Computing the Angularity Tolerance*

(extended abstract)

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1 Introduction

Manufactured objects are always approximations to some ideal object: parts that are supposed to be flat will not be perfectly flat, round parts will not be perfectly round, and so on. In many situations, however, it is important that the manufactured object is very close to the ideal object. In such cases the specification of an object includes a description of how far the manufactured object is allowed to deviate from the ideal one. The field of dimensional tolerancing [2] provides the language for this. Given a specification, one must test whether the manufactured object meets it, which is the area of study called computational metrology. The objects are often tested as follows. Suppose for simplicity that we want to manufacture a flat surface. First, a so-called Coordinate Measuring System (CMS) 'measures' the manufactured surface. The output of the CMS is a set of points in 3-dimensional space that are on the manufactured surface. The second step is to compute two parallel planes at minimum distance to each other that have all the measured points in between them. In other words, one wants to compute the width of the point set. The surface meets the requirement if the width is below the specified threshold. Computing the width of a point set can be done in $O(n \log n)$ time in the plane [3] and in $O(n^{3/2+\varepsilon})$ expected time in 3-dimensional space [1].

We study another problem from computational metrology, which arises when one wants to manufacture an object with two flat surfaces that make a specified angle with each other. Testing whether the manufactured object meets the specifications leads to the *angularity problem*: given a set of points, compute a thinnest wedge whose legs make a given angle with each other and that contains all the points. We show that in the plane this problem can be solved in $O(n^2 \log n)$ time. In 3-dimensional space we study a simpler variant, where all the measured points come from one of the two surfaces; the other surface, the so-called *datum plane*, is assumed to be in a known orientation. The problem is now to find the thinnest 'sandwich' (that is, two parallel planes) that makes a given angle with the datum plane and contains all the points. In other words, we want to compute the width under the restriction that the planes make a given angle with the datum plane. We solve this problem in $O(n \log n)$ time. Both in the planar case and in the 3-dimensional case we also study variants where the points have uncertainty regions associated with them.

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2 Point sets in two dimensions

We start by studying the angularity problem in the plane. In the simplest version we are given a datum line, a set of n points (which are on one side of the line), and an angle θ. The problem then is to compute the thinnest strip (or, *sandwich*) that contains all the points and makes an angle θ with the datum line. In the plane this simple version is not so interesting; it can easily be solved in linear time by computing the extrema of the point set in the direction perpendicular to θ. Therefore we concentrate on the case where the datum plane is not given. In this setting we are only given a set S of n points and an angle θ, and we want to compute the thinnest θ-wedge that contains all the points, where a θ-wedge is defined as follows:

Definition 1 *A θ-wedge of width δ is the closed area bounded by four directed half lines b_1, b_2, l_1*

and l_2 such that

- *b_1 is parallel to and to the right of b_2 and l_1 is parallel to and to left of l_2*
- *b_1 and l_1, as well as b_2 and l_2 have a common starting point*
- *the angle measured in counter clockwise direction between b_1 and l_1 is equal to θ*
- *the distance between b_1 and b_2 and between l_1 and l_2 is δ*

Figure 1 shows a wedge containing all points shown. The minimum δ such that there is a θ-wedge of width δ containing S is called the *tolerance* of S (with respect to θ-wedges).

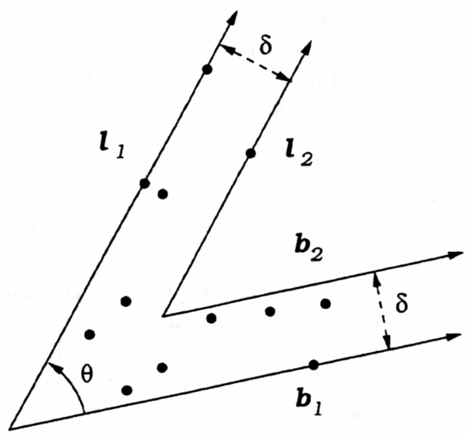

Figure 1: A θ-wedge of width δ.

Toussaint and Ramaswami [6] have solved a simpler variant of the problem, where it is known in which of the two 'legs' of the wedge each input point lies: one is given an angle θ and two sets of points, each of which has to be enclosed in a strip such that the angle between the two strips is θ.

Let $W(\phi)$ be a θ-wedge of minimal width, such that the bisector of b_1 and l_1 has direction ϕ and the wedge contains S. If there is no point of S on b_1 we can move $W(\phi)$ so that at least one point of S is on b_1, while S remains contained in $W(\phi)$. So without loss of generality we may assume that there is at least one point of S on b_1 and, similarly, at least one on l_1. It is now easy to see that $W(\phi)$ is unique for each value of ϕ. Let $A(\phi)$ be the apex of $W(\phi)$. Let OC, the outer curve, be the collection of all points $A(\phi)$ for $0 \leq \phi \leq 2\pi$. Let $B(\phi)$ be the common starting point of b_2 and l_2 of wedge $W(\phi)$, and define the inner curve IC as the collection of all points $B(\phi)$. Our algorithm to compute the thinnest θ-wedge containing S starts by computing the curves OC and IC. The next two lemmas state how these curves look, and how much time we need to compute them. The proof of these lemmas is omitted in this extended abstract.

Lemma 1 *The collection OC is a closed curve of piece-wise circular arcs, has a linear combinatorial complexity, and can be computed in $O(n \log n)$ time.*

Lemma 2 *The collection IC is a closed curve of piece-wise circular arcs, has a combinatorial complexity of $O(n^2)$, and can be computed in $O(n^2 \log n)$ time.*

After computing OC and IC it is easy to compute the thinnest wedge: We split the range $[0 : 2\pi]$ of possible orientations of the wedge into subranges where both the outer curve and the inner curve are attained by a single (piece of a) circular arc. The previous two lemmas imply that there are $O(n^2)$ subranges. (In fact, the subranges correspond exactly to the arcs of IC.) For each subrange we can then compute the thinnest wedge in constant time. This leads to the following result.

Theorem 1 *Given an angle θ and a set of n points S, the thinnest θ-wedge containing S can be found in $O(n^2 \log n)$ time.*

The running time of our algorithm is dominated by the time to compute IC. One might hope to improve this by showing a better bound on the complexity of IC. The next theorem shows that this is not possible. The example proving the theorem is omitted in this extended abstract.

Theorem 2 *The worst-case complexity of IC is $\Theta(n^2)$.*

3 Uncertainty regions in two dimensions

In computational metrology the sample points normally are not exact but come with some uncertainty: rather than a set of points, we get a set U of uncertainty regions $\{u_1, ..., u_n\}$. For each region u_i there is a point $p_i \in u_i$ that lies on the surface of the manufactured object, but due to the inaccuracy in the measuring process the point p_i is not known. In this case one would like to compute upper and lower bounds on the tolerance.

To illustrate the definitions, we first look at the simple version of the problem, where we are given a datum line, a set U of uncertainty regions, and an angle θ. Define a θ-sandwich to be a sandwich (that is, a strip) that makes an angle θ with the datum line. For a set S of points, define $\delta(\theta, S)$ to be the tolerance of S, that is, $\delta(\theta, S)$ is the width of the thinnest θ-sandwich (strip) that contains S. An upper bound on the tolerance of any set $S = \{p_1, \ldots, p_n\}$ of points within the uncertainty regions of $U = \{u_1, \ldots, u_n\}$ is given by the quantity

$$\max\{\delta(\theta, S) : p_i \in u_i \text{ for } 1 \leq i \leq n\}. \quad (1)$$

Unfortunately this quantity is hard to compute. Therefore we compute a more conservative upper bound, $\delta_{\max}(\theta, U)$, defined as follows:

$$\delta_{\max}(\theta, U) = \text{minimum width of any } \theta\text{-sandwich containing all uncertainty regions in } U.$$

The value $\delta_{\max}(\theta, U)$ is called the maximum tolerance of U. Notice that $\delta_{\max}(\theta, U)$ is also the width of the thinnest θ-sandwich that is guaranteed to contain all points of any set $S = \{p_1, \ldots, p_n\}$ with $p_i \in u_i$. At first glance it might seem that $\delta_{\max}(\theta, U)$ is the same as the upper bound given by equation (1), but this is not true.

Theorem 3 *For some sets U, the value $\delta_{\max}(\theta, U)$ is greater than the upper bound on the tolerance as given by equation (1).*

Proof: We get a trivial example by taking a set U consisting of only one region, say the unit circle. In this case we have $\delta_{\max}(\theta, U) = 2$ and $\max \delta(\theta, S) = 0$. But also for a larger number of uncertainty regions $\delta_{\max}(\theta, U)$ can be greater than $\max \delta(\theta, S)$.

Consider the example with two regions shown in Figure 2. In the example $\theta = \pi/3$ and U consists of two unit circles. In this example $\delta_{\max}(\theta, U) = 2 + \sqrt{3}$. However, $\delta(\theta, S) \leq 2\sqrt{3}$ for all choices of S. (If we place the points of S where the thinnest sandwich for U touches the two circles, then we can obtain a thinner θ-sandwich by 'flipping' the sandwich, so that the angle with the datum line (here the x-axis) is no longer given by the angle with the positive x-axis but with the negative x-axis. In the planar case one could argue that the angle of the flipped sandwich is not θ but $\pi - \theta$. In 3-dimensional space, however, a similar example applies, and there it is natural to allow 'rotating' the sandwich while keeping the angle with the datum plane fixed.) ∎

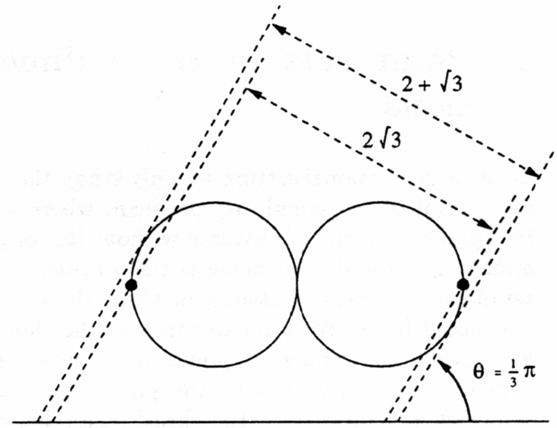

Figure 2: Two regions of uncertainty

When we define a lower bound on the tolerance, then we do not get these problems; we define

$$\delta_{\min}(\theta, U) = \min\{\delta(\theta, S) : p_i \in u_i \text{ for } 1 \leq i \leq n\}.$$

The value $\delta_{\min}(\theta, U)$ is called the minimum tolerance of U. The minimum tolerance is the same as the width of the thinnest sandwich which contains at least one point from each uncertainty region u_i. In order to compute sandwiches containing uncertainty regions, we make the assumption that given a direction ϕ and an uncertainty region u_i, we can compute the two tangents of u_i with direction ϕ in constant time. Computing the maximum and minimum tolerance of a given set of uncertainty regions, when we are given a datum line and an angle θ is trivial to do in linear time.

Now consider the case where no datum line is given. The definitions of maximum and minimum tolerance readily carry over.

In order to compute the thinnest wedge containing a set of uncertainty regions, we can proceed as in the previous section. The combinatorial complexity of the curves OC and IC depends on the shape of the uncertainty regions. For example if all regions are equal size circles, the curve IC does not consist of circular arcs, but it has at most a quadratic complexity.

Theorem 4 *Given a set of n uncertainty regions U consisting of equal size circles and an angle θ, the value $\delta_{\max}(\theta, U)$ can be found in $O(n^2 \log n)$ time.*

It is an open problem to determine which other shapes of uncertainty regions permit an $O(n^2 \log n)$ algorithm.

4 Point sets in three dimensions

In the 3-dimensional setting we only study the simple variant of the angularity problem, where a datum plane is given. We assume without loss of generality that the datum plane is the xy-plane. The set of points, which we assume lie above the x-plane, is denoted by S. We want to compute the thinnest θ-sandwich containing all points in S, where θ is a given angle. (Similar to the planar case, a θ-sandwich is defined to be the closed region between two parallel planes that make an angle θ with the datum plane.) The width of the sandwich is the distance between the two planes. We denote the plane bounding the sandwich from above by h_1 and the plane bounding it from below by h_2.

To find the pair h_1 and h_2 bounding the thinnest sandwich, we transform the problem into a 2-dimensional problem as follows. (We could also work directly in 3-space, but we feel that the transformation makes the algorithm easier to understand, especially in the case of uncertainty regions, which is studied later.) For a point $p_i \in S$ let C_i be the cone pointing upwards with apex p_i (thus, p_i is the highest point of the cone) and apex angle $\pi - 2\theta$, that is, the sides make an angle of θ with the xy-plane. Now p_i lies below or on h_1 if and only if C_i lies below or on h_1. Similarly, p_i lies above h_2 if and only if C_i does not lie completely below or on h_2. Each cone C_i intersects the xy-plane in a circle c_i.

Let h_1 and h_2 be the two planes that form a θ-sandwich containing S. The plane h_1 intersects the xy-plane in a line l_1, and h_2 intersects it in a line l_2 parallel to l_1. We direct the lines l_1 and l_2 so that l_2 is to the left of l_1. The cone C_i lies below h_1 if and only if the circle c_i lies to the left of l_1. Similarly C_i lies below h_2 if and only if circle c_i is to the left of l_2. This leads to the following 2-dimensional reformulation of the the problem: Given a set of circles $\{c_1, ..., c_n\}$, determine two parallel directed lines l_1 and l_2 such that

i) All circles lie to the left of or on l_1.

ii) No circle completely lies to the left of l_2.

iii) Among all pairs of lines that satisfy i) and ii) the distance between l_1 and l_2 is minimal.

It is easy to verify that this is indeed the same problem. Figure 3 shows an example of a solution for such a 2-dimensional problem. We can now solve

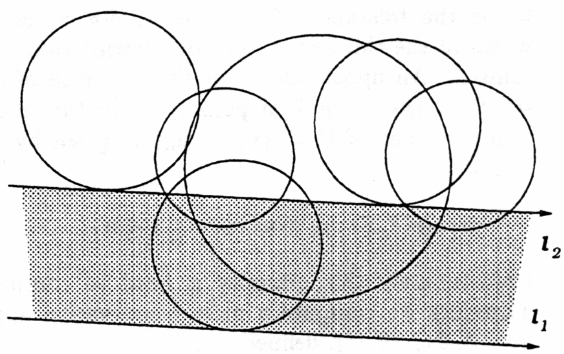

Figure 3: Solution to the planar problem.

the problem as follows. Consider a valid pair of lines of minimum width, where we fix the slope of the lines to be, say, zero. For each region, take the lowest point on its boundary. Now the lines l_1 and l_2 go through the lowest and the highest point, respectively, of all such lowest points. When we start increasing the slope of the lines from zero to 2π, then the extreme points defining l_1 and l_2 move along the boundaries of the regions on which they lie. At some point they will switch from one region to another one. Which two regions define the two extreme points for a given slope ϕ can be determined by computing suitably defined lower and upper envelopes. The thinnest sandwich is then determined by the minimum distance between these two envelopes. Details are given in the full paper. This leads to the following result.

Theorem 5 *Given a set of points S in 3-space and an angle θ, the minimum width θ-sandwich that contains S can be found in $O(n \log n)$ time.*

5 Uncertainty regions in three dimensions

We define the maximum and minimum tolerance of a set U of uncertainty regions with respect to

θ-sandwiches similar to the planar case—see Section 3. We first show how the thinnest θ-sandwich can be computed that contains a set of uncertainty regions U. As before we can transform the 3-dimensional problem into a 2-dimensional problem. Let P be a plane that has an angle θ with the xy–plane (which is again assumed to be the datum plane) and is tangent to region u_i, such that u_i is below P. We define the generalised cone C_i as the intersection of the half spaces below all such planes P. The intersection of C_i and the xy–plane is the region c_i. For example, if u_i is a sphere with positive z-coordinates, then C_i is an upwards pointing cone and c_i is a circle, as shown in Figure 4. Similarly, let Q be a plane which has an angle θ

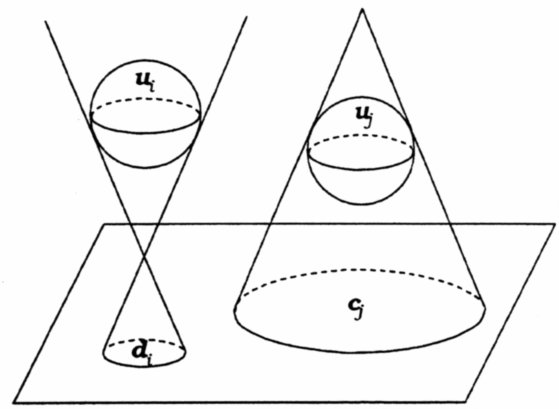

Figure 4: Turning 3-dimensional regions into 2-dimensional convex objects

with the xy–plane and is tangent to u_i, such that u_i is above Q. The generalised cone D_i is the intersection of the half spaces below all such planes Q. The intersection of D_i and the xy–plane is the region d_i. So for all i we have $d_i \subset c_i$. Without loss of generality we can assume that all regions u_i have sufficiently large z-coordinates so that none of the regions d_i is empty.

Notice that an uncertainty region c_i or d_i in the xy-plane is convex and has exactly one tangent through each point on its boundary. We assume that the uncertainty regions c_i and d_i satisfy the following properties.

- Given a direction ϕ in the xy-plane and a region we can compute the two tangents of the region with direction ϕ in constant time.
- Given two non-intersecting regions, their common inner and outer tangents can be computed in constant time.
- The boundaries of two intersecting regions intersect each other at most k times where k is a constant and their common outer tangents can be computed in $O(1)$ time.
- For any connected part c of the boundary of a region c_i and any connected part d of the boundary of a region d_j we can find in constant time two parallel tangents through a point of c and d respectively which have minimal distance to each other.

Any two parallel planes that have an angle θ with the xy-plane and that are guaranteed to contain all points from U correspond to two directed parallel lines m_1 and m_2, such that the following holds:

i) All regions c_i lie to the left of or on m_1,

ii) No region d_i is completely to the left of m_2.

We can now find the thinnest θ-sandwich that contains all regions in U, that is, the maximum tolerance of U, with an algorithm that is similar to the algorithm we used for points.

A similar approach can be used to compute the minimum tolerance of U. We get the following result.

Theorem 6 *Given a set of n uncertainty regions U and an angle θ, the maximum and the minimum tolerance of U with respect to θ-sandwiches can be found in $O(\lambda_k(n) \log n)$ time.*

6 Conclusions

We solved the sandwich problems in two and three dimensions, for point sets as well as uncertainty regions. The problem of finding a thinnest wedge is only solved for the 2-dimensional case. Solving the wedge problems for uncertainty regions and for point sets in 3-space dimensions remains open.

7 Acknowledgements

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