On a problem of immobilizing polygons.

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Abstract: For a given polygon P and a set I of points (pins) in the plane, we say that I immobilizes P if any rigid motion of P in the plane forces at least one pin of I to penetrate the interior of P. We give an O(n) algorithm checking whether a given set of n pins, different from convex vertices of P, immobilizes it. We also prove that from any set of immobilizing pins we can choose four of them which will also immobilize P.

1.Introduction.

Let P be a polygon and I a set of n points (pins) in the plane. We say that I immobilizes P if any rigid motion of P in the plane forces at least one of the pins of I to penetrate the interior of P. As for small motion only pins from the boundary of P may penetrate it, we will suppose that all pins actually belong to the boundary of the polygon. A rigid motion of polygon P in the plane is a mapping M (different from identity) from the set t×P (t represents time) to the plane, continuous with respect to its first coordinate, such that for every pair of points u, v∈P the distance between their images remains constant for all t and M(0, u)=u for every element of P. A set of points I immobilizes the shape P if the only motion of P which does not allow the penetration of some element of I to the interior of P is the identity M(t, u)=u for all t and u.

Problems of immobilization were introduced by W. Kuperberg [K] and some of them were later reported in [O]. These problems generally concerned two classes of figures: shapes and polygons. We call a shape any figure bounded by a Jordan curve which is different from a circle. The circle is obviously excluded from consideration since any number of pins on its boundary leave it free to rotate. For polygons, some interesting algorithmic questions arise.

- Are four pins sufficient to immobilize any shape?
- Do three pins suffice for all smooth convex shapes?
- Find all classes of convex shapes for which three pins are not sufficient.
- Design an algorithm finding a set of immobilizing pins for a given polygon.
- Design an algorithm deciding whether a set of n given pins immobilizes a given polygon.

Some of this questions may be partially answered using known results from grasping. For piecewise smooth shape P, Mishra, Schwartz and Sharir [MSS] as well as Markenscoff, Ni and Papadimitriou [MNP] studied the problem of closure grasp, i.e. the ability to respond to any external force or torque by applying appropriate forces at the grasp points. They proved that there exists the force-torque closure grasp of the shape P, using a minimal set S of four finger points. The set S may be found in O(k) time, for the shape S being a polygon of k vertices, see [MSS]. It follows that any rigid velocity of P causes at least one of point of S to have an instantaneous velocity strictly directed towards the interior of P (see [MS]). As a consequence, the set S of four pins would immobilize P. However, such set S of four grasping fingers is not the smallest possible for the immobilizing purpose. Czyzowicz, Stojmenovic and Urrutia [CSU2] proved that for most polygons, including all polygons without parallel sides, three immobilizing pins may be always found. They propose an O(n) algorithm finding these three pins. They also have exact combinatorial bounds for higher dimensions. For the case of smooth and convex shapes, using tools of differential geometry, Montejano and Urrutia [MU] proved that three immobilizing pins are always sufficient. Finally Czyzowicz, Stojmenovic and Urrutia [CSU1] settled the question for general shapes, proving that four immobilizing pins are always sufficient.

In this paper we propose an algorithm determining whether a given set of n pins immobilizes the polygon P. If the positions of the pins are different from convex vertices of P we have an O(n) algorithm to check whether I immobilizes P. If I does not immobilize P this algorithm may also output a possible motion of P so that no pin of I penetrates the polygon. When k pins are situated at the convex vertices of P, this algorithm must be adjusted correspondingly and, it may be proved, that its complexity increases to $O(n^2)$, $O(n \cdot 2^k)$ or $O((n+k^2) \log n)$. If the set I immobilizes P we prove that we can choose four pins of I which will also immobilize P.

The idea of the main algorithm is the following. Any position of the polygon P may be expressed as a composition of a translation and a rotation, applied to the original position of P. This new position of the polygon may be expressed as the point in the three-dimensional space. A motion of P is then represented as a curve in this space (called motion space). For each pin p_i, we compute the region (called feasible region) of the motion space representing the positions of P when p_i does not penetrate the interior of P. If the intersection of such regions computed for all the pins of I is not degenerated to a single point (i.e. the original position of P) then there exists a motion of P. In most cases, it is not necessary to compute the intersection of all feasible regions. The algorithm will determine first, whether this intersection is a zero-, one-, two- or three-dimensional set. Then, depending on the case, the algorithm will act accordingly.

1. Preliminaries.

Theorem 1.1. (Meggido [Me83]): Two dimensional linear programming can be solved in linear time with respect to the number of constrains.

Meggido's theorem implies the following:

Corollary 1.1. Given a set H of n halfplanes in the plane we can check in O(n) time whether their intersection is:

a) empty.
b) a point.
c) one-dimensional (segment, semiline or line).
d) non-degenerated two dimensional convex region.

Theorem 1.2. ([CSU]). Let be given a polygon P and three points X, Y and Z belonging to the interior of three edges x, y and z of P, respectively. X, Y and Z immobilize P, if and only if two following conditions are verified:

1) perpendiculars to x, y and z at X, Y and Z respectively, meet at a common point,
2) three halfplanes bounded by x, y and z, containing the interior of P in the neighborhood of X, Y and Z respectively, have a nonempty, finite intersection.

From the proof of this theorem it followed

Corollary 1.2. If the three perpendiculars do not meet at a common point and no two of them are parallel then there exists a rotation of P, such that no pin penetrates the interior of P.

Theorem 1.3. (Helly [H]) There exists a point which intersects a finite family of convex sets in E^d if and only if the intersection of every d+1 sets is non-empty.

2. Releasing motions and feasible regions.

In order to decide whether a set of pins immobilize a given polygon we have to check if there exists a *releasing motion*, i.e. a motion with no pin penetrating the interior of the polygon. In this section we will observe some facts concerning these motions, essential to find an efficient algorithm.

For polygon P and a set I of pins on its boundary we choose as $\varepsilon(P,I)$ a value smaller than a distance between any two pins of I and smaller than a distance between any pin and any side of P not meeting this pin. As P is simple, we can always choose $\varepsilon(P,I)>0$. A motion of P will be called *small* if the distance between any point of P and its image after the motion is smaller that $\varepsilon(P,I)$. In the sequel a motion refers to a small motion. Indeed, as motion any of P is a continuous function, the existence of a motion for given I implies the existence of a small motion. For small motions, a pin may penetrate the interior of polygon P only through the side of P containing it; penetration of the interior of P by a pin is then equivalent to the penetration of the halfplane determined by the side of P containing the pin. The complement of such halfplane of penetration will be called *halfplane of release*.

The definition of motion M in the introduction implies, that for every t, the mapping $M(t,.)$ is an even isometry of the plane. As every even isometry of the plane is a composition of a rotation around any point O and some translation, we may consider the following alternative definition of motion.

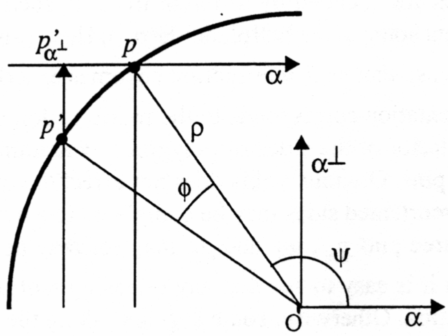
Choose a point O in the plane. Motion in the plane is a mappingF(t)= $(\Phi(t), X(t), Y(t))$; $t\in[0..1]$, where $F(0)=(0,0,0)$, and $\Phi(t)$, X(t) and Y(t)) are continuous functions. F(t') represents here the position P' of the polygon at the time moment t'. The position P' is obtained, by applying to the original position P_o, a rotation around O by an angle $\Phi(t')$, $0\leq\Phi(t')\leq\pi$, followed by the translation by a vector (X(t'), Y(t')). F(t) will be called a *motion curve* in the three-dimensional *motion space M*. Obviously, the mapping M(t,.) is continuous if and only if $\Phi(t)$, X(t) and Y(t)) are all continuous.

We start by few lemmas. Their are either easy, or they are omitted in this abstract.

Lemma 2.1. Let be given n points on the boundary of a given polygon P but not at the convex vertices of P. In O(n) time we can decide if there exists a releasing translation of P.

Lemma 2.2. Suppose that there exists a releasing (small) motion of the polygon P moving it to the new position P'. If P' is the image of some translation of P then this translation is also a releasing motion.

For any particular pin on the boundary of a given polygon consider the motions of this polygon to the new positions with the pin not penetrating its interior. All such positions form a region in the motion space M which we will call the *feasible region*. Any releasing motion for a particular pin must be represented by a motion curve belonging entirely to this region. To find the feasible region for a pin p suppose that the polygon is given as a sequence of sides oriented clockwise, and p belongs to the interior of a side of the polygon. Let α be the vector of this side and $\alpha^{\perp}$ vector orthogonal to α, oriented towards the exterior of the polygon. Let ψ denote an angle,



$$\mathrm{dist}(p'_{\alpha^\perp}, p') = \rho(\sin\psi - \sin(\phi+\psi))$$

Figure 1.

oriented counterclockwise, between α and the vector (O,p) and $\rho = |(O,p)|$. Rotate the pin p counterclockwise around the point O by an angle ϕ (see Fig. 1). This may be interpreted as the rotation of the polygon clockwise by ϕ with the pin remaining fixed. What must be the translation following that rotation to keep the pin p outside the interior of the polygon? The distance between p', the new position of the pin, and the side of the polygon which contained p is equal to $\rho|\sin\psi - \sin(\phi+\psi)|$. In the situation from Fig.1, p' is inside the polygon, so the coordinate of the vector of translation in the direction α is at least equal to $\rho(\sin\psi - \sin(\phi+\psi))$. Observe that this value is positive when p' is inside the polygon and negative when it is outside, independently on the position of O with respect to the pin p and the side containing it. The feasible region $\mathbf{R}(p)$ is now given by

$\mathbf{R}(p) = (\phi, (\rho(\sin\psi - \sin(\phi+\psi)) + t) * \alpha^\perp/|\alpha^\perp| + s * \alpha/|\alpha|), \ t \geq 0$

For any fixed ϕ this region is a halfplane bounded by the line parallel to α. Cutting $\mathbf{R}(p)$ with the plane orthogonal to α we get a sinusoid

$$\alpha^\perp(\phi) = \rho(\sin\psi - \sin(\phi+\psi))/|\alpha^\perp|.$$

Lemma 2.3. Let x be the point in one of the halfspaces $\phi > 0$ or $\phi < 0$ and not belonging to the plane tangent to $\mathbf{R}(p)$ at point O. If x lies on the same side of tangent plane as the interior of $\mathbf{R}(p)$ in the neighborhood of O, then some part of the segment (O,x) in the neighborhood of O lies entirely in the interior of $\mathbf{R}(p)$.

3. Main theorem.

If each of n pins of I lies on the boundary, but not at a vertex of P, a releasing motion exists if and only if the common intersection of n feasible regions contains some curve starting at O.

Lemma 3.1. The feasible region for the pin placed at a reflex (respectively convex) vertex is the intersection (respectively union) of two feasible regions: each one arising from the pin being placed on one of the segments adjacent to the vertex.

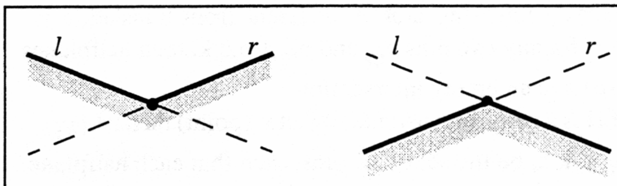


Figure 2a. Figure 2b.

This lemma will permit to extend the consideration from pins placed in the interior of polygon edges to pins which may be placed at reflex vertices as well. When checking immobilization by a set of pins, a pin at a reflex vertex is equivalent to two pins, one on each of the two adjacent sides.

The following two lemmas relate to the special cases of the placement of the set of pins.

Lemma 3.2. Let be given a collection of n pins placed on the parallel sides of the polygon P and belonging to a common orthogonal to these sides. There exists a releasing motion different from translation if and only if no two halfplanes of release have an empty intersection.

Lemma 3.3. Suppose that for a set of n pins, each belonging to the interior of a side of a polygon P, such that the lines orthogonal to these sides at the pins are concurrent. There exists a releasing motion different from translation if and only if n releasing halfplanes have a nonempty intersection.

According to an earlier observation, there exists a releasing motion different from translation, if and only if, in the neighborhood of the origin O for Φ 0, (or forΦ 0), the intersection of all feasible regions $\mathbf{R}(p_i)$, i=1,...,n, is nonempty. Take the case Φ 0 (by symmetry all the considerations will extend to the case Φ 0). As we do not really need to compute this intersection, we will estimate its existence by approximating the boundary of each feasible region $\mathbf{R}(p_i)$ by a plane tangent to it at O. Thus each feasible region is approximated in the neighborhood of O, by a halfspace bounded by this tangent plane. Cut these halfspaces by the plane $\Phi = \phi_0$, $\phi_0 > 0$, parallel to (X,Y). Let $H(p_i)$ denote the halfplane being the intersection of this cut $\Phi = \phi_0$ and the halfspace approximating $\mathbf{R}(p_i)$. Consider the intersection of all the halfplanes $H(p_i)$, i=1,...,n, resulting from this cut. The following theorem will be used to decide if there exists a releasing motion which is different from translation.

Theorem 3.1.

a) Let H denote the intersection of the halfplanes $H(p_i)$, i=1,...,n, defined as above. If H is a two-dimensional object (polygon) then there exists a releasing motion different from translation.

b) if H is a one-dimensional object (line, semiline or line segment) then let $\{p_{i1}, p_{i2}, ..., p_{ik}\}$, k 2, be the set of all pins, such that each halfplane $H(p_{ij})$, 1 j k, have H on its boundary.

There exists a releasing motion different from translation if and only if for any two pins p_{ip} and p_{iq},1 p,q k, their halfplanes of release have an empty intersection.

c) if H is a zero-dimensional object x (point) then let $\{p_{i1}, p_{i2},...,p_{ik}\}$, k 2, be the set of all pins, such that each halfplane $H(p_{ij})$, 1 j k, contains$x$ on its boundary. There exists a releasing motion different from translation if and only if the intersection of all the halfplanes of release for the pins $\{p_{i1}, p_{i2},...,p_{ik}\}$ is nonempty.

d) if H is empty then there exists no releasing motion different from translation.

Proof:

a) Take a point x in the interior of the polygon H. By lemma 2.3, in the neighborhood of O, the line (O,x) belongs to each feasible region. This line is then the releasing motion curve.

b) Take a point x in the interior of the line (semiline, line segment) H. From lemma 2.3 it follows that some cone with the vertex O and axis (O,x) belongs entirely to any region $\mathbf{R}(p)$, such that H does not lie on the boundary of $H(p)$. On the other hand if there exists a releasing motion then there exists a releasing motion inside this cone. As a consequence a releasing motion exists for the original set of pins, if and only if it exists for the set of pins $\{p_{i1}, p_{i2},...,p_{ik}\}$.

Observe that as the planes $H(p_{ij})$, j=1,...,k, are all the same they correspond to the pins placed at the parallel sides of P. We will prove that the pins $p_{i1}, p_{i2},...,p_{ik}$ belong to a common orthogonal to these sides. Observe that the plane $(\phi, \alpha^{\perp})$cuts all the feasible regions $\mathbf{R}(p_{i1}), \mathbf{R}(p_{i2}), . . .,\mathbf{R}(p_{ik})$ at the same angle. Thus the derivative in the direction $\alpha^{\perp}$, that is the value $\rho_{ij} \cos\psi_{ij}$ must be the same for all pins p_{ij}, j=1,...,k. The locus of such points is the line perpendicular to α (see Fig. 1). The claim follows now from lemma 3.2.

c) The same argument as in point b) proves that releasing motion exists if and only if it exists for the set of pins $\{p_{i1}, p_{i2},...,p_{ik}\}$ where each halfplane $H(p_{ij})$, j=1,...,k, contains p on its boundary.

We prove now that all the orthogonals at the points $p_{i1}, p_{i2},...,p_{ik}$ to the corresponding sides of the polygon P meet at a common point. If p lies on the ϕ axis then O belongs to each orthogonal; suppose then that p is outside the ϕ axis. It is sufficient to prove that every three orthogonals meet a common point. Take any three pins p_x , p_y and p_z.. If among the halfplanes $H(p_x)$, $H(p_y)$ and $H(p_z)$ there are two (or three) bounded by the same line, then, as in point b), the corresponding pins lie on the common orthogonal and the three orthogonals have a nonempty intersection. We may suppose then that the three halfplanes are bounded by different lines. In such a case we may reorient some of the halfplanes $H(p_x)$, $H(p_y)$ and $H(p_z)$, if necessary, so that their intersection becomes exactly point p. This reorientation corresponds to the reorientation of the interior and exterior of the sides of polygon P containing the corresponding pins. Obviously this does not affect the orthogonals and the reoriented sides may be completed to a new polygon P'. The three pins p_x , p_y and p_z. may (or may not) immobilize P', but it is easy to see that any releasing motion must not be a rotation. Otherwise, some segment along the ϕ axis starting at O would have to belong to all feasible regions $\mathbf{R}(p_x)$, $\mathbf{R}(p_y)$ and $\mathbf{R}(p_z)$. As p does not lie on the ϕ axis this cannot happen. Observe, that this is true for both halfspaces ϕ 0 and ϕ 0, that is for clockwise and counterclockwise rotations. However, by corollary 1.2, this is possible only when the three orthogonals at p_x, p_y and p_z to the corresponding sides of P' meet at a common point.

d) By Helly's theorem (1.3) there are three halfplanes with empty intersection. There are two cases:

Case 1. Two lines bounding these halfplanes, say $H(p_1)$ and $H(p_2)$, are parallel. The plane containing some line located between these two and parallel to them as well as point O will separate $\mathbf{R}(p_1)$ and $\mathbf{R}(p_2)$ for small values of $\phi>0$.

Case 2. Three halfplanes, say $H(p_1)$, $H(p_2)$ and $H(p_3)$, are bounded, respectively, by three lines l_1, l_2 and l_3 forming a triangle. Choose a point x in the interior of this triangle and three lines l_1', l_2' and l_3', such that l_i' is parallel to l_i and such that l_i' separates l_i from x, i=1,2,3. In some neighborhood of O the halfspace bounded by the plane containing l_i' and O, and not including x, contains $\mathbf{R}(p_i)$. As these three halfspaces have only O in common, the three feasible regions $\mathbf{R}(p_1)$, $\mathbf{R}(p_2)$ and $\mathbf{R}(p_3)$ have empty intersection in the neighborhood of O.

Theorem 3.2. In a set of pins immobilizing a polygon there exists a subset of four pins which also immobilize it.

Proof: omitted in the abstract. It's idea in general case is to apply Helly's theorem to the set of open halfspaces, each bounded by a halfplane tangent to a feasible region.

Theorem 3.3. The smallest set of pins immobilizing polygon P may be found in $O(n^3)$ time.

Idea of the proof: It is possible to check in $O(n^2)$ time if P admits two immobilizing pins, because one pin must be at a reflex vertex, and another one on a side or at another reflex vertex.

We use corollary 1.2 and lemma 3.3 to check whether three pins, each placed in the interior of some side of P, immobilize it. For every triple of sides of P, we check whether three halfplanes of release have a nonempty intersection. We can also check in constant time existence of a point in the plane whose perpendicular projections lie, respectively, on actual sides (rather that their extensions). Such projections would be the positions of immobilizing pins. More technical and longer analysis is needed to deal with pins which may be placed at the vertices of P. This step is completed within $O(n^3)$ time.

If three immobilizing pins are not found, four pins are found applying O(n) algorithm resulting from [MSS], [MNP] and [MS] (see introduction).

4. The Algorithm.

Lemma 2.2 permits to investigate the existence of the motion by considering only:

a) translations (motion curve being a segment in the plane $\Phi=0$),

b) motions whose curves lie entirely in the halfspace $\Phi\geq 0$ or in the halfspace $\Phi\leq 0$.

Indeed, suppose that $F(t)= (\Phi(t), X(t), Y(t))$ is a releasing motion. If the curve F(t) crosses the plane $\Phi=0$ in every neighborhood of the point O, then we can find a position *P'* of the polygon which is the image of a (small) translation of *P*. According to lemma 2.2 this translation is also a releasing motion. In other case, in some neighborhood of O the motion curve F(t) lies entirely in one of the halfspaces $\Phi>0$ or $\Phi<0$. The following algorithm will search for the releasing motion in the halfspace $\Phi>0$ (the same algorithm applies in the symmetric case).

Algorithm 4.1.:

Input: Set I of n pins on the oriented sides of the polygon *P*.

Output: Decision whether there exists a releasing motion of *P* in the halfspace $\Phi>0$, which is different from translation.

1. **for** each pin p_i on the side s_i **do** compute $\mathbf{R}(p_i)$;
2. **for** each $\mathbf{R}(p_i)$ **do**
   compute $H(p_i)$ the halfplane arising from intersection of the plane $\Phi=\phi_0$, ($\phi_0>0$), with the halfspace approximating $\mathbf{R}(p_i)$, bounded by the plane tangent to $\mathbf{R}(p_i)$ at the point O.
3. decide if the intersection H of $H(p_i)$, i=1,2,...,n, is a nondegenerated polygon, line, semiline, point or empty.

4.1. **if** H is a nondegenerated polygon
  then there exists a releasing motion

4.2. **else if** H is a segment, semiline or line **then**
  begin
   $S_1 := \{\}$; $S_2 := \{\}$;

4.2.1. **for** each p_i **do**
   if $H(p_i)$ is bounded by the line containing H **then**
    if $H(p_i)$ is oriented upwards
     then $S_1 := S_1 \cup \{p_i\}$
     else $S_2 := S_2 \cup \{p_i\}$;

4.2.2. compute R_1 - intersection of halfplanes of release of all pins in S_1 and R_2 - intersection of halfplanes of release of all pins in S_2;

4.2.3. **if** $R_1 \cap R_2$ is non-empty
   then there exists a releasing motion
   else all motions for $\Phi>0$ cause penetration
  end

4.3. **else if** H is a point **then**
  begin
   $S := \{\}$;

4.3.1. **for** each p_i **do**
   if $H(p_i)$ contains H on its boundary
    then $S := S \cup \{p_i\}$;

4.3.2. decide whether R - intersection of halfplanes of release of all pins in S is empty;

4.3.3. **if** R is nonempty
   then there exists a releasing motion
   else all motions for $\Phi>0$ cause penetration
  end

4.4. **else** all motions for $\Phi>0$ cause penetration.

Theorem 4.1. It can be decided in O(n) time whether the polygon P is immobilized by a given set I of n pins, different from convex vertices of *P*.

Proof: Observe first that if the pin *p* is at a reflex vertex of *P*, according to lemma 3.1, it may be treated as two pins, belonging to two sides of *P* adjacent at *p*. To find a releasing motion we check first in O(n) time, following lemma 2.1 whether there exists a releasing translation. Then we run Algorithm 4.1. twice (once for $\Phi>0$ and second time for $\Phi<0$). The first two steps of the algorithm obviously take O(n) time. Using linear programming, steps 3 and 4.3.3 take O(n) times following corollary 1.1. Each iteration of the for loops from step 4.2.1 and 4.2.3. takes a constant time. Steps 4.2.2 and 4.2.3 are also linear as all the halfplanes of release are bounded by the parallel lines. §

An interested reader may observe that if some pin *p* is placed at a convex vertex of *P*, according to lemma 3.1, H(*p*) is a union of two halplanes rather than a single halfplane. Steps 2 and 3 of Algorithm 4.1 must be adjusted accordingly. As region H from step 3 is now nonconvex and not necessarily connected, linear programming algorithm can no longer be used here (note that H was not in fact computed in the algorithm). Instead, we can apply, for example, the $O(n^2)$ topological sweep (see [EG]). The treatment in steps 4.2 and 4.3 must be also modified. Other $O(n \cdot 2^k)$ or $O((n+k^2) \log n)$ algorithms (k is the number of pins placed at convex vertices) may be also possible here.

5. Conclusions and open problems

One natural way of extension of this work is to consider higher-dimensional case. Although the general case (nondegenerated or empty H) may still be treated in O(n) time, it is not clear how to deal with degenerated cases.

We conjecture that theorem 3.2 will also extend to d-dimensional polytopes and that from the set of n immobilizing pins we can choose 2d of them which will be also sufficient to immobilize.

It is also interesting to give a good algorithm dealing with pins placed at convex vertices. However, as H may then split into $\Omega(k^2)$ connected components, we suspect that a linear time algorithm may be impossible to find. It is unclear whether theorem 3.2 may be extended to the case when pins may be placed at convex vertices.

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