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Partitioned Neighborhood Spanners of Minimal Outdegree[1]

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Abstract

A geometric spanner with vertex set $P \subset \mathbb{R}^D$ is a sparse approximation of the complete Euclidean graph determined by P. We introduce the notion of *partitioned neighborhood graphs* (PNGs), unifying and generalizing most constructions of spanners treated in literature. Two important parameters characterizing their properties are the outdegree $k \in \mathbb{N}$ and the stretch factor $f > 1$ describing the 'quality' of the approximation. PNGs have been throughly investigated with respect to small values of f. We present in this work results about small values of k. The aim of minimizing k rather than f arises from two observations:

a) k determines the amount of space required for storing PNGs.

b) Many algorithms employing a (previously constructed) spanner have running times depending on its outdegree.

Our results include, for fixed dimensions D as well as asymptotically, upper and lower bounds on this optimal value of k. The upper bounds are shown constructively and yield efficient algorithms for actually computing the corresponding PNGs even in degenerate cases.

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1 Motivation

Spanners allow for an efficient solution of many geometric problems. For given finite set $P \subset \mathbb{R}^D$, such a graph $G = (P, E)$ approximates the complete Euclidean graph up to some factor $f > 1$. f-spanners enabled Rao and Smith to construct a FPTAS (Fully Polynomial Time Approximation Scheme) for the Euclidean Travelling Salesperson Problem [21]. Further applications are closest point queries [25], motion planning [7] as well as many range searching problems [1, 18].

For example, the objective of a *circular range query* is reporting all those points p of P lying within a circle of given radius r and center c. Having constructed an f-spanner G for $P \subset \mathbb{R}^D$ of outdegree k, queries with centers $c \in P$ can be answered in nearly output sensitive running time, i.e., $\mathcal{O}(fkm)$ independent of $|P|$ and m close to the number of points reported [14].
More precisely, this kind of geometric searching problems occurring in interactive virtual reality animations requires G to have only a *weakened* spanning property: The 'radius' of a path from s to t, rather than its total length, needs to be bounded by a factor f^*. In particular, every (strong) f-spanner is a weak f^*-spanner for some f^* at most as large as f, usually substantially smaller.

Starting with Yao [28], spanners for given P are usually computed by a generalization of *proximity graphs* [17]: Partition $\mathbb{R}^D$ into $k \in \mathbb{N}$ convex cones $C_0, \ldots, C_{k-1}$. Then, from vertex $p \in P$, draw directed edges (*arcs*) to the closest point of P lying in the translated cone $p + C_j$; do this for $j = 0 \ldots k \Leftrightarrow 1$. The resulting graph is called a *partitioned neighborhood graph* (PNG). Its properties strongly depend on the number and shape of the cones $\{C_0, \ldots, C_{k-1}\} =: \mathcal{C}$, but also on the norm $d(\cdot)$ inducing the (not necessarily Euclidean) notion of 'closest': A disadvantageous choice for the latter may result not only in big values of f but even fail to produce stronly connected graphs! However, every PNG $G = (P, E)$ is sparse with $|E| \leq kn = \mathcal{O}(n)$, $n := |P|$ and benefits from the simple construction principle, numerical robustness [2], fast computability (in time optimal up to a polylogarithmic factor [6]) and locality properties that allow for incremental dynamic updates [14]. Furthermore, PNGs have applications in min-cost perfect matching [27] and answering cone range queries in output sensitive time.

Given D and small $\varepsilon > 0$, sufficient conditions on $\mathcal{C}$ and $d(\cdot)$ have been investigated in the literature in order to ensure that, for any point set P, the according PNG is an $(1+\varepsilon)$-spanner. Indeed, many applications — like the TSP-FPTAS mentioned above — rely on $\varepsilon \to 0$.

However, there are cases where the outdegree $k = |\mathcal{C}|$ of G is of equal importance as its stretch factor: Consider the mentioned circular range query with running time proportional to $f \cdot k$. But even for other algorithms that do *not* depend on k, a small outdegree may be more crucial than a small f when it comes to actually implementing it. Suppose, for example, that one can choose between a spanner $G_{f,k}$ of small f, but large outdegree k and one of small outdegree $\tilde{k}$, but large stretch factor $\tilde{f}$, called $G_{\tilde{f},\tilde{k}}$. On the one hand, the algorithm will run faster with $G_{f,k}$, but this graph requires more memory and access to secondary storage (e.g., a disk) being about 1000 times slower. $G_{\tilde{f},\tilde{k}}$ on the other hand entirely fits into one's computer's main memory so that eventually it still outperforms $G_{f,k}$, even if $\tilde{f}$ is 500 times bigger than f!

We therefore aim to determine the minimal value of k (together with its dependence on dimension D) such that PNGs of this outdgree still are spanners/weak spanners. Upper bounds on this extremal problem in combinatoral geometry are not only of theoretical interest but also lead to efficient algorithms for constructing such graphs. Particular emphasis is laid on ensuring that these also work for degenerate cases. Matching lower bounds prove their optimality.

In Section 2, we give formal definitions for the notions *spanner*, *PNG*, and the goal we aim for. A survey of both previous and new results can be found in Section 3, together with a tabular compilation of the actual bounds induced thereof. Proofs of theorems leading to lower bounds are collected in Section 5 whereas Section 7 contains those for upper bounds. The part describing algorithms which construct PNGs without requiring general position have been put one Section in advance since most readers will probably be more interested in actually computing the optimal graphs that realize our upper bounds. For similar reasons, conclusions and open problems are exposed in Section 4.

2 Definitions

2.1 Spanners

Fix dimension $D \in \mathbb{N}$ and some norm $|\cdot|$ on $\mathbb{R}^D$. Given a path $s = p_0 \leadsto p_1 \leadsto \ldots \leadsto p_m = t$ from $s \in P$ to $t \in P \subset \mathbb{R}^D$ in some geometric graph $G = (P, E)$, the numbers

$$f(p_0, \ldots, p_m) \quad := \quad \sum_{i=1}^{m} |\, p_{i-1} \Leftrightarrow p_i \,| \;/\; |\, p_0 \Leftrightarrow p_m \,| \tag{1}$$

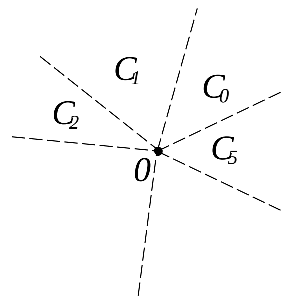
$$f^*(p_0, \ldots, p_m) \quad := \quad \max_{i=1..m} |\, p_0 \Leftrightarrow p_i \,| \;/\; |\, p_0 \Leftrightarrow p_m \,| \tag{2}$$

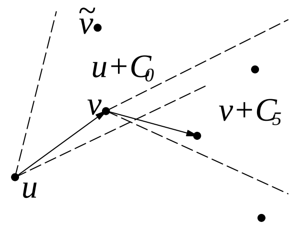
are called its *stretch factor* and *weak stretch factor*, respectively. An *f-spanner* for P is a graph which for all $s, t \in P$ contains a path from s to t of stretch factor at most f; similarly for a *weak f^*-spanner*...

Recall that *every* (strongly) connected graph trivially comprises an f-spanner for some $f < \infty$, simply by finiteness of P. But of course, the goal is to construct f-spanners with f being independent of P. This is reflected by calling graphs forming a family $\mathcal{G} = \{G(P) : P \subset \mathbb{R}^D \text{ finite}\}$ to be *uniform* f-spanners iff each $G(P)$ is an f-spanner; call them *uniform spanners* if there exists $f < \infty$ such that they are uniform f-spanners. Respective notions will be used for weak spanners.

Let us remark that, by topological equivalence of any two norms on $\mathbb{R}^D$ (Claim 5.5), transition from $|\cdot|$ to $\widetilde{|\cdot|}$ affects f by merely a constant factor. In particular, the notion of 'uniform spanners' does not depend on the chosen norm.

2.2 Partitioned Neighborhood Graphs





To formalize PNGs, consider some family $\mathcal{C} = \{C_0, C_1, .., C_{k-1}\}$ of convex cones [15] forming a *partition* of $\mathbb{R}^D$ in the sense that it covers the whole space and is 'almost' disjoint:

$$\bigcup_{j=0}^{k-1} C_j \;=\; \mathbb{R}^D, \qquad \forall i \neq j: \quad C_j \cap C_i \;\subset\; \{0\}.$$

In this context, $C \subset \mathbb{R}^D$ is said to be a *convex cone* if $\lambda(u+v) \in C$ for all $u, v \in C$ and $\lambda \geq 0$. Accordingly, we need a family $\mathcal{D} = \{d_0, d_1, \ldots, d_{k-1}\}$ of k norms d_j. Then, for finite $P \subset \mathbb{R}^D$, the partitioned neighborhood graph $G(\mathcal{C}, \mathcal{D}; P) = (P, E)$ is defined by choosing, to each vertex $u \in P$ and each $0 \leq j < k$, one neighbor v in $(C_j + u) \cap P \setminus \{u\} =: P_j(u)$ nearest to u with respect to d_j. More precisely, the edges E of $G(\mathcal{C}, \mathcal{D}; P)$ are characterized by three conditions:

$$\forall u \in P \quad \forall j: \qquad P_j(u) = \emptyset \;\vee\; \exists v \in P_j(u): \; (u,v) \in E \tag{3}$$

$$(u,v),(u,w) \in E, v \neq w \quad \Longrightarrow \quad \forall j: \quad v \notin P_j(u) \;\vee\; w \notin P_j(u) \tag{4}$$

$$(u,v) \in E \quad \Longrightarrow \quad \exists j: \; v \in P_j(u) \;\wedge\; \forall \tilde{v} \in P_j(u): \; d_j(v \Leftrightarrow u) \leq d_j(\tilde{v} \Leftrightarrow u) \tag{5}$$

To define the *greedy path* from s to t in PNG $G = (P, E)$, consider the unique $C_j \in \mathcal{C}$ such that $t \in s + C_j$. Then, since $t \in P_j(s) \neq \emptyset$, there exists (3) at least and (4) at most one $v \in P_j(s)$ such that $(s,v) \in E$. Take $s \rightsquigarrow v$ as the first step and repeat from v to t.

2.3 Measures of Distance

In the previous paragraph, proximity of two points was gauged with respect to some norm d_j. But in fact, our considerations do not rely on its symmetry property. d_j may therefore be a more general distance function $d: \mathbb{R}^D \to [0, \infty) \subset \mathbb{R}$ which is

positively linear	$d(\lambda v) = \lambda d(v)$	$v \in \mathbb{R}^D$, $\lambda \geq 0$
nondegenerate	$d(v) \neq 0$	$\mathbb{R}^D \ni v \neq 0$
and convex.	$d(u+v) \leq d(u) + d(v)$	$u, v \in \mathbb{R}^D$ (6)

It is well known that such mappings uniquely correspond to the compact and convex subsets K of $\mathbb{R}^D$ with 0 in their interior: According to Claim 5.5, the unit sphere $\{v \in \mathbb{R}^D : d(v) \leq 1\}$ is such a set and, vice versa, K's so called *Minkowsky functional* μ_K fulfills the three conditions above,

$$\mu_K(v) \quad = \quad \inf\{\mu > 0 \;:\; v/\mu \in K\} \quad = \quad \min\{\mu \geq 0 \;:\; \mu K \ni v\}.$$

For dealing with cases where two points $v, \tilde{v} \in P_j(u)$ are both closest to u, we permit the distance function d_j to include a *rule for breaking ties*, i.e., a total (or *linear*) order $\tilde{d_j} \subset C_j \times C_j$ that extends the partial order[2] $\{(u,v) \;:\; u, v \in C_j,\; u = v \vee d_j(u) < d_j(v)\}$

[2] $\mathcal{O} := \{(u,v) : u, v \in C_j, d_j(u) \leq d_j(v)\}$ is no order: It violates axiom "$(u,v),(v,u) \in \mathcal{O} \Rightarrow u = v$".

induced by d_j on C_j in the sense that

$$\forall v,w \in C_j: \qquad d_j(v) < d_j(w) \quad \Longrightarrow \quad (v,w) \in \tilde{d}_j.$$

By the axiom of choice, such $\tilde{d}_j$ always exists [26] and will be called an *extended norm*. Equation (5) then has to be replaced by

$$(u,v) \in E \quad \Longrightarrow \quad \exists j:\ v \in P_j(u) \ \wedge\ \forall \tilde{v} \in P_j(u):\ (v \Leftrightarrow u, \tilde{v} \Leftrightarrow u) \in \tilde{d}_j \tag{7}$$

2.4 Our goal

So, each choice of $\mathcal{C}$ and $\tilde{\mathcal{D}} = \{\tilde{d}_j : j = 0 \ldots k \Leftrightarrow 1\}$ induces a family

$$\mathcal{G}(\mathcal{C}, \tilde{\mathcal{D}}) = \big\{ G(\mathcal{C}, \tilde{\mathcal{D}}; P) : P \subset \mathbb{R}^D \text{ finite} \big\}$$

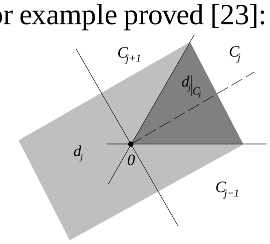
of graphs with outdegree $|\mathcal{C}|$, and we aim to determine for different dimensions D the quantity $\;k(D) = \min\big\{k \in \mathbb{N} \mid f(D,k) < \infty\big\}\;$ where

$$f(D,k) := \inf\big\{ f > 1 \quad \big| \quad \begin{array}{l} \exists\, \mathcal{C} \text{ disjoint partition of } \mathbb{R}^D \text{ into } k \text{ convex cones} \\ \exists\, \tilde{\mathcal{D}} \text{ collection of } k \text{ extended norms} \\ \forall\, P \subset \mathbb{R}^D \text{ finite}\ :\ G(\mathcal{C}, \mathcal{D}; P) \text{ is } f\text{-spanner}\big\} \end{array} \tag{8}$$

In other words, $k(D)$ is the least number of cones required such that this family consists of uniform spanners and their corresponding Euclidean stretch factors. Similarly, we investigate on the correspondingly defined numbers $k^*(D)$ and $f^*(D,k)$ for uniform weak spanners.
Since any f-spanner is a weak f-spanner as well, inequalities $k^*(D) \leq k(D)$ and $f^*(D,k) \leq f(D,k)$ are obvious. Furthermore, $f(D,k) \geq f(D,\tilde{k})$ and $f^*(D,k) \geq f^*(D,\tilde{k})$ hold whenever $k \leq \tilde{k}$: simply choose $\tilde{k} \Leftrightarrow k$ cones empty.

3 Results

There already exist works which, in spite of focussing on $f \to 1$, showed specific choices for $\mathcal{C}$ and $\tilde{\mathcal{D}}$ to yield partitioned neighborhood f-spanners. In that way, they imply upper bounds on $k(D)$ and $f(D,k)$. Ruppert and Seidel for example proved [23]:



3.1 Theorem: Suppose every $C \in \mathcal{C}$ has angular diameter $\sphericalangle(C) := \sup\big\{\sphericalangle(a,b) : a,b \in C\big\}$ at most $\theta < \pi/3$. Consider (arbitrary total extension $\tilde{d}_j$ of) the norm d_j with unit sphere depicted to the right. Then, each step $p_m \rightsquigarrow p_{m+1}$ of the greedy path (see 2.2) in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ from $s = p_0$ to $t = 0$ has

$$\big|p_m\big|_2 \Leftrightarrow \big|p_{m+1}\big|_2 \geq \big(1 \Leftrightarrow 2\sin(\theta/2)\big) \cdot \big|p_{m+1} \Leftrightarrow p_m\big|_2 \tag{9}$$

Since $k = 7$ equally sized wedges do form such a partition $\mathcal{C}$ in dimension $D = 2$, $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ is an Euclidean f-spanners for

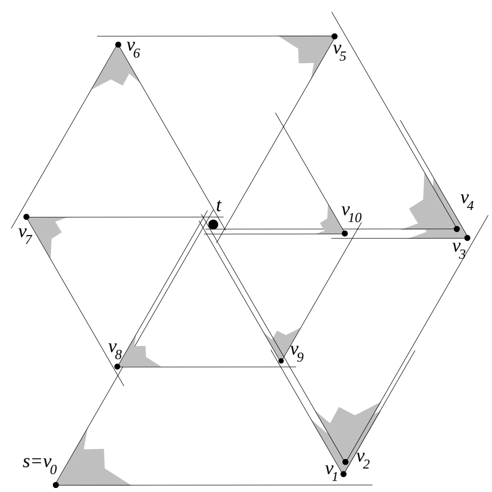
$$f \ := \ \frac{1}{1 \Leftrightarrow 2\sin(\theta/2)}\bigg|_{\theta = 2\pi/7} \approx\ 7.57\ \geq\ f(2,7), \qquad \text{thus} \quad k(2) \leq 7. \quad \square$$

Combining Theorem 3.1 with the following result from Coding Theory due to Hardin, Sloane, and Smith [16] implies $k(3) \leq 20$ and $f(3,20) \leq 88.1$:

3.2 Theorem: There exists a covering of the unit sphere $\mathcal{S}^3 \subset \mathbb{R}^3$ with $k = 20$ caps of angular diameter $\theta \approx 59.25°$. □

The table below shows a compilation of such results as well as the improved upper bounds presented in this paper. Note that we are the first to prove lower bounds!

dim	reference	bound	bound
$D = 2$	Keil, Gutwin 1991	$k(2) \leq 9$	$k^*(2) \leq 9$
	Ruppert, Seidel 1992	$k(2) \leq 7$ $f(2,7) \leq 7.57$	$k^*(2) \leq 7$ $f^*(2,7) \leq 7.57$
	Fischer, Meyer a.d. Heide, Strothmann '97		$k^*(2) \leq 6$ $f^*(2,6) \leq 2$
	Fischer, Lukovszki, Ziegler 1998		$k^*(2) \leq 4$ $f^*(2,4) \leq 2.29$
	new	$k^*(2) \geq 4$	$k^*(2) = 4$
	conjecture	$k(2) = 4$	
$D = 3$	Hardin, Sloane, Smith 1994	$k(3) \leq 20$ $f(3,20) \leq 88.1$	$k^*(3) \leq 20$ $f^*(3,20) \leq 88.1$
	new		$k^*(3) \leq 8$ $f^*(3,8) \leq 2.53$
	new	$k^*(3) \geq 5$	$k^*(3) \geq 5$
$D \to \infty$	Rogers 1963	$k(D) \leq 2^{\mathcal{O}(D)}$	$k^*(D) \leq 2^{\mathcal{O}(D)}$
	new	$k(D) \geq D+2$	$k^*(D) \geq D+2$



For $60° \leq \theta \leq 90°$, greedy paths become unboundedly long (see figure) but remain of bounded diameter. As for degenerate point sets P this may include the possibility of cycling infinitely without ever reaching t, it does not necessarily imply obtaining a weak spanner. By carefully choosing the cones' boundaries to be open or closed and by employing sophisticated extensions of norms, these cases can be taken care of without some 'general position' presumption. However, doing so becomes a singular combinatorial challenge. The planar cases have been treated in [14] and [12]:

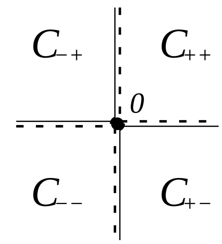
3.3 Theorem: Let $D = 2$, $k \geq 7$, and $\mathcal{C}$ consist of k consequtive wedges

$$C_j \quad = \quad \left\{(r\cos\varphi, r\sin\varphi) : r > 0, \frac{2\pi}{k} j \leq \varphi < \frac{2\pi}{k}(j+1)\right\}, \qquad j = 0 \ldots k \Leftrightarrow 1.$$

Then, for $\tilde{\mathcal{D}}$ as in Theorem 3.1, $G(\mathcal{C},\tilde{\mathcal{D}};P)$ is a weak Euclidean f^*-spanner for

$$f^* \;\le\; \max\left\{\sqrt{1+48\sin^4(\pi/k)},\,\sqrt{5\Leftrightarrow 4\cos(2\pi/k)}\right\}. \quad \square$$

3.4 Theorem: Let $D=2$ and $\mathcal{C}=\{C_{++},C_{+-},C_{-+},C_{--}\}$ the four canonical quadrants with boundaries open/closed as shown to the right. Let $\tilde{\mathcal{D}}=\{\tilde{d}_{++},\dots,\tilde{d}_{--}\}$, $\tilde{d}_j$ arbitrary total extension of



$$\begin{aligned}\big\{(v,w)\;:\;& v,w\in C_j,\\ &(|v|_\infty<|w|_\infty)\quad\vee\quad(|v|_\infty=|w|_\infty\,\wedge\,|v|_0<|w|_0)\;\big\}\end{aligned}$$

i.e. the lexicographical order on C_j induced by $v\mapsto\big(|v|_\infty,|v|_0\big)$,

$$|v|_p=\Big(\sum_i|v_i|^p\Big)^{1/p},\qquad |v|_\infty=\max_i|v_i|,\qquad |v|_0=\min_i|v_i|$$

Then potential function $\Phi(s)=\big(|s\Leftrightarrow t|_\infty,\varphi(s\Leftrightarrow t)\big)$, $\varphi(x,y)=|x+y|$, strictly decreases in each step of the greedy path.
In particular, the latter does reach t with (not necessarily strictly) decreasing $|\cdot\Leftrightarrow t|_\infty$. So, $\mathcal{G}(\mathcal{C},\tilde{\mathcal{D}})$ are uniform weak spanners of Euclidean weak stretch

$$f^*\quad\le\quad\Big\{|a\Leftrightarrow b|_2/|a|_2:|a|_\infty=1=|b|_\infty\Big\}\qquad=\quad\sqrt{3+\sqrt{5}}.\quad\square$$

3.5 Theorem: Given $P\subset\mathbb{R}^2$, $n:=\#P$, the graph $G(\mathcal{C},\tilde{\mathcal{D}};P)$ of Theorem 3.3 can be computed by sequentially performing k sweep line algorithms, each of time $\mathcal{O}(n\cdot\log n)$. The graph of Theorem 3.4 can be computed in the same magnitude of time. $\square$

If $P\subset\mathbb{R}^D$ is restricted to contain no two points which coincide in any coordinate (i.e., all projections $\Pi_i:P\to\mathbb{R}$, $u\mapsto u_i$ are injective), then $k^*(D)\le 2^D$ and the corresponding PNG can be computed in time $\mathcal{O}\big(n\cdot\log^{D-1}n\big)$. However, like in the above result, we want degenerate cases to work as well. The first of our contributions achieves this for $D=3$. Again, special attention has to be paid to open/closed boundaries. Observe that the potential function Φ maps to a lexicographically ordered set of triples instead of tuples:

3.6 Theorem: Let $D=3$ and consider the 8 canonical octants. Turn them into a partition by including each of their common boundaries to one of them and excluding it from the others in the following way: $\mathcal{C}:=\big\{C_{\bar\iota}:\bar\iota\in\{+,\Leftrightarrow\}^3\big\}$,

$$C_{\bar\iota}:=\big\{q\in\mathbb{R}^3,q\neq 0,\,\bar f(\operatorname{sgn}q_x,\operatorname{sgn}q_y,\operatorname{sgn}q_z)=\bar\iota\big\}\tag{10}$$

for $\bar{f} = (f_x, f_y, f_z) : \{+, 0, \Leftrightarrow\}^3 \to \{+, \Leftrightarrow\}^3$, given by $\bar{f}\big|_{\{+,-\}^3}$ =identity and otherwise

x	0	0	0	0	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	0	0	0	0	$+$	$\Leftrightarrow$	0
y	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	0	0	0	0	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	0	0	$+$	$\Leftrightarrow$	0	0	0
z	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	0	0	0	0	$+$	$\Leftrightarrow$	0	0	0	0	0
f_x	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$
f_y	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$
f_z	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$+$	$\Leftrightarrow$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$	$\Leftrightarrow$	$+$

Furthermore, be the partial lexicographical order induced by $C_{\bar{\iota}} \ni v \mapsto \big(|v|_\infty, |v|_1\big)$ extended to a total one $d_{\bar{\iota}}$. Then, $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ has weak stretch factor 2 with respect to $|\cdot|_\infty$ and Euclidean weak stretch $f^* \leq \sqrt{(7+\sqrt{33})/2}$. $\square$

3.7 Theorem: Given $P \subset \mathbb{R}^3$, $n := \#P$, the graph $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ of Theorem 3.6 can be computed in time $\mathcal{O}(n \cdot \log^2 n)$ from 48 sweep plane passes.

The lower bounds mentioned are immediate consequences of the following two results:

3.8 Theorem: In the planar case $D = 2$, no choice of $\mathcal{C}$ and $\tilde{\mathcal{D}}$ of size $k \leq 3$ makes $\mathcal{G}(\mathcal{C}, \tilde{\mathcal{D}})$ a family of uniform weak spanners, since there exists $P \subset \mathbb{R}^2$ and $s, t \in P$ such that no path from s to t is present at all. $\square$

3.9 Theorem: In cases $D \geq 3$, no choice of $\mathcal{C}$ and $\tilde{\mathcal{D}}$ of size $k < D+2$ makes $\mathcal{G}(\mathcal{C}, \tilde{\mathcal{D}})$ a family of strongly connected graphs (and thus neither of uniform weak spanners). More precisely, inequalities $k(D) \geq k(D \Leftrightarrow 1) \Leftrightarrow 1$ and $k^*(D) \geq k^*(D) \Leftrightarrow 1$ hold. $\square$

This does not rule out the possibility to obtain (weak) spanners for $D = 3$, $k = 6$. We can, however, exclude the choice of 6 convex cones arising canonicaly from the faces of a cube ($\mathring{C}$ and $\overline{C}$ denote topological interior and closed hull of C, respectively):

3.10 Theorem: Suppose $\mathcal{C} = \big\{\tilde{C}_{\bar{\iota}} : \bar{\iota} = (i,s) \in \{x,y,z\} \times \{+, \Leftrightarrow\}\big\}$ and

$$\mathring{C}_{\bar{\iota}} \subset \tilde{C}_{\bar{\iota}} \subset \overline{C}_{\bar{\iota}}, \qquad C_{\bar{\iota}} := \big\{q \in \mathbb{R}^3, q \neq 0, |q|_\infty \leq s \times q_i\big\}.$$

Then to any collection $\tilde{\mathcal{D}}$ of 6 extended norms there exists $P \subset \mathbb{R}^3$ such that $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ is not strongly connected. $\square$

4 Conclusions/Open Problems

We presented upper and lower bounds for the numbers $k(d)$ and $k^*(d)$, i.e., the minimally achievable outdegree such that partitioned neighborhood graphs (PNGs) still form spanners and weak spanners, respectively.

The notion of PNGs we suggested is very general since we allow for *arbitrary* partitions of space into convex cones. Furthermore, the neighbor needs not be 'nearest' in the Euclidean sense but with respect to *any* nondegenerate convex distance function

d which may be different for each cone and even be equipped with a rule for breaking ties between equally distant points. In particular, most existing constructions of spanners are PNGs.

We do not aim to find the minimal outdegree of arbitrary spanners $\big($this is well known, anyway: 3. See [8]$\big)$ but of those which can be constructed in nearly linear time $\mathcal{O}(n \cdot \text{polylog} n)$. Our upper bounds are constructive and yield practical algorithms of this optimal time complexity. We obtained lower bounds by proving that for smaller outdegree, the corresponding PNGs will in general be not only of unbounded length and diameter but even disconnected (an important observation, see below!).

This was done by a new technical tool which took care of the vast range of possible choices for the distance functions. This allowed us to reduce the topological part of the problem. The remaining challenge of considering all partitions of space into k convex cones was still difficult enough: finding so called cycles, a simultaneously combinatorial and geometric property of a family of cones.

For $k^*(2) = 4$, our bounds are tight. Concerning the gap between 4 and 7 for $k(2)$, we conjecture that the actual value is 4, too. In order to prove the appropriate upper bound, greedy paths do not suffice any more.
In higher dimensions, we believe $k(3) = k^*(3) = 8$ and $k(D) = k^*(D) = \Theta\big(2^D\big)$. PNGs then would have the interesting property that

- they are are either disconnected or
- permit paths of uniformly bounded length.

The other cases

- connected but unbounded diameter and
- bounded diameter but unbounded length

could not occur by themselves. This is different for arbitrary families of geometric graphs!

Apart from filling the remaining gaps by tightening the upper and lower bounds, another direction of research seems promising: What happens if the notion of 'closest' is not deduced from an extended norm $\tilde{d}$ but from an *arbitrary* total order $\preceq$ of cone C? Perhaps there exists a choice of $\preceq$ that yields PNG-spanners of lower outdegree ...

Even in case $\preceq$ is required to be compatible with the cone's operations "$\cdot$" and "$+$" in the sense of [15], i.e.,

$$u \preceq v, \quad \lambda \geq 0, \quad w \in C \qquad \Longrightarrow \qquad \lambda \cdot u \ \preceq\ \lambda \cdot v \quad \wedge \quad u + w \ \preceq\ v + w,$$

we have no idea whether this actually affects the values $k(d)$ and $k^*(d)$ or does not.

The authors would like to thank Artur Czumaj for many seminal discussions and suggestions.

5 Proofs of lower bounds

For proving a lower bound, *every* choice of $\mathcal{C}$ and $\tilde{\mathcal{D}}$ has to be taken into consideration. The following important result allows us to get at least rid off the norms:

5.1 Definition: Be $\mathcal{C}$ a collection of (not necessarily disjoint neither covering) convex cones $C \subset \mathbb{R}^D$. A *cycle* of $\mathcal{C}$ is a finite sequence $(c_0, c_1, \ldots, c_{L-1}, c_L = c_0)$ of nonzero points $c_l \in \mathbb{R}^D$ such that

$$\forall l = 0 \ldots L \Leftrightarrow 1 \quad \exists C \in \mathcal{C} : \qquad 0 \in c_l + \overset{\circ}{C} \quad \wedge \quad c_{l+1} \in c_l + C \tag{11}$$

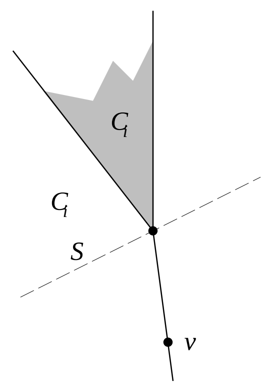
5.2 Proposition: Fix some partition $\mathcal{C}$ of $\mathbb{R}^D$ into convex cones. Suppose there exists subspace S and nonzero vector $v \in \mathbb{R}^D$ such that

$$\mathcal{C}' := \left\{ C \cap S : C \in \mathcal{C}, v \in \overline{C} \right\} \tag{12}$$

contains a cycle. Then for *any* choice of extended norms $\tilde{\mathcal{D}}$ there exists $P \subset \mathbb{R}^D$ such that $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ is not strongly connected. Here we identify S with $\mathbb{R}^{D'}$, $D' < D$. $\square$

Proof of Theorem 3.8: In case $k = 3$,

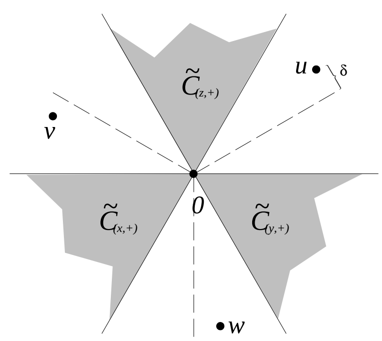
$$\sum_{i=0}^{k-1} \sphericalangle(C_i) = 360^\circ \qquad \Longrightarrow \qquad \exists i : \sphericalangle(C_i) \leq 120^\circ < 180^\circ.$$



The tangent line S at C_i through 0 therefore intersects precisely the other two cones. Choose $v \neq 0$ from their common boundary. Identifying S with $\mathbb{R}^d$, we have $\mathcal{C}' = \left\{ (\Leftrightarrow\infty, 0], [0, +\infty) \right\}$ with obvious cycle $(\Leftrightarrow 42, +42, \Leftrightarrow 42)$.
In case $k = 2$, both cones are halfspaces. Choose v from their boundary and S perpendicular to v. Case $k = 1$ is trivial. $\square$

Proof of Theorem 3.10: Consider $v = (1, 1, 1)$. $S = \left\{ u \in \mathbb{R}^3 : u \perp v \right\} \cong \mathbb{R}^2$ via vectorspace isometry

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \Leftrightarrow\sqrt{1/2}x \Leftrightarrow \sqrt{1/6}y \\ +\sqrt{1/2}x \Leftrightarrow \sqrt{1/6}y \\ +\sqrt{2/3}y \end{pmatrix}$$



The collection of cones $\mathcal{C}'$ induced by $\mathcal{C}$ is shown to the right; boundaries may be open or closed. Now, let $0 < \delta < 30^\circ$ arbitrary. Then, points $\tilde{u}, \tilde{v}, \tilde{w} \in S$,

$$\begin{aligned}
\tilde{u} &= (\Leftrightarrow\sqrt{2/3}\cos\delta, +\sqrt{1/6}\cos\delta \Leftrightarrow \sqrt{1/2}\sin\delta, +\sqrt{1/6}\cos\delta + \sqrt{1/2}\sin\delta) \\
\tilde{v} &= (+\sqrt{1/6}\cos\delta + \sqrt{1/2}\sin\delta, \Leftrightarrow\sqrt{2/3}\cos\delta, +\sqrt{1/6}\cos\delta \Leftrightarrow \sqrt{1/2}\sin\delta) \\
\tilde{w} &= (+\sqrt{1/6}\cos\delta \Leftrightarrow \sqrt{1/2}\sin\delta, +\sqrt{1/2}\sin\delta + \sqrt{1/6}\cos\delta, \Leftrightarrow\sqrt{2/3}\cos\delta)
\end{aligned}$$

corresponding to

$$\begin{array}{rcl} u &=& \big(\cos(\ 30^\circ+\delta),\sin(\ 30^\circ+\delta)\big) \\ v &=& \big(\cos(150^\circ+\delta),\sin(150^\circ+\delta)\big) \qquad\qquad \in \mathbb{R}^2 \\ w &=& \big(\cos(270^\circ+\delta),\sin(270^\circ+\delta)\big) \end{array}$$

obviously form a cycle (u,v,w,u) of $\mathcal{C}'$. $\square$

5.3 Claim: If $C \subset \mathbb{R}^D$ is convex, $a \in \overline{C}$, $b \in \mathring{C}$, then $a+b \in \mathring{C}$.

Proof: Let $a_n \in C$ be a sequence convergent to a. $b \in \mathring{C}$, therefore exists a ball B around b such that $B \subset C$. For each $\tilde{b} \in B$ and each n, $a_n + \tilde{b} \in C$ by convexity and, letting $n \to \infty$, $a + \tilde{b} \in \overline{C}$. This proves that the whole ball $a+B$ around $a+b$ lies within $\overline{C}$, so $a+b \in \mathring{C}$. $\square$

Proof: (Proposition 5.2) Be $\big(c_0, c_1, \ldots, c_{L-1}, c_L = c_0\big)$ a cycle of $\mathcal{C}'$, i.e., $c_{l+1} \in c_l + \big(C_l \cap v^\perp\big) \subset c_l + C_l$ and $\Leftrightarrow c_l \in \mathring{C}_l$. $v \in \overline{C}_l$, therefore $t := \mu v \in \overline{C}_l \subset \mathbb{R}^D$ for any $\mu > 0$. Application of Claim 5.3 to $a = t$, $b = \Leftrightarrow c_l$ ensures $t \in c_l + C_l$; $c_{l+1} \in c_l + C_l$, anyway. Now let $\tilde{d}_l \in \tilde{\mathcal{D}}$ belong to $C_l \in \mathcal{C}$, d_l the distance function which $\tilde{d}_l$ extended to. Since $d_l(c_{l+1} \Leftrightarrow c_l)$ is independent of μ and $d_l\big(t \Leftrightarrow c_l\big) \overset{(*)}{\leq} \mu d_l(v) \Leftrightarrow d_l(c_l) \to \infty$ as $\mu \to \infty$ (Claim 5.5),

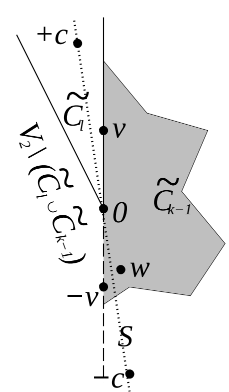
$$\exists \lambda > 0: \qquad \forall l = 0, \ldots, L\Leftrightarrow 1: \quad c_{l+1}, t \in c_l + C_l, \quad d_l\big(c_{l+1} \Leftrightarrow c_l\big) < d_l\big(t \Leftrightarrow c_l\big)$$

Letting $P = \{t, c_0, \ldots, c_{L-1}\}$, no c_l will therefore have an arc to t in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$. $\square$

Proof of Theorem 3.9: Suppose $\mathcal{C} = \big\{C_0, \ldots, C_{k-1}\big\}$, $\tilde{\mathcal{D}} = \big\{\tilde{d}_0, \ldots, \tilde{d}_{k-1}\big\}$ for $k < D+2$. Consider the case $D = 3$. Since C_{k-1} is convex, we can find at boundary point 0 some tangent hyperplane H not touching C_{k-1} other than in 0. The intersections with and restrictions to this $(D\Leftrightarrow 1)$-dimensional subspace

$$\begin{array}{llll} C_0' = C_0 \cap H & C_1' = C_1 \cap H & \ldots & C_{k-2}' = C_{k-2} \cap H \\ \tilde{d}_0' = \tilde{d}_0\big|_H & \tilde{d}_1' = \tilde{d}_1\big|_H & \ldots & \tilde{d}_{k-2}' = \tilde{d}_{k-2}\big|_H \end{array}$$

therefore form a partition $\mathcal{C}'$ of $H \cong \mathbb{R}^2$ into $k \Leftrightarrow 1 < 4$ convex cones and a family $\tilde{\mathcal{D}}'$ of extended norms thereon. Now, take the counter example $P \subset \mathbb{R}^2$ from Theorem 3.8 and place it onto $B \subset \mathbb{R}^3$: The resulting PNGs are disconnected. $\surd$
In cases $D > 3$, employ the same argument as induction step.



Attentive readers might have remarked that in some degenerate cases, C_{k-1} may include angles as large as 180° *and* be closed. Here, we cannot guarantee the tangent hyperplane to be even 'almost' disjoint to C_{k-1}. Fortunately, the subsequent Claim permits a characterization of these particularities! So if H with the required property does not exist, take $v \in C_{k-1}$, $\Leftrightarrow v \in \overline{C_{k-1}}$. Suppose first that $\Leftrightarrow v \notin C_{k-1}$. Then, to $\varepsilon = |v|/2 > 0$ we can find $w \in C_{k-1}$ such that $|w \Leftrightarrow (\Leftrightarrow v)| < \varepsilon$ and in particular w not colinear to $\Leftrightarrow v, v$. Plane $V_2 := \mathrm{span}\{\Leftrightarrow v, v, w\}$ has the property that $\tilde{C}_{k-1} := V_2 \cap C_{k-1}$ is a halfopen wedge of 180°.

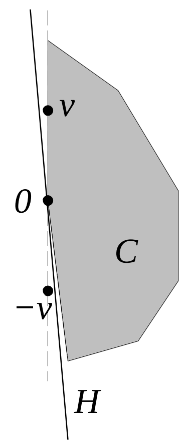
The partition $\tilde{\mathcal{C}} = \{C \cap V_2 : C \in \mathcal{C}\}$ induced on V_2 by $\mathcal{C}$ will therefore look like to the right: Since $\tilde{C}_{k-1}$ is closed at v, the adjacent wedge $\tilde{C}_l \in \tilde{\mathcal{C}}$ must form a nonzero angle (whereas the one containing $\Leftrightarrow v$ could perhaps be nothing more than a ray). Once again refer to the Claim below to understand the existence of a line S through 0 which does not touch $V_2 \setminus (\tilde{C}_l \cup \tilde{C}_{k-1})$. Points $c \neq 0$ and $\Leftrightarrow c$ on this line then form a cycle $(+c, \Leftrightarrow c, +c)$ of

$$\mathcal{C}' = \{C \cap S : C \in \tilde{\mathcal{C}}, v \in \overline{C}\} = \{C \cap S : C \in \mathcal{C}, v \in \overline{C}\}.$$

Application of Proposition 5.2 completes this case. In case $\Leftrightarrow v \in C_{k-1}$, $S := \operatorname{span}\{v\}$ and $(+v, \Leftrightarrow v, +v)$ similarly forms a cycle. $\square$

5.4 Claim: Be $C \subset V_D = \mathbb{R}^D$ convex, $p \in \partial C$. Then, there exists

- either a $(D \Leftrightarrow 1)$-dimensional hyperplane $H_{D-1} \ni p$ such that $H_D \cap C \subset \{p\}$
- or $v \in V_D$ such that $p + v \in C$ but $p \Leftrightarrow v \notin \overline{C}$.



Proof: W.l.o.g. $p = 0$ and presume $\Leftrightarrow v \notin \overline{C}\ \ \forall v \in C$. The claim that H with the required properties exists is trivial for $D = 1$ and obvious in dimension 2 (see sketch to the right). Proceed now by induction to $D+1$. Be $q \in C$ arbitrary. Consider some D-dimensional subspace $V_D \subset V_{D+1}$ containing q. And consider the plane V_2 going through q and 0 perpendicular to V_D, i.e. $V_2 \cap V_D$ is one-dimensional.
$0 \notin C \cap V_2 =: C_2$ is a convex set in V_2 with 0 at is boundary fulfilling $\Leftrightarrow v \notin \overline{C_2}\ \ \forall v \in C_2$. For this reduction to $D = 2$, a 1-dimensional disjoint hyperplane H_1 (simply a line) through 0 is already known to exist. Now consider the projection of C parallel to this line onto V_D,

$$\Pi(H_1, V_D; C) = \{L \cap V_D : L \text{ line through } c \text{ parallel to } H_1, c \in C\}.$$

$0 \notin C_D := \Pi(H_1, V_D; C) \subset V_D$ too is convex (since projection $\Pi(H_1, V_D; \cdot)$ linear mapping) and 0 a boundary point of C_D (as Π is continuous). Furthermore, $\Leftrightarrow \tilde{v} \notin \overline{C_D}$ $\forall \tilde{v} \in C_D$!
Indeed, be $\tilde{v} = \Pi(v)$, $v \in C$ and $\Leftrightarrow \tilde{v} = \Pi(w) \in \overline{C_D}$, $w \in C$. Be definition of Π, lines A and B through $v, \tilde{v}$ and $w, \Leftrightarrow \tilde{v}$, respectively, are parallel to H_1. A, B and H_1 therefore lie on a common twodimensional subspace $\tilde{V}_2$. Line $C \subset \tilde{V}_2$ through v, w however is not parallel to H_1 $\big($ otherwise $\Pi(v) = \Pi(w)$ $\big)$ and so intersects H_1 in some point u which, by convexity, contains to C as well. But $u \in H_1 \cap C$ contradicts the choice of H_1 to be disjoint to C.
Induction hypothesis is thus applicable to C_D and supplies a $(D \Leftrightarrow 1)$-dimensional hyperplane H_{D-1} through 0 disjoint to it.
$H_D := H_{D-1} + H_1$ then will do the job: Suppose $c \in C \cap H_D$. Then its projection $\Pi(c)$ will be on $\Pi(C) \cap H_{D-1}$ contradiction that H_{D-1} is disjoint to C_D. $\square$

5.5 Claim: (Topological Equivalence) Be d_a and d_b nondegenerate convex distance functions on $\mathbb{R}^D$. Then there exist real numbers $\ 0 < \lambda < \Lambda < \infty\ $ such that

$$\forall v \in \mathrm{I\!R}^D: \qquad \lambda \cdot d_a(v) \quad \leq \quad d_b(v) \quad \leq \quad \Lambda \cdot d_a(v).$$

Proof: Denote $e^{(i)}$ the i-th canonical unit vector of $\mathrm{I\!R}^D$, i.e., $e^{(i)}_j = \delta_{ij}$ for $1 \leq i, j \leq D$. We start with the case $d_a = |\cdot|_1 : v = \sum_i v_i e^{(i)} \mapsto \sum_i |v_i|$. Define $\Lambda := \max_i d_b\big(\pm e^{(i)}\big)$.

$$\begin{array}{rcl} \text{Then} \quad d_b(v) & = & d_b\big(\sum_i v_i e^{(i)}\big) \quad = \quad d_b\Big(\sum_{i:v_i>0} |v_i|(+1)e^{(i)} + \sum_{i:v_i<0} |v_i|(\Leftrightarrow 1)e^{(i)}\Big) \\ & \overset{(6)}{\leq} & \sum_{i:v_i>0} d_b\big(|v_i|(+1)e^{(i)}\big) + \sum_{i:v_i<0} d_b\big(|v_i|(\Leftrightarrow 1)e^{(i)}\big) \\ & = & \sum_{i:v_i>0} |v_i| d_b\big(+e^{(i)}\big) + \sum_{i:v_i<0} |v_i| d_b\big(\Leftrightarrow e^{(i)}\big) \\ & \leq & \sum_{i:v_i>0} |v_i|\Lambda + \sum_{i:v_i<0} |v_i|\Lambda \qquad = \quad \big|v\big|_1 \cdot \Lambda \end{array}$$

This in turn implies that d_b is continuous: Let $v^{(n)}$ a sequence in $\mathrm{I\!R}^D$ converging to v.

$$\begin{array}{rcccccl} \text{Then} \quad d_b\big(v^{(n)}\big) \Leftrightarrow d_b\big(v\big) & \overset{(*)}{\leq} & d_b\big(v^{(n)} \Leftrightarrow v\big) & \leq & \Lambda\big|v^{(n)} \Leftrightarrow v\big|_1 & \to 0 \\ \text{and} \quad d_b\big(v\big) \Leftrightarrow d_b\big(v^{(n)}\big) & \overset{(*)}{\leq} & d_b\big(v \Leftrightarrow v^{(n)}\big) & \leq & \Lambda\big|v \Leftrightarrow v^{(n)}\big|_1 & \to 0, \end{array}$$

inequalities (*) coming from

$$d_b(a) \Leftrightarrow d_b(b) = d_b(a \Leftrightarrow b + b) \Leftrightarrow d_b(b) \overset{(6)}{\leq} d_b(a \Leftrightarrow b) + d_b(b) \Leftrightarrow d_b(b) = d_b(a \Leftrightarrow b).$$

Now consider the unit sphere $\mathcal{S}^D = \big\{u \in \mathrm{I\!R}^D : |u|_2 = 1\big\} \subset \mathrm{I\!R}^D$, well known to be compact. Continuous $d_b\big|_{\mathcal{S}^D}$ therefore attains its minimal value $\tilde{\lambda} := \inf_{u \in \mathcal{S}^D} d_b(u)$ on some $u^{(0)} \in \mathcal{S}^D$. $d_b\big(u^{(0)}\big) = \tilde{\lambda} = 0$ contradicts the nondegeneracy of d_b, thus $\tilde{\lambda} > 0$. This means that for arbitrary $v \in \mathrm{I\!R}^D$, $u := v/|v|_2$:

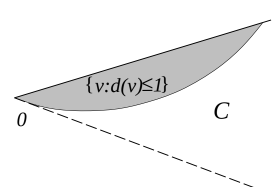
$$d_b(v) \quad = \quad d_b\big(|v|_2 \cdot u\big) \quad = \quad |v|_2 \cdot d_b(u) \quad \geq \quad |v|_2 \cdot \tilde{\lambda} \quad \geq \quad |v|_1 \cdot \tilde{\lambda}\sqrt{D}$$

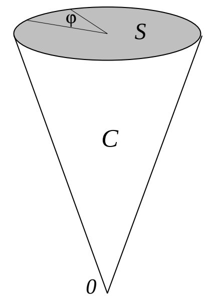
and thus, $\lambda := \sqrt{D} \cdot \tilde{\lambda}$ will do the job.

In the general case, the above considerations show that we find λ_a, Λ_a and λ_b, Λ_b to bound d_a and d_b against $|\cdot|_1$ in the sense that $\lambda_a|\cdot|_1 \leq d_a \leq \Lambda_a|\cdot|_1$ and $\lambda_b|\cdot|_1 \leq d_b \leq \Lambda_b|\cdot|_1$. $\Lambda := \Lambda_b/\lambda_a$ and $\lambda := \lambda_b/\Lambda_a$ have the required property:

$$\frac{\lambda_b}{\Lambda_a} d_a \quad \leq \quad \lambda_b|\cdot|_1 \quad \leq \quad d_b \quad \leq \quad \Lambda_b|\cdot|_1 \quad \leq \quad \frac{\Lambda_b}{\lambda_a} d_a \qquad \square$$

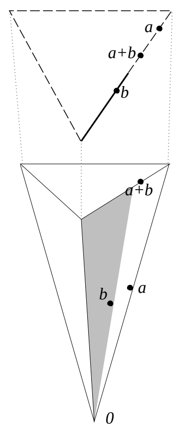
5.6 Remark: If C is not closed, $\mathcal{S}^D \cap C$ is not compact. Claim 5.5 therefore does not hold if the distance function is defined only on a convex cone $C \subset \mathrm{I\!R}^2$. The figure to the right depicts the unit sphere of such a $d : C \to \mathrm{I\!R}$ which cannot be bounded from above by $|\cdot|_2$.





But even in case C is closed, there exist counter examples as illustrated to the right: Be $C \subset \mathbb{R}^3$ with circular cross section. For $v \in \overset{\circ}{C}$, let $d(v) = |v|_2$. For boundary points $v \in \partial C$, denote $\varphi(v) \in [0, 2\pi)$ the angle according to the drawing. Then define

$$d(v) = \Lambda\big(\varphi(v)\big) \cdot |v|_2, \qquad \Lambda(\varphi) = \frac{2\pi}{2\pi \Leftrightarrow \varphi}.$$

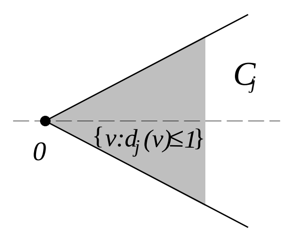


It is important to observe that d indeed fulfills triangle inequality (6) on whole $\overline{C}$: This is due to the fact that points on the boundary of sphere S (the cross section of C) cannot be represented as sum of two other points in S.
As a consequence, convex $d : C \to \mathbb{R}$ possesses in general no convex extension to the whole space $\mathbb{R}^D$!

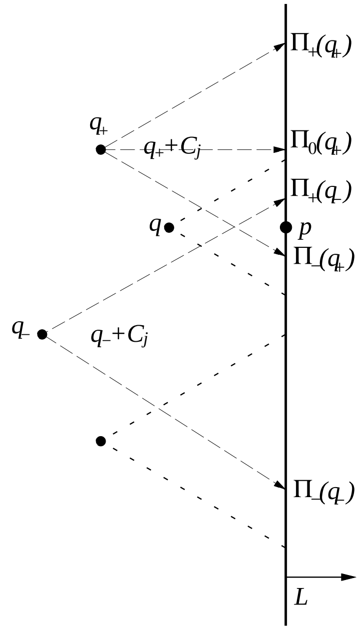
Concerning Claim 5.3, the prerequisite $b \in \overset{\circ}{C}$ is crucial, too: For $a \in \overline{C}$ and $b \in C$, $a + b$ in general does not lie in C any more!
To this end, consider $C \subset \mathbb{R}^3$ with triangular cross section as sketched to the right. Points a and b are on the same face of C, but a lies in the open part of it. And so does $a + b$. $\square$

6 Constructing PNGs

Proof of Theorem 3.5, first part: Fix j. We will describe an algorithm to compute those arcs (u, v) of $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ with $v \in P_j(u)$. According to Equation (5), it then suffices to repeat this process for each $j = 0, 1, \ldots, k \Leftrightarrow 1$.



For notational convenience, be the coordinate system such that the symmetry axis of C_j coincides with the x-axis. Then $d_j(u) = u_x$ for $u \in C_j$, as a look to the unit sphere of d_j depected in Theorem 3.1 reveals. Sort the points of P in ascending order with respect to their x coordinate — time $\mathcal{O}(n \log n)$ — and let the vertical sweep line L proceed from left to right. We maintain a data structure S for storing all those $u \in P$ lying on the left of L which have not yet got a neighbor $v \in u + C_j$. Whenever L hits a vertex $p \in P$, we will insert p to S, query the data structure about all $q \in S$ such that $p \in q + C_j$, create according edges (q,p), and remove q from S: p indeed is closest to q. For, suppose $d_j(\tilde{p} \Leftrightarrow q) = \tilde{p}_x \Leftrightarrow q_x < p_x \Leftrightarrow q_x = d_j(p \Leftrightarrow q)$. Then the line which sweeps P in increasing order of x would have hit $\tilde{p}$ before p, thereby having provided q with an edge and removed it from S, a contradiction.



Take as S some realization of a dynamic sorted array of m elements (e.g., a balanced binary tree) supporting operations LOCATE, INSERT, and DELETE in (amortized) time $\mathcal{O}(\log m)$.
Each of the n points $p \in P$ is inserted exactly once, hence $m \leq n$, adding to a total time for insertions of $\mathcal{O}(n \log n)$. After any of the n insertion, the above algorithm performs a query of $\mathcal{O}(\log m) + \mathcal{O}(\text{\#elements reported})$, summing up to another $\mathcal{O}(n \log n) + \mathcal{O}(n)$. And finally, $q \in P$ gets deleted at most once: $\mathcal{O}(n \log n)$.
Vertices u which still are in S after the sweeping have $P_j(u) = \emptyset$ and remain without outgoing edge.

Let us now explain how to answer the two-dimensional cone stabbing queries $Q(p) = \{q \in S : p \in q + C_j\}$ required above by means of the one-dimensionally ordered data structure S. To this end, be the elements of S sorted with respect to their y-coordinates, i.e., the projection $\Pi_0(q)$ of p parallel to the x axis onto the sweep line. Π_0 has the advantage that is does not change while the sweep line moves and thus can be maintained by data structure S. The latter two, on the other hand, do change but they permit to solve the query

$$Q(p) = \{q \in S : \Pi_-(q) \leq \Pi_-(p)\} \cap \{q \in S : \Pi_+(q) \geq \Pi_+(p)\}$$

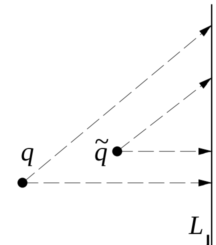
as follows:

- Find the biggest (w.r.t. Π_-) $q \in S$ which is still smaller than p. Call this q_+.
- Find the smallest (w.r.t. Π_+) $q \in S$ which is still bigger than p. Call this q_-.
- Report all vertices $q \in S$ between q_+ and q_- (w.r.t. Π_0).

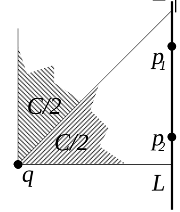
Performing a binary search with respect to one order within items sorted with respect to another usually fails badly. Here, on the contrary, Claim 6.1 guarantees that it does work. The first two steps can therefore be performed in $\mathcal{O}(\log m)$ and the last one indeed returns the elements of $Q(p)$ in output sensitive time. ☐

6.1 Claim: With notions as above, $C \subsetneq \{0\}$, the orders induced by Π_- and Π_+ are weaker than the one induced by Π_0 in the sense that for $q, \tilde{q} \in S$,

$$\begin{array}{lllll} \Pi_0(q) \leq \Pi_0(\tilde{q}) & \Longrightarrow & \Pi_-(q) \leq \Pi_-(\tilde{q}) & \wedge & \Pi_+(q) \leq \Pi_+(\tilde{q}) \\ \Pi_0(q) \geq \Pi_0(\tilde{q}) & \Longrightarrow & \Pi_-(q) \geq \Pi_-(\tilde{q}) & \wedge & \Pi_+(q) \geq \Pi_+(\tilde{q}) \end{array}$$



Proof: We consider $\leq$ and suppose $\Pi_0(q) \leq \Pi_0(\tilde{q})$ but $\Pi_+(q) > \Pi_+(\tilde{q})$. From the definition of Π_0 and Π_+ as center and upper boundary of C, this implies $\tilde{q} \in q + C$. But then, $q \in S$ would have received the neighbor $\tilde{q}$ and been removed from S at that very moment when sweep line L hit $\tilde{q}$ — a contradiction. $\square$



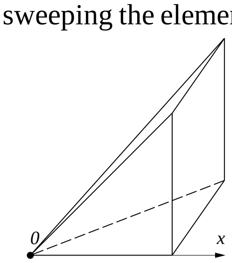
Proof of Theorem 3.5, second part: Constructing the PNG of Theorem 3.4 is more difficult for three reasons: Formerly, we could (within C_j) identify the line shaped boundary of the distance function's unit sphere with the sweep line and therefore in order of increasing d_j process all vertices in one pass.
This time, *two* lines are needed to cover that boundary. Therefore, divide the quadrant along its diagonal axis: Within each part $C/2$, d_j now has only one segment boundary and can be treated as before. The resulting graph temporarily has outdegree 8, but a subsequent $\mathcal{O}(n)$ processing will compare for each u its two neighbors corresponding to the two parts of C and keep only that arc to the closer one.

The other problem to obey is the boundary of quadrant C and whether it is belongs the the cone or not. This can be taken care of by choosing q_+ biggest but smaller *or equal* in the above algorithm and q_- correspondingly.

And third, the tie-breaking-rule (total order) must be applied in case two points are equally close. The latter comes into play when the sweep line simultaneously hits two (or more) vertices p_1 and p_2: Each $q \in Q(p_1) \cap Q(p_2)$ requires to decide which of $|p_1 \Leftrightarrow q|_0$ and $|p_2 \Leftrightarrow q|_0$ is smaller and create either arc (p_1, q) or (p_2, q) accordingly. Luckily, the quadratic time for comparing each $p \in L$ to each $q \in Q(p)$ can be reduced: W.l.o.g. consider the lower $C/2$, the upper one being similar. Now, if the queries $Q(p)$ for different $p \in L$ are processed in increasing order of p_y, this will automatically obey the $|\cdot|_0$ condition!
Indeed, the shape of $C/2$ implies that $q_y \leq p_y$ for each $q \in Q(p)$. Furthermore, $|v|_0 = \min\big\{|v_x|, |v_y|\big\} = |v_y| = v_y$ for $v \in C/2$. Together, this yields

$$\big|p_2 \Leftrightarrow q\big|_0 = p_{1,y} \Leftrightarrow q_y < p_{2,y} \Leftrightarrow q_y = \big|p_2 \Leftrightarrow q\big|_0 \qquad \text{for} \quad p_1, p_2 \in L, p_{1,y} < p_{2,y}. \quad \square$$

Proof of Theorem 3.7: Like in the two dimensional case, our algorithm will work in phases, one for each cone $C \in \mathcal{C}$ of the covering to compute those arcs (q, p) with $p \in C + q$. Instead of a sweep line L, we will employ a plane H, sweeping the elements of P in order of increasing x-coordinate.
Again, we have to subdivide each cone C in such a way that within each part, the distance function's unit sphere has a planar boundary, i.e., $d\big|_C$ is a projection. To this end, cut octant $C = C_{(+,+,+)}$ into three congruent subcones $C/3 = \big\{v \in C : v_y \leq v_x \wedge v_z \leq v_x\big\}$ sketched to the right.



And again, too, the rule $|\cdot|_1$ for breaking ties in case H simultaneously hits several vertices p_1, p_2 will automatically be fulfilled if these are processed in order of increasing $p_y + p_z$. Put differently, let H sweep P sorted lexicographically with respect to $(x, y+z)$.

It now remains to find a dynamic data structure S for efficiently answering the dimensional cone stabbing queries $Q(p) = \{q \in S : p \in q + C/3\}$. Unfortunately, there is no three dimensional analogon to Claim 6.1: Denote $\Pi_+(q)$ the projection of q parallel to the upper boundary plane of $C/3$ onto sweep plane H, i.e. the horizontal line $H \cap (q + \partial^+ C/3)$ and correspondingly $\Pi_-(q)$ for the lower boundary.
Then there exist points $q, \tilde{q}$ such that $\Pi_+(q) < \Pi_+(\tilde{q})$, $\Pi_-(q) > \Pi_-(\tilde{q})$ but neither $q \in \tilde{q} + C$ nor $\tilde{q} \in q + C$: Take the two-dimensional example sketched in Claim 6.1 and choose the third coordinates of q and $\tilde{q}$ so very different that they do not lie in each other's cone any more!

We will give it another try and analyze the applicability of *range trees* [1]: These dynamic data structures can efficiently answer D-dimensional orthogonal range queries parallel to the axises

$$\{q \in S \,:\, a_i \leq q_i < b_i,\ i = 1, \ldots, D\} \quad = \quad S \cap \bigtimes_{i=1}^{D} [a_i, b_i) \ =: \ S \cap [a, b)$$

in time $\mathcal{O}\big(\log^D m\big) + \mathcal{O}(\#\text{elements reported})$. Now consider the four faces of $C/3$ and the planes they lie in. Be $u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}$ their normal vectors, oriented in direction of $C/3$, that is

$u^{(1)} = (0,\ 0,\ 1)$	lower boundary plane of $C/3$
$u^{(2)} = (1,\ 0, \Leftrightarrow 1)/\sqrt{2}$	upper boundary plane of $C/3$
$u^{(3)} = (0,\ 1,\ 0)$	front boundary plane of $C/3$
$u^{(4)} = (1, \Leftrightarrow 1,\ 0)/\sqrt{2}$	back boundary plane of $C/3$

Assign to each vertex $p \in P$ the 4-tuple p^* of its distances to these planes

$$p^* = \Big(\sum_i p_i u_i^{(1)}, \sum_i p_i u_i^{(2)}, \sum_i p_i u_i^{(3)}, \sum_i p_i u_i^{(4)} \Big)$$

and observe that $v \in C/3$ if and only if $v^* \in \big[(0,0,0,0), (\infty,\infty,\infty,\infty)\big)$. Thus,

$$q \in Q(p) \quad \Leftrightarrow \quad q \in S \cap (p \Leftrightarrow C/3) \quad \Leftrightarrow \quad q^* \in S^* \cap \big[\Leftrightarrow p^*, (\Leftrightarrow\infty, \Leftrightarrow\infty, \Leftrightarrow\infty, \Leftrightarrow\infty,)\big).$$

So, a four dimensional range tree S^* can be employed to answer the query $Q(p)$. This gives a sweep plane algorithm of time complexity $\mathcal{O}(n \log^4 n)$ — two magnitudes of $\log n$ slower than claimed.

One factor can be removed with the well known *fractional cascading* technique [5, 19]. For the other one, once again subdivide the cone $C/3$ by triangulating its quadratic cross section: The two resulting $C/6$ will have only three boundary planes. Hence, q^* and S^* are three dimensional instead of four. ☐

6.2 Scholium:[3] Be $\mathcal{C}$ a partition of $\mathbb{R}^D$ into k convex cones and $\mathcal{D}$ a family of norms d_j, $j = 0, \ldots, k \Leftrightarrow 1$. Suppose that each d_j equals the maximum of finitely many projections or, equivalently, its unit sphere is polyhedral.
Then C_j can be subdivided into $\delta_j < \infty$ subcones C_j/δ_j such that for each one,

[3] A scholium is a corollary not to a theorem but to a proof …

- its cross section forms a $(D \Leftrightarrow 1)$ dimensional simplex
- and $d_j\big|_{C_j/\delta_j}$ is a projection.

Furthermore, graph $G(\mathcal{C},\mathcal{D};P)$ can be computed from input $P \subset \mathbb{R}^D$ of size $n = \#P$ from performing $\sum_{j=0}^{k-1} \delta_j$ sweep hyperplane passes, each of time $\mathcal{O}\big(n \cdot \log^{D-1} n\big)$. $\square$

7 Proof of Theorem 3.6

We begin with a

7.1 Remark: concerning the mapping $\bar{f} : \{+,0,\Leftrightarrow\}^3 \to \{+,\Leftrightarrow\}^3$ and its higher dimensional generalizations: This represents a convinient way of specifying for points that are common to the boundary of several octants $\big($in general: hyperquadrants $C_{\bar{\imath}}$, $i \in \{+,\Leftrightarrow\}^D\big)$ to which one it belongs, thereby turing the covering into a partition. Each possible argument $\bar{k} \in \{+,0,\Leftrightarrow\}^D$ assigns to a whole face or subface

$$F_{\bar{k}} \;=\; \big\{u \in \mathbb{R}^D : \operatorname{sgn} u = \bar{k}\big\}, \qquad \operatorname{sgn}(u_1,\ldots,u_d) := (\operatorname{sgn} u_1,\ldots,\operatorname{sgn} u_d),$$

one hyperquadrant $C_{\bar{f}(\bar{k})}$. Denote $\#_0\bar{k} = \operatorname{Card}\{i = 1,\ldots D : k_i = 0\}$, then $F_{\bar{k}}$ has dimension $d \Leftrightarrow \#_0\bar{k}$.
Alas, not every $\bar{f}$ is admissible for this purpose: The d-dimensional (improper) face $F_{\bar{k}} = \mathring{C}_{\bar{k}}$, $k \in \{+,\Leftrightarrow\}^D$ must of course be mapped to $C_{\bar{k}}$.
And for example in two dimensions, face $F_{(+,0)}$ — the positive x-axis — may not be assigned to the upper left quadrant $C_{(-,+)}$ since it does not belong to its boundary: $\bar{f}(+,0)$ must be either $(+,+)$ or $(+,\Leftrightarrow)$. This indicates that only zero components of arguments are to be modified. The non-zero ones, $\bar{f}$ must leave unchaned:

$$k_i \neq 0 \qquad \Longrightarrow \qquad f_i(\bar{k}) = k_i \tag{13}$$

As a generalization to this we require that, if a face $F_{\bar{k}}$ is mapped to one hyperquadrant $C_{\bar{\imath}}$ then all faces $F_{\bar{l}}$ lying w.r.t. inclusion between $F_{\bar{k}}$ and $C_{\bar{\imath}}$ are so, too:

$$s := f_i(\bar{k}) \qquad \Longrightarrow \qquad \bar{f}(\bar{k}, i = s) = \bar{f}(\bar{k}) \tag{14}$$

with notation $\big(\bar{k}, i = s\big) \;=\; \big(k_1,\ldots,k_{i-1},s,k_{i+1},\ldots,k_d\big)$. Condition (14) for example says that if the positive x-axis $F_{(+,0,0)}$ belongs to $C_{(+,+,+)}$ it is not allowed to assign the xz-plane $F_{(+,0,+)}$ (the relative topological closure of which $F_{(+,0,0)}$ belongs to) to, lets say, $C_{(+,-,+)}$. In our proof of Theorem 3.6, this kind of sub-/face compatibility condition will ensure the monotony of potential function Φ_F to hold not only on a $(D \Leftrightarrow 1)$-dimensional face F but also on its boundary, confer Lemma 7.3.

Now each $\bar{f}$ fulfilling the above conditions (13) and (14) induces a permissible partition $\mathcal{C}$ of space into hyperquadrants and vice versa. But $\mathcal{C}$ must also be such that it produces (weak) spanners. A necessary condition to this is, according to Proposition 5.2, that not $\mathcal{C}'$ contains a cycle of length 2. We claim that the latter is equivalent to $\bar{f}$ being antisymmetric:

$$\bar{f}\big(\Leftrightarrow\bar{k}\big) \;=\; \Leftrightarrow\bar{f}\big(\bar{k}\big) \qquad \forall \bar{k} \in \{+,0,\Leftrightarrow\}^D,\ \bar{k} \neq \bar{0}. \tag{15}$$

You will easily verify that the $\bar{f}$ we proposed for $D = 3$ indeed complies with all the above conditions. This can also be seen from its formula representation (17) on page 24. On the other hand, absence of 2-cycles is only necessary: Our proof that $\mathcal{C}$ <u>does</u> yield weak spanners begins with Lemma 7.3.

7.2 Claim: The following are equivalent:

a) For each $i = 1, \dots D$, $s \in \{+, \Leftrightarrow\}$ does $\mathcal{C}'$ as induced by $\mathcal{C}$, $v = (\bar{0}, i = s)$, $S = \{u : u_i = 0\}$ according to Equation (12) in Proposition 5.2, contain no 2-cycle (a, b, a).

b) For each $i = 1, \dots, D$, $s \in \{+, \Leftrightarrow\}$, $\bar{k} \in \{+, 0, \Leftrightarrow\}^D$,

$$\bar{f}\big(+\bar{k}, i = 0\big) \neq \bar{f}\big(+\bar{k}, i = s\big) \qquad \vee \qquad \bar{f}\big(\Leftrightarrow\bar{k}, i = 0\big) \neq \bar{f}\big(\Leftrightarrow\bar{k}, i = s\big)$$

c) $\bar{f}$ is antisymmetric in the sense of (15).

Proof:

"c $\Rightarrow$ a" : Take i, s and suppose that (a, b, a) is a cycle of $\mathcal{C}'$, that is there exist $\bar{A}, \bar{B} \in \{+, \Leftrightarrow\}^D$ such that $\qquad v \in \overline{C_{\bar{A}}}, \overline{C_{\bar{B}}}, \qquad a, b \in S,$

$$0 \in a + \big(C_{\bar{A}} \cap S\big)^\circ \subset C_{\bar{A}}, \quad b \in a + C_{\bar{A}}, \qquad 0, a \in b + C_{\bar{B}}.$$

The first implies $A_i = s = B_i$. The latter, by definition of $C_{\bar{\iota}}$ in Equation (10), requires $\bar{A} = \bar{f}\big(\operatorname{sgn}(b \Leftrightarrow a)\big)$ and $\bar{B} = \bar{f}\big(\operatorname{sgn}(b \Leftrightarrow a)\big)$. Due to prerequisite (15), $\bar{A} = \Leftrightarrow\bar{B}$ and in particular $s = A_i = \Leftrightarrow B_i = s$, a contradiction.

"a $\Rightarrow$ b" : Given i, s, and $\bar{k}$. Without loss of generality, $k_i = 0$. Let $v := (\bar{0}, i = s)$, $S = \big\{u \in \mathbb{R}^D : k_i = 0 \Rightarrow u_i = 0\big\}$, $a := (\Leftrightarrow\bar{k}, i = 0) \in S \ni (+\bar{k}, i = 0) =: b$, $\bar{A} = \bar{f}\big(+\bar{k}, i = s\big)$, $\bar{B} = \bar{f}\big(\Leftrightarrow\bar{k}, i = s\big)$. Note that $v \in \overline{C_{\bar{A}}}, \overline{C_{\bar{B}}}$ as $A_i = s = B_i$. Suppose b) does not hold. Then

$$\bar{A} = \bar{f}\big(\bar{k}, i = s\big) \overset{(*)}{=} \bar{f}\big(\bar{k}, i = 0\big) = \bar{f}\big(\operatorname{sgn}(\Leftrightarrow a)\big) = \bar{f}\big(\operatorname{sgn}\underbrace{(b \Leftrightarrow a)}_{=2\bar{k}}\big)$$

and hence $b \Leftrightarrow a, \Leftrightarrow a \in C_{\bar{A}}$. Since S is of dimension $d \Leftrightarrow \#_0\bar{k}$, it even even follows that $\Leftrightarrow a \in (C_{\bar{A}} \cap S)^\circ$. Similarly, $a \Leftrightarrow b \in C_{\bar{B}}$, $\Leftrightarrow b \in (C_{\bar{B}} \cap S)^\circ$. So, (a, b, a) forms a 2-cycle of $\mathcal{C}'$ in contradiction to a).

"b $\Rightarrow$ c" : Suppose that component f_i is not antisymmetric. From all $\bar{k}$ with $f_i\big(\Leftrightarrow\bar{k}\big) = f_i\big(+\bar{k}\big)$ take one of minimal $\#_0$, i.e., the least number of zeros. Since

$$k_i \neq 0 \qquad \Longrightarrow \quad +k_i \overset{(13)}{=} f_i\big(+\bar{k}\big) \overset{(*)}{=} f_i\big(\Leftrightarrow\bar{k}\big) \overset{(13)}{=} \Leftrightarrow k_i,$$

necessarily $k_i = 0$. Set $s := f_i\big(+\bar{k}\big))$ and verify

$$f_i\big(+\bar{k}, i = 0\big) = f_i\big(+\bar{k}\big) = s \overset{(14)}{=} f_i\big(+\bar{k}, i = s\big)$$
$$\text{and } f_i\big(\Leftrightarrow\bar{k}, i = 0\big) = f_i\big(\Leftrightarrow\bar{k}\big) \overset{(*)}{=} f_i\big(+\bar{k}\big) = s \overset{(13)}{=} f_i\big(\Leftrightarrow\bar{k}, i = s\big)$$

This contradicts b) unless there exists $j \neq i$ such that

$$f_j\big(+\overline{k}, i=0\big) \neq f_j\big(+\overline{k}, i=s\big) \quad \vee \quad f_j\big(\Leftrightarrow\overline{k}, i=0\big) \neq f_j\big(\Leftrightarrow\overline{k}, i=s\big).$$

Again necessarily $k_j = 0$. This time, set $\tilde{s} := f_j\big(+\overline{k}\big)$.

In case f_j is not antisymmetric for this $\overline{k}$ either, we will find a third component $\tilde{j}$ different from i and j such that $k_{\tilde{j}} = 0$, and so on. This process obviously terminates after at most D steps, simply because then there are not components left: $\overline{k} = 0$ in contrast to the prerequisite of Equation (15). So without loss of generality be f_j antisymmetric:

$$f_j\big(\Leftrightarrow\overline{k}, i=0, j=0\big) = f_j\big(\Leftrightarrow\overline{k}\big) = \Leftrightarrow f_j\big(+\overline{k}\big) = \Leftrightarrow f_j\big(+\overline{k}, i=0, j=0\big) = \Leftrightarrow\tilde{s}.$$

$$\begin{aligned}\Longrightarrow \quad & f_i\big(+\overline{k}, j=\tilde{s}, i=0\big) \overset{(14)}{=} f_i\big(+\overline{k}\big) = s = \\ & = s \overset{(*)}{=} f_i\big(\Leftrightarrow\overline{k}\big) = f_i\big(\Leftrightarrow\overline{k}, j=0, i=0\big) \overset{(14)}{=} f_i\big(\Leftrightarrow\overline{k}, j=\Leftrightarrow\tilde{s}, i=0\big).\end{aligned}$$

f_i is therefore not antisymmetric at argument $(\overline{k}, j=\tilde{s})$, neither. But $\#_0(\overline{k}, j = \tilde{s}) = \#_0\overline{k}\Leftrightarrow 1$ contradicts the minimality of $\overline{k}$. $\square$

7.3 Lemma: Given $\mathcal{C}$, $\tilde{\mathcal{D}}$ as in the prerequisites of Theorem 3.6, $s \in P$ and w.l.o.g. $t = 0 \in P$, $|s|_\infty = 1$. The greedy path in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ from s to t has nonincreasing $|\cdot|_\infty$. And, while staying on one face F of this cube $Q := \big\{p : |p|_\infty \leq 1\big\}$, it is even strictly decreasing with respect to some potential function Φ_F. More precisely, be $a \rightsquigarrow b$ one greedy step and $|a|_\infty = 1 = |b|_\infty$. Then

$$\begin{aligned}
a_x = +1 = b_x &\implies (+b_y, +b_z) < (+a_y, +a_z) \geq (0,0)\\
a_y = +1 = b_y &\implies (+b_z, +b_x) < (+a_z, +a_x) \geq (0,0)\\
a_z = +1 = b_z &\implies (+b_x, +b_y) < (+a_x, +a_y) \geq (0,0)\\
a_x = \Leftrightarrow 1 = b_x &\implies (\Leftrightarrow b_y, \Leftrightarrow b_z) < (\Leftrightarrow a_y, \Leftrightarrow a_z) \geq (0,0)\\
a_y = \Leftrightarrow 1 = b_y &\implies (\Leftrightarrow b_z, \Leftrightarrow b_x) < (\Leftrightarrow a_z, \Leftrightarrow a_x) \geq (0,0)\\
a_z = \Leftrightarrow 1 = b_z &\implies (\Leftrightarrow b_x, \Leftrightarrow b_y) < (\Leftrightarrow a_x, \Leftrightarrow a_y) \geq (0,0) \quad \square
\end{aligned}$$

7.4 Lemma: The greedy path will at most once change[4] to a different face

$$F_{\overline{\iota}} = \big\{q \in \mathbb{R}^3 : |q|_\infty = 1, \operatorname{sgn}(q_i) = s\big\}, \qquad \overline{\iota} = (i,s) \in \{x,y,z\} \times \{+, \Leftrightarrow\}$$

of Q. More formally, suppose $a \rightsquigarrow b \rightsquigarrow c \rightsquigarrow d$ are subsequent steps of this path with $a, b, c \in \partial Q$, $a \notin F_{\overline{\iota}}$, $b \in F_{\overline{\iota}}$. Then $|d|_\infty < 1$.

Proof of Theorem 3.6: Denote $\oplus$ addition modulo 3. Within each face $F_{(i,s)}$, the potential function

$$\Phi_{(i,s)}(v) = \big(|v|_\infty, s \cdot v_{i\oplus 1}, s \cdot v_{i\oplus 2}\big) \qquad \text{lexicographically}$$

[4]Observe that $F_{\overline{\iota}}$ is relatively closed. Therefore, changing can mean "entering new, then leaving old one" in two steps, or "already lying in two faces; leave one, then enter another", or in one step "leave old and enter new".

strictly decreases. The only 'escape' — changing to another face — can occur at most once. The greedy path thus finally does reach t and remains in Q. This implies a weak stretch factor of 2 w.r.t. $|\cdot|_\infty$, and the Euclidean weak stretch facter is at most

$$f^* \quad = \quad \Big\{ |a \Leftrightarrow b|_2 / |a|_2 : |a|_\infty = 1 = |b|_\infty \Big\}. \tag{16}$$

Equivalence of norms (c.f. Claim 5.5) $|a|_\infty \le |a|_2 \le \sqrt{D}|a|_\infty$ implies $f^* \le 2\sqrt{3}$, but this bound is not tight. For a better one, square both sides of (16) and note that, for symmetry reasons (simultaneously permuting or inverting components of a and b), the maximum is w.l.o.g. attained in $a_z = +1$, $0 \le a_x, a_y \le 1$. The extremal location of b is thus $b = (\Leftrightarrow 1, \Leftrightarrow 1, \Leftrightarrow 1)$ and $a_x = a_y$. It therefore remains to maximize the one parameter function

$$[0,1] \ni \lambda \;\mapsto\; \frac{|a \Leftrightarrow b|_2^2}{|a|_2^2}\Bigg|_{\substack{a=(\lambda,\lambda,1)\\ b=(-1,-1,-1)}} \; = \; 1 + \frac{4\lambda + 5}{2\lambda^2 + 1}$$

via highschool calculus, obtaining $\lambda_0 = (\Leftrightarrow 5 + \sqrt{33})/4$ and $f^* = \sqrt{(7+\sqrt{33})/2} \approx 2.524$. □

7.5 Scholium: Suppose that $\overline{f} : \{+, 0, \Leftrightarrow\}^D \to \{+, \Leftrightarrow\}^D$ is admissible in the sense of Equations (13), (14),(15) and $\tilde{\mathcal{D}}$ a family of 2^D total orders extending the norm $|\cdot|_\infty$ such that in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$, greedy paths visit no vertex more than once.
Then this graph has Euclidean weak stretch

$$\begin{aligned}
f^* \quad &\le \quad \sqrt{\max_{0\le\lambda\le 1} \frac{\big|a \Leftrightarrow b\big|_2^2}{\big|a\big|_2^2}\Bigg|_{\substack{a=(\lambda,\dots,\lambda,1)\\ b=(-1,\dots,-1,-1)}}} \\
&= \quad \sqrt{1 + \frac{2(d \Leftrightarrow 1)\lambda + d + 2}{1 + (d \Leftrightarrow 1)\lambda^2}}\Bigg|_{\lambda = \left(\sqrt{d(d+8)} - d - 2\right)/2(d-1)} \\
&= \quad \sqrt{\frac{\sqrt{d(d+8)} \Leftrightarrow 4 + d}{\sqrt{d(d+8)} \Leftrightarrow 2 \Leftrightarrow d}} \qquad \simeq \quad \sqrt{d}
\end{aligned}$$

7.6 Claim: Let $a, b \in \mathbb{R}^3 \setminus \{0\}$, $a \neq b$ and $\mathcal{C}$ as in (10). There exists $C \in \mathcal{C}$ with $0 \in a + C$ <u>and</u> $b \in a + C$ iff for each $i = 0, 1, 2$ one if the following holds:

a) $a_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \ge (0,0) \quad \wedge \quad (b_i \Leftrightarrow a_i) \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \le (0,0,0)$

b) $a_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) < (0,0) \quad \wedge \quad (b_i \Leftrightarrow a_i) \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) < (0,0,0)$

where inequalities are to be understood with respect to lexicographical order and multiplication performed componentwise.

7.7 Claim: Let $a \rightsquigarrow b$ be a greedy step, $|a|_\infty = 1$.

a) $|a \Leftrightarrow b|_\infty \leq 1$. $\quad |a \Leftrightarrow b|_\infty = 1$, then $|a \Leftrightarrow b|_1 \leq |a|_1$.

b) $a_z \neq 1, \quad b_z = 1$. $\quad$ Then $a_z = 0, \quad |a_x \Leftrightarrow b_x| + |a_y \Leftrightarrow b_y| \leq |a_x| + |a_y| \Leftrightarrow 1$.

c) $a_z = 1, \quad \Leftrightarrow 1 < a_x < 0, \quad 0 \neq a_y \neq +1$. $\quad$ Then $|b|_\infty < 1$.

d) $a_z = 1, \quad \Leftrightarrow 1 \leq a_x < 0, \quad 0 < a_y < 1$. $\quad$ Then $|b|_\infty < 1$.

The same holds for coordinates (x,y,z) exchanged with (y,z,x), (z,x,y), $(\Leftrightarrow x, \Leftrightarrow y, \Leftrightarrow z)$, $(\Leftrightarrow y, \Leftrightarrow z, \Leftrightarrow x)$, $(\Leftrightarrow z, \Leftrightarrow x, \Leftrightarrow y)$.

Proof of Lemma 7.4: Since Claim 7.6 and Claim 7.7 are invariant under cyclic permutation and inversion of coordinates, we may assume without loss of generality that $\overline{\iota} = (z,+)$, $b_z = 1$. According to Claim 7.7b), $a_z = 0$. Consider the 25 cases $a_x, a_y \in \{\Leftrightarrow 1\}, (\Leftrightarrow 1, 0), \{0\}, (0,+1), \{+1\}$:

a) $\Leftrightarrow 1 < a_x < 0, \quad 0 < a_y < 1$ $\quad$ contradicts $a \in \partial Q$.

b) $\Leftrightarrow 1 < a_x < 0, \ a_y = 0$; $\quad \Leftrightarrow 1 < a_x < 0, \ \Leftrightarrow 1 < a_y < 0$; $\quad a_x = 0, \ 0 < a_y < 1$; $a_x = 0, \ a_y = 0$; $\quad a_x = 0, \ \Leftrightarrow 1 < a_y < 0$; $\quad 0 < a_x < 1, \ 0 < a_y < 1$; $\quad 0 < a_x < 1$, $a_y = 0$; $\quad 0 < a_x < 1, \ \Leftrightarrow 1 < a_y < 0$ $\quad$ similarly.

c) $a_x = 1, a_y = \Leftrightarrow 1$:
$a \rightsquigarrow b$ greedy, so by definition $\exists C \in \mathfrak{C}$: $b, 0 \in a + C$. Since $a_z \cdot (\operatorname{sgn} a_x, \operatorname{sgn} a_y) = (0,0)$, case a) of Claim 7.6 for $i = 2$ implies lexicographically:

$$(0,0,0) \geq (b_z \Leftrightarrow a_z) \cdot (\operatorname{sgn} a_z, \operatorname{sgn} a_x, \operatorname{sgn} a_y) = (0, b_z \Leftrightarrow a_z, a_z \Leftrightarrow b_z)$$

Therefore $b_z \leq a_z$, a contradiction: this case does not occur.

d) $a_x = 1, \Leftrightarrow 1 < a_y < 0$; $\quad a_x = 1, a_y = 0$; $\quad a_x = 1, 0 < a_y < 1$; $\quad a_x = 1, a_y = 1$; $0 < a_x < 1, a_y = 1$; $\quad 0 < a_x < 1, a_y = \Leftrightarrow 1$; $\quad a_x = 0, a_y = 1$ $\quad$ don't either.

e) $a_x = \Leftrightarrow 1, a_y = +1$:
This time, case b) of Claim 7.6 for $i = 1$ holds, ensuring $b_y < a_y = 1$. Furthermore, by Claim 7.7a), $1 \Leftrightarrow b_y \leq |a_y \Leftrightarrow b_y| \leq |a \Leftrightarrow b|_\infty \leq 1$. Similar application of Claim 7.6 for $i = 0$ yields $0 \geq b_x > \Leftrightarrow 1$. Thus

$$\big(1 \Leftrightarrow |b_x|\big) + \big(1 \Leftrightarrow |b_y|\big) = |a_x \Leftrightarrow b_x| + |a_y \Leftrightarrow b_y| \leq |a_x| + |a_y| \Leftrightarrow 1 = 1,$$

the inequality coming from Claim 7.7b). Hence $|b_x| + |b_y| \geq 1$. As we already know $|b_x|, |b_y| < 1$, this means $b_x \neq 0 \neq b_y$, thereby proving

$$b_z = 1, \qquad \Leftrightarrow 1 < b_x < 0, \qquad 0 < b_y < 1.$$

Put this into Claim 7.7c) to see: $\qquad 1 > |c|_\infty \geq |d|_\infty$ $\qquad \surd$

f) $\Leftrightarrow 1 < a_x < 0, a_y = +1$:
Again, Claims 7.6 and 7.7b) say $b_z = 1, \Leftrightarrow 1 < b_x < 0, 0 < b_y < 1$, so $|c|_\infty < 1$.

g) $a_x = \Leftrightarrow 1$, $a_y = 0$:
By $(b_x + 1) + |b_y| \leq 0$ (Claim 7.7b), necessarily $b_x = \Leftrightarrow 1$, $b_y = 0$. Which in turn requires (Claim 7.6) $c_z < 1$, $c_x > \Leftrightarrow 1$, $c_y < 1$. So $|c|_\infty < 1$ unless $c_y = \Leftrightarrow 1$. Analogously to case e), $c_y = \Leftrightarrow 1$ means $\Leftrightarrow 1 < c_x < 0$, $0 < c_z < 1$ and therefore $|d|_\infty < 1$ due to Claim 7.7c) for coordinates (x, y, z) exchanged with $(\Leftrightarrow y, \Leftrightarrow z, \Leftrightarrow x)$. $\surd$

h) $a_x = 0$, $a_y = \Leftrightarrow 1$:
Then necessarily $b = (0, \Leftrightarrow 1, +1)$, $c_x = 1$, $\Leftrightarrow 1 < c_y < 0$, $0 < c_z < 1$, $|d|_\infty < 1$.

i) $\Leftrightarrow 1 < a_x < 0$, $a_y = \Leftrightarrow 1$:
Apply Claim 7.7b) to see $\Leftrightarrow 1 < b_x \leq 0$, $\Leftrightarrow 1 \leq b_y < 0$. If b_x was $\neq 0$, then Claim 7.7c) would mean $|c|_\infty < 1$. Thus, $b_x = 0$ and (Claim 7.7b) $b_y = \Leftrightarrow 1$. As above, $c_x = 1$, $\Leftrightarrow 1 < c_y < 0$, $0 < c_z < 1$, and $|d|_\infty < 1$.

j) $a_x = \Leftrightarrow 1$, $\Leftrightarrow 1 < a_y < 0$ similarly:
$\Leftrightarrow 1 \leq b_x < 0$, $\Leftrightarrow 1 < b_y \leq 0$. For $b_y \neq 0$, we have $|c|_\infty < 1$. And for $b_y = 0$, we have $b_x = \Leftrightarrow 1$, implying (like in g) $c_y = \Leftrightarrow 1$, $\Leftrightarrow 1 < c_x < 0$, $0 < c_z < 1$ and $|d|_\infty < 1$.

k) $a_x = \Leftrightarrow 1$, $0 < a_y < 1$:
Claim 7.6 prohibits $b_x = \Leftrightarrow 1$. Claim 7.7b) then infers $\Leftrightarrow 1 < b_x < 0$, $0 < b_y < 1$. And $|c|_\infty < 1$ by Claim 7.7c).

l) $a_x = \Leftrightarrow 1$, $a_y = \Leftrightarrow 1$:
Then $\Leftrightarrow 1 \leq b_x \leq 0$, $\Leftrightarrow 1 \leq b_y \leq 0$. Consider sub-cases

i) $b_y \neq 0$, $0 \neq b_x \neq \Leftrightarrow 1 \quad \Rightarrow \quad |c|_\infty < 1$ by applying Claim 7.7d).

ii) $b_y \neq 0$, $b_x = \Leftrightarrow 1 \quad \Rightarrow \quad \Leftrightarrow 1 < c_y \leq 0$, $0 \leq c_z < 1$ using Claim 7.6 and Claim 7.7a). $|c|_\infty = 1$ requires $c_x = \Leftrightarrow 1$. But then, neither do b nor c leave the face $F_{(x,-)}$ which a started in, preserving strict decrease of the same $\Phi_{(+,-)}$ all the time due to Lemma 7.3.

iii) $b_x = 0 \quad \overset{7.7b)}{\Longrightarrow} \quad b_y = \Leftrightarrow 1$. Refer to case h) to see: $|d|_\infty < 1$.

iv) $b_y = 0 \quad \overset{7.7b)}{\Longrightarrow} \quad b_x = \Leftrightarrow 1$ which transforms to case h) under change of coordinates $(x, y, z) \mapsto (\Leftrightarrow z, \Leftrightarrow x, \Leftrightarrow y)$, and therefore $c_y = \Leftrightarrow 1$, $0 < c_z < 1$, $\Leftrightarrow 1 < c_x < 0$, $|d|_\infty < 1$ as well. $\square$

Proof of Lemma 7.3: Consider $a_x = 1 = b_x$, the other cases being similar. Since $(b_x \Leftrightarrow a_x) \cdot (\cdots) = (0,0,0)$, we have case a) rather than b) of Claim 7.6. Therefore, $a_x \cdot (\operatorname{sgn} a_y, \operatorname{sgn} a_z) \geq (0,0)$ which means $(a_y, a_z) \geq (0,0)$. Now, apply Claim 7.6 to $i = y$ and get

$$(b_y \Leftrightarrow a_y) \cdot \big(\underbrace{\operatorname{sgn} a_y, \operatorname{sgn} a_z}_{\geq (0,0)}, \underbrace{\operatorname{sgn} a_x}_{=1} \big) \leq (0,0,0),$$

hence $b_y \leq a_y$. If $b_y < a_y$, we are done.
So, be $b_y = a_y$. This requires $a_y \cdot (\operatorname{sgn} a_z, \operatorname{sgn} a_x) \geq (0,0)$. Combining that with $(a_y, a_z) \geq (0,0)$ implies $a_z \geq 0$ for $a_y \neq 0$ and $a_y = 0$. One more time, look at Claim 7.6 to see $(b_z \Leftrightarrow a_z) \cdot \big(\underbrace{\operatorname{sgn} a_z}_{\geq 0}, \underbrace{\operatorname{sgn} a_x}_{=1}, \operatorname{sgn} a_y \big) \leq (0,0,0)$ and $b_z \leq a_z$.

Let's finally exclude the possibility of $b_z = a_z$ by remarking that this means $a \rightsquigarrow b = a$, a self loop. Such, on the other hand, cannot occur in PNGs, as each $C \in \mathcal{C}$ has $0 \notin C$ and thus $u \notin P_j(u)$ in Equation (7). $\square$

Proof of Claim 7.7: Since, as a prerequisite to Theorem 3.6, $\tilde{d_\iota} \in \tilde{\mathcal{D}}$ is an extension of $\big(|\cdot|_\infty, |\cdot|_1\big)$ w.r.t. lexicographical order, *every* arc — greedy or not — in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ trivially obeys a).

For b), $|a|_\infty = 1$ and $a_z \neq 1$ and require $\Leftrightarrow 1 \leq a_z < 1$. $1 \Leftrightarrow a_z = |b_z \Leftrightarrow a_z| \leq |b \Leftrightarrow a|_\infty \overset{a)}{\leq} 1$ further restrict to $0 \leq a_z < 1$. Suppose $a_z > 0$, then application of Claim 7.6 to $i = 3$ yields

$$(1 \Leftrightarrow a_z, any, any) = (b_z \Leftrightarrow a_z) \cdot \big(\operatorname{sgn} a_z, \operatorname{sgn} a_x, \operatorname{sgn} a_y\big) \leq (0,0,0)$$

independent of which of the two cases actually holds. Hence, $a_z \geq 1$: a contradiction. The rest of b) is obtained by inserting $a_z = 0$, $b_z = 1$ into a).

$\Leftrightarrow 1 < a_x < 0$ implies $\Leftrightarrow 1 \overset{7.6}{<} b_x \overset{a)}{<} 1$, therefore $|b_x| < 1$. Similarly, cases $\Leftrightarrow 1 < a_y < 0$ and $0 < a_y < 1$ yield $|b_y| < 1$. Analogous arguments in case $a_y = \Leftrightarrow 1$ only gives $\Leftrightarrow 1 \leq b_y \leq 0$, but $b_y = \Leftrightarrow 1$ is ruled out by part b) of Claim 7.6. Finally, from $a_z = 1$ follows $0 \leq b_z \leq 1$, and again $b_z = 1$ prohibited: $|b_z| < 1$, too. Together $|b|_\infty < 1$, the claim of c).
Finally, part d): $0 < a_y < 1$, so $\Leftrightarrow 1 < b_y < 1$. $\Leftrightarrow 1 < a_x < 0$, so $\Leftrightarrow 1 \leq b_x < \Leftrightarrow 1$. Even for $a_x = \Leftrightarrow 1$, $b_x = \Leftrightarrow 1$ is impossible due to Claim 7.6b). The same holds for $b_z = 1$, so $0 \leq b_z < 1$ and $|b|_\infty < 1$. $\square$

Proof of Claim 7.6: The reader will easily verify that $\bar{f} : \{+, 0, \Leftrightarrow\}^3 \to \{+, \Leftrightarrow\}^3$,

$$f_i : \bar{k} = (k_0, k_1, k_3) \mapsto k_{i \oplus j(\bar{k})}, \quad i \oplus j(\bar{k}) := i + \min\big\{ j = 0,1,2 : k_{i \oplus j} \neq 0 \big\} \tag{17}$$

is exactly the one used in Equation (10) and has the following property:

$$f_i(\bar{k}) = s \in \{+, \Leftrightarrow\} \quad \Leftrightarrow \quad s \cdot (k_i, k_{i\oplus 1}, k_{i \oplus 2}) \geq \bar{0} \quad \Rightarrow \quad s \cdot (k_i, k_{i \oplus 1}) \geq \bar{0}, \tag{18}$$

Remember, that w.r.t. lexicographical order and for $x, y \in \mathbb{R}$, $\bar{u} \in \mathbb{R}^n$, $\bar{v} \in \mathbb{R}^m$:

$$\begin{aligned} (x,y) \gtreqless (0,0) \quad &\Leftrightarrow \quad (\operatorname{sgn} x, y) \gtreqless (0,0) \quad \Leftrightarrow \quad (\operatorname{sgn} x, \operatorname{sgn} y) \gtreqless (0,0) && (19)\\ \bar{u} \geq \bar{0}, \qquad \bar{v} \geq \bar{0} \qquad &\Longrightarrow \qquad \bar{u} \otimes \bar{v} \geq \bar{0} && (20)\\ \bar{u} > \bar{0}, \qquad \bar{u} \otimes \bar{v} \geq \bar{0} \ \vee \ \bar{v} \otimes \bar{u} \geq \bar{0} \quad &\Rightarrow \quad \bar{v} \geq \bar{0} && (21) \end{aligned}$$

$\bar{u} \otimes \bar{v} := \big(u_0 v_0, u_0 v_1, \ldots, u_0 v_m,\, u_1 v_0, u_1 v_1, \ldots, u_1 v_m,\, \ldots\ldots u_n v_0, u_n v_1, \ldots, u_n v_m\big)$.
Thus, for $\bar{s} \in \{+, \Leftrightarrow\}^3$,

$$\begin{aligned} 0, b \in a + C_{\bar{s}} \ &\overset{(10)}{\Longleftrightarrow} \ \forall i = 0,1,2: \ f_i\big(\operatorname{sgn}(b \Leftrightarrow a)\big) = s_i = f_i\big(\operatorname{sgn}(\Leftrightarrow a)\big) \\ &\underset{(19)}{\overset{(18)}{\Longleftrightarrow}} \ \big(b_i \Leftrightarrow a_i, b_{i\oplus 1} \Leftrightarrow a_{i \oplus 1}, b_{i \oplus 2} \Leftrightarrow a_{i \oplus 2}\big) \cdot s_i =: \vec{u}^{\imath} \geq (0,0,0) \qquad (22) \\ &\qquad \wedge \quad \Leftrightarrow \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i \oplus 1}, \operatorname{sgn} a_{i \oplus 2}\big) \cdot s_i =: \vec{v}^{\imath} \geq (0,0,0) \end{aligned}$$

which by (20) yields, considering only the first 3 components of $\vec{u}^{\imath} \otimes \vec{v}^{\imath}$:

$$\Leftrightarrow s_i^2 (b_i \Leftrightarrow a_i) \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i \oplus 1}, \operatorname{sgn} a_{i \oplus 2}\big) \geq (0,0,0) \tag{23}$$

and in particular ($s_i^2 = 1$) part a) of the claim. For b), be $a_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) < (0,0)$. Hence, $0 \neq \Leftrightarrow \operatorname{sgn} a_i = f_i(\Leftrightarrow \operatorname{sgn} a) = s_i = \Leftrightarrow s_{i\oplus 1}$ by definition of f_i and $f_{i\oplus 1}$. It suffices to show $b_i \neq a_i$ since the rest follows from (23).
To this end, suppose on contrary $a_i = b_i$. First component of Equation (22) vanishes:

$$(0,0) \leq s_i \cdot \big(b_{i\oplus 1} \Leftrightarrow a_{i\oplus 1}, b_{i\oplus 2} \Leftrightarrow a_{i\oplus 2}\big) = \Leftrightarrow s_{i\oplus 1} \cdot \big(b_{i\oplus 1} \Leftrightarrow a_{i\oplus 1}, b_{i\oplus 2} \Leftrightarrow a_{i\oplus 2}\big)$$

Application of (22) to $i \oplus 1$ instead of i requires the reversed inequality to hold, too:

$$\Longrightarrow \quad s_{i\oplus 1} \cdot \big(b_{i\oplus 1} \Leftrightarrow a_{i\oplus 1}, b_{i\oplus 2} \Leftrightarrow a_{i\oplus 2}\big) = (0,0), \qquad b_i = a_i \text{ by assumption}$$

implying ($s_{i\oplus 1} \neq 0$) $a = b$ in contradiction to the prerequisites.

Be now valid, for each $i = 0,1,2$, one of cases a) and b); $s_i := f_i\big(\operatorname{sgn}(\Leftrightarrow a)\big)$ and we will prove $\Leftrightarrow a, b \Leftrightarrow a \in C_{\bar{s}}$ by verifying Equation (22). Suppose

$$(b_i \Leftrightarrow a_i) \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) < (0,0,0).$$

Multiply by $s_i^2 = 1 \geq 0$ according to (20) to find out

$$s_i \cdot (b_i \Leftrightarrow a_i) \cdot \underbrace{s_i \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big)}_{=:\bar{u}} < (0,0,0),$$

$\bar{u}$ being $\leq (0,0,0)$ by definition of s_i and (18). As $\bar{u} \neq (0,0,0)$, we have even $\bar{u} < (0,0,0)$ and may conclude $s_i \cdot (b_i \Leftrightarrow a_i) > 0$, yielding Equation (22).
If $(b_i \Leftrightarrow a_i) \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) = (0,0,0)$, necessarily $b_i = a_i$ and (case a)

$$a_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, a_{i\oplus 2}\big) \geq (0,0). \tag{24}$$

If $0 \neq \operatorname{sgn} a_i = \Leftrightarrow s_i$, this means $\Leftrightarrow s_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \geq (0,0)$, and if $0 = \operatorname{sgn} a_i$, $k_i = \Leftrightarrow \operatorname{sgn} a_i = 0$ reduces Equation (18) to $\Leftrightarrow s_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \geq (0,0)$, too. So, take prerequisite w.r.t. $i \oplus 1$

$$(b_{i\oplus 1} \Leftrightarrow a_{i\oplus 1}) \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \leq (0,0)$$

and multiply with $s_i^2 = 1$ to end up at $s_i \cdot (b_{i\oplus 1} \Leftrightarrow a_{i\oplus 1}) \geq (0,0)$. For $a_{i\oplus 1} \neq b_{i\oplus 1}$, this proves (22), so suppose equality. Similar to above, this means (case a)

$$a_{\oplus 1} \cdot \big(\operatorname{sgn} a_{i\oplus 2}, \operatorname{sgn} a_i\big) \geq (0,0) \qquad \text{and} \qquad (b_{i\oplus 2} \Leftrightarrow a_{i\oplus 2}) \cdot \big(\operatorname{sgn} a_{i\oplus 2}, \operatorname{sgn} a_i\big) \leq (0,0),$$

the latter from prerequisite for $i \oplus 2$. Therefore, $s_{i\oplus 1} \cdot (b_{i\oplus 2} \Leftrightarrow a_{i\oplus 2}) \geq (0,0)$. Equation (24) finally requires $s_i = s_{i\oplus 1}$ for both $a_i = 0$ and $a_i \neq 0$. $\square$

References

[1] Pankaj K. Agarwal and Jeff Erickson: “Geometric Range Searching and Its Relatives”, to appear in *Advances in Discrete and Computational Geometry*, (B. Chazelle, J. E. Goodman, and R. Pollack, editors).

[2] Fabrizio d’Amore, Paolo G. Franciosa, Giuseppe Liotta: “A robust region approach to the computation of geometric graphs”, *Proceedings of the Sixth Annual European Symposium on Algorithms* ESA’98 in Venice, Lecture Notes in Computer Science, Vol. **1461**, Springer Verlag, 175-186.

[3] Sunil Arya, Gautam Das, David M. Mount, Jeffrey S. Salowe, Michiel H. Smid: “Euclidean Spanners – Short, Thin, and Lanky”, *Proceedings of the Twenty-Seventh Annual ACM Symposium on the Theory of Computing* 1995, 489-498.

[4] Russel V. Benson: “Euclidean Geometry and Convexity”, McGraw-Hill 1966.

[5] Bernard Chazelle, Leonidas J. Guibas: “Fractional cascading: I. A data structuring technique”, *Algorithmica* **1**, 1986, 133-162.

[6] Danny Z. Chen, Gautam Das, Michiel Smid: “Lower Bounds for Computing Geometric Spanners and Approximate Shortest Paths”, *Proc. 8th Canad. Conf. Comput. Geom.* 1996, 155-160.

[7] Ken L. Clarkson: “Approximation algorithms for shortest path motion planning”, *Proceedings of the 19th ACM SToC*, 1987, 56-65.

[8] Gautam Das, Paul J. Heffernan: “Constructing Degree-3 Spanners with Other Sparseness Properties”, *Proc. 4th Annual Intern. Symp. on Algorithms*, Lecture Notes in Computer Science, Vol. **762**, Springer-Verlag Berlin 1993, 11-20.

[9] D. P. Dobkin, S. J. Friedman, K. J. Supowit: “Delaunay graphs are almost as good as complete graphs”, *Discrete and Computational Geometry* **5**, 1990, 399-408.

[10] David Eppstein: “Spanning Trees and Spanners”, Technical Report 96-16 `http://www.ics.uci.edu/~eppstein/`, Dept. Information and Computer Science, University of California.

[11] J. Mark Keil, Carl A. Gutwin: “Classes of graphs which approximate the complete euclidean graph”, *Discrete and Computational Geometry* **7**, 1992, 13-28.

[12] Matthias Fischer, Tamás Lukovszki, Martin Ziegler: “Geometric Searching in Walk-Through Animations with Weak Spanners in Real Time”, *Proceedings of the Sixth Annual European Symposium on Algorithms* ESA’98 in Venice, Lecture Notes in Computer Science, Vol. **1461**, Springer Verlag, 163-174.

[13] Matthias Fischer, Tamás Lukovszki, Martin Ziegler: “A Network Based Approach for Realtime Walkthrough of Massive Models”, *Electronic Proceedings of the 2nd Workshop on Algorithmic Engineering* WAE98, `http://www.mpi-sb.mpg.de/~wae98/PROCEEDINGS/`

[14] Matthias Fischer, Friedhelm Meyer auf der Heide, Willy B. Strothmann: "Dynamic data structures for realtime management of large geometric scenes", *Proceedings of the Fifth Annual European Symposium on Algorithms*, 1997, 157-170.

[15] B. Fuchssteiner, W. Lusky: "Convex Cones", North-Holland 1981.

[16] R. H. Hardin, Neil J. A. Sloane and Warren D. Smith:
"A library of putatively optimal coverings of the sphere with *n* equal caps", `http://www.research.att.com/~njas/coverings/`

[17] J.W. Jaromczyk and G.T. Toussaint: "Relative Neighborhood Graphs and Their Relatives", *Proceedings of IEEE* **80**, 1992, 1502-1517.

[18] Jiří Matoušek: "Geometric Range Searching", *ACM Computing Surveys* **26(4)**, 1994, 422-461.

[19] Kurt Mehlhorn, S. Näher: "Dynamic Fractional Cascading", *Algorithmica* **5**, 1990, 215-241.

[20] Franco P. Preparata and Michael Ian Shamos: *"Computational Geometry: An Introduction"*, Springer Verlag New York 1985.

[21] Satish B. Rao and Warren D. Smith: "Approximating Geometrical Graphs via 'spanners' and 'banyans'", *Proceedings of the 30th ACM SToC*, 1998, 540-550.

[22] C.A. Rogers: "Covering a Sphere with Spheres", *Mathematika* **10**, 1963, 157-164.

[23] Jim Ruppert, Raimund Seidel: "Approximating the *D*-dimensional complete euclidean graph", *Canadian Conference on Computational Geometry* 1991, 207-210.

[24] Jeffrey S. Salowe,: "On Euclidean Spanner Graphs with Small Degree", *ACM Symposium on Computational Geometry* 1992, 186-191.

[25] Michiel Smid: "Closest Point Problems in Computational Geometry", *Handbook of Computational Geometry* (Jörg-R. Sack and Jorge Urrutia, editors), Elsevier Science, Amsterdam, to appear.

[26] Edward Szpilrajn-Marczewski: "Sur l'extension de l'ordre partiel", *Fundamenta Mathematicae* **16**, 1930, 386-389.

[27] Kasturi R. Varadarajan: "A divide and conquer algorithm for min-cost perfect matching in the plane", *Proceedings of the 39th FoCS*, 1998, 320-329.

[28] Andrew Chi-Chih Yao: "On Constructing Minimum Spanning Trees in *k*-dimensional Spaces and Related Problems", *SIAM J. Comput.* **11(4)**, 1982, 721-736.