

# Contents

<b>1. A Historical Introduction</b>	1
1.1 Motivation	1
1.2 Some Historical Examples	2
1.2.1 Brownian Motion	2
1.2.2 Langevin's Equation	6
1.3 Birth-Death Processes	8
1.4 Noise in Electronic Systems	11
1.4.1 Shot Noise	11
1.4.2 Autocorrelation Functions and Spectra	15
1.4.3 Fourier Analysis of Fluctuating Functions: Stationary Systems	17
1.4.4 Johnson Noise and Nyquist's Theorem	18
<b>2. Probability Concepts</b>	21
2.1 Events, and Sets of Events	21
2.2 Probabilities	22
2.2.1 Probability Axioms	22
2.2.2 The Meaning of $P(A)$	23
2.2.3 The Meaning of the Axioms	23
2.2.4 Random Variables	24
2.3 Joint and Conditional Probabilities: Independence	25
2.3.1 Joint Probabilities	25
2.3.2 Conditional Probabilities	25
2.3.3 Relationship Between Joint Probabilities of Different Orders	26
2.3.4 Independence	27
2.4 Mean Values and Probability Density	28
2.4.1 Determination of Probability Density by Means of Arbitrary Functions	28
2.4.2 Sets of Probability Zero	29
2.5 Mean Values	29
2.5.1 Moments, Correlations, and Covariances	30
2.5.2 The Law of Large Numbers	30
2.6 Characteristic Function	32
2.7 Cumulant Generating Function: Correlation Functions and Cumulants	33
2.7.1 Example: Cumulant of Order 4: $\langle\langle X_1 X_2 X_3 X_4 \rangle\rangle$	35
2.7.2 Significance of Cumulants	35
2.8 Gaussian and Poissonian Probability Distributions	36
2.8.1 The Gaussian Distribution	36

2.8.2	Central Limit Theorem . . . . .	37
2.8.3	The Poisson Distribution . . . . .	38
2.9	Limits of Sequences of Random Variables . . . . .	39
2.9.1	Almost Certain Limit . . . . .	40
2.9.2	Mean Square Limit (Limit in the Mean) . . . . .	40
2.9.3	Stochastic Limit, or Limit in Probability . . . . .	40
2.9.4	Limit in Distribution . . . . .	41
2.9.5	Relationship Between Limits . . . . .	41
<b>3.</b>	<b>Markov Processes . . . . .</b>	<b>42</b>
3.1	Stochastic Processes . . . . .	42
3.2	Markov Process . . . . .	43
3.2.1	Consistency—the Chapman-Kolmogorov Equation . . . . .	43
3.2.2	Discrete State Spaces . . . . .	44
3.2.3	More General Measures . . . . .	44
3.3	Continuity in Stochastic Processes . . . . .	45
3.3.1	Mathematical Definition of a Continuous Markov Process . . . . .	46
3.4	Differential Chapman-Kolmogorov Equation . . . . .	47
3.4.1	Derivation of the Differential Chapman-Kolmogorov Equation . . . . .	48
3.4.2	Status of the Differential Chapman-Kolmogorov Equation . . . . .	51
3.5	Interpretation of Conditions and Results . . . . .	51
3.5.1	Jump Processes: The Master Equation . . . . .	52
3.5.2	Diffusion Processes—the Fokker-Planck Equation . . . . .	52
3.5.3	Deterministic Processes—Liouville’s Equation . . . . .	53
3.5.4	General Processes . . . . .	54
3.6	Equations for Time Development in Initial Time— Backward Equations . . . . .	55
3.7	Stationary and Homogeneous Markov Processes . . . . .	56
3.7.1	Ergodic Properties . . . . .	57
3.7.2	Homogeneous Processes . . . . .	60
3.7.3	Approach to a Stationary Process . . . . .	61
3.7.4	Autocorrelation Function for Markov Processes . . . . .	64
3.8	Examples of Markov Processes . . . . .	66
3.8.1	The Wiener Process . . . . .	66
3.8.2	The Random Walk in One Dimension . . . . .	70
3.8.3	Poisson Process . . . . .	73
3.8.4	The Ornstein-Uhlenbeck Process . . . . .	74
3.8.5	Random Telegraph Process . . . . .	77
<b>4.</b>	<b>The Ito Calculus and Stochastic Differential Equations . . . . .</b>	<b>80</b>
4.1	Motivation . . . . .	80
4.2	Stochastic Integration . . . . .	83
4.2.1	Definition of the Stochastic Integral . . . . .	83
4.2.2	Example $\int_{t_0}^t W(t') dW(t')$ . . . . .	84

4.2.3	The Stratonovich Integral . . . . .	86
4.2.4	Nonanticipating Functions . . . . .	86
4.2.5	Proof that $dW(t)^2 = dt$ and $dW(t)^{2+N} = 0$ . . . . .	87
4.2.6	Properties of the Ito Stochastic Integral . . . . .	88
4.3	Stochastic Differential Equations (SDE) . . . . .	92
4.3.1	Ito Stochastic Differential Equation: Definition . . . . .	93
4.3.2	Markov Property of the Solution of an Ito Stochastic Differential Equation . . . . .	95
4.3.3	Change of Variables: Ito's Formula . . . . .	95
4.3.4	Connection Between Fokker-Planck Equation and Stochastic Differential Equation . . . . .	96
4.3.5	Multivariable Systems . . . . .	97
4.3.6	Stratonovich's Stochastic Differential Equation . . . . .	98
4.3.7	Dependence on Initial Conditions and Parameters . . . . .	101
4.4	Some Examples and Solutions . . . . .	102
4.4.1	Coefficients Without $x$ Dependence . . . . .	102
4.4.2	Multiplicative Linear White Noise Process . . . . .	103
4.4.3	Complex Oscillator with Noisy Frequency . . . . .	104
4.4.4	Ornstein-Uhlenbeck Process . . . . .	106
4.4.5	Conversion from Cartesian to Polar Coordinates . . . . .	107
4.4.6	Multivariate Ornstein-Uhlenbeck Process . . . . .	109
4.4.7	The General Single Variable Linear Equation . . . . .	112
4.4.8	Multivariable Linear Equations . . . . .	114
4.4.9	Time-Dependent Ornstein-Uhlenbeck Process . . . . .	115
<b>5.</b>	<b>The Fokker-Planck Equation . . . . .</b>	<b>117</b>
5.1	Background . . . . .	117
5.2	Fokker-Planck Equation in One Dimension . . . . .	118
5.2.1	Boundary Conditions . . . . .	118
5.2.2	Stationary Solutions for Homogeneous Fokker-Planck Equations . . . . .	124
5.2.3	Examples of Stationary Solutions . . . . .	126
5.2.4	Boundary Conditions for the Backward Fokker-Planck Equation . . . . .	128
5.2.5	Eigenfunction Methods (Homogeneous Processes) . . . . .	129
5.2.6	Examples . . . . .	132
5.2.7	First Passage Times for Homogeneous Processes . . . . .	136
5.2.8	Probability of Exit Through a Particular End of the Interval . . . . .	142
5.3	Fokker-Planck Equations in Several Dimensions . . . . .	143
5.3.1	Change of Variables . . . . .	144
5.3.2	Boundary Conditions . . . . .	146
5.3.3	Stationary Solutions: Potential Conditions . . . . .	146
5.3.4	Detailed Balance . . . . .	148
5.3.5	Consequences of Detailed Balance . . . . .	150
5.3.6	Examples of Detailed Balance in Fokker-Planck Equations . . . . .	155

5.3.7	Eigenfunction Methods in Many Variables— Homogeneous Processes . . . . .	165
5.4	First Exit Time from a Region (Homogeneous Processes) . . . . .	170
5.4.1	Solutions of Mean Exit Time Problems . . . . .	171
5.4.2	Distribution of Exit Points . . . . .	174
<b>6.</b>	<b>Approximation Methods for Diffusion Processes . . . . .</b>	<b>177</b>
6.1	Small Noise Perturbation Theories . . . . .	177
6.2	Small Noise Expansions for Stochastic Differential Equations . . . . .	180
6.2.1	Validity of the Expansion . . . . .	182
6.2.2	Stationary Solutions (Homogeneous Processes) . . . . .	183
6.2.3	Mean, Variance, and Time Correlation Function . . . . .	184
6.2.4	Failure of Small Noise Perturbation Theories . . . . .	185
6.3	Small Noise Expansion of the Fokker-Planck Equation . . . . .	187
6.3.1	Equations for Moments and Autocorrelation Functions . . . . .	189
6.3.2	Example . . . . .	192
6.3.3	Asymptotic Method for Stationary Distributions . . . . .	194
6.4	Adiabatic Elimination of Fast Variables . . . . .	195
6.4.1	Abstract Formulation in Terms of Operators and Projectors . . . . .	198
6.4.2	Solution Using Laplace Transform . . . . .	200
6.4.3	Short-Time Behaviour . . . . .	203
6.4.4	Boundary Conditions . . . . .	205
6.4.5	Systematic Perturbative Analysis . . . . .	206
6.5	White Noise Process as a Limit of Nonwhite Process . . . . .	210
6.5.1	Generality of the Result . . . . .	215
6.5.2	More General Fluctuation Equations . . . . .	215
6.5.3	Time Nonhomogeneous Systems . . . . .	216
6.5.4	Effect of Time Dependence in $L$ , . . . . .	217
6.6	Adiabatic Elimination of Fast Variables: The General Case . . . . .	218
6.6.1	Example: Elimination of Short-Lived Chemical Intermediates . . . . .	218
6.6.2	Adiabatic Elimination in Haken's Model . . . . .	223
6.6.3	Adiabatic Elimination of Fast Variables: A Nonlinear Case . . . . .	227
6.6.4	An Example with Arbitrary Nonlinear Coupling . . . . .	232
<b>7.</b>	<b>Master Equations and Jump Processes . . . . .</b>	<b>235</b>
7.1	Birth-Death Master Equations—One Variable . . . . .	236
7.1.1	Stationary Solutions . . . . .	236
7.1.2	Example: Chemical Reaction $X \rightleftharpoons A$ . . . . .	238
7.1.3	A Chemical Bistable System . . . . .	241
7.2	Approximation of Master Equations by Fokker-Planck Equations . . . . .	246
7.2.1	Jump Process Approximation of a Diffusion Process . . . . .	246
7.2.2	The Kramers-Moyal Expansion . . . . .	245
7.2.3	Van Kampen's System Size Expansion . . . . .	250

7.2.4	Kurtz's Theorem . . . . .	254
7.2.5	Critical Fluctuations . . . . .	255
7.3	Boundary Conditions for Birth-Death Processes . . . . .	257
7.4	Mean First Passage Times . . . . .	259
7.4.1	Probability of Absorption . . . . .	261
7.4.2	Comparison with Fokker-Planck Equation . . . . .	261
7.5	Birth-Death Systems with Many Variables . . . . .	262
7.5.1	Stationary Solutions when Detailed Balance Holds . . . . .	263
7.5.2	Stationary Solutions Without Detailed Balance (Kirchoff's Solution) . . . . .	266
7.5.3	System Size Expansion and Related Expansions . . . . .	266
7.6	Some Examples . . . . .	267
7.6.1	$X + A \rightleftharpoons 2X$ . . . . .	267
7.6.2	$X \rightleftharpoons Y \rightleftharpoons A$ . . . . .	267
7.6.3	Prey-Predator System . . . . .	268
7.6.4	Generating Function Equations . . . . .	273
7.7	The Poisson Representation . . . . .	277
7.7.1	Kinds of Poisson Representations . . . . .	282
7.7.2	Real Poisson Representations . . . . .	282
7.7.3	Complex Poisson Representations . . . . .	282
7.7.4	The Positive Poisson Representation . . . . .	285
7.7.5	Time Correlation Functions . . . . .	289
7.7.6	Trimolecular Reaction . . . . .	294
7.7.7	Third-Order Noise . . . . .	299
<b>8.</b>	<b>Spatially Distributed Systems . . . . .</b>	<b>303</b>
8.1	Background . . . . .	303
8.1.1	Functional Fokker-Planck Equations . . . . .	305
8.2	Multivariate Master Equation Description . . . . .	307
8.2.1	Diffusion . . . . .	307
8.2.2	Continuum Form of Diffusion Master Equation . . . . .	308
8.2.3	Reactions and Diffusion Combined . . . . .	313
8.2.4	Poisson Representation Methods . . . . .	314
8.3	Spatial and Temporal Correlation Structures . . . . .	315
8.3.1	Reaction $X \rightleftharpoons Y$ . . . . .	315
8.3.2	Reactions $B + X \rightleftharpoons C, A + X \rightarrow 2X$ . . . . .	319
8.3.3	A Nonlinear Model with a Second-Order Phase Transition . . . . .	324
8.4	Connection Between Local and Global Descriptions . . . . .	328
8.4.1	Explicit Adiabatic Elimination of Inhomogeneous Modes . . . . .	328
8.5	Phase-Space Master Equation . . . . .	331
8.5.1	Treatment of Flow . . . . .	331
8.5.2	Flow as a Birth-Death Process . . . . .	332
8.5.3	Inclusion of Collisions—the Boltzmann Master Equation . . . . .	336
8.5.4	Collisions and Flow Together . . . . .	339

<b>9. Bistability, Metastability, and Escape Problems</b>	342
9.1 Diffusion in a Double-Well Potential (One Variable)	342
9.1.1 Behaviour for $D = 0$	343
9.1.2 Behaviour if $D$ is Very Small	343
9.1.3 Exit Time	345
9.1.4 Splitting Probability	345
9.1.5 Decay from an Unstable State	347
9.2 Equilibration of Populations in Each Well	348
9.2.1 Kramers' Method	349
9.2.2 Example: Reversible Denaturation of Chymotrypsinogen	352
9.2.3 Bistability with Birth-Death Master Equations (One Variable)	354
9.3 Bistability in Multivariable Systems	357
9.3.1 Distribution of Exit Points	357
9.3.2 Asymptotic Analysis of Mean Exit Time	362
9.3.3 Kramers' Method in Several Dimensions	363
9.3.4 Example: Brownian Motion in a Double Potential	372
<b>10. Simulation of Stochastic Differential Equations</b>	373
10.1 The One Variable Taylor Expansion	374
10.1.1 Euler Methods	374
10.1.2 Higher Orders	374
10.1.3 Multiple Stochastic Integrals	375
10.1.4 The Euler Algorithm	375
10.1.5 Milstein Algorithm	378
10.2 The Meaning of Weak and Strong Convergence	379
10.3 Stability	379
10.3.1 Consistency	381
10.4 Implicit and Semi-implicit Algorithms	382
10.5 Vector Stochastic Differential Equations	383
10.5.1 Formulae and Notation	383
10.5.2 Multiple Stochastic Integrals	384
10.5.3 The Vector Euler Algorithm	386
10.5.4 The Vector Milstein Algorithm	386
10.5.5 The Strong Vector Semi-implicit Algorithm	387
10.5.6 The Weak Vector Semi-implicit Algorithm	387
10.6 Higher Order Algorithms	388
10.7 Stochastic Partial Differential Equations	389
10.7.1 Fourier Transform Methods	390
10.7.2 The Interaction Picture Method	390
10.8 Software Resources	391
<b>References</b>	393
<b>Bibliography</b>	399

<b>Symbol Index</b> . . . . .	403
<b>Author Index</b> . . . . .	407
<b>Subject Index</b> . . . . .	409