Semantic Characterization of Kracht Formulas

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Abstract

Kracht formulas are first-order correspondents of modal Sahlqvist formulas. In this paper we present a model-theoretic characterization of Kracht formulas similar to Van Benthem's theorem saying that a first-order formula is equivalent to a modal formula iff it is invariant under bisimulation. Our characterization yields a method to prove that a given first-order formula is not equivalent to any Kracht formula. In particular, we prove that the first-order formula, expressing the 'cubic property' of a 3-dimensional modal frame does not have a Kracht equivalent.

Keywords: Kracht Formulas, Semantic Characterization, Kracht-simulation

1 Introduction

Sahlqvist theorem on completeness and correspondence [12] is one of the basic tools in modal logic. Completeness and decidability of many modal calculi can be proved using this theorem together with other methods.

Kracht's theorem [8],[9] gives a syntactic characterization of first-order correspondents of Sahlqvist formulas; they are called Kracht formulas. [1] This theorem also describes an algorithm constructing a Sahlqvist correspondent for a given Kracht formula.

However, we do not know any sufficiently general method to decide whether a given first-order formula is equivalent to a Kracht formula. In general this problem is undecidable [3], and even particular cases of this problem may be hard.

For instance, recall a standard argument showing that not all first-order definable modal formulas are Sahlqvist. It is based on the fact that unlike Sahlqvist formulas, the formula $(\Diamond p \to \Diamond\Diamond p) \wedge (\Box\Diamond p \to \Diamond\Box p)$ is locally undefinable.

[1] Correspondence Theory and, in particular, Krachts formulas are applicable to knowledge base query answering, cf. [13].

When generalized Sahlqvist formulas appeared [5], there was a question if they really semantically extend the class of standard Sahlqvist formulas. The question was solved positively by D. Vakarelov and V. Goranko in [6], where they introduced the notion of a-persistence, showed that all standard Sahlqvist formulas are a-persistent and gave an example of a generalized Sahlqvist formula without this property.

In [13] some syntactic extensions of Kracht's fragment were proposed, and there is a question if these new formulas extend Kracht formulas semantically.

This paper proposes a general method for distinguishing Kracht formulas and thus for proofs that certain first-order formulas do not have Kracht equivalents. We point out that unlike [6], we use the methods of classical first-order model theory (elementary chains, ultraproducts and ω-saturated models) and deal with classical first-order formulas.

Our characterization is obtained as a combination of two well-known ideas. The first is Van Benthem's theorem [1] — a first-order formula is equivalent to a modal formula iff it is bisimulation-invariant. The second is preservation of first-order positive formulas under homomorphisms.

2 Kracht formulas.

Let Λ be a set of indices. We use the standard first-order language $\mathcal{L}f_\Lambda$ containing countably many individual variables x_i, binary predicates R_λ for every $\lambda \in \Lambda$, equality, boolean connectives $\wedge, \vee, \rightarrow, \neg$ and quantifiers $\exists x_i$, $\forall x_i$. To avoid subscripts, we often denote individual variables by $x, y, \ldots$.

The definitions from this section are almost the same as in [2] and originate from [8], [9].

Definition 2.1 *We use the following abbreviations*

$$(\forall x_i \rhd_\lambda x_j)A \equiv \forall x_i(x_j R_\lambda x_i \rightarrow A);$$

$$(\exists x_i \rhd_\lambda x_j)A \equiv \exists x_i(x_j R_\lambda x_i \wedge A).$$

$(\forall x_i \rhd_\lambda x_j)$, $(\exists x_i \rhd_\lambda x_j)$ *are called restricted quantifiers.*

To define Kracht formulas, we need a fragment $\mathcal{R}f_\Lambda$ of $\mathcal{L}f_\Lambda$. Informally, to obtain $\mathcal{R}f_\Lambda$ we take the positive fragment of $\mathcal{L}f_\Lambda$, add new relation symbols for compositions of R_λ and use only restricted quantifiers. We consider the symbols $\rhd_\lambda$ as elements of our language.

Formally, the formulas of $\mathcal{R}f_\Lambda$ are defined by recursion [9]:

- $\bot, \top$ are formulas of $\mathcal{R}f_\Lambda$;
- if $\varepsilon = \lambda_1 \ldots \lambda_n$, where $n \geq 0$, $\lambda_i \in \Lambda$, and x_i, x_j are individual variables, then $x_i R^\varepsilon x_j$ is a formula of $\mathcal{R}f_\Lambda$ (if ε is empty, we obtain equality);
- if A, B are formulas of $\mathcal{R}f_\Lambda$, then $(A \wedge B), (A \vee B)$ are formulas of $\mathcal{R}f_\Lambda$;
- if x_i, x_j are individual variables, $\lambda \in \Lambda$ and A is a formula of $\mathcal{R}f_\Lambda$, then $(\forall x_i \rhd_\lambda x_j)A$ and $(\exists x_i \rhd_\lambda x_j)A$ are formulas of $\mathcal{R}f_\Lambda$.

Definition 2.2 *A formula A of $\mathcal{R}f_\Lambda$ is called* clean *if A does not contain variables occurring both free and bound and every two different occurrences of quantifiers in A bind different variables.*

Henceforth we consider only clean formulas.

$\phi[t/x]$ denotes the substitution of the term t for all free occurrences of the variable x in ϕ. In general $\phi[t/x]$ is not necessary clean (even if ϕ is clean), but in this paper all such substitutions generate clean formulas.

Consider an $\mathcal{L}f_\Lambda$-structure $M = (W, (R^M_\lambda : \lambda \in \Lambda))$. $\mathcal{R}f^M_\Lambda$ denotes the language obtained from $\mathcal{R}f_\Lambda$ by adding the constants c_w for all $w \in W$. A formula ϕ of $\mathcal{R}f^M_\Lambda$ is called an $\mathcal{R}f^M_\Lambda$-sentence if ϕ does not contain free variables. The truth of $\mathcal{R}f^M_\Lambda$-sentences in M is defined in a standard way. In particular, the formula $c_w R^\varepsilon c_v$, where $\varepsilon = \lambda_1 \dots \lambda_n$, is true in M iff there is a sequence of points $w_0, w_1, \dots, w_n$ of M such that $w_0 = w$, $w_n = v$ and for all i from 1 to n we have $x_{i-1} R^M_{\lambda_i} x_i$. In particular, if $n = 0$ (i.e. ε is an empty sequence), then $c_w R^\varepsilon c_v \iff w = v$.

Definition 2.3 *A variable x in a formula ϕ is called* inherently universal *for ϕ if either x is free, or x is bound by a universal quantifier, which is not within the scope of an existential quantifier.*

A formula ϕ of $\mathcal{R}f_\Lambda$ is called a parametrized Kracht formula *if in every its atomic subformula of the form $x_i R^\varepsilon x_j$ at least one of the variables x_i and x_j is inherently universal for ϕ.*

A parametrized Kracht formula with a single free variable is called a Kracht formula*.*

Definition 2.4 *Consider an $\mathcal{L}f_\Lambda$-structure $\hat{T} = (W^T, (R^T_\lambda : \lambda \in \Lambda))$, where for all $\lambda \in \Lambda$ $R^T_\lambda \subseteq W^T \times W^T$. A sequence $x_1, \lambda_1, x_2, \lambda_2, \dots x_n$, where $x_i \in W^T$, $\lambda_i \in \Lambda$ and $(x_i, x_{i+1}) \in R^T_{\lambda_i}$ for $1 \le i \le n-1$ is called a* path *from x_1 to x_n in $\hat{T}$. A tuple $T = (\hat{T}, r^T)$ is called a* tree *with a* root *r^T if the following holds:*

- $r^T \in W^T$,
- *W^T is finite;*
- $(R^T_\lambda)^{-1}(r^T) = \emptyset$ *for all* $\lambda \in \Lambda$,
- *for every point $x^T \neq r^T$ there exists a unique path from r^T to x^T.*

Consider a tree $T = (W^T, (R^T_\lambda : \lambda \in \Lambda), r^T)$ and an $\mathcal{L}f_\Lambda$-structure $F = (W^F, (R^F_\lambda : \lambda \in \Lambda))$. A mapping $f : W^T \to W^F$ is called *monotonic* if for all $x, y \in W^T$, $\lambda \in \Lambda$, $x R^T_\lambda y$ implies $f(x) R^F_\lambda f(y)$.

3 Semantic Characterization

Definition 3.1 *Consider two $\mathcal{L}f_\Lambda$-structures $G = (W^G, (R^G_\lambda : \lambda \in \Lambda))$ and $F = (W^F, (R^F_\lambda : \lambda \in \Lambda))$, a tree $T = (W^T, (R^T_\lambda : \lambda \in \Lambda))$, monotonic mappings $g : T \to G$ and $f : T \to F$. A relation $Z \subseteq W^G \times W^F$ is called a* Kracht-simulation *if Z satisfies the following conditions:*

(KB1) For every $t \in W^T$, $(g(t), f(t)) \in Z$;

(KB2) For any $x^G \in W^G$, $x^F \in W^F$, $t \in W^T$, for arbitrary sequence $\varepsilon \in \Lambda^*$ if

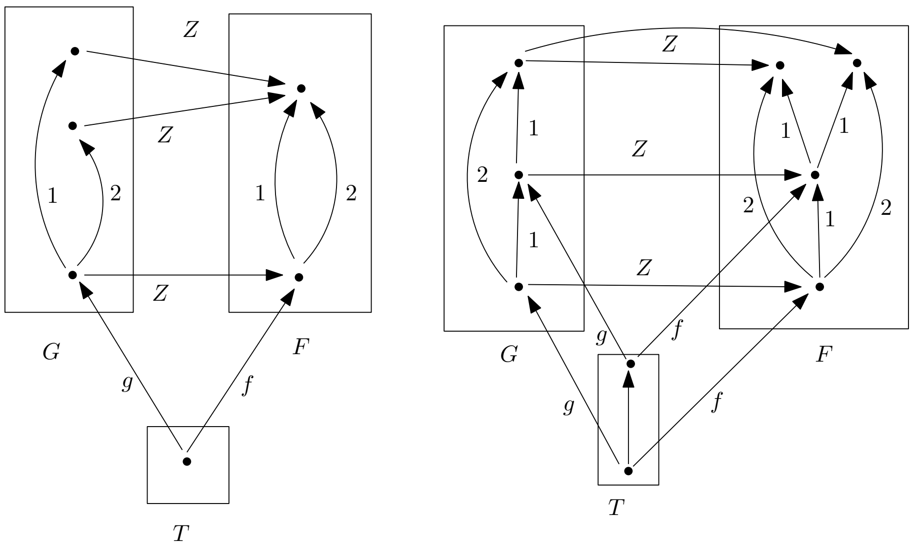


Fig. 1. Some examples of Kracht-simulations.

$(x^G, x^F) \in Z$ and $g(t)(R^G)^\varepsilon x^G$, then $f(t)(R^F)^\varepsilon x^F$.

(KB3) For any points $x^F \in W^F$ and $x^G \in W^G$ such that $(x^G, x^F) \in Z$ for any $(x')^G \in R^G_\lambda(x^G)$ there exists a point $(x')^F \in R^F_\lambda(x^F)$ such that $(x'^G, x'^F) \in Z$.

(KB4) For any points $x^F \in W^F$ and $x^G \in W^G$ such that $(x^G, x^F) \in Z$, for any $(x')^F \in R^F_\lambda(x^F)$ there exists a point $(x')^G \in R^G_\lambda(x^G)$, such that $(x'^G, x'^F) \in Z$.

In this case we say that the triple (G, T, g) is Kracht-reducible to (F, T, f) by Z, in symbols: $(G, T, g) \gg_Z (F, T, f)$.

Note that this notion generalizes modal bisimulations. (KB3) and (KB4) are the classical "forth" and "back" properties of bisimulations. (KB2) replaces the additional property of bisimulation saying that bisimilar worlds satisfy the same proposition letters. Note that unlike classical modal bisimulations, the relation $\gg_Z$ is not symmetric, since (KB2) acts only in one direction.

Examples of Kracht-simulations can be found in Figures 1 and 2.

Definition 3.2 *A tuple $G_\circ = (G, x_0^G)$ is called an $\mathcal{L}f_\Lambda$-structure with a designated point if $G = (W^G, (R^G_\lambda : \lambda \in \Lambda))$ is an $\mathcal{L}f_\Lambda$-structure and $x_0^G \in W^G$.*

Definition 3.3 *Consider two $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ = (G, x_0^G)$ and $F_\circ = (F, x_0^F)$. We say that $G_\circ$ is Kracht-reducible to $F_\circ$ (notation: $G_\circ \ggg F_\circ$) if for any tree $T = (W^T, (R^T_\lambda : \lambda \in \Lambda), x_0^T)$ for all monotonic mappings $f : T \to F$ sending x_0^T to x_0^F, there exists a monotonic mapping $g : T \to G$, sending x_0^T to x_0^G, and a relation $Z \subseteq W^G \times W^F$ such that $(G, T, g) \gg_Z (F, T, f)$.*

The intuition underlying this definition is the following. For $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ$ and $F_\circ$ we can regard a tuple (T, f), where T is a tree, and $f : T \to F$ is a monotonic mapping, sending x_0^T to x_0^F, as a test checking if $F_\circ$ really simulates $G_\circ$. The pair $(G_\circ, F_\circ)$ passes the test if there exist g and Z such that $(G, T, g) \gg_Z (F, T, f)$.

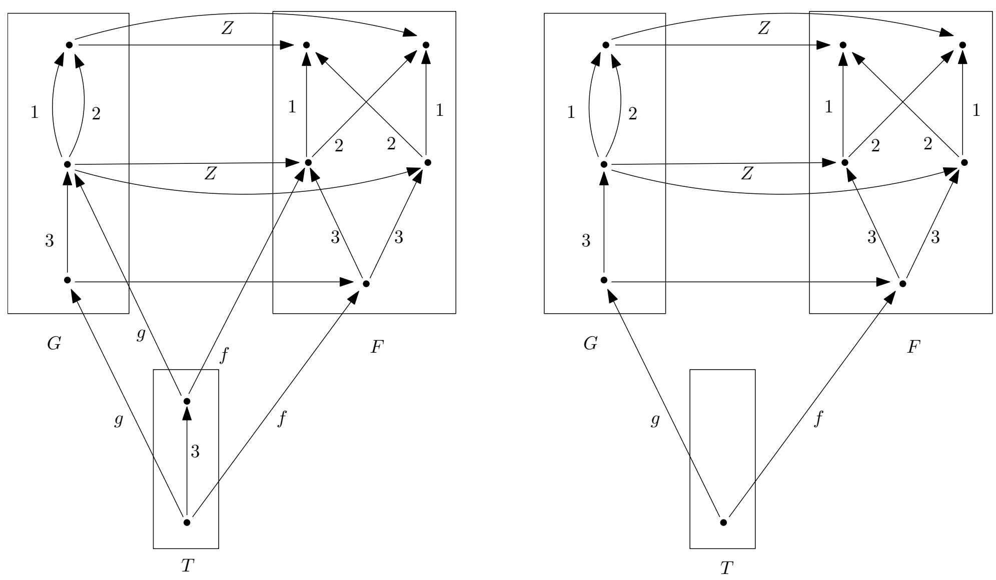


Fig. 2. The left-hand picture is not a Kracht-simulation while the right-hand picture is.

And $G_\circ \ggg F_\circ$ if the pair $(G_\circ, F_\circ)$ passes all possible tests.

Definition 3.4 *We say that a formula $\phi(x_0)$ of $\mathcal{L}f_\Lambda$ is preserved under Kracht-reducibility if for every pair of $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ = (G, x_0^G)$ and $F_\circ = (F, x_0^F)$ if $G_\circ \ggg F_\circ$ and $G \models \phi[x_0^G/x_0]$, then $F \models \phi[x_0^F/x_0]$.*

Theorem 3.5 *A formula $\phi(x_0)$ of $\mathcal{L}f_\Lambda$ is equivalent to a Kracht formula iff $\phi(x_0)$ is preserved under Kracht-reducibility.*

This theorem is proved in Sections 4 – 6.

4 Soundness

The aim of this section is to prove that every Kracht formula is preserved under Kracht-reducibility.

To begin with, we give the definition of a local Kracht formula. The idea of this definition is to characterize a point in any frame 'up to K-reducibility'. To define these formulas we use the following convention. We assume that every local Kracht formula ϕ has a fixed free variable v (however, v may be dummy in ϕ), and v differs from $x_0, x_1, x_2, \ldots$. The notation $\phi(t)$ denotes the substitution of t for v in ϕ.

These formulas are similar to standard translations of modal formulas $ST_x(\chi)$ [2], where x is a fixed variable. To emphasise this similarity, for a point w of an $\mathcal{L}f_\Lambda$-structure $F = (W^F, (R_\lambda^F : \lambda \in \Lambda))$ and a local Kracht formula ϕ it is tempting to write $F, w \models \phi$ instead of $F \models \phi[w/v]$.

But there is yet another difference between local Kracht formulas and the standard translation. In $ST_x(\chi)$ the only free variable is x. Local Kracht formulas may contain some additional free variables $x_0, x_1, \ldots x_n$. Suppose that in addition to an

$\mathcal{L}f_\Lambda$-structure F we have a tree $T = (W^T, (R^T_\Lambda : \lambda \in \Lambda))$ and a monotonic mapping $f : W^T \to W^F$. Suppose also that $W^T = \{x^T_0, x^T_1, \dots, x^T_n\}$. In this case the notation $(F, T, f), w \models \phi(v)$ means $F \models \phi[f(x^T_i)/x_i][w/v]$.

Definition 4.1 *Local Kracht formulas are defined recursively:*

- $\top$, $\bot$ *and* $x_i R^\varepsilon v$ *are local Kracht formulas;*
- *if* $\phi(v)$ *is a local Kracht formula, then* $(\exists v' \rhd_\lambda v)\phi(v')$ *and* $(\forall v' \rhd_\lambda v)\phi(v')$ *are local Kracht formulas. As in the standard translation, here* v' *is a new variable not occurring in* ϕ*. This restriction is crucial, e. g. the formula* $(\forall v_1 \rhd v)(\forall v_2 \rhd v)(v_1 R v_2)$ *is not a local Kracht formula, while* $(\forall v_2 \rhd v)(x_1 R v_2)$ *is.*
- *if* $\phi(v)$ *and* $\psi(v)$ *are local Kracht formulas, then* $\phi(v) \wedge \psi(v)$ *and* $\phi(v) \vee \psi(v)$ *are local Kracht formulas.*

Example 4.2 *Here are some examples of local Kracht formulas:* $x_1 R^\varepsilon v$, $\exists v_1 \rhd_\lambda v(x_1 R^\varepsilon v_1)$, $(\exists v_1 \rhd_\lambda v(x_1 R^{\varepsilon_1} v_1)) \wedge (x_2 R^{\varepsilon_2} v)$.

Lemma 4.3 *If* $(G, T, g) \gg_Z (F, T, f)$ *and* $(x^G, x^F) \in Z$*, then for any local Kracht formula* $\psi(v)$ $(G, T, g), x^G \models \psi(v)$ *implies* $(F, T, f), x^F \models \psi(v)$.

Proof. The proof by induction on the length of ψ.

The base of induction follows from (KB2); the cases $\psi = \psi_1 \wedge \psi_2$ and $\psi = \psi_1 \vee \psi_2$ are trivial.

The case $\psi(v) = \exists v' \rhd_\lambda v\psi'[v'/v]$ follows immediately from (KB3).

Consider the case $\psi(v) = \forall v' \rhd_\lambda v\psi'[v'/v]$. Suppose that $(G, T, g), x^G \models \psi$, but $(F, T, f), x^F \not\models \psi$. The latter means that there exists a point $x'^F \in R^F_\lambda(x^F)$, such that $(F, T, f), x'^F \not\models \psi'$. By $(KB4)$, there is a point $x'^G \in R^G_\lambda(x^G)$, such that $(x'^G, x'^F) \in Z$. By the induction hypothesis, $(G, T, g), x'^G \not\models \psi'$. This contradicts $(G, T, g), x^G \models \psi$. Thus the claim holds. □

Now we reduce arbitrary Kracht formulas to local Kracht formulas.

Definition 4.4 *If a formula* ϕ *is built from formulas of the form* $\psi(x_i)$*, where* ψ *is a local Kracht formula, using* $\wedge$, $\vee$ *and restricted universal quantifiers, we say that* ϕ *is a parametrized decomposed Kracht formula.*

Lemma 4.5 *Let* $\xi(x_1, \dots, x_n, y_1, \dots, y_r)$ *be a formula of* $\mathcal{R}f_\Lambda$*. Suppose that any subformula of* ξ *of the form* $v_1 R^\varepsilon v_2$ *contains at least one* x*-variable. Then there exist local Kracht formulas* $\psi_1(v), \dots, \psi_k(v)$*, not containing* y*-variables, such that* ξ *is equivalent to a formula built from* $\psi_i(x'_i)$ $(x'_i \in \{x_1, \dots, x_n, y_1, \dots, y_r\})$ *using only* $\wedge$ *and* $\vee$.

Proof. At first we replace each subformula of ϕ of the form $zR^\varepsilon x$, where $\varepsilon = \lambda_1 \dots \lambda_n$, and the variable x (but not z) is inherently universal, with an equivalent formula $(\exists z_1 \rhd_{\lambda_1} z)(\exists z_2 \rhd_{\lambda_2} z_1) \dots (\exists z_n \rhd_{\lambda_n} z_{n-1})(x = z_n)$, where all z_i are new variables. This operation gives us a formula with all atoms of the form $x_i R^\varepsilon z$.

Then we argue by induction on the length of ξ.

If $\xi = x_j R^\varepsilon z$, then $\xi = \psi(z)$ for $\psi(v) = x_j R^\varepsilon v$. The cases of booleans are trivial.

Suppose $\xi = (\exists z \rhd_\lambda x_j)\xi'(x_1, \dots, x_n, z)$. By the induction hypothesis and distributivity, ξ' is of the form $K_1 \vee \ldots \vee K_m$, where $K_i = \psi_1^i(x_1^i) \wedge \ldots \wedge \psi_{m_i}^i(x_{m_i}^i)$ ($x_l^i \in \{x_1, \ldots, x_k, z\}$). So, $\xi \equiv (\exists z \triangleright_\lambda x_j)K_1 \vee \ldots \vee (\exists z \triangleright_\lambda x_j)K_m$. But

$$(\exists z \triangleright_\lambda x_j)K_i = \bigwedge_{x_l^i \neq z} \psi_l^i(x_l^i) \wedge \psi(x_j)$$

for the local Kracht formula

$$\psi(v) = (\exists v' \triangleright_\lambda v) \bigwedge_{x_l^i = v} \psi_l^i(v').$$

The case of universal quantifier is proved dually. □

Lemma 4.6 *Every parametrized Kracht formula ϕ is equivalent to a parametrized decomposed formula.*

Proof. By induction on the length of ϕ relying upon Lemma 4.5 for atoms and for formulas beginning with an existential quantifier. □

Example 4.7 *Kracht formula*

$$(\forall x_2 \triangleright_1 x_1)(x_1 R_2 x_2 \vee x_1 R_3 x_2) \wedge (\forall x_3 \triangleright_1 x_1)(\exists x_4 \triangleright_1 x_1) x_3 R_1 x_4$$

is equivalent to a decomposed Kracht formula

$$(\forall x_2 \triangleright_1 x_1)(\psi_1(x_2) \vee \psi_2(x_2)) \wedge (\forall x_3 \triangleright_1 x_1)\psi_3(x_1)$$

for local Kracht formulas $\psi_1(v) = x_1 R_2 v$, $\psi_2(v) = x_1 R_3 v$, $\psi_3(v) = (\exists v' \triangleright_1 v) x_3 R_1 v'$.

Definition 4.8 *Consider a model F and a parametrized decomposed Kracht formula ϕ. Suppose that we substitute points $x_i^F \in W^F$ for all free variables x_i in ϕ. The result of such substitution is called a* Kracht F-sentence*. A* local Kracht F-sentence *is defined in a similar way.*

Definition 4.9 *We say that a set of F-sentences S witnesses the falsity of an F-sentence ϕ if S satisfies the following conditions:*

(i) *$\phi \in S$;*

(ii) *if $\phi_1 \vee \phi_2 \in S$, then $\phi_1 \in S$ and $\phi_2 \in S$;*

(iii) *if $\phi_1 \wedge \phi_2 \in S$, then $\phi_1 \in S$ or $\phi_2 \in S$;*

(iv) *if $(\forall x_i \triangleright_\lambda x_j^F)\phi' \in S$, then there exists a point x_i^F in F such that $x_j^F R_\lambda^F x_i^F$ and $\phi'[x_i^F/x_i] \in S$;*

(v) *if $\psi(v)$ is a local Kracht sentence and $\psi(x_i^F) \in S$, then $F \not\models \psi(x_i^F)$.*

Lemma 4.10 *Let ϕ be a Kracht F-sentence. Then*
$F \not\models \phi \iff$ *there exists a set of Kracht F-sentences S^ϕ witnessing the falsity of ϕ.*

Proof. ($\Longrightarrow$) Suppose that $F \not\models \phi$. We argue by induction on the length of ϕ.

- if ϕ is of the form $\psi(x_k^F)$ for a local Kracht formula $\psi(v)$, put $S^\phi = \{\phi\}$;
- if $\phi = \phi_1 \vee \phi_2$, then both ϕ_1 and ϕ_2 are false, so we put $S^\phi = \{\phi\} \cup S^{\phi_1} \cup S^{\phi_2}$;
- if $\phi = \phi_1 \wedge \phi_2$, then one of ϕ_1 or ϕ_2 is false so we can put either $S^\phi = \{\phi\} \cup S^{\phi_1}$ or $S^\phi = \{\phi\} \cup S^{\phi_2}$ depending on the falsity of ϕ_1 or ϕ_2;
- if $\phi = (\forall x_i \rhd_\lambda x_j^F)\phi' \in S$, then there is a point x_i^F in a model F such that $x_j^F R_\lambda^F x_i^F$ and $F \not\models \phi'[x_i^F/x_i]$, hence we can put $S^\phi = \{\phi\} \cup S^{\phi'[x_i^F/x_i]}$.

($\Longleftarrow$) A trivial induction shows that if S witnesses the falsity of any F-sentence, then for all $\phi' \in S$ $F \not\models \phi'$. But $\phi \in S$. □

Now we are ready to prove that every Kracht formula $\phi(x_0)$ is preserved under Kracht-reducibility, that is the soundness part of Theorem 3.5.

Proof. According to Lemma 4.6, without any loss of generality we can assume that $\phi(x_0)$ is decomposed.

Assume that $\phi(x_0)$ is not preserved under Kracht-reducibility, that is there are $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ$ and $F_\circ$ such that $G_\circ \ggg F_\circ$ and

(1) $$G \models \phi[x_0^G/x_0],$$

but $F \not\models \phi[x_0^F/x_0]$. By Lemma 4.10 there exists a set of F-sentences S_F witnessing the falsity of $\phi[x_0^F/x_0]$ in F.

Let $\Gamma = \{\psi(x_k^F) \mid \psi(v)$ be a local Kracht sentence and $\psi(x_k^F) \in S_F\}$. By W^T we denote the set of all inherently universal variables x_k such that the corresponding constants x_k^F occur in Γ. In other words, W^T consists of those inherently universal variables of ϕ, whose valuations are essential for the falsity of $\phi[x_0^F/x_0]$ in F. Consider the tree $T = (W^T, (R_\lambda^T : \lambda \in \Lambda), x_0)$, where $x_j R_\lambda^T x_i$ iff the variable x_i is bound by a quantifier of the form $(\forall x_i \rhd_\lambda x_j)$.

The construction of S_F gives us a monotonic mapping $f : T \to F$ sending each point $x_i \in W^T$ to a point $x_i^F \in F$. By Definition 3.3, there exists a monotonic mapping $g : T \to G$, sending x_0 to x_0^G such that $(G, T, g) \gg_Z (F, T, f)$ for some Z. Let S_G be the set of G-sentences obtained by replacing all x_i^F with x_i^G in all formulas of S_F.

We claim that S_G witnesses the falsity of the G-sentence $\phi[x_0^G/x_0]$ in G. In fact, the items (i)–(iii) of Definition 4.9 hold by the construction of S_G. The item (v) is true due to Lemma 4.3. Let us show that the item (iv) holds for S_G. Suppose that $(\forall x_i \rhd_\lambda x_j^G)\phi' \in S_G$. This may happen only in the case when $(\forall x_i \rhd_\lambda x_j^F)\phi'[x_k^F/x_k^G] \in S_F$. So there exists $x_i^F \in F$ such that $(F, T, f), x_i^F \not\models \phi'[x_k^F/x_k^G]$. Put $x_i^G = g(x_i)$. By (KB1), $(x_i^G, x_i^F) \in Z$, hence due to (KB2), $(G, T, g), x_i^G \not\models \phi'$. And $x_j^G R_\lambda^G x_i^G$ by the monotonicity of g.

Therefore, by Lemma 4.10, we conclude that $G \not\models \phi[x_0^G/x_0]$. This contradicts our initial assumption (1). □

5 Model-theoretic background

In the next section we assume that the reader is familiar with standard model-theoretic tools such as elementary extensions, ω-saturated models and ultrapowers. However, we recall the latter two notions and their basic properties.

Definition 5.1 *An $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ is called* n-saturated, *if for any set Γ of first-order formulas with at most n free variables $\gamma(x_1, x_2, \ldots, x_n)$ the following holds:*

IF $x_1^0, \ldots, x_{n-1}^0$ is a sequence of points from W such that for all finite $\Delta \subseteq \Gamma$ there is a point $(x_n^0)_\Delta$ such that

$$F \models \gamma(x_1^0, x_2^0, \ldots, x_{n-1}^0, (x_n^0)_\Delta)$$

for all formulas $\gamma \in \Delta$,

THEN there exists a point x_n^0 such that $F \models \gamma(x_1^0, x_2^0, \ldots, x_{n-1}^0, x_n^0)$ for all formulas $\gamma \in \Gamma$.

A model F is called ω-saturated *if it is n-saturated for all n.*

Definition 5.2 *Consider a model $F = (W, (R_\lambda : \lambda \in \Lambda))$ and a non-principal ultrafilter u over the set of all natural numbers $\mathbb{N}$.*

We say that two sequences of points from W $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ and $\bar{\beta} = (\beta_1, \beta_2, \beta_3, \ldots)$ are u-equivalent *(denoted by $\bar{\alpha} \sim_u \bar{\beta}$), if $\{i \mid \alpha_i = \beta_i\} \in u$. The equivalence class of a sequence α is denoted by $\lceil \alpha \rceil$.*

The $\mathcal{L}f_\Lambda$-structure $F = (W', (R'_\lambda : \lambda \in \Lambda))$, where

$$W' = \{ \text{ all sequences of points from } W\} / \sim_u$$

and

$$\lceil \bar{\alpha} \rceil R'_\lambda \lceil \bar{\beta} \rceil \iff \{i \mid \alpha_i R_\lambda \beta_i\} \in u.$$

is called an ultrapower *of F (with respect to u) and denoted by $\prod_u F$.*

Proposition 5.3 *A natural embedding $i : F \to \prod_u F$ such that*

$$i(w) = \lceil (w, w, w, w, \ldots) \rceil,$$

is elementary.

Proposition 5.4 *For any $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ and any non-principal ultrafilter u over $\mathbb{N}$ the ultrapower $\prod_u F$ is ω-saturated.*

6 Completeness

We follow the plan of the proof of van Benthem's theorem from [2]. The keystone of the proof is Lemma 6.5 (an analogue of Detour lemma from [2]). Its proof is carried out step by step in Lemmas 6.1 – 6.4. After that the theorem is proved in a standard way.

Lemma 6.1 *Consider $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ = (G, x_0^G)$ and $F_\circ = (F, x_0^F)$ such that for any Kracht formula $\phi(x_0)$ $G \models \phi[x_0^G/x_0]$ implies $F \models \phi[x_0^F/x_0]$. Suppose there is a tree T and a monotonic mapping $f : T \to F$ sending r^T to x_0^F.*

Then there exists an elementary extension G' of G and a monotonic mapping $g : T \to G'$ sending r^T to x_0^G, such that for any local Kracht formula $\psi(v)$ for any point $t \in W^T$ $G' \models \psi(g(t))$ implies $F \models \psi(f(t))$.

Proof. Let $W^T = \{x_0, x_1, \dots, x_n\}$, let $x_{p(i)}$ be the unique predecessor of x_i in T, and let $x_{p(i)} R_{\lambda_i} x_i$. Suppose that for all $0 \leq i \leq n$

$$\Psi_i = \{\psi(v) \mid \psi(v) \text{ is a local Kracht formula and } F \not\models \psi(f(x_i))\}.$$

We enumerate all formulas of Ψ_i:

$$\Psi_i = \{\psi_1^i(v), \psi_2^i(v), \dots\}.$$

Fix $m \in \mathbb{N}$. Consider the formula

$$\gamma_m = (\exists x_1 \triangleright_{\lambda_1} x_0) \dots (\exists x_n \triangleright_{\lambda_n} x_{p(n)}) \bigwedge_{x_i \in W^T} ((\neg\psi_1^i \wedge \dots \wedge \neg\psi_m^i)(x_i)).$$

It is clear that $F \models \gamma_m[x_0^F/x_0]$ and γ_m is a negation of a Kracht formula.

Hence, due to the assumption of the lemma, $G \models \gamma_m[x_0^G/x_0]$, that is there exist points $g_0^m, \dots, g_n^m \in W^G$ (here $g_0^m = x_0^G$ for all m) such that the mappings $g^m : T \to G$ $g^m(x_i) = g_i^m$ are monotonic and for all $0 \leq i \leq n$

$$G, g_i^m \models (\neg\psi_1^i \wedge \dots \neg\psi_m^i).$$

Let u be a non-principal ultrafilter over $\mathbb{N}$. Put $G' = \Pi_u G$.

This guarantees that G' is an elementary extension of G.

Now define $g(x_i)$ as the equivalence class of the sequence $(g_i^1, g_i^2, g_i^3, \dots)$.

It is clear that if $x_i R_\lambda^T x_j$, then for all m $g_i^m R_\lambda^G g_j^m$, therefore, $g(x_i) R_\lambda^{G'} g(x_j)$.

Now we show that for any local Kracht formula $\psi(v)$ for any point $x_i \in W^T$ if $G' \models \psi(g(x_i))$ then $F \models \psi(f(x_i))$.

In fact, suppose that $F \not\models \psi(f(x_i))$ for some i. Then $\psi \in \Psi_i$, hence there is m_0 such that $\psi = \psi_i^{m_0}$. Then for all $m \geq m_0$ $g_i^m \models \neg\psi$. This contradicts $G' \models \psi(g(x_i))$. □

Lemma 6.2 *Consider $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ$ and $F_\circ$ such that W^F is countable and for any Kracht formula $\phi(x_0)$ $G \models \phi[x_0^G/x_0]$ implies $F \models \phi[x_0^F/x_0]$. Then there is an elementary extension G^* of G, such that*

(2) *for any tree T for any monotonic mapping $f : T \to F$, sending r^T to x_0^F, there exists a monotonic mapping $g : T \to G^*$ sending r^T to x_0^G, such that for any local Kracht formula $\psi(v)$ for any point $x_i \in W^T$ $G^* \models \psi(g(x_i))$ implies $F \models \psi(f(x_i))$*

Proof. Consider all possible pairs (T, f) consisting of a tree T and a monotonic mapping $f : T \to F$ sending r^T to x_0^F. There are countably many such pairs, we enumerate them as $(T_1, f_1), (T_2, f_2), \dots$.

We construct an elementary chain $G_0 \prec G_1 \prec G_2 \prec \ldots$, where $G_0 = G$, and for $i > 0$ G_i is obtained by applying Lemma 6.1 to frames G_{i-1}, F and the pair (T_i, f_i). The condition of Lemma 6.1 holds due to the elementarity of G_{i-1} over G_0.

Put G^* to be the limit of this chain. By the elementary chain principle, G^* is an elementary extension of G_0, so (2) obviously holds. □

Lemma 6.3 *Consider $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ$ and $F_\circ$ such that W^F is countable and for any Kracht formula $\phi(x_0)$, $G \models \phi[x_0^G/x_0]$ implies $F \models \phi[x_0^F/x_0]$. Then there exist elementary embeddings $G \prec \bar{G}$ and $F \prec \bar{F}$ such that $\bar{G}$ and $\bar{F}$ are ω-saturated and satisfy 6.2 (2).*

Proof. Due to Lemma 6.2, there exists an elementary embedding $G \prec G^*$, such that (2) holds.

Take a non-principal ultrafilter u over $\mathbb{N}$ and put $\bar{G} = \Pi_u G^*$, $\bar{F} = \Pi_u F$. Let us verify that (2) still holds for $\bar{G}$ and $\bar{F}$.

In fact, consider $f : T \to \bar{F}$. Suppose that $f(x_i) = \lceil (f_i^1, f_i^2, f_i^3, \ldots) \rceil$. Then for each monotonic mapping $f^j : T \to f$ we find a corresponding monotone mapping $g^j : T \to G^*$, and finally we put $g(x_i) = \lceil (g^1(x_i), g^2(x_i), \ldots) \rceil$. It is clear that g is well defined. Let us prove that for any local Kracht formula $\psi(v)$ for any point t of T $\bar{G} \models \psi(g(x_i))$ implies $F \models \psi(f(x_i))$. If $\bar{G} \models \psi(g(x_i))$, then $A = \{j \mid G \models \psi(g^j(x_i))\} \in u$, therefore $A \subseteq \{j \mid F \models \psi(f^j(x_i))\}$, and so $\{j \mid F \models \psi(f^j(x_i))\} \in u$, hence $\bar{F} \models \psi(f(x_i))$.

Due to Proposition 5.4, the $\mathcal{L}f_\Lambda$-structures $\bar{G}$ and $\bar{F}$ are ω-saturated. □

Lemma 6.4 *Let $\bar{G}$, $\bar{F}$ be ω-saturated $\mathcal{L}f_\Lambda$-structures. Suppose there is a tree T and monotonic mappings $g : T \to \bar{G}$ and $f : T \to \bar{F}$. Suppose also that $W^T = \{x_1, \ldots, x_n\}$. Let us define a relation $Z \subseteq W^G \times W^F$, by putting for $x \in \bar{G}$ and $y \in \bar{F}$*

$$(x, y) \in Z \iff \begin{array}{l} \textit{for any local Kracht formula } \psi(v) \textit{ such that all its free variables} \\ \textit{except } v \textit{ are in } W^T,\ (\bar{G}, T, g) \models \psi(x) \textit{ implies } (\bar{F}, T, f) \models \psi(y). \end{array}$$

Then the relation Z satisfies $(KB2)$–$(KB4)$.

Proof. $(KB2)$ follows readily from the definition of Z.

Let us check $(KB3)$. Suppose that $x^F \in W^F$, $x^G \in W^G$, $x^G Z x^F$ and there exists $(x')^G \in R_\lambda^G(x^G)$. Let Ψ be the set of all local Kracht formulas true at the point $(x')^G$ of the frame G under the valuation $[g(x_i)/x_i]$. Consider the set of formulas $\Psi' = \{x^F R_\lambda v\} \cup \{\psi[f(x_i)/x_i] \mid \psi \in \Psi\}$ in the first-order language $\mathcal{L}f_\Lambda$ enriched with the constants naming the points of W^F. Let us show that every set of the form $\Psi'_m = \{x^F R_\lambda v\} \cup \{\psi_1, \ldots, \psi_m \mid \psi_i \in \Psi\}$ is realized in F. Consider the local Kracht formula $\phi_m = (\exists v' \rhd_\lambda v)(\psi_1(v') \wedge \ldots \wedge \psi_m(v'))$. It is clear that $G \models \phi_m(x^G)$, therefore $F \models \phi_m(x^F)$, that is there exists a point x'^F realizing Ψ_m. Hence, due to the saturation of F, we can conclude that the frame F realizes the whole set Ψ', that is there is a point $(x')^F \in R_\lambda^F(x^F)$ such that $(x'^G, x'^F) \in Z$.

Let us check $(KB4)$. Suppose that $x^F \in W^F$, $x^G \in W^G$, and $x^G Z x^F$. Take a point $(x')^F \in R_\lambda^F(x^F)$.

Let $\Psi = \{\neg\psi \mid \psi$ is a local Kracht formula and $F \models \neg\psi[f(x_i)/x_i][(x')^F/v]\}$. Consider the set of formulas $\Psi' = \{x^G R_\lambda v\} \cup \{\psi[g(x_i)/x_i] \mid \psi \in \Psi\}$ in the first-order language $\mathcal{L}f_\Lambda$ enriched with the constants from W^G. Let us show that any set of the form $\Psi'_m = \{x^G R_\lambda v\} \cup \{\neg\psi_1, \ldots, \neg\psi_m \mid \neg\psi_i \in \psi\}$ is realized in G.

Suppose the contrary, i. e. the formula $\psi_1 \vee \ldots \vee \psi_m$ is true at every point $v \in R_\lambda(x^G)$. Then the local Kracht formula $\phi_m(v) = \forall v' \triangleright_\lambda v(\psi_1 \vee \ldots \vee \psi_m)$ is true at the point x^G . Hence ϕ_m is true at the point x^F of the frame F. This means that the formula $\psi_1 \vee \ldots \vee \psi_m$ is true at $(x')^F$. This contradicts the definition of Ψ.

Due to the saturation of G, we conclude that the whole set Ψ' is realized in G. This means that there is a point $(x')^G \in R^G_\lambda(x^G)$ such that $(x'^G, x'^F) \in Z$. □

Lemma 6.5 *Consider $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ$ and $F_\circ$ such that W^F is countable and for any Kracht formula $\phi(x_0)$ $G \models \phi[x_0^G/x_0]$, implies $F \models \phi[x_0^F/x_0]$. Then there exist elementary embeddings $G \prec \bar{G}$ and $F \prec \bar{F}$, such that $(\bar{G}, x_0^G) \ggg (\bar{F}, x_0^F)$.*

Proof. Apply Lemma 6.3 and then Lemma 6.4. □

Proof. [The proof of theorem 3.5 (completeness)] Consider an arbitrary first-order formula with a single free variable $\phi(v)$ which is preserved under Kracht-reducibility. Let us show that $\phi(v)$ is equivalent to some Kracht formula.

To this end we following the plan in [2], consider the set of first-order formulas with a single free variable v $KC(\phi) = \{\psi(v) \mid \phi(v) \models \psi(v), \psi(v)$ is a Kracht formula$\}$.

(1) Note that if $KC(\phi) \models \phi$, then ϕ is equivalent to a Kracht formula. In fact, if $KC(\phi) \models \phi$, then there exist $\psi_1, \ldots, \psi_n$ such that $\vdash_{PC} \psi_1 \wedge \ldots \wedge \psi_n \to \phi$. Therefore the formula ϕ is equivalent to $\psi_1 \wedge \ldots \wedge \psi_n$.

(2) Let us show that $KC(\phi) \models \phi$. To this end, take a countable model N, a point $y \in N$ and suppose that $N \models KC[y/v]$. Consider the set of formulas

$$NKT(N, y) = \{\neg\delta \mid \delta \text{ is a Kracht formula and } N \models \neg\delta(y)\}.$$

(3) We claim that the set $NKT(N, y) \cup \{\phi\}$ is consistent. In fact, suppose the contrary. Then there is a finite subset $NKT_0 \subset NKT(N, y)$ such that $\vdash_{PC} \phi \to \neg \wedge NKT_0$. This means that there are Kracht formulas $\delta_1, \ldots, \delta_n$ such that $\vdash_{PC} \phi \to \delta_1 \vee \ldots \vee \delta_n$. Then $\delta_1 \vee \ldots \vee \delta_n \in KC(\phi)$, therefore $N, y \models \delta_1 \vee \ldots \vee \delta_n$. This contradicts the fact that $N, y \models \neg\delta_i$ for all i.

(4) Hence, due to the Gödel Completeness Theorem, there is a model M and a point $x \in M$ such that $(M, x) \models NKT(N, y) \cup \{\phi\}$. We claim that for every Kracht formula ψ if $M, x \models \psi$ then $N, y \models \psi$. In fact, if $N, y \models \neg\psi$, then $\neg\psi \in NKT(N, y)$, therefore $M, x \models \neg\psi$. This is the contradiction.

(5) Now we apply Lemma 6.5. It states that there exist elementary extensions $\bar{M}$ and $\bar{N}$ of the models M and N such that $\bar{M} \ggg \bar{N}$. But $M \models \phi$, therefore $\bar{M} \models \phi$, hence $\bar{N} \models \phi$, and so $N \models \phi$. So we have proved that $KC(\phi) \models \phi$. □

7 "Cubic" property

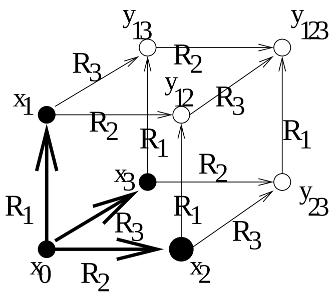
In this section we apply our semantic characterization of Kracht formulas to show that a certain formula fc (well-known in many-dimensional modal logic) does not have a Kracht equivalent.

Consider unimodal Kripke frames $F_1 = (W_1, \hat{R}_1), \ldots, F_n = (W_n, \hat{R}_n)$. Recall that their product $F_1 \times \ldots \times F_n$ is the frame $(W_1 \times \ldots \times W_n, R_1, \ldots, R_n)$, where

$$(x_1, \ldots, x_n)R_i(y_1, \ldots, y_n) \iff x_j = y_j \text{ for } i \neq j \text{ and } x_i\hat{R}_iy_i.$$

All 3-modal frames of the form $F_1 \times F_2 \times F_3$, satisfy [4] the "cubic" formula $\forall x_0 fc(x_0)$, where

$fc(x_0) = \forall x_1 \forall x_2 \forall x_3 (x_0R_1x_1 \wedge x_0R_2x_2 \wedge x_0R_3x_3 \rightarrow$
$\exists y_{12}\exists y_{13}\exists y_{23}\exists y_{123}(x_1R_2y_{12} \wedge x_1R_3y_{13} \wedge$
$\wedge\, x_2R_1y_{12} \wedge x_2R_3y_{23} \wedge x_3R_1y_{13} \wedge$
$x_3R_2y_{23} \wedge y_{23}R_1y_{123} \wedge y_{13}R_2y_{123} \wedge y_{12}R_3y_{123}))$.



In [10] and [11] modifications of the formula fc are used to obtain negative results on axiomatizing modal logics of dimensions ≥ 3.

Theorem 7.1 *The formula $fc(x_0)$ is not equivalent to any Kracht formula.*

To prove Theorem 7.1, we fix $\Lambda = \{1, 2, 3\}$ and construct two $\mathcal{L}f_\Lambda$-structures with designated points $G_\circ$ and $F_\circ$ such that $G_\circ \ggg F_\circ$, $G \models fc[x_0^G/x_0]$, but $F \not\models fc[x_0^F/x_0]$.

Put $F_\circ = (F, r)$ where $F = (W, R_1, R_2, R_3)$, (see Fig. 3)

$$W = \{r\} \cup \{a_1, a_2, a_3\} \cup \{b^i_j \mid i, j = 1, 2, 3\} \cup \{c_1, c_2, c_3\},$$

and R_l are defined by the following conditions:

$$rR_la_i \iff l = i;$$

$$a_iR_lb^j_k \iff i \neq l, i \neq k, k \neq l, j \neq l$$
$$b^j_kR_lc_i \iff k = l \text{ and } (j = i = k \text{ or } (j \neq k \text{ and } i \neq k)).$$

Put $G_\circ = (G, r)$, where $G = (W', R'_1, R'_2, R'_3)$ and

$$W' = W \cup \{\bar{b}^i_j \mid i, j = 1, 2, 3\} \cup \{\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_{123}\},$$

$$R'_l = R_l \cup \{(a_i, \bar{b}^j_k) \mid a_iR_lb^j_k\} \cup \{(\bar{b}^j_k, \bar{c}_i) \mid b^j_kR_lc_i\} \cup \{(\bar{b}^j_j, \bar{c}_{123}) \mid 1 \leq j \leq 3\}.$$

One can see that F is a part of G, that is there exists a natural embedding $\iota : W \to W'$.

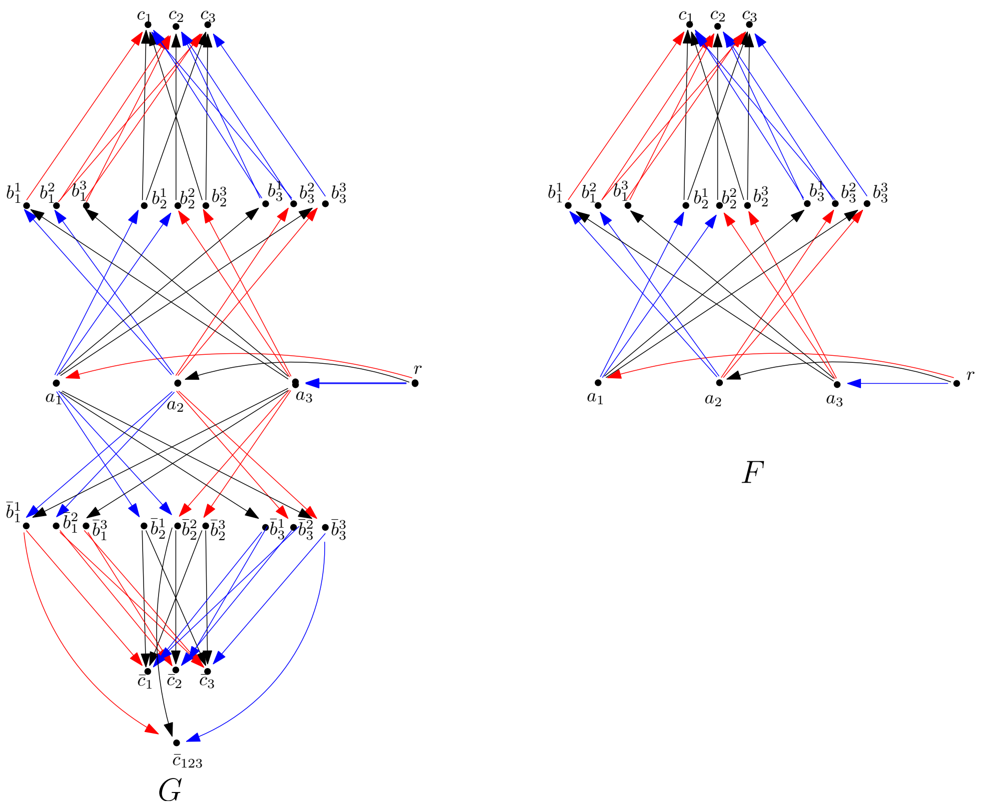


Fig. 3. Frames G and F; here R_1 is shown in red, R_2 in black and R_3 in blue

Lemma 7.2 *In $\mathcal{L}f_\Lambda$-structures F and G for all $i, l \in \{1,2,3\}$ for all $\varepsilon \in \{1,2,3\}^*$*

(3) $$a_i R^\varepsilon c_l \iff \varepsilon = jk, \ \textit{where } |\{i,j,k\}| = 3,$$

(4) $$r R^\varepsilon c_l \iff \varepsilon = ijk, \ \textit{where } |\{i,j,k\}| = 3;$$

In G for all $h \in \{1,2,3,123\}$ for all $\varepsilon \in \{1,2,3\}^*$ in G

(5) $$a_i R^\varepsilon \bar{c}_h \iff \varepsilon = jk \ \textit{and } \{i,j,k\} = \{1,2,3\}.$$

Proof. At first, we prove (3) for F and G simultaneously. Consider the point a_1. One can see that

$$a_1 R_2 b_3^1 R_3 c_1;$$

$$a_1 R_3 b_2^1 R_2 c_1;$$

$$a_1 R_2 b_3^1 R_3 c_2;$$

$$a_1 R_3 b_2^2 R_3 c_2;$$

$$a_1 R_2 b_3^3 R_3 c_3;$$

$$a_1 R_3 b_2^1 R_2 c_3;$$

and $R_1(a_1) = \emptyset$, $R_1((R_2 \cup R_3)(a_1)) = \emptyset$.

For the points a_2, a_3 the statement (3) is true, due to the symmetry of F. The statement (4) follows readily from (3); (5) can be checked similarly. □

Lemma 7.3 *In F*

$$R_3(R_2(a_1) \cap R_1(a_2)) \cap R_2(R_3(a_1) \cap R_1(a_3)) \cap R_1(R_2(a_3) \cap R_3(a_2)) = \emptyset$$

(and hence $F \not\models fc[x_0^F/x_0]$).

Proof. In fact, $R_2(a_1) \cap R_1(a_2) = b_3^3$, $R_3(a_1) \cap R_1(a_3) = b_2^2$, $R_2(a_3) \cap R_3(a_2) = b_1^1$, therefore

$$R_3(R_2(a_1) \cap R_1(a_2)) \cap R_2(R_3(a_1) \cap R_1(a_3)) \cap R_1(R_2(a_3) \cap R_3(a_2)) =$$

$$= R_3(b_3^3) \cap R_2(b_2^2) \cap R_1(b_1^1) = \{c_1\} \cap \{c_2\} \cap \{c_3\} = \emptyset.$$

□

Lemma 7.4 *$G_\circ$ is Kracht-reducible to $F_\circ$.*

Proof. Take a tree T with a root x_0^T and a monotonic mapping $f : T \to F$ sending x_0^T to r. Our goal is to construct a monotonic mapping $g : T \to G$ sending x_0^T to r and a relation $Z \subseteq W' \times W$ such that $(G, T, g) \gg (F, T, f)$.

To this end we use an embedding $\iota : W \to W'$ and put $g = \iota \cdot f$. Then we construct Z (uniformly for all (T, f)).

Put $(\alpha, \beta) \in Z$ if one of the following holds:

(i) $\alpha, \beta \in W$ and $\alpha = \beta$

(ii) $\alpha \in W', \beta \in W$ and $\alpha = \bar{\beta}$;

(iii) $\alpha = \bar{c}_{123}$, $\beta = c_i$, where $i \in \{1, 2, 3\}$.

Now our goal is to check the conditions KB1) – KB4).

KB1) Follows immediately from Item 1 of definition of Z.

KB2) Take $t \in W^T$ and $(\alpha, \beta) \in Z$. We have to ensure that for all $\varepsilon \in \{1, 2, 3\}^*$

(6) if $g(t) R^\varepsilon \alpha$ then $f(t) R^\varepsilon \beta$.

Consider the following two possibilities.

(i) $f(t) \in \{r, a_1, a_2, a_3\}$. Then $g(t) = f(t)$ and the only non-trivial instance of (6) (namely, for $\alpha = \bar{c}_{123}$) holds by (3), (4), and (5).

(ii) $f(t) \in \{b_i^j \mid i, j \in \{1, 2, 3\}\} \cup \{c_i \mid i \in \{1, 2, 3\}\}$. In this case if $g(t) R^\varepsilon \alpha$, then $\alpha \in W$. Hence $\beta = \alpha$, and (6) is evident, since F is a part of G.

KB3) Let $\alpha, \alpha' \in G$, $\alpha R_\lambda \alpha'$, and $\alpha Z \beta$. The only non-trivial case is when $\alpha' = \bar{c}_{123}$ (otherwise, we can take $\beta' = \sigma(\alpha')$, where $\sigma(w) = w$ for $w \in W$ and $\sigma(\bar{b}_i^j) = b_i^j$, $\sigma(\bar{c}_i) = c_i$). In this case β' also depends on α. But in G there are only three points seeing $\bar{c}_{123}$, namely, $\bar{b}_1^1$, $\bar{b}_2^2$ and $\bar{b}_3^3$. So we act as follows: if $\alpha = \bar{b}_k^k$, put $\beta' = c_k$.

KB4) Suppose that $(\alpha, \beta) \in Z$ and $\beta R_\lambda \beta'$. Consider the following two cases.

- If $\alpha = \beta$, then take $\alpha' = \beta'$;
- If $\alpha = \bar{\beta}, \beta \in \{b^i_j \mid i, j \in \{1, 2, 3\}$, then take $\alpha' = \bar{\beta}'$.

□

So due to Lemma 7.4, $G_\circ \ggg F_\circ$. But $G \models fc(r)$ and $F \not\models fc(r)$. Therefore the formula fc is not equivalent to any Kracht formula.

8 Final Remarks

(i) The proposed characterization looks more transparent in terms of games.

Two players, $\forall$ and $\exists$, play over a pair of $\mathcal{L}f_\Lambda$-structures with designated points (G, x_0^G) and (F, x_0^F). The player $\forall$ (he) wants to show that there exists a Kracht formula $\phi(x_0)$ such that $G \models \phi(x_0^G)$, but $F \not\models \phi(x_0^F)$, and the goal of $\exists$ is to prevent it. A position in the game is a triple (T, g, f), where T is a tree with a root x_0^T and $g : T \to G, f : T \to F$ are monotonic mappings sending x_0^T respectively to x_0^G and x_0^F.

When the game starts, $\forall$ announces the number of rounds n and constructs a tree T_0 and a monotonic mapping $f : T_0 \to F$ sending x_0^T to x_0^F. His opponent replies with a monotonic mapping $g : T_0 \to G$ sending x_0^T to x_0^G. Then n round follow indexed by numbers $1, \ldots, n$. Let T_{i-1} be a tree at the position before round i. The round i consists of the following actions. At first, $\forall$ adds a new leaf to T_{i-1}, and obtains a tree T_i. Then he chooses either F or G and extends respectively f or g to T_i. Then $\exists$ extends the other mapping to T_i (if she cannot, she looses). The game is won by $\exists$ if in the final position (T_n, g_n, f_n) for all $t \in T_0, s \in T_n$, $\varepsilon \in \Lambda^*$, if $g_n(t)(R^G)^\varepsilon g_n(s)$, then $f_n(t)(R^F)^\varepsilon f_n(s)$. An infinite generalization of such a game is obvious. In fact, these rules are easily extracted from the very definition of Kracht formulas.

In these terms a Kracht-simulation is nothing but a winning strategy for $\exists$ in a game with infinitely many rounds. Some lemmas from Section 6 can also be reformulated in terms of games. For example, Lemma 6.5 states that if $\exists$ has a winning strategy in every finite game between (G, x_0^G) and (F, x_0^F), then she has a winning strategy in an infinite game between suitable elementary extensions $\bar{G}$ and $\bar{F}$.

(ii) In [7] Kracht's theorem was extended to generalized Sahlqvist formulas of D. Vakarelov and V. Goranko [6]. The semantic characterization from the present paper can be easily transferred to the generalized Kracht formulas from [7].

(iii) Besides our characterization, there is another semantic property of Sahlqvist formulas [6]. A general frame $(W, (R_\lambda : \lambda \in \Lambda), \mathcal{A})$ is called *ample* if $\mathcal{A}$ contains the sets $R^\varepsilon(w)$ for all $\varepsilon \in \Lambda^*$, $w \in W$. A modal formula ϕ is called *locally a-persistent* if for every ample general frame $\mathfrak{F} = (F, \mathcal{A})$, for any world w in F, $\mathfrak{F}, w \models \phi \iff F, w \models \phi$. Clearly, all Sahlqvist modal formulas are a-persistent, and this fact is used in [6] to show that the first-order definable modal formula $p \wedge \Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q$ does not have a Sahlqvist equivalent. It is interesting whether the converse holds:

Question 8.1 *Does a-persistence and first-order definability of a modal formula ϕ imply its frame-equivalence to a Sahlqvist formula?*

Anyway it is interesting to understand how these two properties (i.e., a-persistence of a modal formula and Kracht-reducibility-invariance of its first-order equivalent) are related.

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