Polynomial semantics of probabilistic circuits

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Probabilistic Models

X_1	X_2	Pr
0	0	.1
0	1	.2
1	0	.3
1	1	.4

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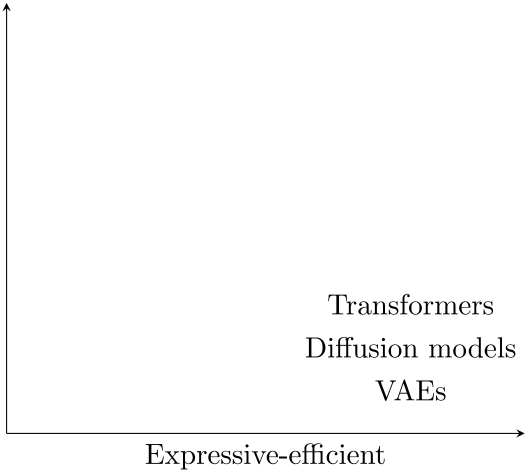

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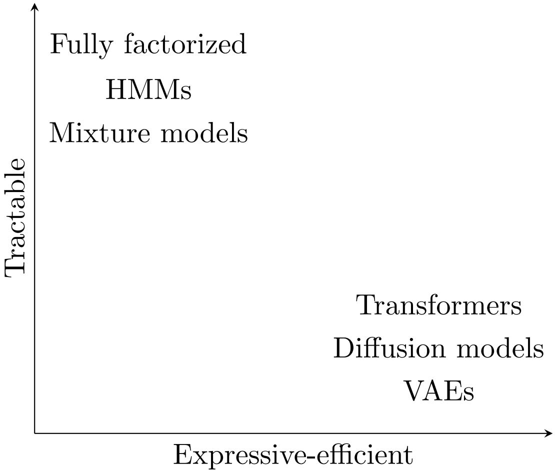

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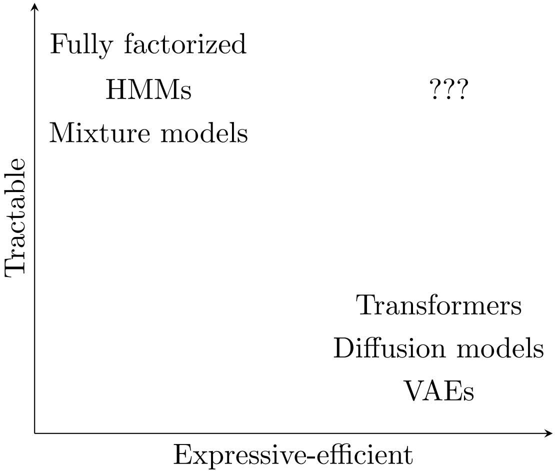

Marginal Inference

X_1	X_2	Pr
0	0	.1
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1	0	.3
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$$\begin{aligned}\Pr[X_1 = 1] &= \Pr[X_1 = 1, X_2 = 0] + \Pr[X_1 = 1, X_2 = 1] \\ &= 0.3 + 0.4 \\ &= 0.7\end{aligned}$$

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Goal: Find maximally expressive-efficient models that support marginal inference in time polynomial in the model size.

Approaches

Bayesian Networks (of bounded treewidth)
Determinantal Point Processes
Characteristic Circuits
Multi-Linear Representations
Probabilistic Generating Circuits
Sum-Product Networks
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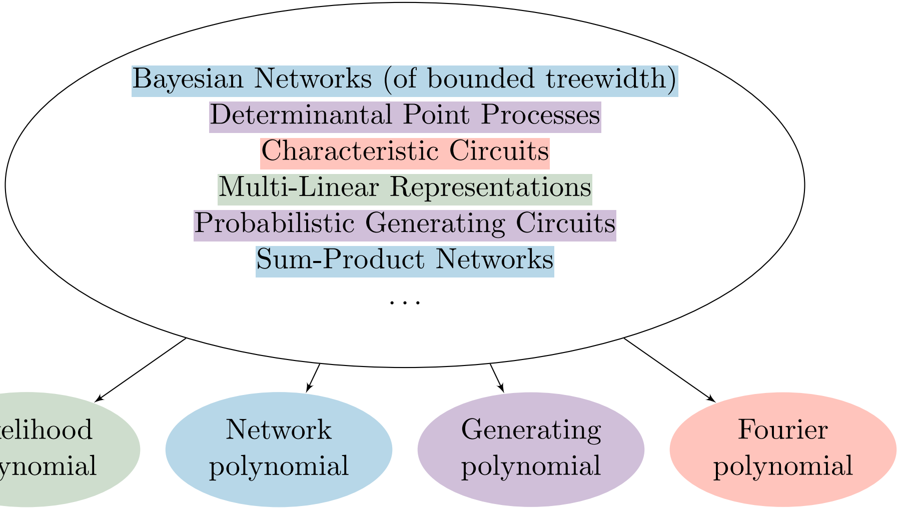
↓

Polynomials!

Approaches



Bayesian Networks (of bounded treewidth)
Determinantal Point Processes
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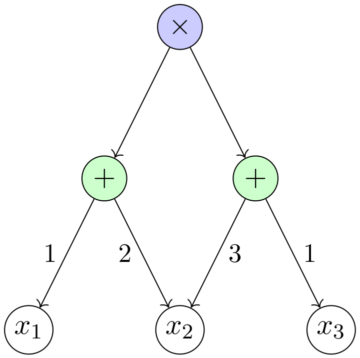
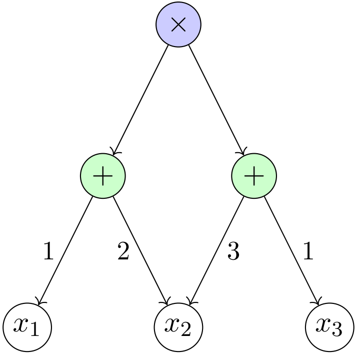
...
Likelihood polynomial
Network polynomial
Generating polynomial
Fourier polynomial

Circuits represent polynomials succinctly



$$3x_1x_2 + x_1x_3 + 6x_2^2 + 2x_2x_3$$

Circuits represent polynomials succinctly



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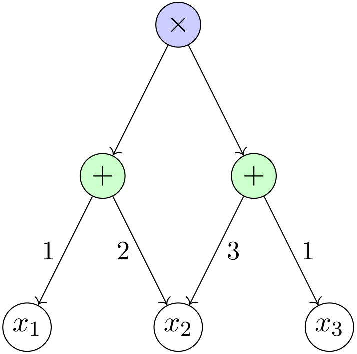
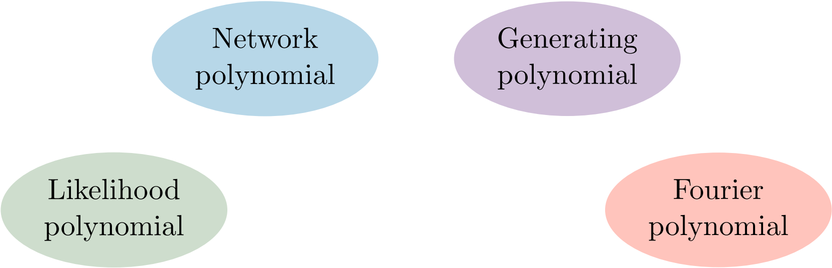


$$3x_1x_2 + x_1x_3 + 6x_2^2 + 2x_2x_3$$

Circuits are *fully expressive*

They can also be *expressive-efficient*

Polynomial Semantics



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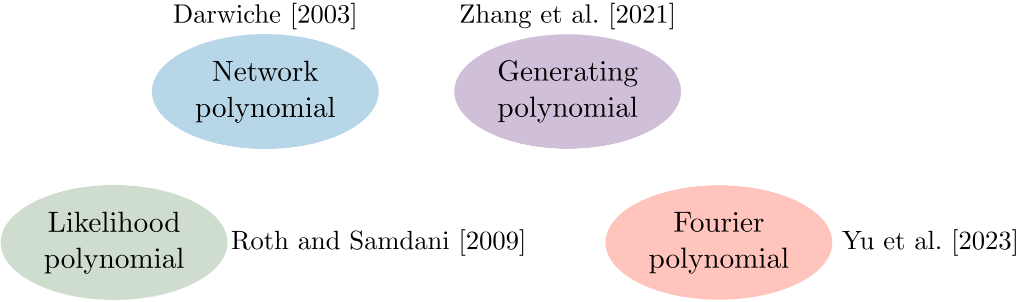

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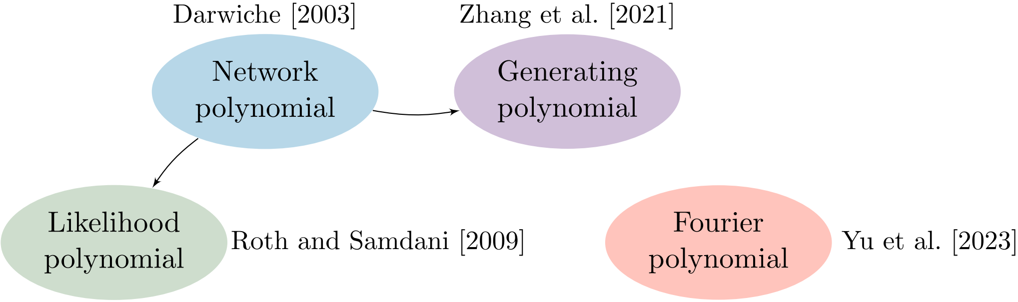

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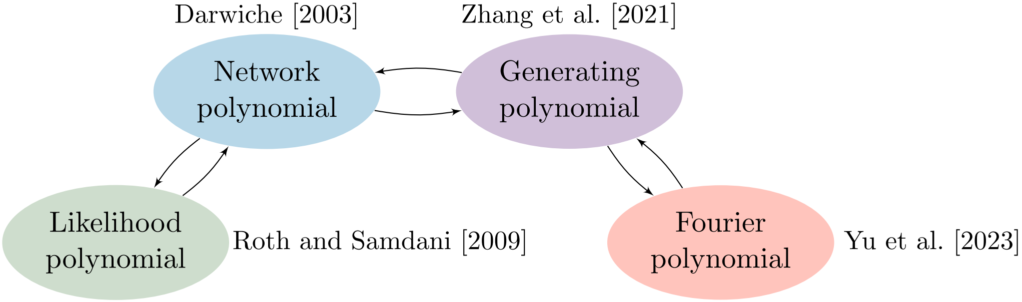

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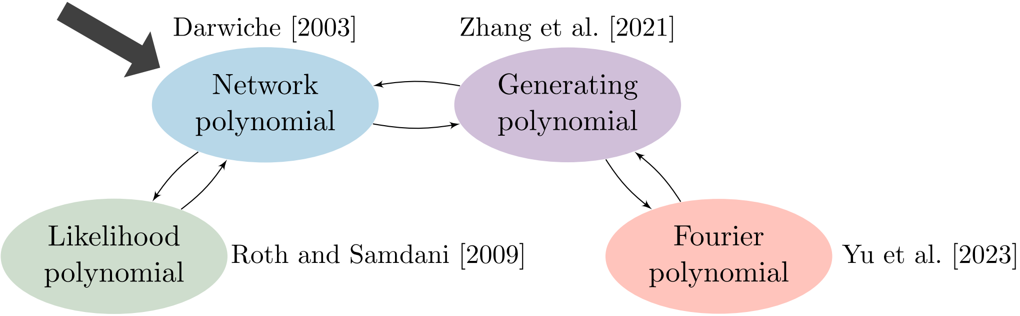

Network polynomial

$$p(x_1, x_2, \bar{x}_1, \bar{x}_2) = .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1 x_2 + .3 x_1 \bar{x}_2 + .4 x_1 x_2$$

X_1	X_2	Pr
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$\Pr[X_1 = 1]$

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$$\begin{aligned}\Pr[X_1 = 1] &= p(1, 1, 0, 1) \\ &= .1(0)(1) + .2(0)(1) + .3(1)(1) + .4(1)(1) \\ &= 0 + 0 + .3 + .4 \\ &= .7\end{aligned}$$

Network polynomial

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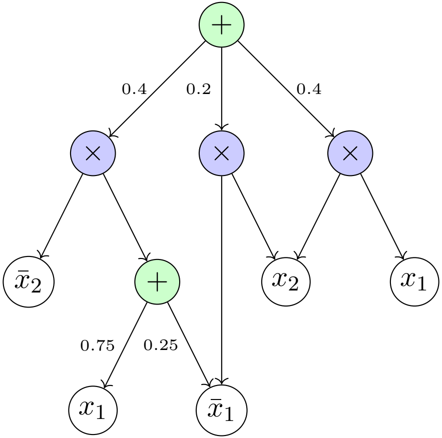

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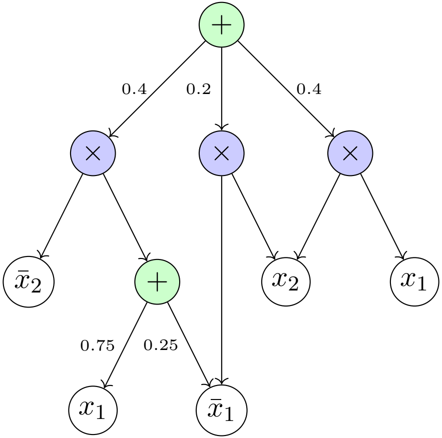

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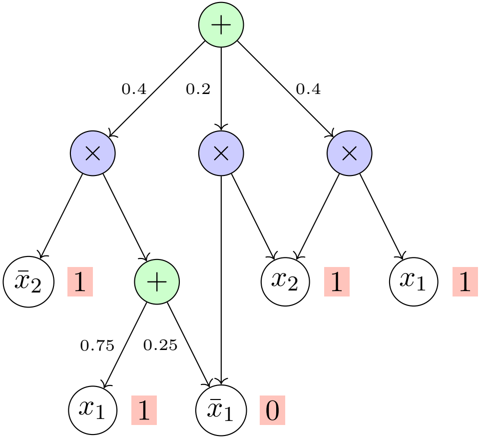

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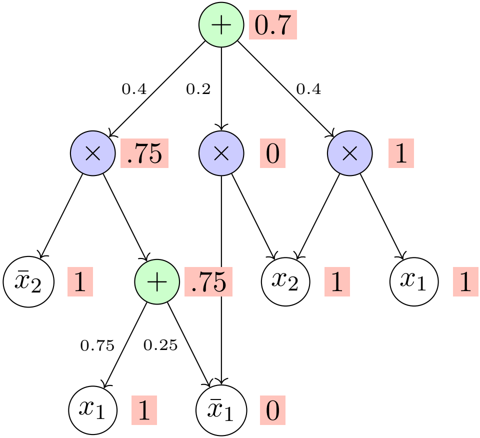

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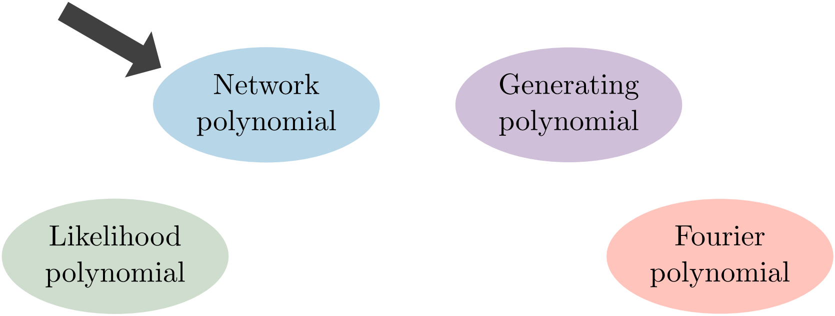
Progress Update



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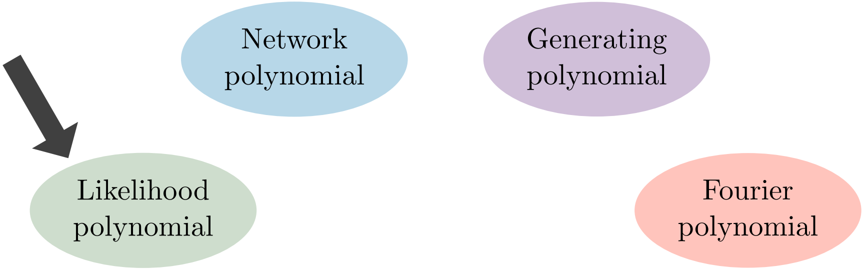

Likelihood polynomial

$$p(x_1, x_2) = .2x_1 + .1x_2 + .1$$

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Marginal inference?
Relation to network polynomial?

Transform network to likelihood:

$$p(x, \bar{x}) = .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1x_2 + .3x_1\bar{x}_2 + .4x_1x_2$$

- Replace $\bar{x}_i$ with $1 - x_i$

Likelihood polynomial

Transform likelihood to network:

$$p(x_1, x_2) = .2x_1 + .1x_2 + .1$$

Likelihood polynomial

Transform likelihood to network:

$$p(x_1, x_2) = .2x_1 + .1x_2 + .1$$

$$(x_1 + \bar{x}_1)(x_2 + \bar{x}_2)\left(.2\frac{x_1}{x_1 + \bar{x}_1} + .1\frac{x_2}{x_2 + \bar{x}_2} + .1\right)$$

Likelihood polynomial

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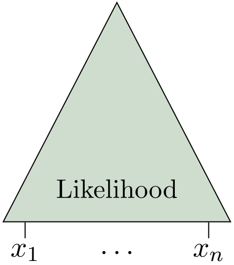
$$= p(x_1, x_2, \bar{x}_1, \bar{x}_2)$$

Likelihood polynomial

Transform likelihood to network:



Likelihood
polynomial

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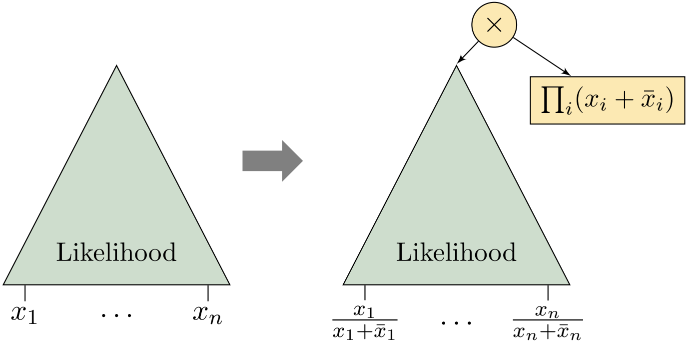

Removing Divisions

Theorem (Strassen [1973]). *You can remove divisions in polynomial time!*

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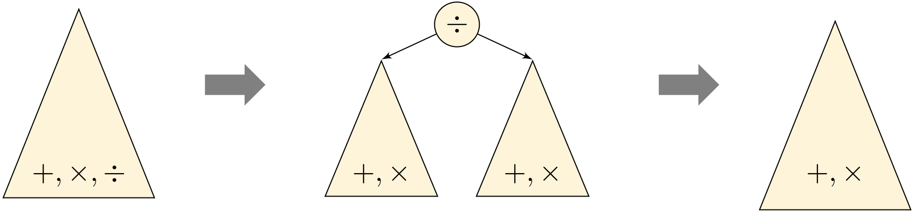

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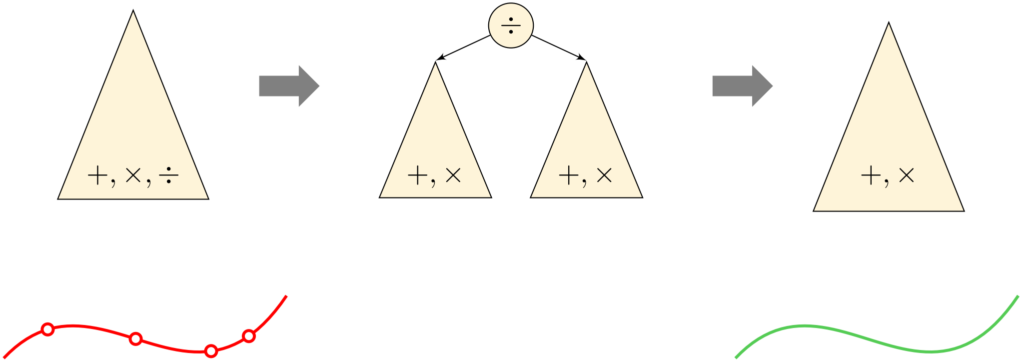
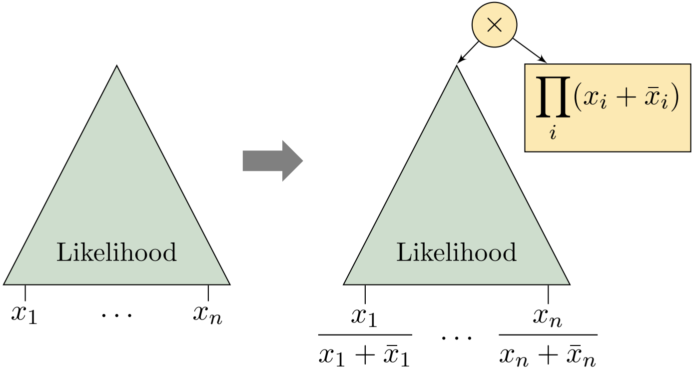


Likelihood polynomial

Transform likelihood to network:



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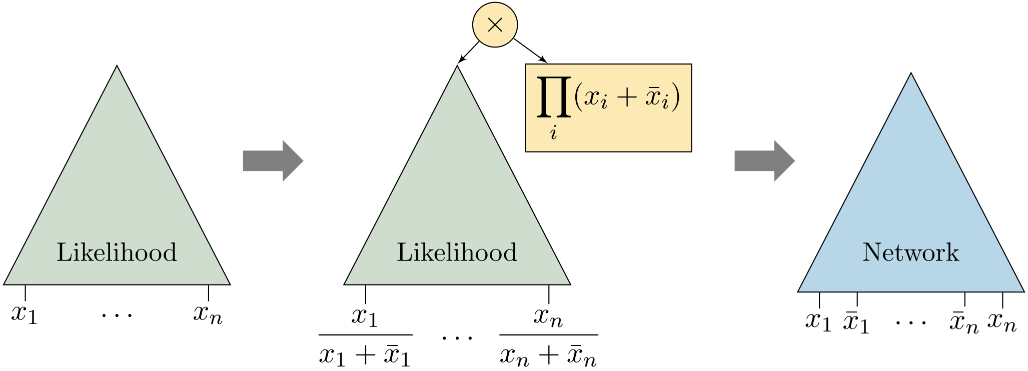

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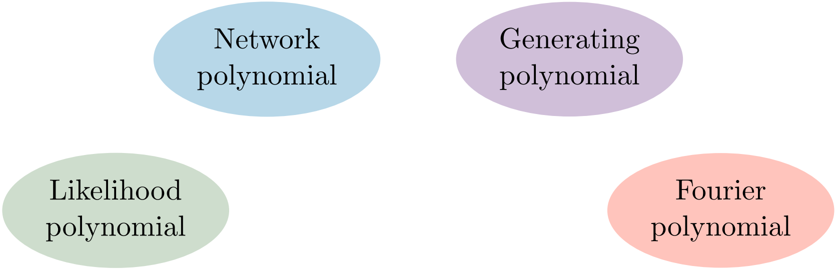

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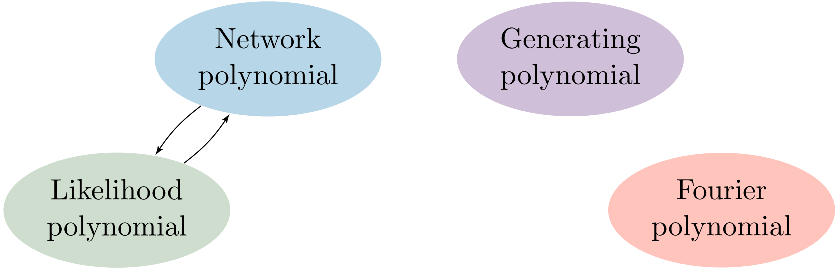

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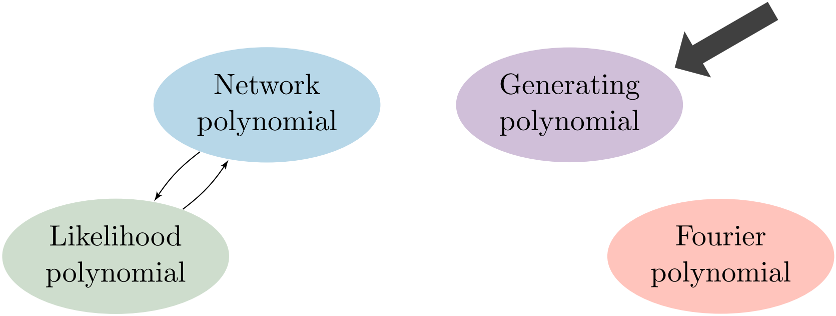
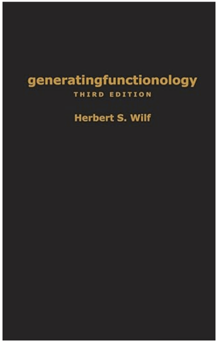


Generating polynomial



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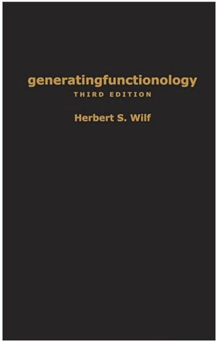
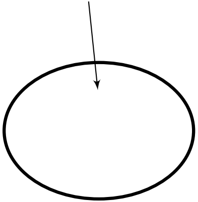


Monotone, decomposable circuits computing network polynomials (SPNs, PCs)



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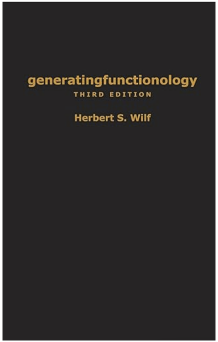

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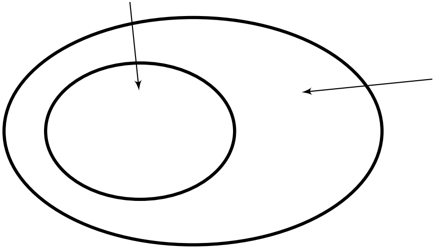
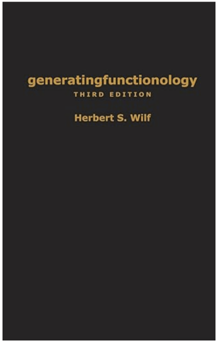


Circuits computing generating polynomials

Generating polynomial



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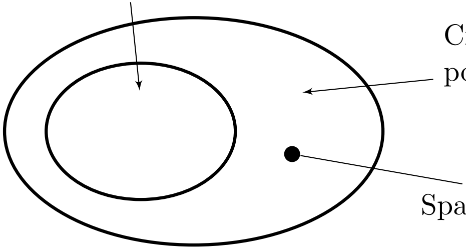

Circuits computing generating polynomials

Spanning tree distribution[a]

[a] Martens and Medabalimi [2015], Zhang et al. [2021]

Generating polynomial

$$g(x) = .1 + .2x_2 + .3x_1 + .4x_1x_2$$

X_1	X_2	Pr
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Generating polynomial

$$g(x) = .1 + .2x_2 + .3x_1 + .4x_1x_2$$

Marginal inference: ✓ [Zhang et al., 2021]

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Relation to network polynomial?

Generating polynomial

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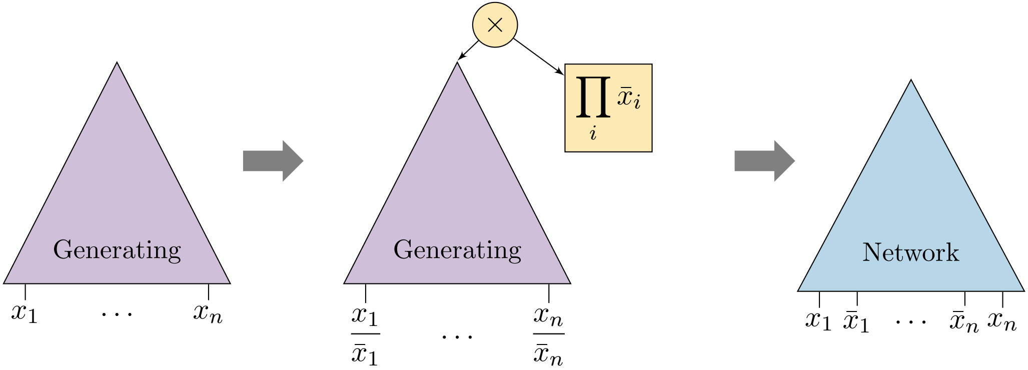
Transform network to generating:

$$p(x_1, x_2, \bar{x}_1, \bar{x}_2) = .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1x_2 + .3x_1\bar{x}_2 + .4x_1x_2$$

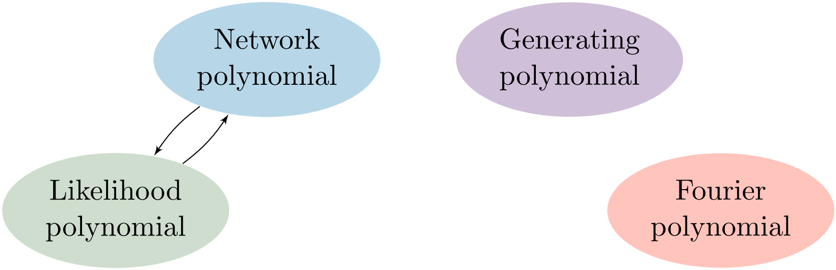
- Replace $\bar{x}_i$ with 1

Generating polynomial

Transform generating to network:



Progress Update



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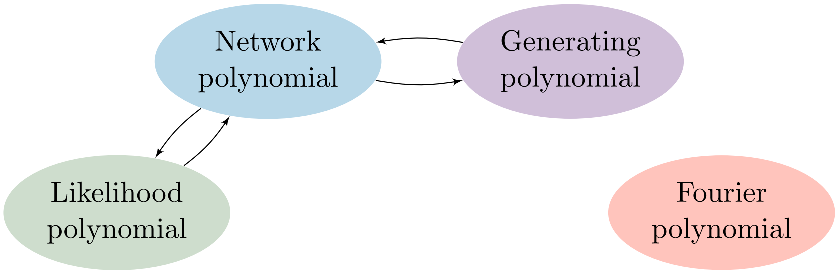

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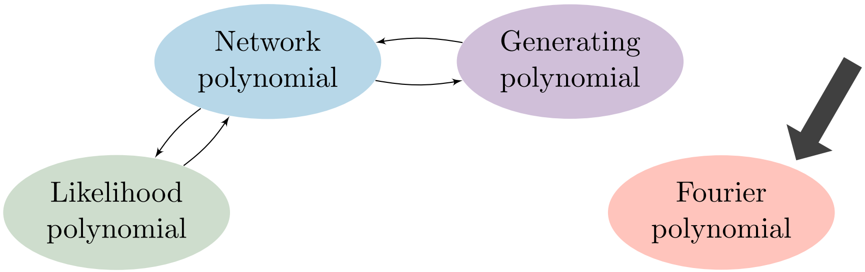
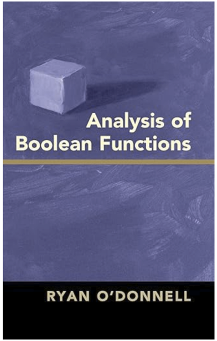


Fourier Polynomial



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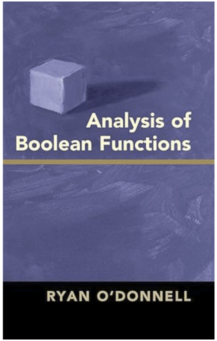
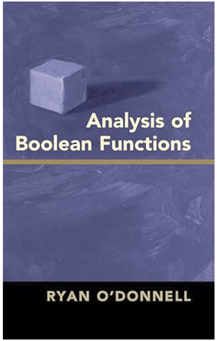


Fourier transform of the probability mass function

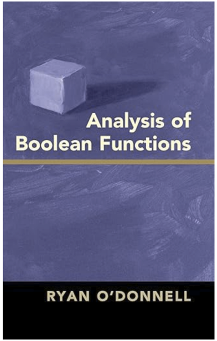
Fourier Polynomial



Fourier transform of the probability mass function

- Graphical model approximate inference
- Characteristic Circuits

Fourier Polynomial

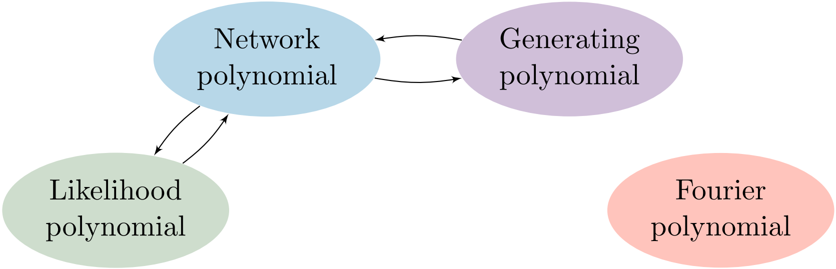


Fourier transform of the probability mass function

- Graphical model approximate inference
- Characteristic Circuits

Proposition. *Generating polynomials and Fourier polynomials compute* ***the same function*** *on respective domains* $\{-1, 1\}^n$ *and* $\{0, 1\}^n$*.*

Progress Update



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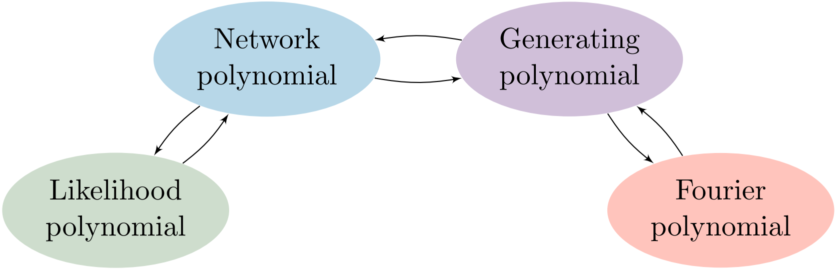
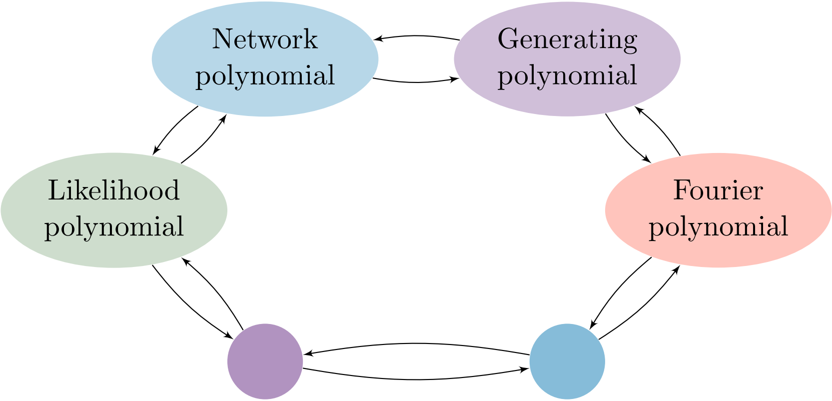


Some New Semantics



Non-binary variables?

X_1	X_2	Pr
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1	3	.3
3	2	.2
$\vdots$	$\vdots$	$\vdots$

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Literature: just use a binary encoding

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Generating polynomial

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Generating polynomial

Theorem. *For $|K| \geq 4$, computing likelihoods on a circuit for $g(x)$ is #P-hard.*

Approach: Reduction from $0,1$-permanent.

Conclusion

What we've done:

- Shown several distinct circuit models are equally expressive-efficient
- Unified existing (and one new) inference algorithms
- Inference is #P-hard in generating polynomials circuits for $k \geq 4$ categories

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Thank you! Questions?