A computational approach to stability problems

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A polytope in $\mathbf{R}^\mathbf{d}$ is said to be *stable* on a facet if and only if the perpendicular to that facet through the *center of gravity* meets the facet itself. One may consider the stability of a body with respect to an arbitrary "center of gravity on the interior of the body [1,3,6], or one may take the center of gravity to be determined by the shape of the body itself. Most typically, the center of gravity is taken to be that point which would be the physical center of gravity were the body composed of a material of uniform density[1,2,4,5,6,7]; however, other possibilities, such as the center of mass determined by a uniform distribution on the n-skeleton of the body for $1 \le n \le d-1$ may also be considered [5].

This type of problem seems to have been first introduced in two papers [1,7] that appeared in *SIAM Review* in the late 1960's. In the second of these, the question of existence of polytopes stable only on one facet was raised. A 19-faceted polyhedron, stable only on one facet. was exhibited; and the construction was modified to yield a 23-faceted polytope stable on only its smallest facet. It was shown that no tetrahedron of uniform density could be monostatic; and it was asked whether any simplex of uniform density could have this property.

This was answered in the affirmative in 1985 by the first author of the present paper, who constructed a 10-dimensional simplex with this property [2]. In the same paper, it was shown that no simplex in $\mathbf{R}^\mathbf{d}$ could be monostatic for $d<7$. To obtain these results, a theorem of Minkowski [8] was used, that characterized the sets of vectors that could be the oriented facet area vectors of a polyhedron.

In particular, Minkowski's theorem showed that any $d+1$ vectors which sum to 0 and span $\mathbf{R}^\mathbf{d}$ with positive coefficients are the facet vectors of a simplex. It may be shown [2] that such a simplex, if of uniform density, will fall from facet i to facet j if and only if the projection of the j^{th} facet vector onto the i^{th} facet vector is longer than the i^{th} facet vector itself. This "projection criterion" turns out to be valuable in rendering simplex stability problems tractable. In [2], the negative result was obtained analytically, using the projection criterion, while the 10-dimensional example was obtained heuristically and verified using that technique.

To obtain the negative result, bounds were found analytically on the projection of a vector $\mathbf{x_n}$ onto an axis, subject to the existence of a chain of vectors $\mathbf{x_n}, \mathbf{x_{n-1}}, \ldots, \mathbf{x_1}, \mathbf{x_0}$ such that $\mathbf{x_i}$ and $\mathbf{x_{i-1}}$ satisfy the projection criterion and $\mathbf{x_0}$ is a unit vector on the specified axis. These bounds were added, and used to show that vectors satisfying the projection criterion could not sum to 0. (It should be noted that there are many possible *falling patterns* in which the facet vectors may satisfy the projection criterion; these can be thought of as rooted directed trees, whose vertices are vectors, with an edge from $\mathbf{x_i}$ to $\mathbf{x_j}$ if the projection criterion is satisfied. It is necessary to eliminate all of these, not merely that in which the simplex falls sequentially across all its facets.)

In a later paper [5], Mak and the present authors extended these results further. On the lower bound side, still using an analytic approach, they showed that any monostatic simplex must have at least 8 dimensions. Here, the sums of pairs of projections were bounded, rather than individual projections. As the location of (say) $\mathbf{x_1}$ that optimizes its projection is not, typically, the location of $\mathbf{x_1}$ which optimizes the projection of $\mathbf{x_n}$, the bound on the projection of $\mathbf{x_1}+\mathbf{x_n}$ is stricter than the sum of the two independent bounds; and the resulting bounds were tight enough to show that a set of 8 vectors obeying the projection criteria in $\mathbf{R}^7$ could not sum to 0.

In higher dimensions, the genetic algorithm was used to search - unsuccessfully - for an example in nine dimensions. However, the GA search did uncover an example, in 11 dimensions, of a simplex that was not only monostatic, but rolled sequentially across all 12 facets from an appropriate starting point.

In [6], and later in [4], the existence of a monostatic simplex in $\mathbf{R}^8$ was ruled out. To do this, sums of projections of entire chains of vectors were optimized. This could not be done analytically, so numerical optimization was carried out.

This was done recursively. For any angle $\theta \in [0,\pi]$, and for $n = 1,2,\ldots 9$, the sum $p(\mathbf{x_n}) + \ldots + p(\mathbf{x_0})$ was optimized, subject to the conditions that:

(i) $\mathbf{x_n}$ is a unit vector at an angle θ to the axis onto which p projects;
(ii) $\mathbf{x_i}$ and $\mathbf{x_{i-1}}$ satisfy the projection criterion.

The bounds $f_n(\theta)$ for each successive n were tabulated at intervals of 0.001 radian, and used to compute the optimal values for n+1.

Bounds were calculated analytically on $df_n/d\theta$ and $d^2f_n/d\theta^2$, and these were used to bound the error in the numerical calculations. Thus, it was possible to prove that the values obtained were accurate to within 10^{-5}. Safe bounds were obtained that permitted it to be shown that no set of 9 vectors in $\mathbf{R}^8$, obeying the projection criterion in any falling pattern, can sum to 0. The same methods were used to show that no simplex in $\mathbf{R}^{10}$ can roll sequentially over all its facets.

References

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