N-DIMENSIONAL GREGORY-BÉZIER FOR N-DIMENSIONAL CELLULAR COMPLEXES

Thery S., Bechmann D., Bertrand Y.

LSIIT UPRES-A ULP-CNRS 7005 - Université Louis Pasteur,
Pôle API - boulevard Sébastien Brant
67400 ILLKIRCH Cedex, France
Email : thery@irit.fr ,bechmann@dpt-info.u-strasbg.fr,bertrand@sic.sp2mi.univ-poitiers.fr
Phone : +33 (0)3 88 65 55 26
Fax : +33 (0)3 88 65 55 01

ABSTRACT

With industrial modelling tools, topological structures and free form surfaces are often managed separately, and patches used for embedding are limited to dimension 2. A new approach is to combine topology structures of any dimension with embedding of same dimension, and use topological operations to modify the shape and the properties of surfaces and volumes. The use of Chains of map as topological structure and Gregory-Bézier as embedding allows the conception of very general objects, made of various dimensional rectangular and triangular patches.

Keywords: modelling tool, topological structure, Bézier-Gregory patch, association.

1. INTRODUCTION

In geometric modelling, there are principally two different kinds of modelling tools. Some are called volume modelling tools, because they can directly model object's volume with CSG trees. Embedding of volumes is implicitly described by equations of basic volumes and composition operations. Other modelling tools, mostly used in CAD/CAM, are called surface modelling tools because most of their operations allow the manipulation of complicated surfaces. Topological structures are often poorly managed.

A recent improvement of surfaces modelling tools is the use of volume topological structures. Complex free form surfaces are associated with faces of objects. Indeed, rectangular and triangular patches are not often combined. The association between topological structures and free form surfaces is often complicated and needs extra data structures.

There are several disadvantages with these modelling methods. Firstly, they do not allow the embedding of 3-dimensional topological cells with volume patches. This can be essential for some applications, such as geological modelling or fluid mechanics simulation. Furthermore, there are also some operations that can be difficult to do with these tools. For example, the chamfering operation gives naturally two kinds of patches, rectangular and triangular. In addition, another problem is the way to associate topological structures with free form surfaces. Modelling tools often use complicated and costly data structures, which imply an increase of the costs in time and space.

We propose a new kind of embedding which avoids these disadvantages. We model free form surfaces with Gregory patches of dimension *n*, triangular or rectangular, strongly associated with chains of maps. The topological structure itself is used to organize embedding data. In addition, this method allows the use of triangular and rectangular patches together and avoids the problem of control points numbering. Finally, the isomorphism between topological structures and free form surfaces implies that there is no need for extra costly data structures to manage the embedding of objects.

We begin in Section 2 and 3 by short recollections about topological and free form models. Then we present the association between chains and patches. Section 4 presents association in dimension 1, the case of dimension 2 is shown in Section 5. Then we show how these associations could be helpful for managing free form surfaces in Section 6. In Section 7 we introduce Gregory-Bézier volumes with a generalization of the association to dimension 3. Before concluding, the case of dimension n is explained in Section 8.

2. TOPOLOGICAL MODELS

The topological model we will use in the remainder of the paper is chains of maps (*n*-Chains). It is an improvement of a more simple model called generalized maps (*n*-G-maps) defined in [lienh88][lienh89][lienh94] and which allow only the modelling of manifold objects (Figure 1). *n*-Chains [elter92][elter93] can be use to define non-manifold (Figure 2) objects which can be orientable or not. For this study, we only give intuitive notions of these topological models, necessary for the comprehension of the remaining of the article.

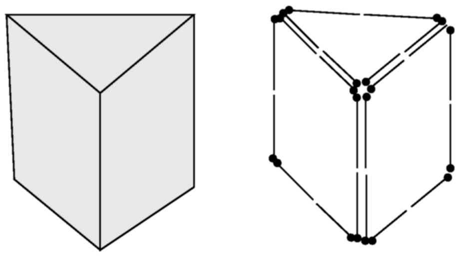


Figure 1: a manifold object, and its topological representation

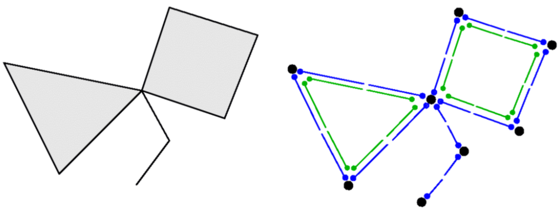


Figure 2: a non manifold object and its topological representation

N-G-MAPS

Notion of G-maps of dimension *n* or *n*-G-map are defined by a unique kind of abstract element, called darts. Darts of G-maps are linked together by involution α_i. On the Figure 3, we can see four G-maps, of dimension 0, 1, 2, and 3. On (a) a vertex, represented by a unique dart. On (b) two darts linked together by α_0 build an edge. On (c) four edges are assembled by α_1 make a quadrangular face (8 darts). And finally on (d) a cube is modelled with 6 faces which darts are linked by α_2 (48 darts).

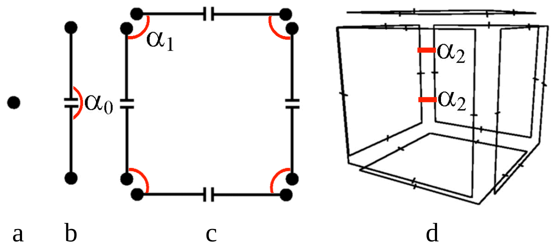


Figure 3: Four cells modelled by G-maps. (a) a vertex, (b) an edge, (c) a face, and (d) a cube.

N-CHAINS OF MAPS

The chains of maps have been introduced to permit the topological modelling of non-manifold objects. With a chain model, each cell is independently represented. A chain is made of several G-maps linked together by mappings. The Figure 4 shows four examples of complexes modelled with chains. On (a) a vertex is modelled as it is with the G-maps representation. On (b), an edge is here modelled with four darts. Three G-maps are linked together, one for the edge itself, and two for its boundaries. On (c) a face modelled with nine G-maps, a G-map of dimension 2 for the face, four 1-G-maps for the edges of the faces, and four 0-G-maps for the vertices. Finally on (d) a cube is modelled with eight 0-G-maps, twelve 1-G-maps, six 2-G-maps and one 3-G-map. G-maps are linked together by mapping σ.

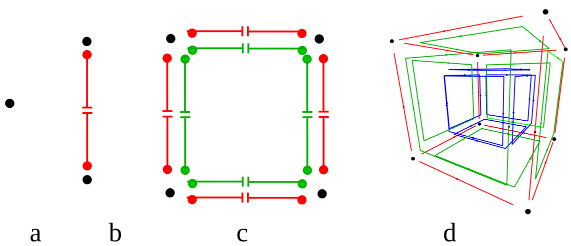


Figure 4: Four cells modelled by chains of maps. (a) a vertex, (b) an edge, (c) a face, and (d) a cube. 0-G-maps are drawn in black, 1-G-maps in red, 2-G-maps in green and the 3-G-map in blue.

3. GREGORY-BÉZIER PATCHES

The embedding models used in this work are Gregory-Bézier patches [Grego74] [Chiyo83] [Takam90], which is an enhancement of Bézier patches [Bezie74] [Farin86] [Farin88]. One of the principal difficulties encountered in the study of patches continuity is the intertwining of constraints around a central point. To bypass this difficulty Gregory-Bézier patches have been introduced. By splitting internal control points, they permit independent twist constraints along two consecutive boundaries. Their definitions apply to rectangular and triangular cases.

A rectangular Gregory-Bézier patch of degree $m \times n$ can be constructed from a Bézier patch of the same degree. Four inner control points have to be split to give a new lattice as shown on Figure 5.

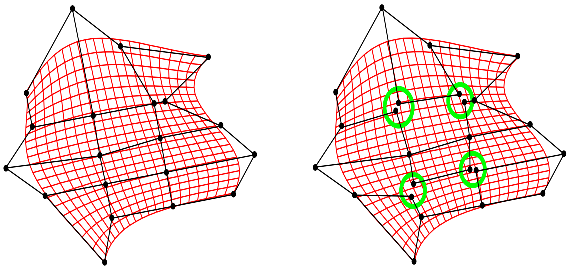


Figure 5: Lattice of control points of a rectangular Bézier patch of degree 3x4 and the corresponding Gregory patch.

Construction of a triangular Gregory-Bézier patch of degree 3 is more complicated. It can be done from a triangular Bézier patch of degree 4 in two steps. First the boundary of the patch is degenerate from degree 3 to degree 4, and then inner control points are split.

The Figure 6 shows Bézier and Gregory-Bézier patches with their lattices of control points.

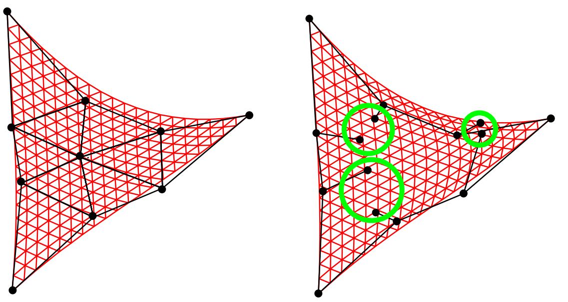


Figure 6: Lattice of control points of a triangular Bézier patch and the corresponding Gregory Bézier patch.

If we look at a corner or at a side of a patch, we can notice that the structure of triangular and rectangular Gregory-Bézier patches are similar. This makes it easier to join and also to control the continuity of rectangular and triangular patches together.

The reader can consult [Grego74] and [Chiyo83] for complete definitions and properties of Gregory-Bézier rectangular patches and [Takam90] for triangular patches.

4. CURVE ASSOCIATION

We want here to associated Bézier curves of degree 3 with 1-chains. The associated *1*-chain is made of four darts, which are linked in a manner which make the structure very similar to the one of the Bézier curve lattice of control points (see Figure 7).

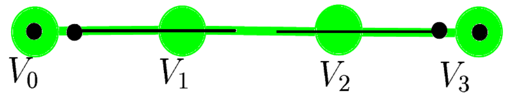


Figure 7: A 1-chain and the associated lattice of control points.

5. SURFACE ASSOCIATIONS

Association between 2-chains and Gregory-Bézier patches is very natural. A simple chain, which models a rectangular face, has as many darts as there are control points in a rectangular Gregory-Bézier patch. Furthermore their structures are very similar. Internal control points can be associated with darts of the face, corner points with darts of the vertices, and others points with darts of the edges (top of Figure 8). The association can also be done with triangular patches, the same remarks about structure of patches and chains can be done (see bottom of Figure 8).

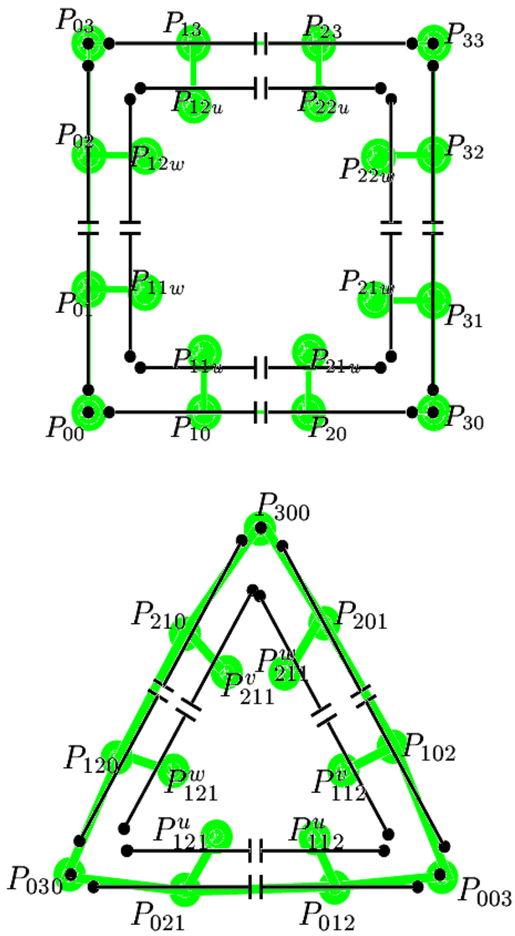


Figure 8: A rectangular and a triangular Gregory-Bézier patch with associated 2-chains in grey.

6. ADVANTAGES OF ASSOCIATIONS

There are several advantages in our model, even when modelling only 2-dimensional objects (surfaces). Firstly, it does not need any numbering and extra data structure to store and access control points. Then, it automatically ensures G^0 continuity between surfaces when joining the topological structures. It improves the access to control points and finally permits the use of rectangular and triangular patches together.

NO GLOBAL NUMBERING AND COSTLY STRUCTURE

In a classical approach, control points of patches are stored in a vector, and a numbering and a function-of-indexes transformation ensure their access. This transformation is easy with Bézier patches, but with the structure of Gregory-Bézier that is not a matrix, the storage of their control points is more complicated. With the association we have defined, there is a one to one mapping between darts and control points. Control points data can then be stored in the dart data structure, and there is no need of extra data structure to manage control points.

AUTOMATIC G^0 CONTINUITY

The most common operation on patches is joining. There are several kinds of joining, which depend on the continuity of surface. G^0 continuity defines a joining with the common boundary curve. A G^1 continuity joining must be G^0 and must verify the continuity of tangent plane along the common boundary. Geometric continuity (G^n) and derivative continuity (C^n) can be defined at any order.

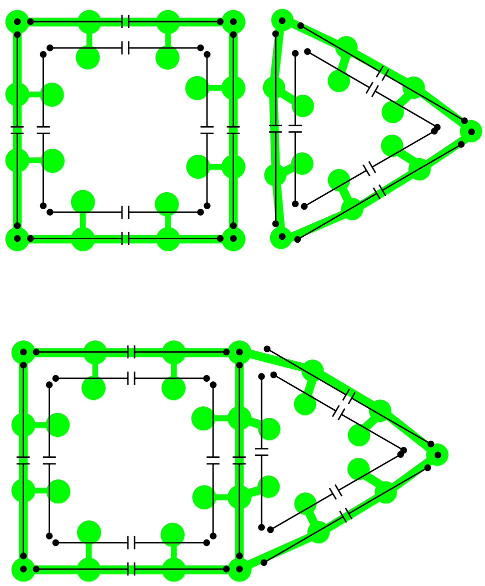


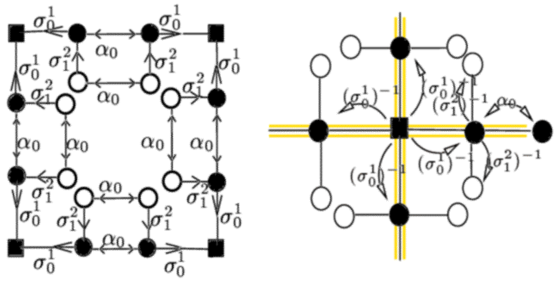
Figure 9: Two patches before and after identification of their boundaries.

The operation of identification permits an automatic G^0 joining of patches. When we identify two 2-chains with a common boundary of dimension 1, it merges the two subset of four darts defining the two initial boundaries. When these 2-chains are associated with Gregory-Bézier patches, the two subsets of darts are associated with the controls points defining the two initial boundaries. Then, the identification merges the two subsets of control points and gives a G^0 continuity joining as it is shown on Figure 9.

To obtain higher order continuity, constraints must be applied on control points. These constraints can no more be automatically obtained from topological operation, but they can be optimised by them. In fact, topological structures can be used to improve access to subset of control points involved in continuity constraints.

IMPROVEMENT OF ACCESS TO CONTROL POINTS

Access to control points can be improved with topological functions that permit direct access to control points. This is possible because there is a relation between topological function (σ and α) and geometrical relative positions of points. Involutions α link darts of G-map. α_0 links the two darts of an edge. α_1 links darts of edges to create a face. In a chain, σ^i_j links a dart of a I-G-map with a dart of a j-G-map. Figure 10 (a) shows the relations between these functions and the possible ways of moving in the mesh of control points.



a b

Figure 10: Different kinds of geometrical relations and the associated topological functions.

This allows us easily to access the subset of control points involved in constraints like continuity. For example, it is easy to find all control points directly adjacent to a vertex *P* common to several patches by applying $(\sigma_0^1)^{-1}$ on *P* (Figure 10 (b), *P* is the square).

MIXING OF RECTANGULAR AND TRIANGULAR PATCHES

Classical modelling tools use currently only one kind of patch, either rectangular (mostly used in industry) or triangular. But surfaces of real objects are often a combination of rectangular and triangular surfaces. In addition, some operations, like edge blending, generate the two kinds of patches as it shown on Figure 11. Our model permits the used of combined rectangular and triangular patches directly (Figure 9). In addition the structure and definition of Gregory-Bézier patches allow us to control continuity of joining between patches whatever their shape (triangular or rectangular).

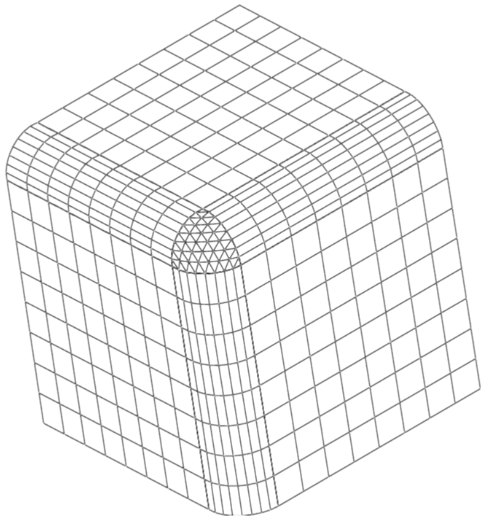


Figure 11: Example of edges and vertex blending of a cube.

7. GENERALIZATION TO DIMENSION 3

Gregory-Bézier patches have been defined from Bézier patches in dimension 2. Bézier patches can be defined very easily at any dimension, but the extension to Gregory-Bézier patches is not so easy.

To generalize rectangular Gregory-Bézier patches, we have to determine which control points have to be split. The problem is more complicated in the triangular case. In dimension 2, we have to take a Bézier patch of degree 4, but in dimension 3 it does not work with a tetrahedron of degree 4.

The use of chains of dimension 3 makes these extensions easier. They permit us to determine the structure of patches directly, because there is a one to one mapping between the darts of a chain and the control points of a Gregory-Bézier.

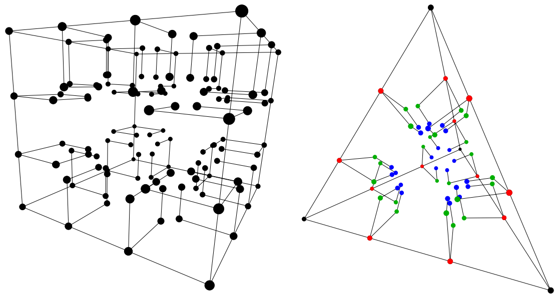


Figure 12: Gregory-Bézier cube and tetrahedron.

We have seen in section 5 that the structure of 2-Chains maps exactly with the one of Gregory-Bézier patches. Then, we can use the structure of 3-Chains to define the shape of Gregory-Bézier volumes of degree 3.

The association says that there is a one to one mapping between darts and control points. Here we take a 3-chain which models a hexahedron to determine Gregory Bézier cube (Figure 13). We know that there are 128 darts in this chain, this gives us the number of control points. Furthermore the structure of the chain gives us the structure of the lattice of control points. Indeed, as in dimension 2 there is a mapping between boundaries of the topological structures and boundaries of free form volumes. The eight darts that model vertices of the chain are associated with the eight corners of the lattice. The twenty-four darts of topological edges are associated with control points that belong to edges of the lattice. The twenty-four darts of topological face are associated with points, which are inside faces. And the forty-eight darts of the topological cube are associated with internal control points. Each vertex of the topological cube is made of six darts, it corresponds with Bézier control point split in six Gregory-Bézier control points.

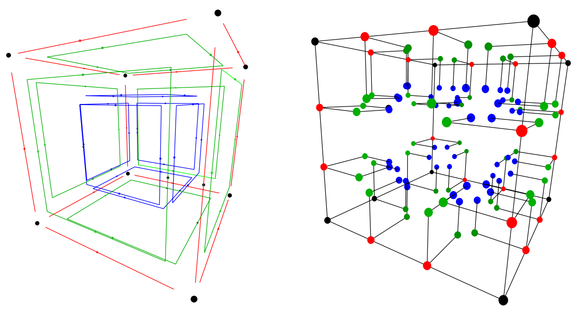


Figure 13: A hexahedral 3-chain and the corresponding Gregory-B\ézier volume.

As in dimension 2 the evaluation is done by going back to Bézier volume. We have to interpolate control points of dimension 2 (points inside the face), and of dimension 3 (points inside the volume).

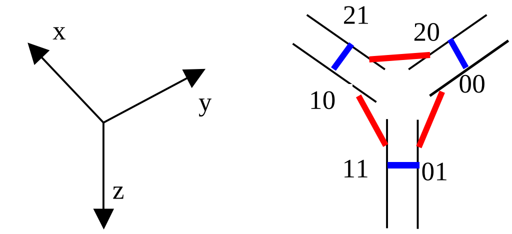


Figure 14: A possible numbering of darts in a corner

Interpolation of points of dimension 2 can be done in the same way as in case of surfaces. For 3 dimensional points, we have to do a double interpolation. First points have to be bilinearly interpolated two by two, and then the three new points can be trilinearly interpolated. Figure 14 shows a possible numbering of dart of one corner of the 3-G-map of the 3-chain, which permit to write the following formula:

$$P = \frac{uP_u + vP_v + wP_w}{u+v+w} \quad \text{with}$$

$$P_u = \frac{wP_{00} + vP_{01}}{v+w},$$

$$P_v = \frac{wP_{10} + uP_{11}}{u+w} \quad \text{and}$$

$$P_w = \frac{vP_{21} + uP_{01}}{u+v}$$

The same association can be done between a 3-chain which models a tetrahedron and a Gregory-Bézier tetrahedron, lattice of control points of a Gregory-Bézier tetrahedron is drawn on Figure 15.

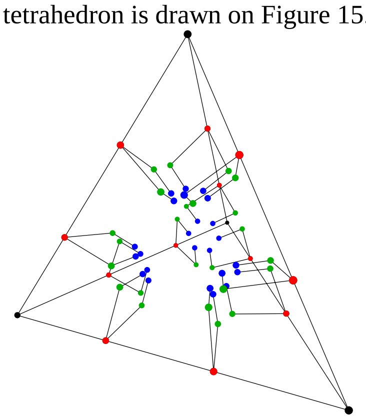


Figure 15: A Gregory tetrahedron

8. GENERALISATION TO DIMENSION N

As seen in section Generalization to dimension 3 the generalization of Gregory-Bézier patches is not easy. To define them in dimension *n*, their association with *n*-chains is useful. It permits us to easily determine the number of control points and the structure of the lattice.

In dimension n, we have to take the n-chain to determine the shape of the *n*-dimensional Gregory-Bézier patch. The *n*-chain that models a cube of dimension n has

$$\sum_{i=0}^{n} \frac{2^n n!}{(n-i)!} \approx 2^n n! e \text{ darts,}$$

which is then the number of control points of the patch. Its internal points are associated with darts of central n-G-map that is made up $2^n n!$ darts. As there is 2^n vertices in a *n*-dimensional cube, internal points can be grouped in 2^n subsets of *n!* vertices (it is verified in dimension 3 : 8 subsets of 6 points).

Since the boundary of an *n*-chain is made of *(n-1)*-chains, boundary of a *n*-dimensional Gregory-Bézier patch is made of *2n (n-1)*-dimensional Gregory-Bézier patches.

To evaluate an *n*-dimensional Gregory-Bézier patch, we have to interpolate subsets of *n!* control points to obtain a Bézier patch. This can be done by interpolating *n* points which are results of *n* interpolations of n-1 points, which are interpolations of *n-2* points, and so on until interpolations of 2 points.

9. CONCLUSION

We have defined an association between a topological and an embedding model. *n*-chains of maps have been chosen as topological models because they permit the modelling of non-manifold objects, which can be opened or closed, orientable or not. Gregory-Bézier patches, which are an extension of Bézier patches, have been chosen, because their association with *n*-chains is very natural. They are also defined in quadrangular and triangular cases, and their continuity can easily be studied.

Associations have been defined for curves and surfaces, and generalized to volumes and *n*-dimensional entities. We show how a topological model can be useful to manage free form surfaces, and how this modelling method allows the creation of complicated objects with strong topological structures and lower cost in time and space than actual modelling tools. Furthermore the method permits the embedding of 3-dimensional topological structures with volume patches. In addition triangular and rectangular patches can be mixed without any trouble. This modelling method has been experimented in a modelling tool called *Multifil* with high level operation which allow us to create objects as shown on Figure 16. These objects have been created by using classical operations on vertices

(rotation, translation, bend …) and three high-level operations, vertex and edge blending, extrusion and thickening. Objects shapes have been improved by applying continuity constraints.
This work has to be continued, by applying continuity constraints on objects. To have perfect continuity on object of this model, we have to study continuity of curves, surfaces and volumes. It will be interesting to study continuity between objects of different dimensions like surface and curves. Definition of Gregory-Bézier volume could be applied on Free Form Deformation. Another application is the study of seismic waves propagation [Fouss97], it uses hexahedral subdivision of space, and needs continuity of joining. This subdivision can be modelled with Gregory-Bézier cubes and structured by 3-chain, then continuity and consistency of lattices will be ensured.

REFERENCES

[Lienh88] Lienhardt P.: Subdivisions of Surfaces and generalized maps, *Eurographic*, 1988.

[Lienh89] Lienhardt P.: Subdivisions of n-dimensional spaces and n-dimensional generalized maps, *Proceedings 5th ACM Symposium on Computational Geometry*, pp 228-236, 1989.

[Lienh94] Lienhardt P.: N-Dimensional Generalized Combinatorial Maps and Cellular Quasi-Manifolds, *International Journal of Computational Geometry and Applications*, Vol. 4, nº 3, pp. 275-324, 1994.

[Elter92] Elter H., Lienhardt P.: Extension of the notion of map for the representation of the topology of cellular complexes, *Proceedings of the 4th Canad. Conf. Computer Geom*., pp 65-70, 1992.

[Elter93] Elter H., Lienhardt P.: Different combinatorial models based on the map concept for the representation of different types of cellular complexes, proceedings of *IFIP TC 5/WG II, Working Conference on Geometric Modeling in Computer Graphics Genova, Italy, july 1993, in Modeling in Computer Graphics* (B. Falcidieno et T.L. Kunii eds., Springer).

[Bezie74] Pierre E. Bézier. Mathematical and practical possibilities of UNISURF. Robert E. Barnhill and Richard F. Riesenfeld, editors, *Computer Aided Geometric Design*, pp 127-152. Academic Press, 1974.

[Farin86] Farin Gerald.: Triangular Bernstein-Bézier patches, *Computer-Aided Geometric Design*, Vol. 3, nº 2, pp 83-128, 1986.

[Farin88] Farin Gerald.: Curves and Surfaces for Computer Aided Geometric Design, *A Practical Guide*, Academic Press, San Diego, CA, 1988.

[Grego74] J.A. Gregory.: Smooth interpolation without twist Constraints, *Computer Aided Geometric Design*, pp 71-87, 1974.

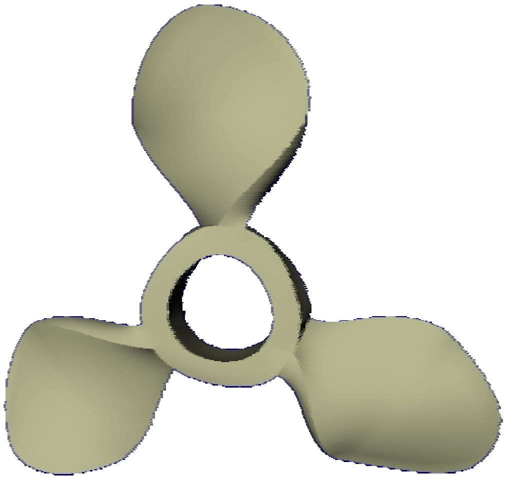
[Chiyo83] Chiyokura Hiroaki, Kimura Fumihiko: Design of Solid with Free-Form Surfaces, *SIGGRAPH'83*, volume 17, number 3, pp. 294-296.

[Takam90] Takamura T., Ohta M., Toriya H., Chiyokura H.: A Method to Convert a Gregory patch and a Rational Boundary Gregory Patch to a Rational Bézier Patch and its Applications, *CG international 1990*, pp 543-562.

[Fouss97] Fousse A., Bertrand Y., Dufourd J.F., Francon F. and Rodrigues D.: Localisation des points d'un maillage généré en vue de calculs en différences finies, *Journées Modélisation du sous-sol*, 3 Février 1997, BRGM - ENSMP -INRIA.

[Guoer94] Guoerguieva S.,Marcheix D.: Non-manifold boundary representation for solid modelling, *Proc. of the International Computer Symposium*, Taiwan ROC, pp 897-904, 1994.

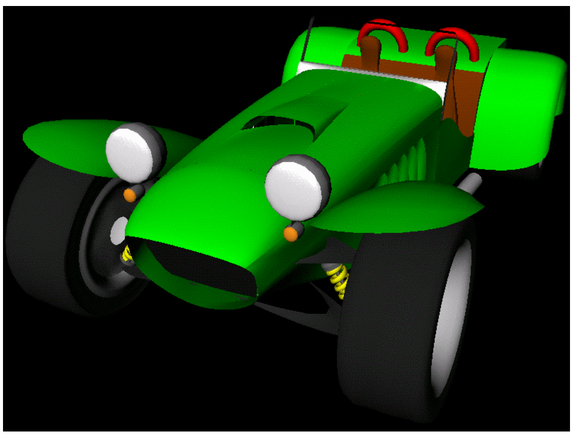
a kitchen



an helix

a Klein bottle

a keyboard



a car

a phone

Figure 16: Examples of objects modelled with associated patches and chains of maps.