Conditionally Knowing What

Yanjing Wang Jie Fan

Department of Philosophy
Peking University
{*y.wang,fanjie*}*@pku.edu.cn*

Abstract

Classic epistemic logic focuses on propositional knowledge expressed by "knowing that" operators. However, there are various types of knowledge used in natural language, in terms of "knowing how", "knowing whether", "knowing what", and so on. In [10], Plaza proposed an intuitive know-what operator which was generalized in [16] by introducing a condition. The latter know-what operator can express natural conditional knowledge such as "I know what your password is, if it is 4-digits", which is not simply a material implication. Essentially this know-what operator packages a first-order quantifier and an S5-modality together in a non-trivial way, thus making it hard to axiomatize. In [16] an axiomatization is given for the single-agent epistemic logic with both know-that and know-what operators, while leaving axiomatizing the multi-agent case open due to various technical difficulties. In this paper, we solve this open problem. The completeness proof is highly non-trivial, compared to the single-agent case, which requires different techniques inspired by first-order intensional logic.

Keywords: knowing what, first-order intensional logic, epistemic logic, conditional knowledge.

1 Introduction

Epistemic Logic (**EL**), since its birth, has been mainly focusing on reasoning about propositional knowledge, the knowledge expressed by "i knows that ϕ". However, besides "knowing that", there are apparently various types of knowledge used in everyday life, expressed by "knowing what" (i knows what d is), "knowing how" (i knows how to do ϕ), "knowing whether"(i knows whether ϕ) and so on. A natural question which keeps philosophers busy is to ask whether these types of knowledge can be reduced to propositional knowledge. For example, there is a long-lasting debate in philosophy on whether "knowing how" can be reduced to "knowing that" ever since the seminal work of Ryle [12]. Compared to this heated discussion, "knowing what" received relatively little attention despite efforts trying to unify "knowing wh-" (what, where, which, who, why) and "knowing how" in terms of "knowing that" (e.g., [13]).

On the other hand, in computer science and AI, "knowing what" plays an important role, as argued by McCarthy [9]. In particular, in a security setting, we need to express that "He knows that she knows [what] her own private key [is], but he does not know what exactly the key is." The literal translation of this sentence in terms of the usual know-that modal operator K_i does not work since $K_1 K_2 p \wedge \neg K_1 p$ is not consistent in the standard epistemic logic with `T` axiom. This fact has lead a number of authors to propose suitable new knowledge operators (e.g., [11,6]). In [10], one of the defining works of dynamic epistemic logic, Plaza introduced a very natural modal operator Kv_i to express knowing what in a dynamic epistemic setting. $Kv_i d$ expresses exactly that "i knows what d is". As for the semantics, $Kv_i d$ is true on a pointed epistemic model with world-dependent assignments for d iff d has the same value on all the epistemically accessible worlds for i. In this setting, it is perfectly possible that $K_1 Kv_2 d \wedge \neg Kv_1 d$ since two kinds of knowledge are treated differently.

In [16], we generalize the Kv_i operator to a *conditional* one and obtain a complete axiomatization of the single-agent public announcement logic with both know-that and know-what operators. [1] The resulting new formula $Kv_i(\phi, d)$ expresses that "agent i knows what d is, given ϕ." For example, as it happens a lot in this internet age, I may forget my own login password for some website, but I know if the password is 4-digit then it must be 1234, since I have never used another 4-digit password (though I have several 6-digit passwords). In such a case people often say "I know my password if it is 4-digit". This can be expressed as $Kv_i(p, d)$ where p denotes the proposition that the password is 4-digit. Note that this *conditional knowledge* is not an implication $p \to Kv_i d$ nor $K_i(p \to Kv_i d)$. The difference is that, according to the semantics, $Kv_i(\phi, d)$ essentially expresses what i *would know* if he were *informed* that ϕ. This distinction will become clear when we define the operator formally. In this light, there is clearly a connection to Public Announcement Logic [10]: to know what d is given ϕ is similar to knowing what d is after the announcement ϕ. [2]

This kind of conditional knowledge has a philosophical connection to the phenomenon of *elusive knowledge* studied by Lewis [8]: "Maybe we do know a lot in daily life; but maybe when we look hard at our knowledge, it goes away." One explanation is that what we claim to know is mainly *conditional knowledge* where the conditions are often implicit, e.g., "I know I have hands" can be viewed as an abbreviation of "I know I have hands, given that I am not a brain in the vat". This holds for all kinds of common sense "knowledge" that we have in every day life, which invites systematic logical study.

Coming back to the technical storyline, note that the original Kv_i operator is a special case of the conditional one since $Kv_i d$ is simply $Kv_i(\top, d)$. Then it is natural to ask whether the epistemic logic extended with conditional Kv_i (call it $\mathbf{ELKv}^r$) is more expressive than the epistemic logic with the standard Kv_i operator (call it $\mathbf{ELKv}$). In [16] we show that $\mathbf{ELKv}^r$ is indeed strictly

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[1] We called the conditional version of Kv_i the *relativized* Kv_i operator in [16], due to the similarity between it and the relativized common knowledge operator introduced in [15]

[2] Though there is still a difference: we do not require ϕ to be truthful in $Kv_i(\phi, d)$: ϕ can be just *hypothetical*.

more expressive than **ELKv**. More interestingly, we show that adding the public announcement operators to $\mathbf{ELKv}^r$ does not increase the expressive power, i.e., $\mathbf{ELKv}^r$ is closed under announcement updates. As in the standard epistemic logic, this is a good property for a logic as a foundation of epistemic reasoning (cf. e.g.,[14]).

To really layout the foundation for reasoning about both knowing that and knowing what, we need to axiomatize $\mathbf{ELKv}^r$. In [16], an interesting system is given to axiomatize the single-agent $\mathbf{ELKv}^r$. The completeness proof relies on a canonical model construction which consists of two copies of each maximal consistent set. However, such a method does not generalize to the multi-agent case thus leaving the axiomatization of the multi-agent $\mathbf{ELKv}^r$ open.

In this paper, we solve this open problem by showing that the multi-agent version of the system proposed in [16] is indeed complete for the multi-agent $\mathbf{ELKv}^r$. The techniques used here are quite different from the single-agent completeness proof and are inspired by the following observation: $\mathbf{ELKv}^r$ can be viewed as a fragment of first-order intensional logic (**FOIL**) proposed and studied in [4,5]. **FOIL** features two kinds of variables: the object (rigid) variables and the intension (non-rigid) variables where the latter variables range over the functions from the set of possible worlds to the set of objects. A technique called *predicate abstraction* is applied to abstract predicates from formulas. Now consider the following fragment ($\mathbf{FOIL}^-$) of **FOIL** where the non-rigid variables are not quantified, the only predicates are unary ones over rigid variables, and the only (implicit) predicate abstraction is applied to equalities between a rigid variable x and a non-rigid variable d: [3]

$$\phi ::= \top \mid Px \mid d = x \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid \forall x\phi$$

$\mathbf{ELKv}^r$ can then be viewed as a small fragment of $\mathbf{FOIL}^-$ by recursively translating $Kv_i(\phi, d)$ into $\exists x K_i(\phi' \to d = x)$ where ϕ' is the $\mathbf{FOIL}^-$-translation of ϕ. Actually, this first-order formulation is also in accordance with the treatment of "knowing-wh" in terms of "knowing that" in [7,13]. In this way we can see clearly that Kv_i packages a first-order quantifier and a modality together.

This observation motivates our construction of the canonical model for multi-agent $\mathbf{ELKv}^r$. However, the method for axiomatizing **FOIL** as in [5] cannot be applied directly here, due to two reasons: first, our language is much weaker and we cannot express the desired first-order axioms in $\mathbf{ELKv}^r$; second, to our knowledge, it is unknown, how to axiomatize **FOIL** on S5 frames due to the diffcuities introduced by symmetry property as explained in [5]. In this work, we found a way to provide *just enough* extra information in the states of the canonical model to encode the "omitted" information expressible by potential $\mathbf{FOIL}^-$ formulas, while keeping it controlled by purely $\mathbf{ELKv}^r$ axioms. We believe that this method can be applied to other similar fragments of first-order intensional logic (over S5 frames).

[3] In Fitting's syntax of **FOIL**, $d = x$ should be formalized as $\langle \lambda y.y = x\rangle(d)$ (cf. [5]). Here, d may also be viewed as a constant since it is never quantified in the language.

In the rest of this paper, we first review the syntax and semantics of multi-agent $\mathbf{ELKv}^r$ and the proof system $\mathbb{ELKV}^r$ in Section 2. In Section 3, we prove our main result that $\mathbb{ELKV}^r$ completely axiomatizes multi-agent $\mathbf{ELKv}^r$. We conclude with future work in Section 4.

2 Preliminaries

Given a countably infinite set of proposition letters $\mathbf{P}$, a countably infinite set of agent names $\mathbf{I}$, and a countably infinite set of (non-rigid) constant symbols $\mathbf{D}$, the language of $\mathbf{ELKv}^r$ is defined as follows:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi \mid Kv_i(\phi, d)$$

where $p \in \mathbf{P}, i \in \mathbf{I}, d \in \mathbf{D}$.

$K_i\phi$ says that the agent i knows that ϕ. $Kv_i(\phi, d)$ says that the agent i knows what d is, given ϕ. More precisely, $Kv_i(\phi, d)$ says that the agent i would know what d is if he were informed that ϕ. The original (unconditional) $Kv_i d$ formulas proposed in [10] can be viewed as $Kv_i(\top, d)$. As usual, we define $\bot, (\phi \vee \psi), (\phi \to \psi), (\phi \leftrightarrow \psi), \hat{K}_i\phi$ as the abbreviations of, respectively, $\neg\top, \neg(\neg\phi \wedge \neg\psi), (\neg\phi \vee \psi), ((\phi \to \psi) \wedge (\psi \to \phi)), \neg K_i \neg\phi$. We omit parentheses from formulas unless confusion results.

$\mathbf{ELKv}^r$ is interpreted on epistemic models with assignments for the elements in $\mathbf{D}$: $\mathcal{M} = \langle S, O, \{\sim_i \mid i \in \mathbf{I}\}, V, V_{\mathbf{D}}\rangle$ where S is a non-empty set of possible worlds, O is a non-empty set of objects, $\sim_i$ is an equivalence relation over S, and V is a valuation function assigning a set of worlds $V(p) \subseteq S$ to each $p \in \mathbf{P}$, and $V_{\mathbf{D}} : \mathbf{D} \times S \to O$ is a function assigning each $d \in \mathbf{D}$ at each world an object. In terms of first-order intensional logic [4], $\mathcal{M}$ is an S5 intensional model with a constant domain O and assignments for non-rigid variables in $\mathbf{D}$. Note that for each $d \in \mathbf{D}$, $V_{\mathbf{D}}(d, \cdot)$ is a function from S to O which can be viewed as an *intension* as in [4]. The semantics is defined as follows:

$\mathcal{M}, s \vDash \top$		always holds
$\mathcal{M}, s \vDash p$	$\Leftrightarrow$	$s \in V(p)$
$\mathcal{M}, s \vDash \neg\phi$	$\Leftrightarrow$	$\mathcal{M}, s \nvDash \phi$
$\mathcal{M}, s \vDash \phi \wedge \psi$	$\Leftrightarrow$	$\mathcal{M}, s \vDash \phi$ and $\mathcal{M}, s \vDash \psi$
$\mathcal{M}, s \vDash K_i\psi$	$\Leftrightarrow$	for all t such that $s \sim_i t : \mathcal{M}, t \vDash \psi$
$\mathcal{M}, s \vDash Kv_i(\phi, d)$	$\Leftrightarrow$	for any $t_1, t_2 \in S$ such that $s \sim_i t_1$ and $s \sim_i t_2$: if $\mathcal{M}, t_1 \vDash \phi$ and $\mathcal{M}, t_2 \vDash \phi$, then $V_{\mathbf{D}}(d, t_1) = V_{\mathbf{D}}(d, t_2)$

Intuitively, $Kv_i(\phi, d)$ is true at s iff all the i-accessible ϕ-worlds agree on the value of d. In other words, i knows what d is given ϕ iff he is sure about d's value on ϕ-worlds. Based on this semantics, we can see clearly that $Kv_i(\phi, d)$ is indeed different from $\phi \to Kv_i d$ and $K_i(\phi \to Kv_i d)$. The condition ϕ restricts the accessible worlds to be considered, and we then check whether d has the same value on these "relative alternatives".

In [16], we give a complete axiomatization of the single agent $\mathbf{ELKv}^r$ and the following is the multi-agent version of that system which is an extension of the multi-modality $\mathbb{S}5$.

System $\mathbb{ELKV}^r$

Axiom Schemas		Rules	
TAUT	all the instances of tautologies		
DISTK	$K_i(\phi \to \psi) \to (K_i\phi \to K_i\psi)$	MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
T	$K_i\phi \to \phi$		
4	$K_i\phi \to K_iK_i\phi$	NECK	$\dfrac{\phi}{K_i\phi}$
5	$\neg K_i\phi \to K_i\neg K_i\phi$		
DISTKv^r	$K_i(\phi \to \psi) \to (Kv_i(\psi, d) \to Kv_i(\phi, d))$	RE	$\dfrac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$
Kv^{r}4	$Kv_i(\phi, d) \to K_iKv_i(\phi, d)$		
Kv$^r\bot$	$Kv_i(\bot, d)$		
Kv$^r\vee$	$\hat{K}_i(\phi \wedge \psi) \wedge Kv_i(\phi, d) \wedge Kv_i(\psi, d) \to Kv_i(\phi \vee \psi, d)$		

where RE is the rule of replacement of equivalents, which plays an important role in the later proofs. In the rest of the paper, we use $\vdash$ to denote the derivation relation within $\mathbb{ELKV}^r$.

Note that Kv_i operators do not behave like modalities in a normal modal logic and the (obvious adaptations of) necessitation rule and the K axiom are not valid for Kv_i. Instead, we have the distribution axiom schema DISTKv^r(note the swap of ψ and ϕ in the consequent). Kv^{r}4 is the counter part of the positive introspection axiom 4, and Kv$^r\bot$ stipulates the effect of the absurd precondition. The most important axiom is Kv$^r\vee$ which handles the composition of the conditions: if all the possible ϕ-worlds agree on what d is and all the possible ψ-worlds also agree on d, then the overlap between ϕ possibilities and ψ possibilities implies that all the $\phi \vee \psi$ possibilities also agree on what d is. We can show that the above system is sound (cf. [16, Theorem 11]).

To facilitate the later proofs, we need the following propositions.

Proposition 2.1 (i) $\vdash \neg K_i\phi \leftrightarrow K_i\neg K_i\phi$

(ii) *The rule (RM):* $\dfrac{\phi \to \psi}{\hat{K}_i\phi \to \hat{K}_i\psi}$ *is admissible in* $\mathbb{ELKV}^r$.

(iii) $\vdash \neg Kv_i(\phi, d) \leftrightarrow K_i\neg Kv_i(\phi, d)$

Proof (i) and (ii) are standard exercises in S5. The $\leftarrow$ direction in (iii) is trivial due to T. We show the other way around:

$K_iKv_i(\phi, d) \leftrightarrow Kv_i(\phi, d)$	T, Kv^{r}4
$\neg K_iKv_i(\phi, d) \to K_i\neg K_iKv_i(\phi, d)$	5
$\neg Kv_i(\phi, d) \to K_i\neg Kv_i(\phi, d)$	RE

□

Note that the $\to$ half of (iii) can be viewed as the Kv_i counterpart of 5 thus we denote it by Kv^{r}5.

Another useful observation is that Kv$^r\vee$ can be generalized to arbitrary finite disjunctions as the following proposition shows.

Proposition 2.2 *For any non-empty finite set U of* $\boldsymbol{ELKv}^r$ *formulas:*

$$\vdash \hat{K}_i(\bigwedge U) \wedge \bigwedge_{\phi\in U} Kv_i(\phi, d) \to Kv_i(\bigvee U, d).$$

Proof If $|U| = 1$ then the statement holds trivially. Suppose $|U| \geq 2$, we prove the statement by an inductive proof on $|U|$. The case of $|U| = 2$ is immediate due to $\mathtt{Kv}^r\vee$. Suppose the claim holds when $|U| = k$. Now consider the case $|U| = k + 1$, and let $U = U' \cup \{\psi\}$ such that $|U'| = k$:

(i)	$\hat{K}_i(\bigwedge U') \wedge \bigwedge_{\phi\in U'} Kv_i(\phi, d) \to Kv_i(\bigvee U', d)$	IH
(ii)	$Kv_i(\bigvee U', d) \wedge Kv_i(\psi, d) \wedge \hat{K}_i(\bigvee U' \wedge \psi) \to Kv_i(\bigvee U, d)$	$\mathtt{Kv}^r\vee$
(iii)	$\hat{K}_i(\bigwedge U) \wedge \bigwedge_{\phi\in U} Kv_i(\phi, d)$	$\mathtt{TAUT}$
	$\to \hat{K}_i(\bigwedge U') \wedge \bigwedge_{\phi\in U'} Kv_i(\phi, d) \wedge Kv_i(\psi, d) \wedge \hat{K}_i(\bigvee U' \wedge \psi)$	$\mathtt{DISTK}, \mathtt{NECK}$
(iv)	$\hat{K}_i(\bigwedge U) \wedge \bigwedge_{\phi\in U} Kv_i(\phi, d) \to Kv_i(\bigvee U, d)$	$(i)(ii)(iii)$

□

The following proposition essentially says if d has the same value over accessible ϕ- and ψ-worlds respectively, and there are a ϕ-world and a ψ-world sharing the same d value, then d has the same value over accessible $\phi\vee\psi$-worlds. This proposition allows us to relax the antecedent of $\mathtt{Kv}^r\vee$ a little bit to make it more useful.

Proposition 2.3 *For any* $\phi, \psi, \chi \in \boldsymbol{ELKv}^r$*, any* $d \in \boldsymbol{D}$*:*
$\vdash \hat{K}_i(\phi \wedge \chi) \wedge \hat{K}_i(\psi \wedge \chi) \wedge Kv_i(\phi, d) \wedge Kv_i(\psi, d) \wedge Kv_i(\chi, d) \to Kv_i(\phi \vee \psi, d)$

Proof

(i)	$\hat{K}_i(\phi \wedge \chi) \wedge Kv_i(\phi, d) \wedge Kv_i(\chi, d) \to Kv_i(\phi \vee \chi, d)$	$\mathtt{Kv}^r\vee$
(ii)	$\hat{K}_i(\psi \wedge \chi) \wedge Kv_i(\psi, d) \wedge Kv_i(\chi, d) \to Kv_i(\psi \vee \chi, d)$	$\mathtt{Kv}^r\vee$
(iii)	$\phi \wedge \chi \to (\phi \vee \chi) \wedge (\psi \vee \chi)$	$\mathtt{TAUT}$
(iv)	$\hat{K}_i(\phi \wedge \chi) \to \hat{K}_i((\phi \vee \chi) \wedge (\psi \vee \chi))$	$\mathtt{RM}, (iii)$
(v)	$\hat{K}_i((\phi \vee \chi) \wedge (\psi \vee \chi)) \wedge Kv_i(\phi \vee \chi, d) \wedge Kv_i(\psi \vee \chi, d)$	
	$\to Kv_i(\phi \vee \psi \vee \chi, d)$	$\mathtt{Kv}^r\vee$
(vi)	$\hat{K}_i(\phi \wedge \chi) \wedge \hat{K}_i(\psi \wedge \chi) \wedge Kv_i(\phi, d) \wedge Kv_i(\psi, d) \wedge Kv_i(\chi, d)$	
	$\to Kv_i(\phi \vee \psi \vee \chi, d)$	$(i)(ii)(iv)(v)$
(vii)	$\phi \vee \psi \to \phi \vee \psi \vee \chi$	$\mathtt{TAUT}$
$(viii)$	$K_i(\phi \vee \psi \to \phi \vee \psi \vee \chi)$	$\mathtt{NECK}, (vii)$
(ix)	$K_i(\phi \vee \psi \to \phi \vee \psi \vee \chi) \to Kv_i(\phi \vee \psi \vee \chi, d)$	
	$\to Kv_i(\phi \vee \psi, d)$	$\mathtt{DISTKv}^r$
(x)	$Kv_i(\phi \vee \psi \vee \chi, d) \to Kv_i(\phi \vee \psi, d)$	$\mathtt{MP}, (viii)(ix)$
(xi)	$\hat{K}_i(\phi \wedge \chi) \wedge \hat{K}_i(\psi \wedge \chi) \wedge Kv_i(\phi, d) \wedge Kv_i(\psi, d) \wedge Kv_i(\chi, d)$	
	$\to Kv_i(\phi \vee \psi, d)$	$(vi)(x)$

□

3 Completeness of multi-agent $\mathbb{ELKV}^r$

To prove the completeness, we need to build a canonical model such that each maximal consistent set of $\mathbb{ELKV}^r$ is satisfied in it. The general difficulty is as in the single-agent case: just using the maximal consistent sets as the states in the canonical model is not enough, more information should be provided in the states of the canonical model. As we mentioned, the $Kv_i(\phi, d)$ formulas can be viewed as $\exists x K_i(\phi \to d = x)$ where x is a rigid variable and d is a non-rigid one. To build a canonical model for such a first-order intensional logic, we need to include atomic formulas such as $d = x$ and modal formulas such as $K_i(\phi \to d = x)$ and control their interactions by axioms. However, those formulas are not expressible in $\mathbf{ELKv}^r$ since we simply cannot say what d exactly is. Therefore, in the canonical model, we need to equip the $\mathbb{ELKV}^r$-maximal consistent sets with information which can function as those atomic formulas. Moreover, we need to specify how such extra information is related to the $\mathbb{ELKV}^r$-maximal consistent sets. Note that since $\mathbb{ELKV}^r$ is very limited we cannot enforce the extra information behave exactly like the intended first-order intensional formulas. The real difficulty is to find the requirements which are "just enough" to make sure the truth lemma holds and this is the most fundamental idea behind our definition of the canonical model. It will also become more clear why the single-agent case is much simpler (cf. Remark 3.6).

3.1 Canonical model

In the sequel we define our canonical model with the set of natural numbers $\mathbb{N}$ as the constant domain of objects. [4]

Definition 3.1 *Let* $\boldsymbol{MCS}$ *be the set of maximal consistent sets w.r.t.* $\mathbb{ELKV}^r$*, and let* $\mathbb{N}$ *be the set of natural numbers. The canonical model* $\mathcal{M}^c$ *of* $\mathbb{ELKV}^r$ *is a tuple* $\langle S^c, \mathbb{N}, \{\sim_i^c \mid i \in \boldsymbol{I}\}, V^c, V^c_{\boldsymbol{D}}\rangle$ *where:*

- S^c *consists of all the triples* $\langle \Gamma, f, g\rangle \in \boldsymbol{MCS} \times \mathbb{N}^{\boldsymbol{D}} \times (\mathbb{N} \cup \{\star\})^{\boldsymbol{I} \times \boldsymbol{ELKv}^r \times \boldsymbol{D}}$ *that satisfy the following three conditions for any* $i \in \boldsymbol{I}$*, any* $\psi, \phi \in \boldsymbol{ELKv}^r$*, and any* $d \in \boldsymbol{D}$:
  (i) $g(i, \psi, d) = \star$ *iff* $Kv_i(\psi, d) \wedge \hat{K}_i\psi \notin \Gamma$*,*
  (ii) *If* $g(i, \phi, d) \neq \star$ *and* $g(i, \psi, d) \neq \star$ *then:* $g(i, \phi, d) = g(i, \psi, d)$ *iff there exists a* χ *such that* $Kv_i(\chi, d)$ *and* $\hat{K}_i(\phi \wedge \chi)$ *and* $\hat{K}_i(\psi \wedge \chi)$ *are in* Γ.
  (iii) $\psi \wedge Kv_i(\psi, d) \in \Gamma$ *implies* $f(d) = g(i, \psi, d)$.
  For any $s \in S^c$*, we write* $\phi \in s$ *if* ϕ *is in the maximal consistent set of* s *and write* $\phi \in s \cap t$ *if* $\phi \in s$ *and* $\phi \in t$*.* f_s *and* g_s *are used as the corresponding functions in* s*, and* $g_s(i)$ *is the function from* $\boldsymbol{ELKv}^r \times \boldsymbol{D}$ *to* $\mathbb{N} \cup \{\star\}$ *induced by* g_s *fixing a particular* $i \in \boldsymbol{I}$.
- $s \sim_i^c t$ *iff* $\{\phi \mid K_i\phi \in s\} \subseteq t$ *and* $g_s(i) = g_t(i)$
- $V^c_{\boldsymbol{D}}(d, s) = f_s(d)$

Remark 3.2 Intuitively, f is roughly functioning as the collection of $d = x$ formulas, and g is roughly functioning as the collection of $K_i(\psi \to d = x)$ formulas. Now for the intuitive ideas behind the three conditions:

[4] Note that this countable set is indeed big enough, since the (countable) language of $\mathbf{ELKv}^r$ can be translated into first-order intensional logic, which can be again translated into 3-sorted first-order logic, which still enjoys Löwenheim-Skolem property (cf. [1]).

- (i): We use $\star$ to mark that the value of $g(i, \psi, d)$ is *irrelevant.* If $Kv_i(\psi, d) \notin \Gamma$ then of course the value of $g(i, \psi, d)$ is irrelevant. If $Kv_i(\psi, d) \in \Gamma$ but $K_i\neg\psi \in \Gamma$, then the condition ψ is never possible for i thus the value of $g(i, \psi, d)$ is also irrelevant. Condition (i) is mainly for the technical convenience.
- (ii): This condition handles how the g values are inter-related. Intuitively, it roughly says that $x = y$ iff ($K_i(\psi \to d = x)$ and $K_i(\phi \to d = y)$, and there are some accessible ψ-world and ϕ-world which share the same value of d).
- (iii): Intuitively, this condition says that if $K_i(\psi \to d = x)$ and ψ is indeed true then $d = x$.

The definition of $\sim_i^c$ is in spirit the same as in the canonical model for the standard epistemic logic. The extra condition $g_s(i) = g_t(i)$ says the i-indistinguishable worlds should satisfy the same $K_i(\psi \to d = x)$ formulas.

Condition (i), (ii), (iii) and the definition of $\sim_i^c$ specify the minimal requirements of the extra information attached to maximal consistent sets. We need to control them without using **FOIL**$^-$ formulas.

To show the above model is indeed an epistemic model, we need the following proposition:

Proposition 3.3 *For any $i \in \boldsymbol{I}$, $\sim_i^c$ is an equivalence relation.*

Proof As a standard exercise in modal logic, by using `T`, `4`, and `5`, we can prove the following claim:

$$\text{(for all } \phi : K_i\phi \in s \text{ implies } \phi \in t\text{) iff (for all } \phi : K_i\phi \in s \text{ iff } K_i\phi \in t\text{)} \qquad (*).$$

Thus $s \sim_i^c t$ iff $\{\phi \mid K_i\phi \in s\} = \{\phi \mid K_i\phi \in t\}$ and $g_s(i) = g_t(i)$. Then it is easy to see $\sim_i^c$ is an equivalence relation.

□

Based on the above claim $(*)$, using `T` and $\texttt{Kv}^r4$, the following is immediate:

Proposition 3.4 *For any two maximal consistent sets Δ and Γ, if $\{\phi \mid K_i\phi \in \Delta\} \subseteq \Gamma$, then the following hold for all ϕ:*

- $K_i\phi \in \Delta$ *iff* $K_i\phi \in \Gamma$
- $\hat{K}_i\phi \in \Delta$ *iff* $\hat{K}_i\phi \in \Gamma$
- $Kv_i(\phi, d) \in \Delta$ *iff* $Kv_i(\phi, d) \in \Gamma$.

3.2 Completeness

In order to establish the truth lemma, the most difficult things are the existence lemmas for K_i and Kv_i operators. Since the states in the canonical models are *not* merely maximal consistent sets, more efforts are required.

We first propose a general method to construct proper successors. This can be viewed as some kind of Lindenbaum's Lemma, though highly non-trivial in this case, if we view g and f as collections of "hidden formulas".

Proposition 3.5 *Given a state $s \in S^c$, an agent $i \in \boldsymbol{I}$, and a maximal consistent set Γ such that $\{\phi \mid K_i\phi \in s\} \subseteq \Gamma$, and any natural number x, we have a deterministic method to construct $t = \langle \Gamma, f, g\rangle$ based on s, Γ, and x such that $t \in S^c$ and $s \sim_i^c t$.*

Proof The construction and the proof are quite involved: we first build f and build g by using finite approximations and show that the constructions are well-defined; then we show that the $\langle \Gamma, f, g\rangle$ satisfy the three conditions of states in S^c and that $s \sim_i^c t$. In the following, we fix a natural number x.

Let $T^0 = \{\langle j, \phi, d\rangle \mid j = i \text{ or } \phi \wedge Kv_j(\phi, d) \in \Gamma\}$. Let $g^0 : T^0 \to \mathbb{N} \cup \{\star\}$ be defined as follows:

$$g^0(j, \phi, d) = \begin{cases} g_s(i, \phi, d) & \text{if } j = i \\ g_s(i, \psi, d) & \text{if } j \neq i \text{ and } \psi \wedge Kv_i(\psi, d) \in \Gamma \text{ for some } \psi \\ x & \text{if otherwise} \end{cases}$$

We need to show that the second case is well-defined: the choice of ψ does not affect the value of $g_s(i, \psi, d)$, namely $\psi \wedge Kv_i(\psi, d) \in \Gamma$ and $\psi' \wedge Kv_i(\psi', d) \in \Gamma$ implies $g_s(i, \psi, d) = g_s(i, \psi', d)$. Suppose that $\psi \wedge Kv_i(\psi, d) \in \Gamma$ and $\psi' \wedge Kv_i(\psi', d) \in \Gamma$, then the following formulas are also in Γ: $\hat{K}_i\psi', \hat{K}_i\psi, \hat{K}_i(\psi' \wedge \psi)$ due to the contrapositive of Axiom `T`. By Proposition 3.4, the following formulas are all in s: $\hat{K}_i(\psi' \wedge \psi), \hat{K}_i\psi'$, $\hat{K}_i\psi$, $Kv_i(\psi', d)$, and $Kv_i(\psi, d)$. Now it is clear that $g_s(i, \psi, d) \neq \star$ and $g_s(i, \psi', d) \neq \star$ due to condition (i) of s. Let $\chi = \psi'$ now we have $Kv_i(\chi, d)$ and $\hat{K}_i(\psi' \wedge \chi)$ and $\hat{K}_i(\psi \wedge \chi)$ are all in s. By condition (ii) of s, $g_s(i, \psi', d) = g_s(i, \psi, d)$.

Now we define f as follows:

$$f(d) = \begin{cases} g^0(j, \phi, d) & \text{if } Kv_j(\phi, d) \wedge \phi \in \Gamma \text{ for some } \phi \text{ and } j \\ x & \text{if otherwise} \end{cases}$$

We need to show that the first case is well-defined: the choices of ϕ and j do not affect the value of $g^0(j, \phi, d)$, namely $\phi \wedge Kv_j(\phi, d) \in \Gamma$ and $\psi \wedge Kv_k(\psi, d) \in \Gamma$ implies $g^0(j, \phi, d) = g^0(k, \psi, d)$. To see this, consider four cases:

- $j = i$ and $k = i$. Then due to the above proof and the first clause of the definition of g^0, we have $g^0(j, \phi, d) = g_s(i, \phi, d) = g_s(i, \psi, d) = g^0(k, \psi, d)$.
- $j \neq i$ and $k \neq i$. If there exists χ such that $\chi \wedge Kv_i(\chi, d) \in \Gamma$, then by the second clause of g^0, we have $g^0(j, \phi, d) = g_s(i, \chi, d) = g^0(k, \psi, d)$; otherwise, by the third clause of g^0, we have $g^0(j, \phi, d) = x = g^0(k, \psi, d)$.
- $j = i$ and $k \neq i$. From the first clause of g^0, it follows that $g^0(j, \phi, d) = g_s(i, \phi, d)$. Due to the fact that $\phi \wedge Kv_i(\phi, d) \in \Gamma$ and the second clause of g^0, we have $g^0(k, \psi, d) = g_s(i, \phi, d)$, thus $g^0(j, \phi, d) = g^0(k, \psi, d)$.
- $j \neq i$ and $k = i$. Similar to the third case.

Now let Δ be the set of the remaining non-i-triples:

$$\Delta = \{\langle j, \phi, d\rangle \mid j \neq i, \phi \wedge Kv_j(\phi, d) \notin \Gamma, j \in \mathbf{I}, \phi \in \mathbf{ELKv}^r, d \in \mathbf{D}\}.$$

Due to the fact that $\mathbf{D}$, $\mathbf{I}$ and $\mathbf{ELKv}^r$ are countable, we can enumerate Δ as $\delta_1, \delta_2, \dots$ and approximate g step by step by extending the domain of g^k with δ_{k+1}. Let Δ^k be $\{\delta_l \mid 1 \leq l \leq k\}$, and in particular $\Delta^0 = \emptyset$. Let Λ^k be $\{g^k(\delta) \mid \delta \in \Delta^k\}$ and let max be the function which assigns to each non-empty finite set of natural numbers its maximum. Let $Dom(g^k)$ be the domain of g^k. For $k \geq 0$, our construction of $g^{k+1} : T^0 \cup \Delta^{k+1} \to \mathbb{N} \cup \{\star\}$ is as follows: let $g^{k+1}(j, \phi, d) = g^k(j, \phi, d)$ if $\langle j, \phi, d \rangle \in Dom(g^k) = T^0 \cup \Delta^k$, and for the only new $\langle j, \phi, d \rangle \notin Dom(g^k)$ (thus $j \neq i$) we have:

$$g^{k+1}(j,\phi,d) = \begin{cases} \star & \text{if } \hat{K}_j\phi \wedge Kv_j(\phi,d) \notin \Gamma \\ g^k(j,\psi,d) & \text{if } \hat{K}_j\phi \wedge Kv_j(\phi,d) \in \Gamma \text{ and there are} \\ & \chi,\psi \text{ such that } \langle j,\psi,d\rangle \in Dom(g^k) \\ & \text{and the following formulas are all} \\ & \text{in } \Gamma:\ \hat{K}_j(\chi\wedge\phi),\ \hat{K}_j(\chi\wedge\psi), \\ & Kv_j(\psi,d),\ Kv_j(\chi,d) \\ max(\Lambda^k \cup \{f(d)\}) + 1 & \text{if otherwise} \end{cases}$$

Note that, we still need to show that the second case in the above definition of g^{k+1} is well-defined. More precisely, we need to show for any $k \geq 0$ any $j \neq i$, the following (1) implies (2):

(1) there exist ψ, ψ', χ, and χ' such that $\langle j, \psi, d \rangle$ and $\langle j, \psi', d \rangle$ are in $Dom(g^k)$ and the following formulas are all in Γ: $\hat{K}_j(\chi \wedge \psi), \hat{K}_j(\chi \wedge \phi), \hat{K}_j(\chi' \wedge \psi'), \hat{K}_j(\chi' \wedge \phi), Kv_j(\psi, d), Kv_j(\psi', d), Kv_j(\chi, d), Kv_j(\chi', d)$.

(2) $g^k(j, \psi, d) = g^k(j, \psi', d)$.

Induction on k:

- $k = 0$: $\langle j, \psi, d \rangle$ and $\langle j, \psi', d \rangle$ are both in $Dom(g^0) = T^0$ then according to the definition of g^0 and the fact that $j \neq i$, $g^0(j, \psi, d) = g^0(j, \psi', d)$.
- Induction Hypothesis: (1) implies (2) holds for all $k \leq n$.
- $k = n + 1$: w.l.o.g, we assume that at least one of (j, ψ, d) and (j, ψ', d) is not in $Dom(g^0)$, for otherwise the case is like the above one. Then we can assume that there exists an $m \leq n$ such that $\langle j, \psi, d \rangle \in Dom(g^m)$, $\langle j, \psi', d \rangle \notin Dom(g^m)$, and $\langle j, \psi', d \rangle$ is added into $Dom(g^{m+1})$ by our construction. Assuming (1), let θ be $\chi \vee \chi'$, we can show $\hat{K}_j(\theta \wedge \psi)$ and $\hat{K}_j(\theta \wedge \psi') \in \Gamma$ since $\hat{K}_j(\chi \wedge \psi)$ and $\hat{K}_j(\chi' \wedge \psi')$ are in Γ. Moreover, since $\hat{K}_j(\chi \wedge \phi), \hat{K}_j(\chi' \wedge \phi), Kv_j(\chi, d), Kv_j(\chi', d), Kv_j(\phi, d) \in \Gamma$, we have $Kv_j(\theta, d) \in \Gamma$ by Proposition 2.3. Now we have $\hat{K}_j(\theta \wedge \psi)$, $\hat{K}_j(\theta \wedge \psi')$, and $Kv_j(\theta, d)$ are all in Γ. According to our construction of g^{m+1}, $g^{m+1}(j, \psi', d) = g^m(j, \psi, d)$ and the induction hypothesis guarantees the uniqueness of $g^m(j, \psi, d)$ since $m \leq n$. Therefore $g^k(j, \psi', d) = g^{m+1}(j, \psi', d) = g^m(j, \psi, d) = g^k(j, \psi, d)$.

Viewing each g^k as a set of pairs $\langle\langle j, \phi, d \rangle, g^k(j, \phi, d)\rangle$, we let g be $\bigcup_{k<\omega} g^k$.

Now we need to verify conditions (i), (ii) and (iii). Condition (iii) is trivial by the definition of f. We verify condition (i) and (ii) below.

For condition (i): we first show that for the fixed i and any $\phi \in \mathbf{ELKv}^r$, any $d \in \mathbf{D}$: $g^0(i, \phi, d) \neq \star$ iff $\hat{K}_i\phi \wedge Kv_i(\phi, d) \in \Gamma$. From right to left: suppose that $g^0(i, \phi, d) = \star$ then we have $g_s(i, \phi, d) = \star$ thus $\hat{K}_j\phi \wedge Kv_i(\phi, d) \notin s$, i.e., $K_i\neg\phi \in s$ or $\neg Kv_i(\phi, d) \in s$. By Proposition 3.4, we have $K_i\neg\phi \in \Gamma$ or $\neg Kv_i(\phi, d) \in \Gamma$, i.e., $\hat{K}_i\phi \wedge Kv_i(\phi, d) \notin \Gamma$. From left to right: suppose that $g^0(i, \phi, d) \neq \star$ then $\hat{K}_i\phi \wedge Kv_i(\phi, d) \in s$ thus by Proposition 3.4 again, $\hat{K}_i\phi \wedge Kv_i(\phi, d) \in \Gamma$.

Now consider $\langle j, \phi, d\rangle \in Dom(g^0)$ where $j \neq i$. By definition, $\phi \wedge Kv_j(\phi, d) \in \Gamma$, thus $\hat{K}_j\phi \wedge Kv_j(\phi, d) \in \Gamma$. By the construction of g^0 it is clear that $g^0(j, \phi, d) \neq \star$, since $x \neq \star$ and the fact that $\hat{K}_i\psi \wedge Kv_i(\psi, d) \in \Gamma$ implies $g^0(i, \psi, d) \neq \star$ which we have just proved. This concludes the proof for the base case: for any $\langle j, \phi, d\rangle \in Dom(g^0)$: $g^0(j, \phi, d) \neq \star$ iff $\hat{K}_j\phi \wedge Kv_j(\phi, d) \in \Gamma$. The inductive case is obvious by the three cases of our construction of g^{k+1}.

Condition (ii) is more complicated to verify and it requires an inductive proof. We first claim the following:

Claim ($\circ$): For each $k \geq 0$, and any $\langle j, \psi, d\rangle$ and $\langle j, \phi, d\rangle$ in $Dom(g^k)$ such that $g^k(j, \psi, d) \neq \star$ and $g^k(j, \phi, d) \neq \star$, the following two are equivalent:

(1) $g^k(j, \phi, d) = g^k(j, \psi, d)$

(2) there exists a χ such that $Kv_j(\chi, d)$, $\hat{K}_j(\phi \wedge \chi)$ and $\hat{K}_j(\psi \wedge \chi)$ are in Γ.

If claim ($\circ$) holds then Condition (ii) holds too, since any $\langle j, \psi, d\rangle$ and $\langle j, \phi, d\rangle$ must both exist in $Dom(g^k)$ for some k. Now we prove the claim ($\circ$).

- If $k = 0$ then both $\langle j, \psi, d\rangle$ and $\langle j, \phi, d\rangle$ are in $Dom(g^0)$. There are two subcases:
  - · If $j = i$ then we have $g(j, \phi, d) = g(j, \psi, d)$ iff $g^0(i, \phi, d) = g^0(i, \psi, d)$ iff $g_s(i, \phi, d) = g_s(i, \psi, d)$ iff there exists a χ such that $Kv_i(\chi, d)$ and $\hat{K}_i(\phi \wedge \chi)$ and $\hat{K}_i(\psi \wedge \chi)$ are all in s (by condition (ii) of s). According to Proposition 3.4, the last statement is equivalent to that there exists a χ such that $\{Kv_i(\chi, d), \hat{K}_i(\phi \wedge \chi), \hat{K}_i(\psi \wedge \chi)\} \subseteq \Gamma$.
  - · If $j \neq i$ then clearly $g(j, \psi, d) = g^0(j, \psi, d) = g^0(j, \phi, d) = g(j, \phi, d)$ by the definition of g^0. Now since $g(j, \psi, d) \neq \star$, $Kv_j(\psi, d) \in \Gamma$ due to condition (i) of Γ which we have just verified. Since $\langle j, \psi, d\rangle$ and $\langle j, \phi, d\rangle$ are both in $Dom(g^0) = T^0$, we have $\phi, \psi \in \Gamma$, thus $\hat{K}_j(\phi \wedge \psi) \in \Gamma$ by axiom `T`. Finally we have $\{Kv_j(\chi, d), \hat{K}_j(\phi \wedge \chi), \hat{K}_j(\psi \wedge \chi)\} \subseteq \Gamma$ given $\chi = \psi$.
- Induction Hypothesis: the claim ($\circ$) holds for $k \leq m$.
- Suppose $k = m+1$ and at least one of $\langle j, \phi, d\rangle$ and $\langle j, \psi, d\rangle$ is not in $Dom(g^m)$, for otherwise it can be handled by IH. Then clearly $j \neq i$ for otherwise both triples are in $Dom(g^0)$ thus in $Dom(g^m)$. W.l.o.g, we can assume that $\langle j, \psi, d\rangle \in Dom(g^m)$ and $\langle j, \phi, d\rangle \notin Dom(g^m)$ but $\langle j, \phi, d\rangle \in Dom(g^{m+1})$, i.e., $\langle j, \phi, d\rangle$ is added at step $m+1$. By assumption $g^{m+1}(j, \phi, d) \neq \star$ and $g^{m+1}(j, \psi, d) \neq \star$, then by condition (i) $\hat{K}_j\phi \wedge Kv_j(\phi, d) \in \Gamma$ and $\hat{K}_j\psi \wedge Kv_j(\psi, d) \in \Gamma$. Now if there exists a χ such that $\hat{K}_j(\chi \wedge \psi) \wedge \hat{K}_j(\chi \wedge \phi) \wedge$

$Kv_j(\chi, d) \in \Gamma$ then by the second clause of the definition of g^{m+1} we have $g^{m+1}(j, \phi, d) = g^m(j, \psi, d) = g^{m+1}(j, \psi, d)$. This proves that (2) implies (1).

For the other direction, suppose that $g^{m+1}(j, \phi, d) = g^{m+1}(j, \psi, d) = g^m(j, \psi, d) \neq \star$, due to the definition of g^{m+1}, $g^{m+1}(j, \phi, d)$ must be constructed according to the second clause, for the third clause can make sure $g^{m+1}(j, \phi, d) \neq g^m(j, \psi, d)$. To see this, note that if $\langle j, \psi, d\rangle \in Dom(g^0)$ then $g^m(j, \psi, d) = g^0(j, \psi, d) = f(d)$ by the definition of g^0 (note that $j \neq i$). The third clause guarantees that $g^{m+1}(j, \phi, d) > f(d) = g^m(j, \psi, d)$. If $\langle j, \psi, d\rangle \notin Dom(g^0)$ then the third clause guarantees that $g^{m+1}(j, \phi, d) > max(\Lambda^m) \geq g^m(j, \psi, d)$.

Now, if $g^{m+1}(j, \phi, d)$ is constructed by the second clause based on $g^m(j, \psi, d)$ then (2) is immediate. Suppose otherwise that $g^{m+1}(j, \phi, d)$ is constructed based on $g^m(j, \theta, d)$ for some $\theta \neq \psi$, such that $g^m(j, \theta, d) = g^m(j, \psi, d) \neq \star$, then there exists a ξ such that $\hat{K}_j(\xi \wedge \theta) \wedge \hat{K}_j(\xi \wedge \phi) \wedge Kv_j(\xi, d) \in \Gamma$. Since $g^m(j, \theta, d) \neq \star$, by condition (i) we also have $Kv_j(\theta, d) \in \Gamma$. Since $g^m(j, \theta, d) = g^m(j, \psi, d) \neq \star$, by IH, there exists ξ' such that $\hat{K}_j(\xi' \wedge \psi) \wedge \hat{K}_j(\xi' \wedge \theta) \wedge Kv_j(\xi', d) \in \Gamma$. Now we have $\hat{K}_j(\xi \wedge \theta), \hat{K}_j(\xi' \wedge \theta), Kv_j(\xi, d), Kv_j(\xi', d)$ and $Kv_j(\theta, d)$ all in Γ. By Proposition 2.3, $Kv_j(\xi \vee \xi', d) \in \Gamma$. Let $\chi = \xi \vee \xi'$. Since $\hat{K}_j(\xi \wedge \phi)$ and $\hat{K}_j(\xi' \wedge \psi)$ are in Γ, we have $\hat{K}_j(\chi \wedge \phi), \hat{K}_j(\chi \wedge \psi)$ and $Kv_j(\chi, d)$ are all in Γ, and this completes the proof of claim ($\circ$). Thus $\langle \Gamma, f, g\rangle$ satisfies the condition (ii).

In sum, $\langle \Gamma, f, g\rangle \in S^c$, and $s \sim^c_i \langle \Gamma, f, g\rangle$ due to the facts that $g(i) = g_s(i)$ (by the construction of g^0) and the assumption that $\{\phi \mid K_i\phi \in s\} \subseteq \Gamma$. □

Remark 3.6 To build an i-successor of s, we need to construct a proper g such that it takes care of the information not only about i but also about $j \neq i$. Note that if $\mathbf{I} = \{i\}$, then $g(i) = g_s(i)$ implies $g = g_s$. In this case we do not need the above construction, thus the single-agent case is much simper.

Important notation In the sequel, we refer to the above construction of f as $F(s, i, \Gamma, x)$ where x is a natural number as a parameter.

Now we are ready to prove two important existence lemmas:

Lemma 3.7 *For any $s \in S^c$, any $i \in \boldsymbol{I}$: $K_i\psi \notin s$ implies there is a world t such that $s \sim^c_i t$ and $\neg\psi \in t$.*

Proof It is a standard exercise in modal logic to show that $X = \{\neg\psi\} \cup \{\phi \mid K_i\phi \in s\}$ is consistent. Then by Lindenbaum Lemma for $\mathbf{ELKv}^r$, there exists an MCS Γ including X. Now from Proposition 3.5 we can equip Γ with some proper f and g, such that $\langle \Gamma, f, g\rangle \in S^c$ and $s \sim^c_i \langle \Gamma, f, g\rangle$. □

Lemma 3.8 *For any $s \in S^c$, any $i \in \boldsymbol{I}$: $\neg Kv_i(\phi, d) \in s$ implies there are two states w, v in S^c such that $s \sim^c_i w$, $s \sim^c_i v$, $\phi \in w \cap v$, and $f_w(d) \neq f_v(d)$.*

The proof of the above lemma is again quite involved, we break it into Proposition 3.9 and Proposition 3.10 below.

Proposition 3.9 *Given any $s \in S^c$ and any $i \in \boldsymbol{I}$, suppose there exist two (possibly identical) maximal consistent sets Γ_1 and Γ_2 such that:*

(a) $\{\psi \mid K_i\psi \in s\} \subseteq \Gamma_1 \cap \Gamma_2$

(b) *for any $Kv_i(\theta, d) \in s$, $\theta \notin \Gamma_1 \cap \Gamma_2$.*

then Γ_1 and Γ_2 can be extended into two states w, v in S^c such that $s \sim_i^c w$, $s \sim_i^c v$ and $f_w(d) \neq f_v(d)$.

Proof By condition (a) and Proposition 3.5, Γ_1 and Γ_2 can be extended into two i-accessible states by using $F(s, i, \Gamma_1, x)$ and $F(s, i, \Gamma_2, y)$. We argue that condition (b) and condition (ii) of s allow us to construct two states in S^c that differ in the value of d. Consider the following cases:

- Suppose that there is no $Kv_i(\chi, d) \wedge \chi \in \Gamma_1$ for any χ. Note that in this case if $f_w = F(s, i, \Gamma_1, x)$ then $f_w(d) = x$. Now let $f_v = F(s, i, \Gamma_2, 0)$ and let $f_w = F(s, i, \Gamma_1, f_v(d) + 1)$. Clearly $f_w(d) = f_v(d) + 1 \neq f_v(d)$. The symmetric case when there is no $Kv_i(\chi, d) \wedge \chi \in \Gamma_2$ for any χ is similar.
- Suppose there exists $Kv_i(\chi, d) \wedge \chi \in \Gamma_1$ for some χ and there exists $Kv_i(\chi', d) \wedge \chi' \in \Gamma_2$ for some χ'. Now let $f_w = F(s, i, \Gamma_1, 0)$ and $f_v = F(s, i, \Gamma_2, 0)$ we have $f_w(d) = g_s(i, \chi, d)$ and $f_v(d) = g_s(i, \chi', d)$. We need to show $g_s(i, \chi, d) \neq g_s(i, \chi', d)$. Towards contradiction suppose $g_s(i, \chi, d) = g_s(i, \chi', d)$ then by condition (ii) of s, there exists θ such that $Kv_i(\theta, d)$ and $\hat{K}_i(\theta \wedge \chi)$ and $\hat{K}_i(\theta \wedge \chi')$ are in s. Note that due to Proposition 3.4, $Kv_i(\chi, d)$ and $Kv_i(\chi', d)$ are in s. Now by Proposition 2.3, $Kv_i(\chi \vee \chi', d) \in s$. However, since $\chi \in \Gamma_1$ and $\chi' \in \Gamma_2$, $\chi \vee \chi' \in \Gamma_1 \cap \Gamma_2$, which contradicts to the assumption (b).

□

In [16], we proved the following proposition in the single agent case. The proof for the multi-agent version is almost the same.

Proposition 3.10 *Given any $s \in S^c$ and any $i \in \boldsymbol{I}$, suppose $\neg Kv_i(\phi, d) \in s$ then there are two (possibly identical) maximal consistent sets Γ_1 and Γ_2 such that:*

(a') $\{\phi\} \cup \{\psi \mid K_i\psi \in s\} \subseteq \Gamma_1 \cap \Gamma_2$

(b) *for any $Kv_i(\theta, d) \in s$, $\theta \notin \Gamma_1 \cap \Gamma_2$.*

Proof Let $Z = \{\psi \mid K_i\psi \in s\} \cup \{\phi\}$ and let $X = \{\neg\theta \mid Kv_i(\theta, d) \in s\}$. Note that due to $\mathtt{Kv}^r\bot$, X is non-empty.[5] We want to build two consistent sets B and C such that $Z \subseteq B \cap C$ and $X \subseteq B \cup C$. Then by a Lindenbaum-like argument over countable language, we can extend B and C into the desired Γ_1 and Γ_2: (a') is guaranteed by $Z \subseteq B \cap C$, and (b) is guaranteed by $X \subseteq B \cup C$

[5] $\mathtt{Kv}^r\bot$ is indispensable in the proof system $\mathbb{ELKV}^r$. We can show that it is not provable in $\mathbb{ELKV}^r - \mathtt{Kv}^r\bot$ by designing an alternative semantics which coincides with the standard semantics for Kv_i-free formulas but falsifies all the $Kv_i(\phi, d)$ formulas for any i, ϕ and d. It is not hard to see that $\mathbb{ELKV}^r - \mathtt{Kv}^r\bot$ is sound w.r.t. this new semantics but $Kv_i(\bot, d)$ is not valid, thus $\mathtt{Kv}^r\bot$ is not provable in $\mathbb{ELKV}^r$.

which says that for any $Kv_i(\theta, d) \in s$, $\neg\theta \in B$ or $\neg\theta \in C$ thus $\theta \notin \Gamma_1$ or $\theta \notin \Gamma_2$. In the following we build B and C.

The idea is straightforward: simply adding the formulas in X one by one into two copies of Z while keeping the consistency. Formally, we enumerate formulas in X as $\neg\theta_0, \neg\theta_1, \dots$ and let $B_0 = Z \cup \{\neg\theta_0\}$ and let $C_0 = Z$ as the starting points. Then we build B_{n+1} and C_{n+1} based on the already defined B_n and C_n by adding $\neg\theta_{n+1}$ into one of them:

(i) if $\neg\theta_{n+1}$ is consistent with B_n then $B_{n+1} = B_n \cup \{\neg\theta_{n+1}\}$ and $C_{n+1} = C_n$;

(ii) if $\neg\theta_{n+1}$ is not consistent with B_n then $B_{n+1} = B_n$ and $C_{n+1} = C_n \cup \{\neg\theta_{n+1}\}$.

Let $B = \bigcup_{n<\omega} B_n$, $C = \bigcup_{n<\omega} C_n$ and we need to show that B and C are consistent. Note that B (C) is consistent iff B_n (C_n) is consistent for each n, since if B (C) is not consistent then there must be an n such that B_n (C_n) is not consistent, due to the finitary nature of logical consistency. In the following we show B_n and C_n are consistent by induction on n.

- $n = 0$: Suppose towards contradiction that B_0 is not consistent, then there exist $\psi_1, \dots, \psi_m \in \{\psi \mid K_i\psi \in s\}$ such that $\vdash \psi_1 \wedge \cdots \wedge \psi_m \wedge \phi \to \theta_0$, i.e., $\vdash \psi_1 \wedge \cdots \wedge \psi_m \to (\phi \to \theta_0)$. Therefore $\vdash K_i\psi_1 \wedge \cdots \wedge K_i\psi_m \to K_i(\phi \to \theta_0)$ by `DISTK`, `NECK` and `RE`. Since $K_i\psi_1, \dots, K_i\psi_m \in s$, $K_i(\phi \to \theta_0) \in s$. Now by `DISTKv`^{r} and the fact that $Kv_i(\theta_0, d) \in s$ (since X is non-empty), it follows that $Kv_i(\phi, d) \in s$, contradiction. Since $C_0 \subseteq B_0$, C_0 is also consistent.
- $n = k + 1$: by the induction hypothesis B_k and C_k are consistent. According to our construction of B_{k+1} we just need to show that if $\neg\theta_{k+1}$ is not consistent with B_k then it is consistent with C_k. Suppose not, then both $B_k \cup \{\neg\theta_{k+1}\}$ and $C_k \cup \{\neg\theta_{k+1}\}$ are inconsistent. In the sequel, to derive a contradiction, we adopt the proof in [16, Lemma 19] for the multi-agent setting .

  Let $\overline{U} = B_k \backslash Z$, $\overline{V} = C_k \backslash Z$, $U = \{\theta \mid \neg\theta \in \overline{U}\}$, and $V = \{\theta \mid \neg\theta \in \overline{V}\}$. Note that $U, V, \overline{U}, \overline{V}$ are all finite and each formula in $\overline{V}$ is not consistent with B_k due to the construction of B_k.

  We claim: there exist $\psi_1, \dots, \psi_l, \psi'_1, \dots, \psi'_m$, $\psi''_1, \dots, \psi''_r \in \{\psi \mid K_i\psi \in s\}$ such that
$$\begin{array}{ll} (i) & \vdash \psi_1 \wedge \cdots \wedge \psi_l \wedge \phi \wedge \bigwedge \overline{U} \to \theta_{k+1}, \\ (ii) & \vdash \psi'_1 \wedge \cdots \wedge \psi'_m \wedge \phi \wedge \bigwedge \overline{V} \to \theta_{k+1}, \\ (iii) & \vdash \psi''_1 \wedge \cdots \wedge \psi''_r \wedge \phi \wedge \bigwedge \overline{U} \to \bigwedge V. \end{array}$$

  (i) and (ii) are immediate from the inconsistency of $B_k \cup \{\neg\theta_{k+1}\}$ and $C_k \cup \{\neg\theta_{k+1}\}$. For (iii), first recall that for any $\theta \in V$, $\{\neg\theta\} \cup B_k$ is inconsistent due to the construction of B_k. Therefore for each $\theta \in V$ there exist $\chi_1, \dots, \chi_h \in \{\psi \mid K_i\psi \in s\}$ such that:
$$\vdash (\chi_1 \wedge \cdots \wedge \chi_h \wedge \phi \wedge \bigwedge \overline{U}) \to \theta$$
  Since V is a finite set, we can collect all such χ for each $\theta \in V$ to obtain (iii).

From $(i) - (iii)$, NECK, DISTK, RE and the fact that

$$K_i\psi_1, \ldots, K_i\psi_l, K_i\psi'_1, \ldots, K_i\psi'_m, K_i\psi''_1, \ldots, K_i\psi''_n \in s,$$

we can show the following:

$$\begin{array}{ll} (iv) & K_i((\phi \wedge \bigwedge \overline{U}) \to \theta_{k+1}) \in s, \\ (v) & K_i((\phi \wedge \bigwedge \overline{V}) \to \theta_{k+1}) \in s, \\ (vi) & K_i((\phi \wedge \bigwedge \overline{U}) \to \bigwedge V) \in s. \end{array}$$

In the following, we will show that $\hat{K}_i(\theta_{k+1} \wedge \bigwedge V) \in s$. First we claim $\hat{K}_i(\phi \wedge \bigwedge \overline{U}) \in s$. Suppose not, then $K_i \neg(\phi \wedge \bigwedge \overline{U}) \in s$, thus $\neg(\phi \wedge \bigwedge \overline{U}) \in B_k$. Due to the construction of B_k we know ϕ and $\overline{U}$ are in B_k, thus B_k is inconsistent, contradicting the assumption. Therefore $\hat{K}_i(\phi \wedge \bigwedge \overline{U}) \in s$ thus by $(iv), (vi)$ we have $\hat{K}_i(\theta_{k+1} \wedge \bigwedge V) \in s$.

By our assumption, for any $\theta \in V \cup \{\theta_{k+1}\}$ we have $Kv_i(\theta, d) \in s$. Now based on this fact and $\hat{K}_i(\theta_{k+1} \wedge \bigwedge V) \in s$, we can use Proposition 2.2, and obtain the following:

$$(vii) \ Kv_i(\theta_{k+1} \vee \bigvee V, d) \in s.$$

Now using $\vdash \neg \bigwedge \overline{V} \leftrightarrow \bigvee V$, let us change the from of (v) to the following:

$$(v') \ K_i(\phi \to (\bigvee V \vee \theta_{k+1})) \in s,$$

Based on $(v'), (vii)$ and DISTKv^r, we have $Kv_i(\phi, d) \in s$, contradiction. Therefore, B_{k+1} and C_{k+1} are consistent and this concludes the the inductive proof.

In sum, B and C are consistent thus can be extended into Γ_1 and Γ_2 satisfying (a') and (b). □

Clearly, (a') in Proposition 3.10 implies (a) in Proposition 3.9, then Lemma 3.8 is immediate.

Now we are ready to prove the truth lemma:

Lemma 3.11 (Truth Lemma) *For any $\phi \in$ **ELKv**^{r} and $s \in S^c$, $\phi \in s$ iff $\mathcal{M}^c, s \vDash \phi$.*

Proof We only show the non-trivial cases of $K_i\psi$ and $Kv_i(\psi, d)$.

- $\phi = K_i\psi$: If $K_i\psi \in s$, then for any t such that $s \sim_i^c t$ we have $\psi \in t$ by the definition of $\sim_i^c$. Now by induction hypothesis (IH), $\mathcal{M}^c, s \vDash K_i\psi$. Now suppose $K_i\psi \notin s$, then by Lemma 3.7 and the IH, we have $\mathcal{M}^c, s \vDash \neg K_i\psi$.
- $\phi = Kv_i(\psi, d)$: Suppose that $Kv_i(\psi, d) \in s$, $s \sim_i^c t$, $s \sim_i^c t'$, $\psi \in t$ and $\psi \in t'$. It is easy to see that $Kv_i(\psi, d) \in t \cap t'$ and $g_t(i) = g_{t'}(i) = g_s(i)$. Since $\psi \in t$ and $\psi \in t'$, according to condition (iii) and the fact that $g_t(i) = g_{t'}(i)$:

$$V_{\mathbf{D}}(d, t) = f_t(d) = g_t(i, \psi, d) = g_s(i, \psi, d) = g_{t'}(i, \psi, d) = f_{t'}(d) = V_{\mathbf{D}}(d, t').$$

Now suppose $Kv_i(\psi, d) \notin s$ then by Lemma 3.8 and IH, $\mathcal{M}^c, s \vDash \neg Kv_i(\psi, d)$.

□

From the above truth lemma, the completeness theorem almost follows. The only missing piece is to show for each maximal consistent set there is indeed at least one corresponding state in $\mathcal{M}^c$.

Lemma 3.12 *For every maximal consistent set* Γ*, there exist* f *and* g *such that* $\langle\Gamma, f, g\rangle \in S^c$*.*

Proof The construction is very similar to the one in the proof of Proposition 3.5, though simpler. The only essential difference is the definition of g^0, thus the proofs related to g^0 need to be adapted.

Let x be a natural number, and let $T = \{\langle j, \phi, d\rangle \mid Kv_j(\phi, d) \wedge \phi \in \Gamma, j \in \mathbf{I}, \phi \in \mathbf{ELKv}^r, d \in \mathbf{D}\}$.[6] Let $g^0 : T \to \mathbb{N} \cup \{\star\}$ be the constant function such that $g^0(j, \phi, d) = x$ for all the triples in T. We define f as the constant function such that $f(d) = x$ for all $d \in \mathbf{D}$. Now redefine Δ as the set of the remaining triples:

$$\Delta = \{\langle j, \phi, d\rangle \mid \phi \wedge Kv_j(\phi, d) \notin \Gamma, j \in \mathbf{I}, \phi \in \mathbf{ELKv}^r, d \in \mathbf{D}\}.$$

As before, we can enumerate Δ and build g^{k+1} by adding the new $\langle j, \phi, d\rangle \notin Dom(g^k)$ into the domain (where Λ^k is defined as before w.r.t. the new Δ):

$$g^{k+1}(j, \phi, d) = \begin{cases} \star & \text{if } \hat{K}_j\phi \wedge Kv_j(\phi, d) \notin \Gamma \\ g^k(j, \psi, d) & \text{if } \hat{K}_j\phi \wedge Kv_j(\phi, d) \in \Gamma \text{ and there are} \\ & \chi, \psi \text{ such that } \langle j, \psi, d\rangle \in Dom(g^k) \\ & \text{and the following formulas are all} \\ & \text{in } \Gamma:\ \hat{K}_j(\chi \wedge \phi),\ \hat{K}_j(\chi \wedge \psi), \\ & Kv_j(\psi, d),\ Kv_j(\chi, d) \\ max(\Lambda^k \cup \{f(d)\}) + 1 & \text{if otherwise} \end{cases}$$

Similar to the corresponding proof of Proposition 3.5, we can show that the second clause in the above definition of g^{k+1} is well-defined ($k = 0$ case is now obvious due to the definition of g^0).

Now let $g = \bigcup_{k\in\mathbb{N}} g^k$, we need to verify conditions (i), (ii) and (iii). Condition (iii) is immediate from the definition of f.

For condition (i), if $\langle j, \phi, d\rangle \in Dom(g^0) = T$, then $g^0(j, \phi, d) = x \neq \star$, and we can see that $Kv_j(\phi, d) \wedge \hat{K}_j\phi \in \Gamma$ since $Kv_j(\phi, d) \wedge \phi \in \Gamma$. Thus we have for any $\langle j, \phi, d\rangle \in Dom(g^0)$, $g^0(j, \phi, d) \neq \star$ iff $Kv_j(\phi, d) \wedge \hat{K}_j\phi \in \Gamma$. The inductive case is obvious by the three clauses of g^{k+1}.

For condition (ii), suppose that $g(j, \phi, d) \neq \star$ and $g(j, \psi, d) \neq \star$, we need to prove claim ($\circ$) inductively as before. For that, we only need to revise the proof for the base case as follows:

Suppose $k = 0$ and thus both $\langle j, \phi, d\rangle$ and $\langle j, \psi, d\rangle$ are in $Dom(g^0)$. By definition of g^0, it is easy to see that $g(j, \phi, d) = g^0(j, \phi, d) = x = g^0(j, \psi, d) = g(j, \psi, d)$. Moreover, we have $\phi \wedge Kv_j(\phi, d) \in \Gamma$ and $\psi \wedge Kv_j(\psi, d) \in \Gamma$. Then

[6] Note that T may be empty. In that case we start from the empty function g^0.

setting $\chi = \phi$ gives us $Kv_j(\chi, d) \wedge \hat{K}_j(\phi \wedge \chi) \wedge \hat{K}_j(\psi \wedge \chi) \in \Gamma$. Then for any $\langle j, \phi, d\rangle$ and $\langle j, \psi, d\rangle$ in $Dom(g^0)$, $g^0(j, \phi, d) = g^0(j, \psi, d)$ iff there exists a χ such that $\{Kv_j(\chi, d), \hat{K}_j(\phi \wedge \chi), \hat{K}_j(\psi \wedge \chi)\} \subseteq \Gamma$.

In sum, $\langle \Gamma, f, g\rangle \in S^c$. □

Based on Lemma 3.12 and Lemma 3.11 we can show the completeness.

Theorem 3.13 $\mathbb{ELKV}^r$ *is sound and strongly complete for multi-agent* ***ELKv***^{r}.

Proof The soundness part can be found in [16, Theorem 11]. For the completeness part, we show that each consistent set of **ELKv**^{r} formulas is satisfiable. Given a consistent set Δ of **ELKv**^{r} formulas, by the Lindenbaum Lemma for the countable language **ELKv**^{r}, there exists a maximal consistent set Γ such that $\Delta \subseteq \Gamma$. Now Lemma 3.12 tells us that there exist f, g such that $\langle \Gamma, f, g\rangle \in S^c$. From Lemma 3.11, it follows that $\mathcal{M}^c, \langle \Gamma, f, g\rangle \vDash \Gamma$ thus $\mathcal{M}^c, \langle \Gamma, f, g\rangle \vDash \Delta$. □

In [16], we also discussed the logic of **ELKv**^{r} extended with public announcement operators (call it **PALKv**^{r}):

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid \langle\phi\rangle\phi$$

As an immediate corollary of the above completeness theorem and Theorem 10 in [16], we can axiomatize multi-agent **PALKv**^{r} by adding the following reduction axioms to $\mathbb{ELKV}^r$ (call the resulting system $\mathbb{PALKV}^r$):

`!ATOM`	$\langle\psi\rangle p \leftrightarrow (\psi \wedge p)$
`!NEG`	$\langle\psi\rangle\neg\phi \leftrightarrow (\psi \wedge \neg\langle\psi\rangle\phi)$
`!CON`	$\langle\psi\rangle(\phi \wedge \chi) \leftrightarrow (\langle\psi\rangle\phi \wedge \langle\psi\rangle\chi)$
`!K`	$\langle\psi\rangle K_i\phi \leftrightarrow (\psi \wedge K_i(\psi \to \langle\psi\rangle\phi))$
`!Kv`^{r}	$\langle\phi\rangle Kv_i(\psi, d) \leftrightarrow (\phi \wedge Kv_i(\langle\phi\rangle\psi, d))$

Corollary 3.14 $\mathbb{PALKV}^r$ *is sound and complete for multi-agent* ***PALKv***^{r}.

4 Future work

In this paper, we showed that $\mathbb{ELKV}^r$ is sound and complete for multi-agent **ELKv**^{r} (over S5 frames). This is just a starting point of an unfolding story about interesting modal fragments of first-order intensional logic.

For future work, the decidability of **ELKv**^{r} deserves a careful investigation. The single-agent case is particularly promising, since we do have a neat canonical model construction which only uses two copies of each maximal consistent set, which may facilitate a finite filtration leading to to the small model property. On the other hand, there are also hints for the undecidablity, for example, in [3], it is shown that the quantifier-free fragment of S5-**FOIL** is undecidable, where arbitrary relation symbols and arbitrary predicate abstractions are allowed. Of course we may study the logic of **ELKv**^{r} on other weaker frame classes, where decidability is more plausible according to [3].

Another natural question to ask is how to axiomatize the logic where Kv_i are the *only* primitive operators (call it **PLKv**^{r}). In $\mathbb{ELKV}^r$, most of the axioms involve interactions between "knowing that" and "knowing what". We are unsure if the system without these axioms can axiomatize $\mathbf{PLKv}^r$, though it is unlikely.

As motivated in the introduction, $\mathbf{ELKv}^r$ can be used in a security setting where the interaction between "knowing what" and "knowing that" is important. To really handle epistemic reasoning in such scenarios, we need to express statements like "I know that the message I just received is indeed the private message that I sent before for authentication", where equality is inevitable. Due to our completeness proof method, we suspect that adding the equality symbol (between $d \in \mathbf{D}$) freely in $\mathbf{ELKv}^r$ may in turn ease the axiomatization, since we have a better grip on the information we need in the canonical model.

Last but not least, on the philosophical side, the conditional versions of other types of knowledge should be studied, probably in the context of relevant alternative theory [2], since they may capture the common sense use of knowledge better.

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