Constrained Empty-Rectangle Delaunay Graphs*

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Abstract

Given an arbitrary convex shape C, a set P of points in the plane and a set S of line segments whose endpoints are in P, a constrained generalized Delaunay graph of P with respect to C denoted $CDG_C(P)$ is constructed by adding an edge between two points p and q if and only if there exists a homothet of C with p and q on its boundary and no point of P in the interior visible to both p and q. We study the case where the empty convex shape is an arbitrary rectangle and show that the constrained generalized Delaunay graph has spanning ratio at most $\sqrt{2}\cdot(2l/s+1)$, where l and s are the length of the long and short side of the rectangle.

1 Introduction

A geometric graph G is a graph whose vertices are points in the Euclidean plane and whose edges are line segments between pairs of points. Every edge is weighted by the Euclidean distance between its endpoints. A geometric graph G is called plane if no two edges intersect properly. The distance between two vertices u and v in G, denoted by $\delta_G(u, v)$, is defined as the sum of the weights of the edges along the shortest path between u and v in G. A subgraph H of G is a t-spanner of G (for $t \geq 1$) if for each pair of vertices u and v, $\delta_H(u, v) \leq t \cdot \delta_G(u, v)$. The smallest value t for which H is a t-spanner is the *spanning ratio* or *stretch factor* of H. The spanning properties of various geometric graphs have been studied extensively in the literature (see [5, 9] for an overview of the topic).

We study this problem in the presence of line segment *constraints*. Specifically, let P be a set of points in the plane and let S be a set of line segments with endpoints in P, with no two line segments intersecting properly. The line segments of S are called *constraints*. Two vertices u and v can *see each other* or *are visible to each other* if and only if either the line segment uv does not properly intersect any constraint or uv is itself a constraint. If two vertices u and v can see each other, the line segment uv is a *visibility edge*. The *visibility graph* of P with respect to a set of constraints S, denoted $Vis(P, S)$, has P as vertex set and all visibility edges as edge set. In other words, it is the complete graph on P minus all edges that properly intersect one or more constraints.

This setting has been studied extensively within the context of motion planning amid obstacles. Clarkson [7] was one of the first to study this problem and showed how to construct a linear-sized $(1 + \epsilon)$-spanner of $Vis(P, S)$. Subsequently, Das [8] showed how to construct a spanner of $Vis(P, S)$ with constant spanning ratio and constant degree. Bose and Keil [4] showed that the Constrained Delaunay Triangulation is a $4\pi\sqrt{3}/9 \approx 2.419$-spanner of $Vis(P, S)$. The constrained Delaunay graph where the empty convex shape is an equilateral triangle was shown to be a 2-spanner [3]. Recently, it was shown that regardless of the empty convex shape C used, the constrained generalized Delaunay graph is a plane spanner with constant spanning ratio, where the spanning ratio depends on the perimeter and the width of C [2].

In this paper, we improve the spanning ratio for the case where the empty convex shape is a rectangle. In the unconstrained setting, Chew [6] showed that the spanning ratio for squares is at most $\sqrt{10} \approx 3.16$. This was later improved by Bonichon *et al.* [1], who showed a tight spanning ratio of $\sqrt{4 + 2\sqrt{2}} \approx 2.61$. We show that in the constrained setting the spanning ratio is at most $\sqrt{2} \cdot (2l/s + 1)$, where l and s are the length of the long and short side of C. For squares (the rectangles that minimize l/s), this implies a ratio of $3\sqrt{2} \approx 4.25$.

2 Preliminaries

Throughout this paper, we fix a convex shape C. We assume without loss of generality that the origin lies in the interior of C. A *homothet* of C is obtained by scaling C with respect to the origin, followed by a translation. Thus, a homothet of C can be written as

$$x + \lambda C = \{x + \lambda z : z \in C\},$$

for some scaling factor $\lambda > 0$ and some point x in the interior of C after translation. We refer to x as the *center* of the homothet $x + \lambda C$.

For a given set of vertices P and a set of constraints S, we now define the constrained generalized Delaunay graph. Given any two visible vertices p and q, let $C(p, q)$ be any homothet of C with p and q on its boundary. The

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constrained generalized Delaunay graph contains an edge between p and q if and only if there exists a $C(p,q)$ such that there are no vertices of P in the interior of $C(p,q)$ visible to both p and q. Note that this implies that constraints are *not* necessarily edges of the constrained generalized Delaunay graph. We assume that no four points lie on the boundary of any homothet of C.

2.1 Auxiliary Lemmas

Next, we present three auxiliary lemmas that are needed to prove our main results. First, we reformulate a lemma that appears in [10].

Lemma 1 *Let C be a closed convex curve in the plane. The intersection of two distinct homothets of C is the union of two sets, each of which is either a segment, a single point, or empty.*

We say that a region R *contains* a vertex v if v lies in the interior or on the boundary of R. We call a region *empty* if it does not contain any vertex of P. Though the following lemma was applied to constrained θ-graphs in [3], the property holds for any visibility graph.

Lemma 2 *Let u, v, and w be three arbitrary points in the plane such that uw and vw are visibility edges and w is not the endpoint of a constraint intersecting the interior of triangle uvw. Then there exists a convex chain of visibility edges from u to v in triangle uvw, such that the polygon defined by uw, wv and the convex chain is empty and does not contain any constraints.*

Finally, we re-introduce a definition and lemma from [2]. Let p and q be two vertices that can see each other and let $C(p,q)$ be a convex polygon with p and q on its boundary. We look at the constraints that have p as an endpoint and the edge(s) of $C(p,q)$ on which p lies, and extend them to half-lines that have p as an endpoint (see Figure 1a). Given the cyclic order of these half-lines around p and the line segment pq, we define the clockwise neighbor of pq to be the half-line that minimizes the strictly positive clockwise angle with pq. Analogously, we define the counterclockwise neighbor of pq to be the half-line that minimizes the strictly positive counterclockwise angle with pq. We define the *cone* C_q^p that contains q to be the region between the clockwise and counterclockwise neighbor of pq. Finally, let $C(p,q)_q^p$, the *region of $C(p,q)$ that contains q with respect to p*, be the intersection of $C(p,q)$ and C_q^p (see Figure 1b).

Lemma 3 *Let p and q be two vertices that can see each other and let $C(p,q)$ be any convex polygon with p and q on its boundary. If there is a vertex x in $C(p,q)_q^p$ (other than p and q) that is visible to p, then there is a vertex y (other than p and q) in $C(p,q)$ that is visible to both p and q and triangle pyq is empty.*

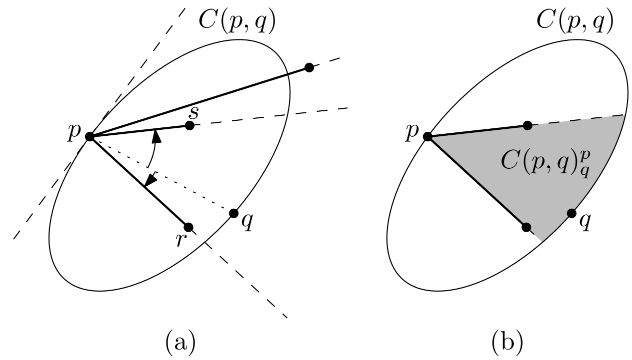

Figure 1: Defining the region of $C(p,q)$ that contains q with respect to p: (a) The clockwise and counterclockwise neighbor of pq are the half-lines through pr and ps, (b) $C(p,q)_q^p$ is marked in gray.

3 The Constrained Empty-Rectangle Delaunay Graph

We look at the case where the empty convex shape is an arbitrary rectangle. We assume without loss of generality that the rectangle is axis-aligned. We do not, however, assume anything about the ratio between the height and width of the rectangle. We first show that if two visible vertices cannot see any vertices in $C(p,q)$ on one side of pq, then no vertex in $C(p,q)$ on the opposite side of pq can see any vertices beyond pq either.

Lemma 4 *Let p and q be two vertices that can see each other, such that pq is not vertical, and let $C(p,q)$ be any convex polygon with p and q on its boundary. If the region of $C(p,q)$ below pq does not contain any vertices visible to p and q, then no point x in $C(p,q)$ above pq can see any vertices in $C(p,q)$ below pq.*

Proof. We prove the lemma by contradiction, so assume that there exists a vertex y in $C(p,q)$ below pq that is visible to x, but not to p and q. Since $C(p,q)$ is a convex polygon and x and y lie on opposite sides of pq, the visibility edge xy intersects pq. Let z be this intersection (see Figure 2).

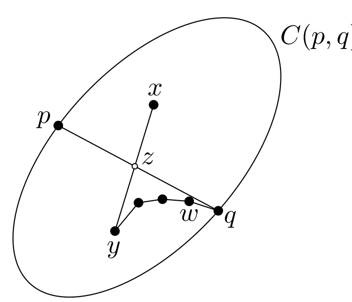

Figure 2: If x can see a vertex below pq, then so can q.

Hence, zy and zq are visibility edges. Since z is not a vertex, it is not the endpoint of any constraints intersecting the interior of triangle yzq. It follows from Lemma 2 that there exists a convex chain of visibility edges between y and q and this chain is contained in yzq. However, this implies that w, the neighbor of q along this chain, is visible to q and lies in $C(p,q)$ below pq. Next, we apply Lemma 2 on triangle pqw and find that the neighbor of p along the chain from p to w is visible to both p and q and lies in $C(p,q)$ below pq, contradicting that this region does not contain any vertices visible to p and q. □

Next, we introduce some notation for the following lemma. Let p and q be two vertices of the constrained generalized Delaunay graph that can see each other. Let R be a rectangle with p and q on its West and East boundary and let a, b, and r be the Northwest, Northeast, and Southwest corner of R. Let $m_1, ..., m_{k-1}$ be any $k-1$ points on pq in the order they are visited when walking from p to q (see Figure 3). Let $m_0 = p$ and $m_k = q$. Consider the homothets S_i of R with m_i and m_{i+1} on their respective boundaries, for $0 \leq i < k$, such that $|pa|/|ra| = |m_i a_i|/|r_i a_i|$, where a_i, b_i, r_i are the Northwest, Northeast, and Southwest corner of S_i.

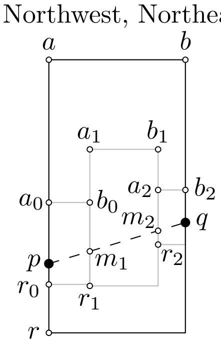

Figure 3: The total length of the sides of the rectangles S_i equals that of $C(p,q)$.

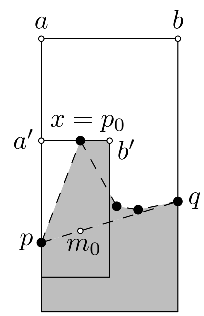

Figure 4: An inductive path from p to q.

Lemma 5 *We have*

$$\sum_{i=0}^{k-1} \left(|m_i a_i| + |a_i b_i| + |b_i m_{i+1}|\right) = |pa| + |ab| + |bq|.$$

Proof. Let $c = (|pa|+|ab|+|bq|)/|pq|$. Since for every S_i we have that $|pa|/|ra| = |m_i a_i|/|r_i a_i|$, we have $(|m_i a_i| + |a_i b_i| + |b_i m_{i+1}|)/|m_i m_{i+1}| = c$, for $0 \leq i < k$. Hence, we get

$$\begin{aligned}\sum_{i=0}^{k-1} \left(|m_i a_i| + |a_i b_i| + |b_i m_{i+1}|\right) &= \sum_{i=0}^{k-1} \left(c \cdot |m_i m_{i+1}|\right) \\ &= c \cdot |pq| \\ &= |pa| + |ab| + |bq|,\end{aligned}$$

proving the lemma. □

Before we prove the bound on the spanning ratio of the constrained generalized Delaunay graph, we first bound the length of the spanning path between vertices p and q for the case where the rectangle $C(p,q)$ is partially empty. We call a rectangle $C(p,q)$ *half-empty* when $C(p,q)$ contains no vertices in $C(p,q)^p_q$ below pq that are visible to p and $C(p,q)$ contains no vertices in $C(p,q)^q_p$ below pq that are visible to q. We denote the x- and y-coordinate of a point p by p_x and p_y.

Lemma 6 *Let p and q be two vertices that can see each other. Let $C(p,q)$ be a rectangle with p and q on its boundary, such that it is half-empty. Let a and b be the corners of $C(p,q)$ on the non-half-empty side. The constrained generalized Delaunay graph contains a path between p and q of length at most $|pa| + |ab| + |bq|$.*

Proof. We prove the lemma by induction on the rank of $C(x,y)$ when ordered by size, for any two visible vertices x and y, such that $C(x,y)$ is half-empty. We assume without loss of generality that p lies on the West boundary, q lies on the East boundary and that $C(p,q)$ is half-empty below pq. This implies that a and b are the Northwest and Northeast corner of $C(p,q)$. We also assume without loss of generality that the slope of pq is non-negative, i.e. $p_x < q_x$ and $p_y \leq q_y$ (see Figure 4).

We note that the case where p lies on the West boundary, q lies on the North boundary and $C(p,q)$ is half-empty below pq can be viewed as a special case of the one above: We shrink $C(p,q)$ until one of p and q lies in a corner. This point can now be viewed as being on both sides defining the corner and hence p and q are on opposite sides. An analogous statement holds for the case where p lies on the West boundary, q lies on the North boundary and $C(p,q)$ is half-empty above pq.

Let r be the Southwest corner of $C(p,q)$. Let R be a homothet of $C(p,q)$ that is contained in $C(p,q)$ and whose West boundary is intersected by pq. Let a', b', r' be the Northwest, Northeast, and Southwest corner of R and let m be the intersection of $a'r'$ and pq. We call homothet R *similar* to $C(p,q)$ if and only if $|pa|/|ra| = |ma'|/|r'a'|$.

Base case: If $C(p,q)$ is a rectangle of smallest area, then $C(p,q)$ does not contain any vertices visible to both p and q: Assume this is not the case and grow a rectangle R similar to $C(p,q)$ from p to q. Let x be the first vertex hit by R that is visible to p and lies in $C(p,q)^p_q$. Note that this implies that R is contained in $C(p,q)$. Therefore, R is smaller than $C(p,q)$. Furthermore, R is half-empty: By Lemma 4, the part below the line through p and q does not contain any vertices visible to p or x in $C(p,q)^p_q$, and the part between the line through p and x and the line through p and q does not contain any vertices visible to p or x since x is the first visible vertex hit while growing R. However, this contradicts that $C(p,q)$ is the smallest half-empty rectangle.

Hence, $C(p,q)$ does not contain any vertices visible to both p and q, which implies that pq is an edge of the constrained generalized Delaunay graph. Therefore the length of the shortest path from p to q is at most $|pq| \leq |pa| + |ab| + |bq|$.

Induction step: We assume that for all half-empty rectangles $C(x, y)$ smaller than $C(p, q)$ the lemma holds. If pq is an edge of the constrained generalized Delaunay graph, the length of the shortest path from p to q is at most $|pq| \leq |pa| + |ab| + |bq|$.

If pq is not an edge of the constrained generalized Delaunay graph, there exists a vertex in $C(p, q)$ that is visible from both p and q. We grow a rectangle R similar to $C(p, q)$ from p to q. Let x be the first vertex hit by R that is visible to p and lies in $C(p, q)^p_q$ and let a' and b' be the Northwest and Northeast corner of R (see Figure 4). Note that this implies that R is contained in $C(p, q)$. We also note that px is not necessarily an edge in the constrained generalized Delaunay graph, since if it is a constraint, there can be vertices visible to both p and x above px. However, since R is half-empty and smaller than $C(p, q)$, we can apply induction on it and we obtain that the path from p to x has length at most $|pa'| + |a'b'| + |b'x|$ when x lies on the East boundary of R, and that the path from p to x has length at most $|pa'| + |a'x|$ when x lies on the North boundary of R.

Let m_0 be the projection of x along the vertical axis onto pq. Since m_0 is contained in R, x can see m_0. Since xm_0 and m_0q are visibility edges and m_0 is not the endpoint of a constraint intersecting the interior of triangle xm_0q, we can apply Lemma 2 and obtain a convex chain $x = p_0, p_1, ..., p_k = q$ of visibility edges (see Figure 4). For each of these visibility edges p_ip_{i+1}, there is a homothet R_i of $C(p, q)$ that falls in one of the following three types (see Figure 5): (i) p_i lies on the North boundary and p_{i+1} lies in the Southeast corner, (ii) p_i lies on the West boundary and p_{i+1} lies on the East boundary and the slope of p_ip_{i+1} is negative, (iii) p_i lies on the West boundary and p_{i+1} lies on the East boundary and the slope of p_ip_{i+1} is not negative. Let a_i and b_i be the Northwest and Northeast corner of R_i. We note that by convexity, these three types occur in the order Type (i), Type (ii), and Type (iii).

Let m_i be the projection of p_i along the vertical axis onto pq, let C_i be the homothet of $C(p, q)$ with m_i and m_{i+1} on its boundary that is similar to $C(p, q)$, and let a'_i and b'_i be the Northwest and Northeast corner of C_i. Using these C_i, we shift Type (ii) and Type (iii) rectangles down as far as possible: We shift R_i down until either p_i or p_{i+1} lies in one of the North corners or the South boundary corresponds to the South boundary of C_i. In the latter case, R_i and C_i are the same rectangle.

Since all rectangles R_i are smaller than $C(p, q)$, we can apply induction, provided that we can show that R_i is half-empty. For Type (i) visibility edges, the part of the rectangle that lies below the line through p_i and p_{i+1} is contained in R, which does not contain any visible

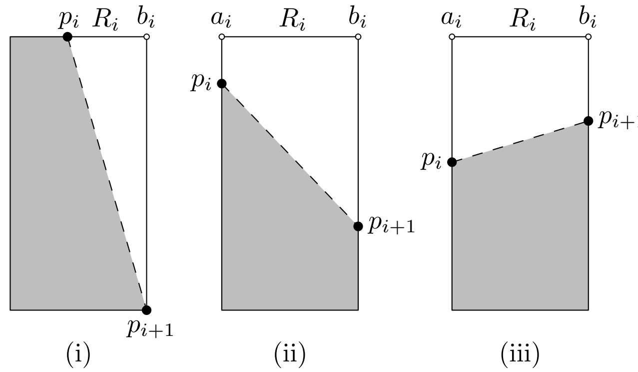

Figure 5: The three types of rectangles along the convex chain.

vertices, and the region of $C(p, q)^p_q$ below the convex chain, which is empty. For Type (ii) and Type (iii) visibility edges, the part of the rectangle that lies below the line through p_i and p_{i+1} is contained in the region of $C(p, q)^p_q$ below the convex chain, which is empty, and the region of $C(p, q)$ below the line through p and q, which does not contain any visible vertices by Lemma 4. Hence, all R_i are half-empty and we obtain an inductive path of length at most: (i) $|p_ib_i| + |b_ip_{i+1}|$, (ii) $|p_ia_i| + |a_ib_i| + |b_ip_{i+1}|$, (iii) $|p_ia_i| + |a_ib_i| + |b_ip_{i+1}|$.

To bound the total path length, we perform case distinction on the location of x on R and whether the convex path from x to q goes down: (a) x lies on the East boundary of R and the convex path does not go down, (b) x lies on the East boundary of R and the convex path goes down, (c) x lies on the North boundary of R and the convex path does not go down, (d) x lies on the North boundary of R and the convex path goes down.

Case (a): The vertex x lies on the East boundary of R and the convex path does not go down. Recall that the length of the path from p to x is at most $|pa'| + |a'b'| + |b'x|$, which is at most $|pa'| + |a'b'| + |b'm_0|$. Since the convex chain does not go down, it cannot contain any Type (i) or Type (ii) visibility edges. Furthermore, since x lies on the East boundary of R, R and all C_i are disjoint. Thus, Lemma 5 implies that the boundaries above pq of R and all C_i sum up to $|pa| + |ab| + |bq|$. Hence, if we can show that, for all R_i, $|p_ia_i| + |a_ib_i| + |b_ip_{i+1}| \leq |m_ia'_i| + |a'_ib'_i| + |b'_im_{i+1}|$, the proof of this case is complete.

By convexity, the slope of p_ip_{i+1} is at most that of pq and m_im_{i+1}. Hence, when p_{i+1} lies in the Northeast corner of R_i, we have $p_{i+1} = b_i$ and $|p_ia_i| + |a_ip_{i+1}| \leq |m_ia'_i| + |a'_ib'_i| + |b'_im_{i+1}|$. If p_{i+1} does not lie in the Northeast corner, $R_i = C_i$. Hence, since p_i and p_{i+1} lie above pq, we have that $|p_ia_i| + |a_ib_i| + |b_ip_{i+1}| \leq |m_ia'_i| + |a'_ib'_i| + |b'_im_{i+1}|$.

Case (b): The vertex x lies on the East boundary of R and the convex path goes down. Recall that the length of the path from p to x is at most $|pa'|+|a'b'|+|b'x|$. Let

p_j be the lowest vertex along the convex chain. Since p_j lies above pq and pq has non-negative slope, the descent of the convex path is at most $|xm_0|$. Hence, when we charge this to R, we used $|pa'| + |a'b'| + |b'm_0|$ of its boundary (see Figure 6).

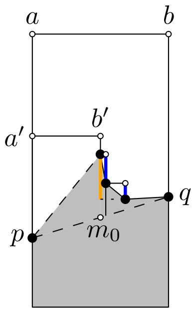

Figure 6: Going down along the convex chain (blue) is charged to R (orange).

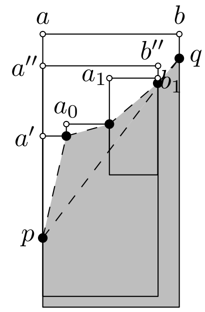

Figure 7: Charging the path from p to p_j to $C(p, p_j)$.

Like in the Case (a), since x lies on the East boundary of R, R and all C_i are disjoint. Thus, Lemma 5 implies that the boundaries above pq of R and all C_i sum up to $|pa|+|ab|+|bq|$. Hence, if we can show that, for all R_i, the inductive path length is at most $|m_i a'_i|+|a'_i b'_i|+|b'_i m_{i+1}|$, the proof of this case is complete.

For Type (i) visibility edges, we have already charged $|b_i p_{i+1}|$ to R, so it remains to show that $|p_i b_i| \leq |m_i a'_i| + |a'_i b'_i| + |b'_i m_{i+1}|$. This follows, since m_i and m_{i+1} are the vertical projections of p_i and p_{i+1}, which implies that $|p_i b_i| = |a'_i b'_i|$.

For Type (ii) visibility edges, we already charged $|b_i p_{i+1}| - |p_i a_i|$ to R, so we can consider $p_i p_{i+1}$ to be horizontal and it remains to charge the remaining $2 \cdot |p_i a_i| + |a_i b_i|$. If p_i lies in the Northwest corner of R_i, it follows that $|p_i a_i| = 0$ and we have that $|p_i b_i| = |a'_i b'_i| \leq |m_i a'_i| + |a'_i b'_i| + |b'_i m_{i+1}|$. If p_i does not lie in the Northwest corner, R_i is the same as C_i. Hence, since we can consider $p_i p_{i+1}$ to be horizontal and p_i and p_{i+1} lie above pq, it follows that $2 \cdot |p_i a_i| + |a_i b_i| \leq |m_i a'_i| + |a'_i b'_i| + |b'_i m_{i+1}|$.

Finally, Type (iii) visibility edges are charged as in Case (a), hence we have that $|p_i a_i| + |a_i b_i| + |b_i p_{i+1}| \leq |m_i a'_i| + |a'_i b'_i| + |b'_i m_{i+1}|$, completing the proof of this case.

Case (c): Vertex x lies on the North boundary of R and the convex path does not go down. Recall that the length of the path from p to x is at most $|pa'| + |a'x|$. Since the convex chain does not go down, it cannot contain any Type (i) or Type (ii) visibility edges. Let p_j be the first vertex along the chain, such that R_{j-1} is the same as C_{j-1}. Since q lies on the East boundary of $C(p, q)$, this condition is satisfied for the last visibility edge along the convex chain, hence p_j exists.

Let $C(p, p_j)$ be the homothet of $C(p, q)$ that has p and p_j on its boundary and is similar $C(p, q)$. Let a'' and b'' be the Northwest and Northeast corners of $C(p, p_j)$ (see Figure 7). Since p_j is first vertex along the convex chain that does not lie in the Northeast corner of R_{j-1}, we have that along the path from p to p_j the projections of $a'x$, all $a_i p_{i+1}$, and $a_{j-1} b_{j-1}$ onto $a''b''$ are disjoint and the projections of pa', all $p_i a_i$, and $p_{j-1} a_{j-1}$ onto pa'' are disjoint. Hence, their lengths sum up to at most $|pa''| + |a''b''|$. Finally, since $|b_{j-1} p_j| \leq |b'' p_j|$, the total length of the path from p to p_j is at most $|pa''|+|a''b''|+|b'' p_j|$, which is at most $|pa''| + |a''b''| + |b'' m_j|$.

All Type (iii) visibility edges following p_j are charged as in Case (a), hence we have that $|p_i a_i| + |a_i b_i| + |b_i p_{i+1}| \leq |m_i a'_i| + |a'_i b'_i| + |b'_i m_{i+1}|$. We now apply Lemma 5 to $C(p, p_j)$ and all C_i following p_j and obtain that the total length of the path from p to q is at most $|pa| + |ab| + |bq|$.

Case (d): Vertex x lies on the North boundary of R and the convex path goes down. Recall that the length of the path from p to x is at most $|pa'|+|a'x|$ and that p_1 is the neighbor of x along the convex chain. Let $C(p, p_1)$ be the homothet of $C(p, q)$ that has p and p_1 on its boundary and is similar to $C(p, q)$. Let a'' and b'' be the Northwest and Northeast corners of $C(p, p_1)$. Since p_1 lies to the right of R and lower than x, it lies on the East boundary of $C(p, p_1)$. We first show that the length of the path from p to p_1 is at most $|pa''| + |a''b''| + |b'' p_1|$.

If xp_1 is a Type (i) visibility edge, the length of the path from x to p_1 is at most $|xb_0|+|b_0 p_1|$. Hence we have a path from p to p_1 of length at most $|pa'|+|a'x|+|xb_0|+|b_0 p_1| = |pa'| + |a''b''| + |b_0 p_1|$. Since $|pa'| \leq |pa''|$ and $|b_0 p_1| \leq |b'' p_1|$, this implies that the path has length at most $|pa''|+|a''b''|+|b'' p_1|$. If xp_1 is a Type (ii) visibility edge and x lies in the Northwest corner an analogous argument shows that the path from p to p_1 is at most $|pa''| + |a''b''| + |b'' p_1|$. If xp_1 is a Type (ii) visibility edge and $R_0 = C_0$, we have that the projections of $a'x$ and $a_0 b_0$ onto $a''b''$ are disjoint and the projections of pa' and xa_0 onto pa'' are disjoint. Hence, their total lengths sum up to at most $|pa''| + |a''b''|$. Finally, since $|b_0 p_1| \leq |b'' p_1|$, the total length of the path from p to p_1 is at most $|pa''| + |a''b''| + |b'' p_1|$.

Next, we observe, like in Case (b), that starting from p_1 the convex path cannot go down more than $|p_1 m_1|$. Hence, when we charge this to $C(p, p_1)$, we used $|pa''| + |a''b''| + |b'' m_1|$ of its boundary. Finally, we use arguments analogous to the ones in Case (b) to show that each inductive path after p_1 has length at most $|m_i a'_i| + |a'_i b'_i| + |b'_i m_{i+1}|$. We now apply Lemma 5 to $C(p, p_1)$ and all C_i following p_1 and obtain that the total length of the path from p to q is at most $|pa| + |ab| + |bq|$. □

Lemma 7 *Let p and q be two vertices that can see each other. Let $C(p, q)$ be the rectangle with p and q on its boundary, such that p lies in a corner of $C(p, q)$.*

Let l and s be the length of the long and short side of $C(p,q)$. The constrained generalized Delaunay graph contains a path between p and q of length at most $\left(\frac{2l}{s}+1\right)\cdot(|p_x-q_x|+|p_y-q_y|)$.

Proof. We assume without loss of generality that p lies on the Southwest corner and q lies on the East boundary. Note that this implies that the slope of pq is non-negative, i.e. $p_x < q_x$ and $p_y \le q_y$. We prove the lemma by induction on the rank of $C(x,y)$ when ordered by size, for any two visible vertices x and y, such that x lies in a corner of $C(x,y)$. In fact, we show that the constrained generalized Delaunay graph contains a path between x and y of length at most $c\cdot(q_x-p_x)+d\cdot(q_y-p_y)$ and derive bounds on c and d.

Base case: If $C(p,q)$ is the smallest rectangle with p in a corner, then $C(p,q)$ does not contain any vertices visible to both p and q: Let u be a vertex in $C(p,q)$ that is visible to both p and q. Let $C(p,u)$ be the rectangle with p in a corner and u on its boundary. Since u lies in $C(p,q)$, $C(p,u)$ is smaller than $C(p,q)$, contradicting that $C(p,q)$ is the smallest rectangle with p in a corner. Hence, $C(p,q)$ does not contain any vertices visible to both p and q, which implies that pq is an edge of the constrained generalized Delaunay graph. Hence, the constrained generalized Delaunay graph contains a path between p and q of length at most $|pq| \le (q_x-p_x)+(q_y-p_y) \le c\cdot(q_x-p_x)+d\cdot(q_y-p_y)$, provided that $c \ge 1$ and $d \ge 1$.

Induction step: We assume that for all rectangles $C(x,y)$, with x in some corner of $C(p,q)$, smaller than $C(p,q)$ the lemma holds. If pq is an edge of the constrained generalized Delaunay graph, by the triangle inequality, the length of the shortest path from p to q is at most $|pq| \le |p_x-q_x|+|p_y-q_y|$.

If there is no edge between p and q, there exists a vertex u in $C(p,q)$ that is visible from both p and q. We first look at the case where u lies below pq. Let g be the intersection of the South boundary of $C(p,q)$ and the line though q parallel to the diagonal of $C(p,q)$ through p, and let h be the Southeast corner of $C(p,q)$ (see Figure 8). If u lies in triangle pgq, by induction we have that the path from p to u has length at most $c\cdot(u_x-p_x)+d\cdot(u_y-p_y)$ and the path from u to q has length at most $c\cdot(q_x-u_x)+d\cdot(q_y-u_y)$. Hence, there exists a path from p to q via u of length at most $c\cdot(q_x-p_x)+d\cdot(q_y-p_y)$.

If u lies in triangle ghq, by induction we have that the path from p to u has length at most $c\cdot(u_x-p_x)+d\cdot(u_y-p_y)$ and the path from q to u has length at most $d\cdot(q_x-u_x)+c\cdot(q_y-u_y)$. When we take c and d to be equal, this implies that there exists a path from p to q via u of length at most $c\cdot(q_x-p_x)+d\cdot(q_y-p_y)$.

If there does not exist a vertex below pq that is visible to both p and q, than Lemma 3 implies that there are no vertices in $C(p,q)^p_q$ below pq that are visible to p and

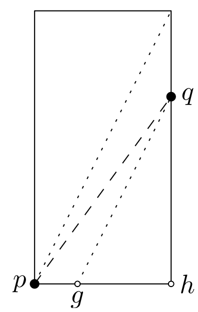

Figure 8: Rectangle $C(p,q)$ with points g and h.

that there are no vertices in $C(p,q)^q_p$ below pq that are visible to q. Hence, we can apply Lemma 6 and obtain that there exists a path between p and q of length at most $|pa|+|ab|+|bq|$, where a and b are the Northwest and Northeast corner of $C(p,q)$. Since $|ab|$ is (q_x-p_x) and $|bq| \le |pa| \le \frac{l}{s}\cdot(q_x-p_x)$, we can upper bound $|pa|+|ab|+|bq|$ by $c\cdot(q_x-p_x)$ when c is at least $\left(\frac{2l}{s}+1\right)$. Hence, since c and d need to be equal, we obtain that all cases work out when $c=d=\left(\frac{2l}{s}+1\right)$. □

Finally, since $(|p_x-q_x|+|p_y-q_y|)/|pq|$ is at most $\sqrt{2}$, we obtain the following theorem.

Theorem 8 *The constrained generalized Delaunay graph using an empty rectangle as empty convex shape has spanning ratio at most $\sqrt{2}\cdot\left(\frac{2l}{s}+1\right)$.*

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