Graphs and Combinatorics (2010) 26:71–76
DOI 10.1007/s00373-010-0908-y

ORIGINAL PAPER

Long Monochromatic Berge Cycles in Colored 4-Uniform Hypergraphs

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Received: 5 February 2008 / Revised: 21 November 2009 / Published online: 17 February 2010

Abstract Here we prove that for $n \geq 140$, in every 3-coloring of the edges of $K_n^{(4)}$ there is a monochromatic Berge cycle of length at least $n-10$. This result sharpens an asymptotic result obtained earlier. Another result is that for $n \geq 15$, in every 2-coloring of the edges of $K_n^{(4)}$ there is a 3-tight Berge cycle of length at least $n-10$.

Keywords Colored complete uniform hypergraphs · Monochromatic Hamiltonian Berge-cycles

1 Introduction

Let $\mathcal{H}$ be an r-uniform hypergraph (a family of some r-element subsets of a set). The *shadow graph* of $\mathcal{H}$ is defined as the graph $\Gamma(\mathcal{H})$ on the same vertex set, where two vertices are adjacent if they are covered by at least one edge of $\mathcal{H}$. A coloring of the edges of an r-uniform hypergraph $\mathcal{H}$, $r \geq 2$, induces a multicoloring on the edges of the shadow graph $\Gamma(\mathcal{H})$ in a natural way; every edge e of $\Gamma(\mathcal{H})$ receives the color of

A. Gyárfás was supported in part by OTKA Grant No. K68322. G. N. Sárközy was supported in part by the National Science Foundation under Grant No. DMS-0456401, by OTKA Grant No. K68322 and by a János Bolyai Research Scholarship.

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all hyperedges containing e. We shall denote by $c(x, y)$ the color set of the edge xy in $\Gamma(\mathcal{H})$. A subgraph of $\Gamma(\mathcal{H})$ is *monochromatic* if the color sets of its edges have a nonempty intersection. Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices.

In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \dots, v_\ell$, such that for each set $(v_i, v_{i+1}, \dots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge e_i of $\mathcal{H}$ that contains these t vertices and the edges e_i are all distinct for i, $1 \leq i \leq \ell$ where $\ell + j \equiv j$. This notion was introduced in Ref. [3] to generalize *Berge-cycles* ($t = 2$,[1]) and the *tight cycle* ($t = r$ see e.g. [11] or [14]). A Berge-cycle of length n in a hypergraph of n vertices is called a Hamiltonian Berge-cycle. It is important to keep in mind that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a Berge-cycle $C_\ell^{(r,t)}$, is not determined uniquely, it is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

In this paper, continuing investigations from Refs. [3,7,8] and [9], we study long Berge-cycles in hypergraphs. In Ref. [3] (by generalizing an earlier conjecture from Ref. [7]) the following conjecture was formulated.

Conjecture 1 *For any fixed* $2 \leq c$, $2 \leq t \leq r$ *satisfying* $c + t \leq r + 1$ *and sufficiently large n, if we color the edges of* $K_n^{(r)}$ *with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

In Ref. [3] it was proved that if the conjecture is true it is best possible, since for any values of $2 \leq c, t \leq r$ satisfying $c + t > r + 1$ the statement is not true. The conjecture was proved for $r = 3$ in Ref. [7]. The asymptotic form of the conjecture was proved for $r = 4$ and $t = 2$ in Ref. [7] and for every r and $t = 2$ in Ref. [9]—in both papers the Regularity Lemma was used. In this paper we apply an elementary approach and we study the $r = 4$ case. We prove the conjecture in both cases ($c = 3$, $t = 2$ and $c = 2$, $t = 3$) with a constant error term. It seems that the methods applied in this paper fail for $r \geq 5$.

Theorem 1 *Suppose that a* 3*-coloring is given on the edges of* $K_n^{(4)}$, *where* $n \geq 140$. *Then there is a monochromatic Berge-cycle of length at least* $n - 10$.

This sharpens the asymptotic result obtained earlier for $r = 4$ in Ref. [7].

Theorem 2 *Suppose that a* 2*-coloring is given on the edges of* $K_n^{(4)}$, *where* $n \geq 15$. *Then there is a monochromatic* 3*-tight Berge-cycle of length at least* $n - 10$.

2 Proofs

Proof of Theorem 1 Suppose that c is a 3-coloring on the edges of $\mathcal{K} = K_n^{(4)}$, where $n \geq 140$. Color $i \in c(x, y)$ on the edge xy of $G = \Gamma(\mathcal{K})$ is a *good color* if at least 3 edges of color i contain $\{x, y\}$ in $\mathcal{K}$. We consider G with a new coloring c^* where $c^*(x, y) \subseteq c(x, y)$ is the set of good colors on xy. Assuming that $\binom{n-2}{2} > 6$, i.e. $n > 6$, every edge of $\mathcal{K}$ has at least one color in c^*.

Suppose first that some edge xy of $G = \Gamma(\mathcal{K})$ is colored (under c^*) with a single color, say with color 1. We claim that there is a Hamiltonian Berge cycle in $\mathcal{K}$ in color 1. Indeed, the definition of xy implies that at most four edges of $\mathcal{K}$ containing $\{x, y\}$ are not colored with 1. Since for $n > 10$ we have $n - 6 > (n - 2)/2$, the color 1 subgraph of $H = G \setminus \{x, y\}$ satisfies Dirac's condition (see Ref. [12]), and thus one can easily find a Hamiltonian path $P = \{y_1, \dots y_{n-2}\}$ of color 1 in H such that there are two extra edges $y_1 y_p$ and $y_{n-2} y_k$ of color 1 from the endpoints of P with $2 < p, k < n - 3$. Now the cyclic ordering $x, y_1, y_2, \dots, y_{n-2}, y$ defines a Hamiltonian Berge-cycle in color 1 with the following edge assignments. For x, y_1 assign $e_n = \{x, y_1, y_p, y\}$. For y_j, y_{j+1} $(1 \leq j \leq n - 3)$ assign $e_j = \{x, y, y_j, y_{j+1}\}$, for y_{n-2}, y assign $e_{n-2} = \{y_{n-2}, y, y_k, x\}$, and finally for x, y we can assign e_{n-1} as any edge of color 1 containing x, y and different from all other e_i-s.

Now we may assume that c^* colors all edges of G with one of the four color sets: 12, 13, 23, 123.

Lemma 1 *Assume that there is a monochromatic Hamiltonian cycle C in G under coloring c^*. Then there is a Hamiltonian Berge-cycle in $\mathcal{K}$ under coloring c.*

Proof Assume that $C = x_1, x_2, \dots, x_n$ is a Hamiltonian cycle of G in color 1 (under c^*). Then, following the cyclic order of vertices on C, let A_j be the set of edges of $\mathcal{K}$ in color 1 containing x_j, x_{j+1}. Since each A_j has at least three elements and no element of A_j covers more than three consecutive pairs of C, Hall's theorem ensures a one-to one correspondence from the consecutive pairs to the sets A_j. This clearly defines the required Hamiltonian Berge-cycle. □

We need some observations on the structure of the coloring c^*. Let x be an arbitrary vertex, define $U_{12}(x), U_{13}(x), U_{23}(x), U_{123}(x)$ as the sets to which x is connected in color sets 12, 13, 23, 123 respectively. When x is implicit, for simplicity we omit the dependence on x. Define

$$B_i = \{x \in V(G) | U_{ij} = U_{ik} = \emptyset, U_{jk} \neq \emptyset\},$$

where i, j, k are the elements of $\{1, 2, 3\}$ in some order. Observe that the B_i's are pairwise disjoint, within the B_i's every edge of G has color set $\{j, k\}$ or 123, and for $j \neq i$, an edge of G from B_i to B_j has color set 123. Set $B_4 = \{x \in V(G) || U_{123}| \geq n/2\}$.

Lemma 2 *Suppose that $\cup_{i=1}^4 B_i = V(G)$. Then there is a monochromatic Hamiltonian cycle G under the coloring c^*.*

Proof Suppose w.l.o.g that $|B_1| \leq |B_2| \leq |B_3|$. We show that there is a Hamiltonian cycle in color 1. Denoting the degree of a vertex v in color i by $d_i(v)$, we have that $d_1(v) \geq |B_2| + |B_3| \geq |B_2| + |B_1|$ if $v \in B_1$, $d_1(v) = n - 1$ if $v \in B_2 \cup B_3$ and $d_1(v) \geq \frac{n}{2}$ if $v \notin \cup_{i=1}^3 B_i$ (since in the latter case $v \in B_4$). These conditions immediately imply—through either Pósa's or Chvátal's condition (see Ref. [12]) that there is a Hamiltonian cycle. □

Thus, we may assume that there exists $x \in V(G) \setminus \cup_{i=1}^4 B_i$ (otherwise Lemmas 1 and 2 would finish the proof). Set $U = V(G) \setminus (\{x\} \cup U_{123})$ and assume w.l.o.g.

$|U_{23}| \le |U_{12}| \le |U_{13}|$. Since $x \notin B_2$ we have $U_{12} \ne \emptyset$ and $x \notin B_4$ implies that $|U| \ge \lfloor n/2 \rfloor$.

We show that $|U_{23}| \le 1$. Indeed, otherwise we may select two two-element sets $A_{23} \subseteq U_{23}$, $A_{12} \subseteq U_{12}$ and a five-element set $A_{13} \subseteq U_{13}$. (The condition $|U| \ge \lfloor n/2 \rfloor$ implies that $|U_{13}| \ge \frac{\lfloor n/2 \rfloor}{3} \ge 5$ so A_{13} can be defined.) For every fixed $u_{23} \in A_{23}$ there are at most two edges of color 1 among the edges of $\mathcal{K}$ in the form $\{x, u_{23}, x_{12}, x_{13}\}$ where $x_{12} \in A_{12}, x_{13} \in A_{13}$ are arbitrary. Repeating this argument for fixed u_{12}, u_{13} we get that there are at most $4+4+10 = 18$ edges of $\mathcal{K}$ in the form $\{x, x_{23}, x_{12}, x_{13}\}$. However, there are $2 \times 2 \times 5 = 20$ such edges giving a contradiction.

Now we fix $y \in U_{12}, z \in U_{13}$ and define a graph H on the vertices of $V(G) \setminus (U_{23} \cup \{x, y, z\})$ as follows. Let $uv \in E(H)$ be an edge of H in the following cases: (1) $u \in U_{13}$, $c(\{x, y, u, v\}) = 1$, in this case the edge is called an xy-edge; (2) $u \in U_{12}, c(\{x, z, u, v\}) = 1$, now the edge is called an xz-edge. Set $|V(H)| = N$ and note that $N \ge n - 4$.

Lemma 3 *The graph H has a cycle C of length at least $N - 6$ in color* 1.

Proof Set

$$T_{12} = U_{12} \cap V(H), \quad T_{13} = U_{13} \cap V(H), \quad T = U \cap V(H), \quad T_{123} = U_{123}.$$

Consider an arbitrary vertex $u \in T_{12} \cup T_{13}$. Set $w = z$ if $u \in T_{12}$ otherwise set $w = y$. Apart from at most four choices of $v \in V(H)$ the edge $\{x, u, w, v\}$ of $\mathcal{K}$ is of color 1. Thus, every vertex of $T \subseteq V(H)$ has degree at least $N - 5$ in H. Consider the set $S \subseteq T_{123}$ of vertices whose degrees are at most 11 in the bipartite subgraph $[T, T_{123}]$ of H. Observe that

$$|T|(|T_{123}| - 5) \le |E[T, T_{123}]| \le (|T_{123}| - |S|)|T| + 11|S|$$

implying that $|S| \le 6$ if $66 \le |T|$ and this is true since $|T| > \lfloor n/2 \rfloor - 4 > 65$. Now consider the subgraph F of H induced by $T \cup (T_{123} \setminus S)$. In fact, we may assume that $|S| = 6$ since deleting $6 - |S|$ vertices does not influence the following observation: each vertex $v \in T$ has degree at least $N - 11$ in F and each vertex $v \in T_{123} \setminus S$ has degree more than 11. Now we can apply Chvátal's condition (see Ref. [12]) to prove that there is a Hamiltonian cycle in $F \subset H$. Indeed, with $M = |V(F)|$, we have to show that $d_k \le k < \frac{M}{2}$ implies that $d_{M-k} \ge M - k$ where $d_1 \le d_2 \le \cdots \le d_M$ is the degree sequence of F. This is immediate because the number of vertices with possibly small degrees (i.e. $v \in T_{123} \setminus S$) is at most

$$|U_{123}| - 6 \le \left\lfloor \frac{n}{2} \right\rfloor - 6 \le \left\lfloor \frac{N+4}{2} \right\rfloor - 6 = \left\lfloor \frac{M+10}{2} \right\rfloor - 6 = \left\lfloor \frac{M}{2} \right\rfloor - 1. \quad (1)$$

Indeed, let us take a k for which $d_k \le k < \frac{M}{2}$. $11 < d_k \le k$ implies that $k > 11$. But then from (1) we get

$$d_{M-k} \ge d_{\lceil \frac{M}{2} \rceil} \ge N - 11 \ge M - 11 > M - k,$$

as desired. □

To finish the proof of Theorem 1, observe that the cycle C obtained from Lemma 3 defines a Berge-cycle if its xy-edges and xz-edges are extended (with $\{x, y\}$ or with $\{x, z\}$ to edges of $\mathcal{K}$. Thus, we have a Berge-cycle of length $N - 6 \geq n - 10$ as required. □

Proof of Theorem 2. Suppose that a 2-coloring c is given on the edges of $\mathcal{K} = K_n^{(4)}$. Let V be the vertex set of $\mathcal{K}$ and observe that c defines a 2-multicoloring on the complete 3-uniform hypergraph $\mathcal{T}$ with vertex set V by coloring a triple T with the colors of the edges of $\mathcal{K}$ containing T. We say that $T \in \mathcal{T}$ is *good in color i* if T is contained in at least two edges of $\mathcal{K}$ of color i $(i = 1, 2)$.

Lemma 4 *Let $G = \Gamma(\mathcal{K})$. Every edge $xy \in E(G)$ is in at least $n - 4$ good triples of the same color.*

Proof Consider an edge xy in G and the edges of $\mathcal{K}$ containing both x and y. Coloring c induces a 2-coloring c' on $W = V \setminus \{x, y\}$. Applying a result of Bollobás and Gyárfás [2], there exists a subgraph H with at least $|W| - 2 = n - 4$ vertices such that H is 2-connected and monochromatic under c', say in color 1. In particular, every vertex of H has degree at least two in color 1. Thus, for every vertex z of H, $\{x, y, z\}$ is a good triple in color 1. □

Using Lemma 4, we can define a 2-coloring c^* on the shadow graph $G = \Gamma(\mathcal{K})$ by coloring $xy \in E(G)$ with the color of the (at least $n - 4$) good triples containing xy. Using a well-known result about the Ramsey number of even cycles ([4,15]) there is a monochromatic even cycle C of length $2t$ where $2t = \lceil \frac{2n}{3} \rceil - 6$ or $2t = \lceil \frac{2n}{3} \rceil - 7$. (In fact there is a bit longer cycle, but that is too long for our purposes.) Assume that C is in color 1. Label the edges of C as $e_j = \{p_j, p_{j+1}\}$, $j = 1, 2, \ldots, 2t$. We use here index arithmetic mod $2t$.

We will find a large Berge-cycle in color 1 with a greedy procedure as follows. By Lemma 4, for each $i \in [2t]$ there is a set $A_i \subset V$ such that $|A_i| \geq n - 4$ and the triple $T_i = \{p_i, p_{i+1}, x\}$ is good in color 1 for every $x \in A_i$.

We claim that we can find a set $\{v_j \in A_{2j-1} \setminus V(C)\}$ of t distinct vertices for $j \in [t]$ with the following property: for every $j \in [t]$,

$$v_j \in A_{2j-2} \cap A_{2j-1} \cap A_{2j}. \tag{2}$$

To prove the claim, assume that for some h, $1 \leq h < t$ we have a set $R = \{v_1, \ldots, v_h\}$ of h distinct vertices such that $\{v_j \in A_{2j-1} \setminus V(C)\}$ for $1 \leq j \leq h$, satisfying (2) and there are at least seven vertices in $S = V \setminus (V(C) \cup R)$. We show that v_{h+1} can be defined so that property (2) is preserved. Indeed, each of the three sets $A_{2h}, A_{2h+1}, A_{2h+2}$ intersects S in at least $|S| - 2$ elements, therefore $|S| \geq 7$ implies that $U = S \cap A_{2h} \cap A_{2h+1} \cap A_{2h+2} \neq \emptyset$. Thus, we can select $v_{h+1} \in U$. Now we only have to observe that at each step of the whole process defining $\{v_1, \ldots, v_t\}$,

$$|S| \geq n - 3t \geq n - \frac{3}{2}\left(\left\lceil \frac{2n}{3} \right\rceil - 6\right) \geq 7,$$

and the claim is proved.

Now we finish the proof by claiming that the cyclic permutation

$$P = p_1, v_1, p_2, p_3, v_2, p_4, \ldots, p_{2t-1}, v_t, p_1$$

determines a Berge-cycle. Indeed, from the definition of v_j, every triple of three consecutive vertices on P is good in color 1. Therefore, at least two edges $\mathcal{K}$ of color 1 are available to cover a consecutive triple. However, no edge of $\mathcal{K}$ can cover more than two consecutive triples of P. Thus, by Hall's theorem, there is a matching from the consecutive triples of P to the set of color 1 edges of $\mathcal{K}$ containing them. The length of this Berge-cycle is $3t \geq \frac{3}{2}(\lceil \frac{2n}{3} \rceil - 7) \geq n - 10$. □

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