geometric simplex*, its canonical linear embedding in $\mathbb{R}^d$. The set of those geometric simplices, with their incidence relations, is called the *geometric dual*. The geometric dual is generally not an embedded complex.

In two dimensions, the geometric dual includes, for each Voronoi vertex v, a dual triangle whose vertices are the three sites that compose the label of v. There is no reason why these triangles should form a triangulation. The two issues to be considered are the combinatorial planarity of the graph, which depends on the connectivity of the cells, and the ability to stretch its edges without crossing, which depends on the curvature of the bisectors.

The goal of the meshing algorithm is to refine the anisotropic Voronoi diagram by inserting new sites, so that its geometric dual becomes a triangulation, with well-shaped triangles. By well-shaped triangles, we mean triangles with no small angles, as seen by any point of the triangle. Furthermore, a set of *constrained segments*, i.e. segments required to appear as a union of edges in the final mesh, can be given as part of the input data. These constrained segments may be split by the insertion of new sites. In such a case, the resulting parts are called *constrained subsegments*. In particular, the edges of the boundary $\partial\Omega$ of the domain Ω are assumed to be constrained segments.

2.3 Original Approach by Labelle and Shewchuk

Labelle and Shewchuk represent the Voronoi diagram as the lower envelop of a set of paraboloids over the domain Ω. Upon the insertion of a new site, this lower envelop is computed in a lazy way, which amounts to computing only the connected component of the cell that contains the new site.

The *wedge* between two sites p and q is the locus of points x such that the angle $\angle_p xpq$ and the angle $\angle_q xqp$ are less than $\pi/2$. A Voronoi edge e is called *wedged* if e is included in the wedge of the pair of sites defining it. The refinement process in dimension 2 aims at enforcing this property for all edges of the diagram. Labelle and Shewchuk prove that once all edges are wedged, every cell is connected and the dual of the diagram is indeed a triangulation. This fact validates their lazy computation of the diagram.

Labelle and Shewchuk's algorithm consists in incrementally inserting points on non-wedged Voronoi edges and at the center of triangles that do not have the same orientation as the three Voronoi cells around their dual Voronoi vertices (this reverse orientation results from a non-wedged edge incident to these vertices), or are badly shaped, or are too large.

3 Our Approach

3.1 Power Diagram and Anisotropic Voronoi Diagram

We now present a way to compute the anisotropic Voronoi diagram in any dimension.

A power diagram is defined for a set of spheres. Given a sphere σ centered at y and of radius r, the *power* of a point x with respect to σ is defined as $\pi_\sigma(x) = \|x - y\|^2 - r^2$.

The cells of the *power diagram* of a set of spheres Σ are defined in the following way: the cell of a sphere σ of Σ is $\mathrm{Pow}(\sigma) = \{x \in \mathbb{R}^d, \pi_\sigma(x) \leq \pi_\tau(x), \forall \tau \in \Sigma\}$. The power diagram is the subdivision induced by the cells of spheres. Its restriction to a manifold X is the subdivision of X induced by the cells intersected by X.

Let $D = d(d+3)/2$. Associate to each point $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ the point $\tilde{x} = (x_r x_s, 1 \leq r \leq s \leq d) \in \mathbb{R}^{\frac{d(d+1)}{2}}$ and the point $\hat{x} = (x, \tilde{x}) \in \mathbb{R}^D$.

In the following, we name $\mathcal{P}$ the d-manifold of $\mathbb{R}^D$ $\left\{\hat{x} \in \mathbb{R}^D, x \in \mathbb{R}^d\right\}$. As before, $S = \{p_1, \ldots, p_n\}$ denotes a finite set of sites in $\mathbb{R}^d$. To each point p_i of S, we attach a symmetric positive definite matrix $M_i = (M^{r,s})_{0 \leq r,s \leq d}$ and we define the point $q_i = (q^{r,s}, 1 \leq r \leq s \leq d) \in \mathbb{R}^{\frac{d(d+1)}{2}}$ by $q^{r,r} = -\frac{1}{2} M^{r,r}$, for $1 \leq r \leq d$ and $q^{r,s} = -M^{r,s}$, for $1 \leq r < s \leq d$.

Then we note $\hat{p_i}$ the point $(M_i p_i, q_i)$ and $\sigma(p_i)$ the sphere with center $\hat{p_i}$ and radius $\sqrt{\|\hat{p_i}\|^2 - p_i^t M_i p_i}$.

Lemma 1 *Let Π be the projection $(x, \tilde{x}) \in \mathbb{R}^D \mapsto x \in \mathbb{R}^d$. The anisotropic Voronoi diagram of S is the image by Π of the restriction of the power diagram of $\Sigma = \{\sigma(p), p \in S\}$ to the manifold $\mathcal{P}$.*

The previous lemma gives a construction of the anisotropic Voronoi diagram, where the quadric bisectors are replaced by affine ones in higher dimension. As is well-known, computing a power diagram in $\mathbb{R}^D$ reduces to computing a lower convex hull in $\mathbb{R}^{D+1}$. Hence, computing a 2-dimensional anisotropic Voronoi diagram reduces to computing a 6-dimensional convex hull and intersecting the corresponding power diagram with a 2-dimensional manifold. Our meshing algorithm computes only the vertices of the anisotropic Voronoi diagram. This will be sufficient for our purpose. Computing these vertices is achieved by computing the intersection of 3-faces of a 5-dimensional power diagram with a 2-manifold.

3.2 Description of our Algorithm

From now on, let Ω be a simply connected polygonal domain of the plane, whose boundary is denoted by $\partial\Omega$. A field of positive definite matrices over Ω is given. We denote by C the set of constrained segments and by S a finite set of sites in Ω. We assume that the edges of $\partial\Omega$ belong to C and that the vertices of $\partial\Omega$ belong to S. Refining the Voronoi diagram consists in adding sites to the set S. We assume that the quadratic form associated to any point of Ω can be obtained.

3.2.1 Local Embedding

We now define some properties of the dual triangles of the Voronoi vertices that the algorithm will aim to enforce. These properties are tailored so that, once they are verified, the dual triangles define a triangulation of the domain they cover.

We consider a set of non-flat triangles T such that

1. the set of vertices of the triangles in T is exactly S;

2. each edge of $\partial\Omega$ is the edge of exactly one triangle in T;

3. if e is the edge of some triangle in T and is not an edge of $\partial\Omega$, e belongs to exactly two triangles in T, which do not overlap.

We prove, under those assumptions, that the triangles of T are inside Ω and that T is a triangulation of Ω. To prove the last fact, we glue the triangles of T along their common edges and vertices to build a surface: we denote by $\mathcal{T} = \{(x,t) \in \Omega \times T \mid x \in t\}$ the set of points associated to their respective triangles, and we define on $\mathcal{T}$ the equivalence relation $\sim$ by setting $(x,t) \sim (x',t')$ if $x = x'$ and $x \in \partial t$ and $x' \in \partial t'$. The final glued space is denoted by $\mathcal{G} = \mathcal{T}/\sim$.

Let $h : (x,t) \in \mathcal{G} \mapsto x$ be the first projection, mapping $\mathcal{G}$ to Ω. The correctness of the triangulation is equivalent to h being a homeomorphism. Let Ω_p be the punctured space obtained by removing from Ω the vertices of the triangles of T, and let $\mathcal{G}_p$ be $h^{-1}(\Omega_p)$.

From assumption 3, the restriction h_p of h to $\mathcal{G}_p$ is a local homeomorphism. Thus h_p is a cover of Ω_p. As the points near $\partial\Omega$ have only one pre-image, from assumption 2, $h_p : \mathcal{G}_p \to \Omega_p$ has only one sheet and is in fact a homeomorphism. Moreover, h_p may be extended to $\mathcal{G}$ as an homeomorphism. Thus, Ω is triangulated by T.

3.2.2 Refinement Algorithm

In this section, we present our algorithm, which mainly apply the result of the previous section to ensure the validity of the returned triangulation.

Let v be a Voronoi vertex of an anisotropic Voronoi diagram, and let $t_v = abc$ be its dual triangle. The *radius* of v is $r(v) = d_a(a,v) = d_b(b,v) = d_c(c,v)$. We denote the shortest edge of t_v by $\delta(t_v)$. The *radius-edge ratio* of v is $\beta(v) = r(v)/\delta(t_v)$. For a given *shape bound* B, a vertex v is considered to be *badly-shaped* if $\beta(v) > B$.

In the following, a constrained subsegment $e = (a,b)$ is said to be *encroached* by a site $p \notin \{a,b\}$ if $\mathrm{Vor}(p) \cap [a,b] \neq \emptyset$. We are given a shape bound B. At each step of the algorithm, we maintain the set T of the triangles dual to the Voronoi vertices lying in Ω (see section 3.1). The algorithm inserts points iteratively, applying the following rules. Rule i is applied only if no rule j with $j < i$ applies. The conditional insertion of a site x appearing in rules 2, 3, 4 and 5 is the following procedure: if x encroaches no constrained subsegment, insert x, but if x encroaches some constrained subsegment e, insert a site on e instead.

1. if a constrained subsegment $e \in C$ does not appear as the edge of some dual triangle because it is encroached, insert a site on e;

2. if a constrained subsegment $e \in C$ does not appear as the edge of some dual triangle because its dual Voronoi edge is a complete ellipse, conditionally insert a site on this ellipse;

3. if a Voronoi vertex is badly shaped, conditionally insert a site located at that vertex;

4. if a triangle is the dual of several Voronoi vertices, conditionally insert a site located at one of them;

5. if two triangles sharing an edge overlap, conditionally insert a site located at the dual Voronoi vertex of one of them.

The algorithm runs until no rule applies anymore. We prove that if the algorithm terminates, every constrained subsegment appears as an edge of some dual triangle. Moreover, we prove that, upon termination of the algorithm, any edge of a dual triangle that is not an edge of $\partial\Omega$ belongs to exactly two dual triangles. Section 3.2.1 then show that T is a triangulation of Ω. It remains to prove that the algorithm terminates.

4 Termination of our Algorithm

We now provide conditions that ensure that the algorithm terminates. This conditions depend on the shape bound B and on the geometry of the set of constrained segments C.

4.1 Distortion and overlapping

In this subsection, we prove that two dual triangles cannot overlap if the relative distortion between adjacent sites is small enough. In the following, abc and abd are two adjacent triangles that are dual to Voronoi vertices q_c and q_d. We define $\gamma = \max(\gamma(x,y))$ where the maximum is taken over all edges $\{x, y\}$ of the two triangles. We denote by $\delta(abc)$ the length of the shortest edge of triangle abc and by $\delta(a,b,c,d)$ the length $\max(\delta(abc), \delta(abd))$. Let $r = (1+4\gamma)B\delta(a,b,c,d)$. We consider the zone $Z = B(a,r) \cap B(b,r) \cap B(c,r) \cap B(d,r)$, where the ball $B(x,r)$ is the set of points y such that $d_x(x,y) \leq r$. The four sites a, b, c and d are in Z, as well as the two centers q_c and q_d. We denote by V_Z the Voronoi diagram of the set $\{a,b,c,d\}$ restricted to Z.

Lemma 2 *For $B > 1$ and $(\gamma^2-1)(1+\gamma)^2(1+4\gamma)^2B^4 \leq 1$, all the edges of V_Z are wedged.*

Under the conditions of lemma 2, a slight adaptation of the proofs of Labelle and Shewchuk [3] (recalled in section 2.3) allows to show that the dual of the restricted Voronoi diagram V_Z is a valid triangulation.

4.2 Minimal Interdistance

We now consider the algorithm at some stage during its execution. We have a shape bound B and a distortion coefficient G, chosen so that $(G^2 - 1)(1+G)^2(1+4G)^2B^4 \leq 1$. Let $d_{\min}$ (resp. $d^w_{\min}$) be the minimal distance between adjacent sites, before (resp. after) inserting site w. We prove that, whatever may be the rule applied to insert w,

$$d^w_{\min} \geq \min \left(\frac{d_{\min}}{G^2\sqrt{G^4-1}}, \frac{Bd_{\min}}{G^2\sqrt{G^2+1}}, \frac{\mathrm{bdr}_{\min}(G)}{(G^5+G^3)\sqrt{G^2+1}}, \frac{\mathrm{lfs}_{\min}}{G} \right).$$

Here $\mathrm{bdr}_{\min}(G)$ is the upper bound on the distance r such that: $d(p,q) < r \Rightarrow \gamma(p,q) < G$ and $\mathrm{lfs}_{\min}$ is the lower bound of the local feature size on Ω, as defined in [3].

We finally obtain that if B and G verify $(G^2 - 1)(1+G)^2(1+4G)^2B^4 \leq 1$, $G^2\sqrt{G^4-1} \leq 1$ and $G^2\sqrt{G^2+1} \leq B$ and if $\mathrm{bdr}_{\min}(G) > 0$ (this condition is always true if the metric field is regular enough), the algorithm has a positive minimal inter-distance. Moreover, if $(G^2-1)B^2 < 1$, the shape condition parametrized by B may be translated into a condition in terms of a lower bound on the angles of the triangles, as measured by any point inside the triangle (see Corollary 10 in [3]). We can find B and G satisfying all those conditions. A classical volume argument then proves that the algorithm terminates.

5 Conclusion

The approach we have presented is built upon the work of Labelle and Shewchuk. Instead of using a lower envelop of paraboloids, we rely on a power diagram in higher dimension. Moreover, we present the algorithm by focusing on the overlapping condition, thus minimizing the dependence over the Voronoi diagram itself, apart from the computation of the Voronoi vertices. As an aside, we also rely only on the Voronoi vertices that are inside the domain Ω, while Labelle and Shewchuk compute the whole diagram.

This algorithm has been implemented using the Computational Geometry Algorithms Library [7].

A similar algorithm can be considered in three dimensions. However, we currently cannot prove that this meshing algorithm terminates in three dimensions because sliver tetrahedra may overlap their neighbors, without inducing a large insertion distance for the new refining point. This may happen even in the case of low distortion of the metric field.

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