A Appendix: Supplementary Material

A.1 Proof of Lemma 2

Proof: Let k be fixed and $n_k = n$. Let $(X_t)_{t=1}^n$ denote the sequence of rewards associated with arm k, where we recall that $X_t \sim \mathcal{N}(\boldsymbol{\mu}, \sigma_k^2 \boldsymbol{I}_2/2)$. Then $\bar{\boldsymbol{x}}_{k,n} = \frac{1}{n}\sum_{t=1}^n X_t \sim \mathcal{N}(\boldsymbol{\mu}, \sigma_k^2 \boldsymbol{I}_2/(2n))$, implying that $\operatorname{cov}\{(X_t - \bar{\boldsymbol{x}}_{k,n}), \bar{\boldsymbol{x}}_{k,n}\} = \frac{1}{n^2}(n\sigma^2\boldsymbol{I} - n\sigma^2\boldsymbol{I}) = 0$. It follows that, conditioned on σ_k^2 and $\boldsymbol{\mu}_k$, $\bar{\boldsymbol{x}}_{k,n}$ and $(X_t - \bar{\boldsymbol{x}}_{k,n})$ are statistically independent for every $t \in \{1, \dots, n\}$, so $\bar{\boldsymbol{x}}_{k,n}$ and $S_{k,n}$ are also independent.

On the other hand, because $\operatorname{Re} X_t$ and $\operatorname{Im} X_t$ have equal variances, $S_{k,n}$ satisfies

$$\begin{aligned} S_{k,n} &= \sum_{t=1}^n \left\| X_t - \frac{1}{n}\sum_{i=1}^n X_i \right\|^2 \\ &= \sum_{t=1}^n X_t^\top X_t - \frac{1}{n}\sum_{i,j=1}^n X_i^\top X_j, \end{aligned} \tag{18}$$

hence, defining

$$\boldsymbol{A} := \begin{bmatrix} \left(\frac{\lambda}{n} + \frac{1}{\sigma^2} - \lambda\right)\boldsymbol{I}_2 & -\frac{\lambda}{n}\boldsymbol{I}_2 & \cdots & \frac{\lambda}{n}\boldsymbol{I}_2 \\ \frac{\lambda}{n}\boldsymbol{I}_2 & \left(\frac{\lambda}{n} + \frac{1}{\sigma^2} - \lambda\right)\boldsymbol{I}_2 & \cdots & \frac{\lambda}{n}\boldsymbol{I}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda}{n}\boldsymbol{I}_2 & \frac{\lambda}{n}\boldsymbol{I}_2 & \cdots & \left(\frac{\lambda}{n} + \frac{1}{\sigma^2} - \lambda\right)\boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{19}$$

it follows that

$$\begin{aligned} \mathbb{E}\left\{e^{\lambda S_{k,n}}\right\} &= \frac{1}{(\pi\sigma_k^2)^n}\int_{\mathbb{R}^{2n}} e^{\lambda\left(\sum_{t=1}^n X_t^\top X_t - \frac{1}{n}\sum_{t=1}^n X_t^\top X_i\right) - \frac{1}{\sigma_k^2}\sum_{t=1}^n X_t^\top X_t} dX_1\cdots dX_n \\ &= \frac{1}{(\pi\sigma_k^2)^n}\int_{\mathbb{R}^{2n}} e^{[X_1^\top \cdots X_n^\top]\boldsymbol{A}\begin{bmatrix} X_1 \\ \vdots \\ X_n\end{bmatrix}} dX_1\cdots dX_n \\ &= \frac{\sqrt{\det \boldsymbol{A}^{-1}}}{\sigma_k^{2n}} \\ &= \frac{\det^{-1/2}\left\{\left[\frac{1}{\sigma_k^2} - \lambda\right]\boldsymbol{I}_{2n} + \frac{\lambda}{n}\mathbf{1}_n\mathbf{1}_n^\top \otimes \boldsymbol{I}_2\right\}}{\sigma_k^{2n}} \\ &= \frac{1}{\sigma_k^{2n}}\left(\frac{1}{\sigma_k^2} - \lambda\right)^{-(2n)/2}\left(\left(1 + \frac{\lambda n}{n\left(\frac{1}{\sigma^2} - t\right)}\right)^2\right)^{-1/2} \\ &= \frac{1}{(1 - \sigma_k^2\lambda)^{n-1}}, \quad \lambda < 1/\sigma_k^2, \end{aligned} \tag{20}$$

where $\otimes$ denotes Kronecker's product. Thus, $S_{k,n}/(\sigma_k^2/2) \sim \chi^2_{2(n-1)}$ because of uniqueness of the moment-generating function, so its pdf is given by

$$f_{S_{k,n}|\boldsymbol{\mu},\sigma_k^2}(s) = \frac{s^{n-2}}{\Gamma(n-1)}\frac{e^{-s/\sigma_k^2}}{\sigma_k^{2(n-1)}}, \tag{21}$$

and the likelihood of $(\bar{\boldsymbol{x}}_{k,n}, S_{k,n})$ is given by

$$\begin{aligned} f_{S_{k,n},\bar{\boldsymbol{x}}_{k,n}|\boldsymbol{\mu}_k,\sigma_k^2}(s, \boldsymbol{x}) &= f_{S_{k,n}|\boldsymbol{\mu}_k,\sigma_k^2}(s) f_{\bar{\boldsymbol{x}}_{k,n}|\boldsymbol{\mu}_k,\sigma_k^2}(\boldsymbol{x}) \\ &= \frac{n}{\pi\sigma_k^2}e^{-\frac{n}{\sigma_k^2}\|\boldsymbol{x}-\boldsymbol{\mu}\|^2}\frac{s^{n-2}}{\Gamma(n-1)}\frac{e^{-s/\sigma_k^2}}{\sigma_k^{2(n-1)}} \\ &= \frac{ns^{n-2}}{\pi\Gamma(n-1)}\frac{e^{-\frac{1}{\sigma^2}(s+n\|\boldsymbol{x}-\boldsymbol{\mu}\|^2)}}{\sigma_k^{2n}}. \end{aligned} \tag{22}$$

It now follows that, for a uniform (improper) prior over $(\boldsymbol{\mu}_k, \sigma_k^2)$, for every arm $k \in \{1, \dots, K\}$,

$$
\begin{aligned}
f_{\boldsymbol{\mu}_k,\sigma_k^2|\bar{\boldsymbol{x}}_{k,n}=\boldsymbol{x},S_{k,n}=s}(\boldsymbol{\mu}_k,\sigma_k^2) &= \frac{f_{S_{k,n},\bar{\boldsymbol{x}}_{k,n}|\boldsymbol{\mu}_k,\sigma_k^2}(s,\boldsymbol{x})\cdot 1}{\int_0^\infty \int_{\mathbb{R}^2} f_{S_{k,n},\bar{\boldsymbol{x}}_{k,n}|\boldsymbol{\mu}_k,\sigma_k^2}(s,\boldsymbol{x})\cdot 1 d\boldsymbol{\mu}_k\, d(\sigma_k^2)} \\
&= \frac{\frac{ns^{n-2}}{\pi\Gamma(n-1)\sigma_k^{2n}} \mathrm{e}^{-\frac{1}{\sigma_k^2}(s+n\|\boldsymbol{x}-\boldsymbol{\mu}_k\|^2)}}{\int_0^\infty \left(\frac{ns^{n-2}}{\pi\Gamma(n-1)\sigma_k^{2n}} \mathrm{e}^{-s/\sigma_k^2} \int_{\mathbb{R}^2} \mathrm{e}^{\frac{-n}{\sigma_k^2}\|\boldsymbol{x}-\boldsymbol{\mu}_k\|^2} d\boldsymbol{\mu}_k \right) d(\sigma_k^2)} \\
&= \frac{\frac{ns^{n-2}}{\pi\Gamma(n-1)\sigma_k^{n}} \mathrm{e}^{-\frac{1}{\sigma^2}(s+n\|\boldsymbol{x}-\boldsymbol{\mu}_k\|^2)}}{\int_0^\infty \frac{ns^{n-2}}{\pi\Gamma(n-1)\sigma_k^{2n}} \mathrm{e}^{-\frac{s}{\sigma^2}} \frac{\sigma_k^2 \pi}{n} d(\sigma_k^2)} \\
&\overset{(a)}{=} \frac{\frac{ns^{n-2}}{\pi\sigma_k^{2n}} \mathrm{e}^{-\frac{1}{\sigma_k^2}(s+n\|\boldsymbol{x}-\boldsymbol{\mu}_k\|^2)}}{\int_0^\infty u^{n-3}\mathrm{e}^{-u} du} \\
&= \frac{ns^{n-2}}{\pi\Gamma(n-2)\sigma_k^{2n}} \mathrm{e}^{-\frac{1}{\sigma_k^2}(s+n\|\boldsymbol{x}-\boldsymbol{\mu}_k\|^2)},
\end{aligned} \tag{23}
$$

where (a) follows from $u = s/\sigma_k^2$. Therefore, the posterior distribution for the mean $\boldsymbol{\mu}_k$ is

$$
\begin{aligned}
f_{\boldsymbol{\mu}_k|\bar{\boldsymbol{x}}_{k,n}=\boldsymbol{x},S_{k,n}=s}(\boldsymbol{\mu}_k) &= \int f_{\boldsymbol{\mu},\sigma^2|\bar{\boldsymbol{x}}_{k,n}=x,S_{k,n}=s}(\boldsymbol{\mu}_k,\sigma_k^2) d(\sigma_k^2) \\
&= \frac{ns^{n-2}}{\pi\Gamma(n-2)} \int_0^\infty \frac{\mathrm{e}^{-\frac{1}{\sigma_k^2}(s+n\|\boldsymbol{x}-\boldsymbol{\mu}_k\|^2)}}{\sigma_k^{2n}} d(\sigma_k^2) \\
&= \frac{ns^{n-2}}{\pi\Gamma(n-2)} \left(s + n\left\|\boldsymbol{x}-\boldsymbol{\mu}_k\right\|^2 \right)^{-n+1} \int_0^\infty \mathrm{e}^{-u} u^{n-2} du \\
&= \frac{n(n-2)}{\pi s} \left(1 + \frac{n\left\|\boldsymbol{x}-\boldsymbol{\mu}_k\right\|^2}{s} \right)^{-n+1},
\end{aligned} \tag{24}
$$

where (b) follows from $u = (s + n\|\boldsymbol{x} - \boldsymbol{\mu}_k\|^2)/\sigma_k^2$. ■

A.2 Lemma 5

Lemma 5 *Under the conditions of Theorem 1,*

$$
\mathbb{E}\left\{ \sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{ k^{\mathrm{TS}}(t) = k, \mathcal{A}(t), \mathcal{B}_k(t) \right\} \right\} \le \frac{\log T}{\log\left(1 + \frac{(\|\boldsymbol{\mu}_1\| - \|\boldsymbol{\mu}_k\| - 2\epsilon)^2}{\sigma_k^2 + \epsilon}\right)} + 3. \tag{25}
$$

Proof: Firstly, the fact that $\mathbb{1}\left\{k^{\mathrm{TS}}(t) = k, \mathcal{A}(t), \mathcal{B}_k(t)\right\} = 1$ implies that $\|\tilde{\boldsymbol{\mu}}^\star(t)\| = \|\tilde{\boldsymbol{\mu}}_k(t)\| \ge \|\boldsymbol{\mu}_1\| - \epsilon$ under $\mathcal{B}_k(t)$. Recall also the fact that $k^{\mathrm{TS}}(t) = k \implies N_k(t+1) \ge N_k(t) + 1$. Then, for every $n > 0$ it holds that

$$
\begin{aligned}
\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{ k^{\mathrm{TS}}(t) = k, \mathcal{A}(t), \mathcal{B}_k(t) \right\} &= \sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{ k^{\mathrm{TS}}(t) = k, \|\tilde{\boldsymbol{\mu}}_k(t)\| \ge \|\boldsymbol{\mu}_1\| - \epsilon, \mathcal{B}_k(t) \right\} \\
&\le \mathbb{1}\left\{ k^{\mathrm{TS}}(t) = k, \|\tilde{\boldsymbol{\mu}}_k(t)\| \ge \|\boldsymbol{\mu}_1\| - \epsilon, \mathcal{B}_k(t), N_k(t) \ge n \right\} \\
&\quad + \sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{ k^{\mathrm{TS}}(t) = k, N_k(t) \le n \right\} \\
&\le n + \sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{ \|\tilde{\boldsymbol{\mu}}_k(t)\| \ge \|\boldsymbol{\mu}_1\| - \epsilon, \mathcal{B}_k(t), N_k(t) \ge n \right\}.
\end{aligned} \tag{26}
$$

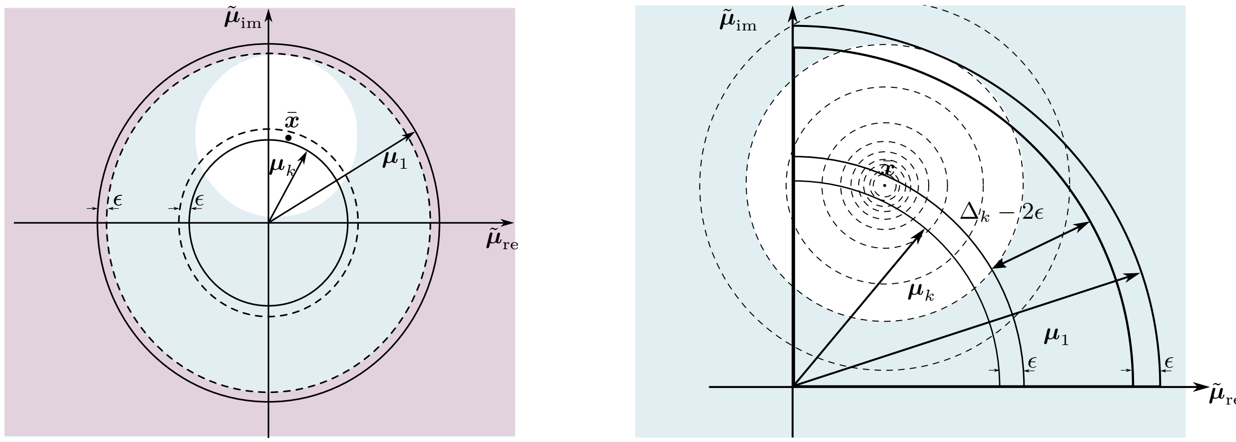

Figure 6: Upper-bounding the probability of $\|\tilde{\boldsymbol{\mu}}_k(t)\| \leq \|\tilde{\boldsymbol{\mu}}_1(t)\| - \epsilon$, given $\hat{\theta}_{k,n}$. On the left side, the red area is upper bounded by the points outside the white circle centered at $\bar{\boldsymbol{x}}$. The right figure shows that the symmetric distribution around $\bar{\boldsymbol{x}}$ and an upper bound for the area outside the circle of radius $\|\boldsymbol{\mu}_1\| - \epsilon$ is upper bounded by any circle of radius $\Delta_k - 2\epsilon$ centered at $\bar{\boldsymbol{x}}$, whenever $\|\bar{\boldsymbol{x}}_k\| \leq \|\boldsymbol{\mu}_k\| - \epsilon$.

Secondly, we can upper bound $\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_k(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \mathcal{B}_k(t), N_k(t) = n\right\}$ as depicted in Fig. 6. From Lemma 4, it follows that

$$\begin{aligned}
&\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_k(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \mathcal{B}_k(t), N_k(t) = n\right\} \\
&\qquad\qquad \leq \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_k(t) - \bar{\boldsymbol{x}}_{k,n}\| \geq \|\boldsymbol{\mu}_1\| - \|\boldsymbol{\mu}_k\| - 2\epsilon \,|\, \mathcal{B}_k(t), N_k(t) = n\right\} \\
&\qquad\qquad \leq \left(1 + \frac{(\|\boldsymbol{\mu}_1\| - \|\boldsymbol{\mu}_k\| - 2\epsilon)^2}{\sigma_k^2 + \epsilon}\right)^{-n+2}.
\end{aligned} \tag{27}$$

Now notice that (27) is decreasing in n, which means that $\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_k(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \mathcal{B}_k(t), N_k(t) \geq n\right\} \leq \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_k(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \mathcal{B}_k(t), N_k(t) = n\right\}$, for every $n > 0$. Then, the expected value of (26) yields

$$\begin{aligned}
\mathbb{E}\left\{\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\text{TS}}(t) = k, \mathcal{A}(t), \mathcal{B}_k(t)\right\}\right\} &\leq n + \sum_{t=\bar{T}+1}^{T} \mathrm{P}\left\{k^{\text{TS}}(t) = k, \mathcal{A}(t), \mathcal{B}_k(t), N_k(t) \geq n\right\} \\
&\leq n + \sum_{t=\bar{T}+1}^{T} \mathrm{P}\left\{k^{\text{TS}}(t) = k, \mathcal{A}(t), \mathcal{B}_k(t), N_k(t) = n\right\} \\
&\leq n + T\left(1 + \frac{(\|\boldsymbol{\mu}_1\| - \|\boldsymbol{\mu}_k\| - 2\epsilon)^2}{\sigma_k^2 + \epsilon}\right)^{-n+2}.
\end{aligned}$$

In particular, for $n = 2 + \frac{\log T}{\log(1 + (\|\boldsymbol{\mu}_1\| - \|\boldsymbol{\mu}_k\| - 2\epsilon)^2/(\sigma_k^2 + \epsilon))} > 0$ we have that

$$\mathbb{E}\left\{\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\text{TS}}(t) = k, \mathcal{A}(t), \mathcal{B}_k(t)\right\}\right\} \leq \frac{\log T}{\log\left(1 + \frac{(\|\boldsymbol{\mu}_1\| - \|\boldsymbol{\mu}_k\| - 2\epsilon)^2}{\sigma_k^2 + \epsilon}\right)} + 3, \tag{28}$$

which concludes the proof. ■

A.3 Lemma 6

Lemma 6 *Under the conditions of Theorem 1, and for every* $k \in \{2, \ldots, K\}$*:*

$$\mathbb{E}\left\{\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\text{TS}}(t) = k, \mathcal{B}_k(t)^c\right\}\right\} \leq \mathrm{O}(\epsilon^{-2}). \tag{29}$$

Proof: We start by noting that the event $B_k^c(t)$ is independent of t whenever $N_k(t)$ is known. Then,

$$\begin{aligned}\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\mathrm{TS}}(t)=k, \mathcal{B}_k^c(t)\right\} &= \sum_{n=\bar{T}/K}^{T} \mathbb{1}\left\{\bigcup_{t=\bar{T}+1}^{T}\left\{k^{\mathrm{TS}}(t)=k, \mathcal{B}_k^c(t), N_k(t)=n\right\}\right\} \\ &\leq \sum_{n=\bar{T}/K}^{T} \mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{n,k}\| \geq \|\boldsymbol{\mu}_k\|+\epsilon \quad \text{or} \quad S_{n,k} \geq n(\sigma_k^2+\epsilon)\right\}. \end{aligned} \tag{30}$$

Then, by means of Lemma 3, the expected value of (30) yields

$$\begin{aligned}\mathbb{E}\left\{\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\mathrm{TS}}(t)=k, \mathcal{B}_k^c(t)\right\}\right\} &\leq \sum_{n=\bar{T}/K}^{T} \mathrm{P}\left\{\|\bar{\boldsymbol{x}}_{n,k}\| \geq \|\boldsymbol{\mu}_k\|+\epsilon\right\} + \mathrm{P}\left\{S_{n,k} \geq n(\sigma_k^2+\epsilon)\right\} \\ &\leq \sum_{n=\bar{T}/K}^{T} \left(\mathrm{e}^{-n\epsilon^2/\sigma^2} + \left(1+\frac{\epsilon}{\sigma_k^2}\right)^{-1} \mathrm{e}^{-nh(\epsilon/\sigma_k^2)}\right) \\ &\leq \frac{1}{1-\mathrm{e}^{-\epsilon^2/\sigma^2}} + \left(1+\frac{\epsilon}{\sigma_k^2}\right)^{-1} \frac{1}{1+\mathrm{e}^{-h(\epsilon/\sigma^2)}}. \\ &= \mathrm{O}(\epsilon^{-2}) + \mathrm{O}(\epsilon^{-2}). \end{aligned} \tag{31}$$

■

A.4 Lemma 7

Lemma 7 *Under the conditions of Theorem 1,*

$$\mathbb{E}\left\{\Delta_{\max} \sum_{t=\bar{T}+1}^{T} \mathbb{E}\left\{\mathbb{1}\left\{k^{\mathrm{TS}}(t) \neq 1, \mathcal{A}^c(t)\right\}\right\}\right\} \leq \mathrm{O}(\epsilon^{-6}). \tag{32}$$

Proof: Note that

$$\begin{aligned}\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\mathrm{TS}}(t) \neq 1, \mathcal{A}^c(t)\right\} &= \sum_{t=\bar{T}+1}^{T}\sum_{n=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\mathrm{TS}}(t) \neq 1, \mathcal{A}^c(t), N_1(t)=n\right\} \\ &= \sum_{n=\bar{T}+1}^{T}\sum_{m=1}^{T} \mathbb{1}\left\{m \leq \sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\mathrm{TS}}(t) \neq 1, \mathcal{A}^c(t), N_1(t)=n\right\}\right\}, \end{aligned} \tag{33}$$

since, for a fixed t, $\mathbb{1}\left\{k^{\mathrm{TS}}(t) \neq 1, \mathcal{A}^c(t), N_1(t)=n\right\}=1$ only for that value of n being exactly $N_1(t)$. Observe that $k^{\mathrm{TS}}(t) \neq 1$ means that $\|\tilde{\boldsymbol{\mu}}_1(t)\| \leq \|\tilde{\boldsymbol{\mu}}^\star(t)\|$, and $\mathcal{A}^c(t)$ means $\|\tilde{\boldsymbol{\mu}}^\star(t)\| \leq \|\boldsymbol{\mu}_1\|-\epsilon$. Then, (33) implies that, for a fixed n, the event $\|\tilde{\boldsymbol{\mu}}_1(t)\| \leq \|\boldsymbol{\mu}_1\|-\epsilon$ has taken place, at least, m times. Let $\nu_n := \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\|-\epsilon \,|\, \hat{\theta}_{1,n}\right\}$. This implies that

$$\begin{aligned}\mathbb{E}\left\{\sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\mathrm{TS}}(t) \neq 1, \mathcal{A}^c(t)\right\}\right\} &= \sum_{n=\bar{T}+1}^{T}\sum_{m=1}^{T} \mathrm{P}\left\{m \leq \sum_{t=\bar{T}+1}^{T} \mathbb{1}\left\{k^{\mathrm{TS}}(t) \neq 1, \mathcal{A}^c(t), N_1(t)=n\right\}\right\} \\ &\leq \mathbb{E}\left\{\sum_{n=\bar{T}+1}^{T}\sum_{m=1}^{T} \left(1-\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\|-\epsilon \,|\, \hat{\theta}_{1,n}\right\}\right)^m\right\} \\ &\leq \sum_{n=\bar{T}+1}^{T} \mathbb{E}\left\{\frac{1-\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\|-\epsilon \,|\, \hat{\theta}_{1,n}\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\|-\epsilon \,|\, \hat{\theta}_{1,n}\right\}}\right\}. \end{aligned} \tag{34}$$

Now, when $\|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \epsilon$ the symmetry of $p_{k,n}(\tilde{\boldsymbol{\mu}}_1)$ (defined in (10)) around $\bar{\boldsymbol{x}}_{1,n}$ guarantees that $\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\} \geq 1/2$, and then it follows that

$$\begin{aligned}1 - \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\} &\leq \frac{1}{2} \\ \frac{1}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}} &\leq 2.\end{aligned} \tag{35}$$

This argument allows us to split (34) as

$$\begin{aligned}&\mathbb{E}\left\{\frac{1 - \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}}\right\} \\ &= \mathbb{E}\left\{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \epsilon\right\}\right\} + \mathbb{E}\left\{\frac{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \leq \|\boldsymbol{\mu}_1\| - \epsilon\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}}\right\} \\ &= \mathbb{E}\left\{\mathbb{1}\left\{\|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2} \geq \|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \epsilon\right\}\right\} + \mathbb{E}\left\{\frac{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \leq \|\boldsymbol{\mu}_1\| - \epsilon\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}}\right\} \\ &\quad + \mathbb{E}\left\{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}, S_{k,n} \geq 2\sigma_1^2 n\right\}\right\} \\ &\quad + 2\mathbb{E}\left\{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}, S_{k,n} \leq 2\sigma_1^2 n\right\}\left(1 - \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}\right)\right\}.\end{aligned} \tag{36}$$

We now proceed to upper bound each term in (36). For the first term, we have that

$$\begin{aligned}\mathbb{E}\left\{\mathbb{1}\left\{\|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2} \geq \|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \epsilon\right\}\right\} &\leq \mathrm{P}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \leq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}\right\} \\ &= \int_{\|\boldsymbol{z}\| \leq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}} \frac{n}{\pi\sigma_1^2} \mathrm{e}^{\frac{-n}{\sigma_1^2}\|\boldsymbol{z} - \boldsymbol{\mu}_1\|^2} d\boldsymbol{z} \\ &\leq \int_{\|\boldsymbol{z}\| \leq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}} \frac{n}{\pi\sigma_1^2} \mathrm{e}^{\frac{-n}{\sigma_1^2}(\|\boldsymbol{\mu}_1\| - \|\boldsymbol{z}\|)^2} d\boldsymbol{z} \\ &\leq \int_{\|\boldsymbol{z}\| \leq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}} \frac{n}{\pi\sigma_1^2} \mathrm{e}^{\frac{-n}{4\sigma_1^2}\epsilon^2} d\boldsymbol{z} \\ &= \left(\|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}\right)^2 \frac{n}{\sigma_1^2} \mathrm{e}^{\frac{-n\epsilon^2}{4\sigma_1^2}} \\ &\leq \|\boldsymbol{\mu}_1\|^2 \frac{n}{\sigma_1^2} \mathrm{e}^{\frac{-n\epsilon^2}{4\sigma_1^2}}.\end{aligned} \tag{37}$$

For the second term, observe that, given $\|\bar{\boldsymbol{x}}_{1,n}\| \leq \|\boldsymbol{\mu}_1\| - \epsilon$ and $S_{1,n} = s$:

$$\begin{aligned}\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\} &\geq \int_{-\alpha}^{\alpha} \int_{\frac{\|\boldsymbol{\mu}_1\| - \|\bar{\boldsymbol{x}}_{1,n}\| - \epsilon}{\cos\alpha}}^{\infty} \frac{n(n-2)}{\pi s} \left(1 + \frac{nr^2}{s}\right)^{-n+1} r\, dr\, d\phi \\ &= \frac{\alpha}{\pi} \left(1 + \frac{n\left(\|\boldsymbol{\mu}_1\| - \|\bar{\boldsymbol{x}}_{1,n}\| - \epsilon\right)^2}{s\cos^2\alpha}\right)^{-n+2},\end{aligned} \tag{38}$$

for every $\alpha \in (0, \pi/2)$. It then follows that

$$\mathbb{E}\left\{\frac{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \leq \|\boldsymbol{\mu}_1\| - \epsilon\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}}\right\}$$

$$
\begin{aligned}
&= \int_0^\infty \int_{\|\boldsymbol{x}\|\le\|\boldsymbol{\mu}_1\|-\epsilon} \frac{ns^{n-2}}{\pi\Gamma(n-1)} \frac{\mathrm{e}^{-\frac{1}{\sigma^2}(s+n\|\boldsymbol{x}-\boldsymbol{\mu}\|^2)}}{\sigma_k^{2n}} \frac{\pi}{\alpha} \left(1+\frac{n\left(\|\boldsymbol{\mu}_1\|-\|\boldsymbol{x}\|-\epsilon\right)^2}{s\cos^2\alpha}\right)^{n-2} d\boldsymbol{x}ds \\
&\overset{(a)}{\le} \frac{n\mathrm{e}^{-n\epsilon^2/\sigma_1^2}}{\alpha\Gamma(n-1)\sigma_1^{2n}} \int_0^\infty s^{n-2}\mathrm{e}^{-s/\sigma_1^2} \int_{\|\boldsymbol{x}\|\le\|\boldsymbol{\mu}_1\|-\epsilon} \mathrm{e}^{\frac{-n(\|\boldsymbol{\mu}_1\|-\|\boldsymbol{x}\|-\epsilon)^2}{\sigma_1^2}} \left(1+\frac{n\left(\|\boldsymbol{\mu}_1\|-\|\boldsymbol{x}\|-\epsilon\right)^2}{s\cos^2\alpha}\right)^{n-2} d\boldsymbol{x}ds \\
&\overset{(b)}{=} \frac{2\pi n\mathrm{e}^{-n\epsilon^2/\sigma_1^2}}{\alpha\Gamma(n-1)\sigma_1^{2n}} \int_0^\infty s^{n-2}\mathrm{e}^{-s/\sigma_1^2} \int_0^{\|\boldsymbol{\mu}_1\|-\epsilon} \mathrm{e}^{\frac{-n(\|\boldsymbol{\mu}_1\|-v-\epsilon)^2}{\sigma_1^2}} \left(1+\frac{n\left(\|\boldsymbol{\mu}_1\|-v-\epsilon\right)^2}{s\cos^2\alpha}\right)^{n-2} v\,dvds \\
&\overset{(c)}{=} \frac{2\pi n\mathrm{e}^{-n\epsilon^2/\sigma_1^2}}{\alpha 2^{n-1}\Gamma(n-1)\sigma_1^{2n}} \int_0^\infty s^{n-2}\mathrm{e}^{-s/\sigma_1^2} \int_\epsilon^{\|\boldsymbol{\mu}_1\|} \mathrm{e}^{\frac{-n(r-\epsilon)^2}{\sigma_1^2}} \left(1+\frac{n\left(r-\epsilon\right)^2}{s\cos^2\alpha}\right)^{n-2} (\|\boldsymbol{\mu}_1\|-r)\,drds \\
&\le \frac{2\pi\|\boldsymbol{\mu}_1\|n\mathrm{e}^{-n\epsilon^2/\sigma_1^2}}{\alpha\Gamma(n-1)\sigma_1^{2n}} \int_0^\infty s^{n-2}\mathrm{e}^{-s/\sigma_1^2} \int_\epsilon^\infty \mathrm{e}^{\frac{-n(r-\epsilon)^2}{\sigma_1^2}} \left(1+\frac{n\left(r-\epsilon\right)^2}{s\cos^2\alpha}\right)^{n-2} drds,
\end{aligned} \tag{39}
$$

where inequality (a) follows from $\|\boldsymbol{\mu}_1-\boldsymbol{x}\|^2 \ge (\|\boldsymbol{\mu}_1-\boldsymbol{x}\|-\epsilon)^2+\epsilon^2 \ge (\|\boldsymbol{\mu}_1\|-\|\boldsymbol{x}\|-\epsilon)^2$ whenever $\|\boldsymbol{x}\|\le\|\boldsymbol{\mu}_1\|-\epsilon$. Equality in (b) follows from a change of variables $\boldsymbol{x}$ to polar coordinates (v,θ), while (c) is consequence of changing v via $r=\|\boldsymbol{\mu}_1\|-v$. We now introduce a new change of variables in (39):

$$
\left.\begin{aligned} r &= \epsilon-\cos\alpha\sqrt{\tfrac{zw}{n}} \\ s &= z(1-w) \end{aligned}\right\} \implies dr\,ds = \left|\det\begin{bmatrix} \frac{\cos\alpha}{2}\sqrt{\frac{w}{nz}} & (1-w) \\ \frac{\cos\alpha}{2}\sqrt{\frac{z}{nw}} & -z \end{bmatrix}\right| dw\,dz = \frac{\cos\alpha}{2}\sqrt{\frac{z}{nw}}dw\,dz,
$$

allowing us to rewrite the double integral in (39) as

$$
\begin{aligned}
&\int_0^\infty z^{n-2}\sqrt{z}\,\mathrm{e}^{\frac{-z}{\sigma^2}} \int_0^1 \mathrm{e}^{-z/\sigma_1^2} w^{\frac{-1}{2}} \frac{\cos\alpha}{2\sqrt{n}} dw\,dz \\
&\qquad = \frac{\cos\alpha}{2\sqrt{n}} \int_0^\infty z^{n-2}\sqrt{z}\,\mathrm{e}^{-z/\sigma_1^2} \frac{\sigma_1}{\sqrt{z}\sin\alpha} \mathrm{e}^{\frac{z\sin^2\alpha}{\sigma_1^2}} D\left(\sqrt{z}\frac{\sin\alpha}{\sigma_1^2}\right) dz \\
&\qquad \overset{(d)}{\le} \frac{\sigma_1}{2\sqrt{n}\tan\alpha} \int_0^\infty z^{n-2}\mathrm{e}^{-z\cos^2\alpha/\sigma_1^2} dz \\
&\qquad = \frac{\sigma_1}{2\sqrt{n}\tan(\alpha)} \frac{\sigma_1^{2(n-1)}}{\cos^{2(n-1)}(\alpha)} \Gamma(n-1),
\end{aligned} \tag{40}
$$

where (d) follows from Dawson's function D [27] being upper bounded by 1. Finally, using this upper bound in (39) yields

$$
\mathbb{E}\left\{\frac{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\|\le\|\boldsymbol{\mu}_1\|-\epsilon\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\|\ge\|\boldsymbol{\mu}_1\|-\epsilon\,|\,\hat{\theta}_{1,n}\right\}}\right\} \le \frac{\pi\|\boldsymbol{\mu}_1\|\cos^3\alpha}{\sigma_1\sin\alpha}\sqrt{n}\left(\frac{\mathrm{e}^{-\epsilon^2/\sigma_1^2}}{\cos^2\alpha}\right)^n, \tag{41}
$$

for every $\alpha\in(0,2\pi)$. In particular, when α is small, the sequence in n defined by the right-hand side of (41) converges to zero for α satisfying $\cos^2\alpha \approx 1-\alpha^2/2 = \mathrm{e}^{-\epsilon^2/(2\sigma_1^2)}$, *i.e.*, for $\alpha=\sqrt{2\left(1-\mathrm{e}^{\epsilon^2/(2\sigma_1^2)}\right)} = \mathrm{O}(\epsilon)$. With this particular choice, and for small ϵ, the bound in (41) becomes

$$
\mathbb{E}\left\{\frac{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\|\le\|\boldsymbol{\mu}_1\|-\epsilon\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\|\ge\|\boldsymbol{\mu}_1\|-\epsilon\,|\,\hat{\theta}_{1,n}\right\}}\right\} \le \frac{\pi\|\boldsymbol{\mu}_1\|}{\alpha^2\sigma_1}\sqrt{n}\left(\mathrm{e}^{-\epsilon^2/(2\sigma_1^2)}\right)^n. \tag{42}
$$

From Lemma 3, the third term in (36) is upper bounded as

$$
\mathbb{E}\left\{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\|\ge\|\boldsymbol{\mu}_1\|-\frac{\epsilon}{2}, S_{1,n}\ge n2\sigma_1^2\right\}\right\} \le \mathrm{P}\left\{S_{1,n}\ge n2\sigma_1^2\right\} \le 2\left(1+\frac{\epsilon}{\sigma_k^2}\right)^{-1}\mathrm{e}^{-nh(1)}. \tag{43}
$$

To upper bound the fourth term in (36), denote $\mathcal{C}_{1,n} := \{\|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}, S_{1,n} \leq 2\sigma_1^2 n\}$. Then by introducing a polar-coordinates change of variable:

$$
\begin{aligned}
1 - \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \mathcal{C}_{1,n}\right\} &= \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \leq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \mathcal{C}_{1,n}\right\} \\
&\leq \int_{-\pi}^{\pi}\int_{\frac{\epsilon}{2}}^{\infty} \frac{n(n-2)}{\pi s}\left(1 + \frac{nr^2}{s}\right)^{-n+1} r\, dr d\phi, \quad (\cdot, s) \in \mathcal{C}_{1,n}, \\
&\leq \left(1 + \frac{\epsilon^2}{8\sigma_1^2}\right)^{-n+2}, \qquad (44)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\mathbb{E}\left\{\mathbb{1}\left\{\|\bar{\boldsymbol{x}}_{1,n}\| \geq \|\boldsymbol{\mu}_1\| - \frac{\epsilon}{2}, S_{k,n} \leq n2\sigma_1^2\right\}\left(1 - \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}\right)\right\} \\
&\qquad \leq \left(1 + \frac{\epsilon^2}{8\sigma_1^2}\right)^{-n+2}. \qquad (45)
\end{aligned}
$$

Putting together (37),(42),(43) and (45) together with (34) lead us to

$$
\begin{aligned}
&\sum_{n=\bar{T}+1}^{T} \mathbb{E}\left\{\frac{1 - \mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}}{\mathrm{P}\left\{\|\tilde{\boldsymbol{\mu}}_1(t)\| \geq \|\boldsymbol{\mu}_1\| - \epsilon \,|\, \hat{\theta}_{1,n}\right\}}\right\} \\
&\leq \sum_{n=\bar{T}+1}^{T} \|\boldsymbol{\mu}_1\|^2 \frac{n}{\sigma_1^2} \mathrm{e}^{\frac{-n\epsilon^2}{4\sigma_1^2}} + \frac{\pi\|\boldsymbol{\mu}_1\|}{\alpha^2\sigma_1}\sqrt{n}\left(\mathrm{e}^{-\epsilon^2/(2\sigma_1^2)}\right)^n + 2\left(1 + \frac{\epsilon}{\sigma_k^2}\right)^{-1} \mathrm{e}^{-nh(1)} + \left(1 + \frac{\epsilon^2}{8\sigma_1^2}\right)^{-n+2} \\
&\leq \frac{\|\boldsymbol{\mu}_1\|^2}{\sigma_1^2} \frac{\mathrm{e}^{-\epsilon^2/(4\sigma_1^2)}}{1 - \mathrm{e}^{-\epsilon^2/(4\sigma_1^2)}} + \frac{\pi\|\boldsymbol{\mu}_1\|}{\alpha^2\sigma_1} \frac{\mathrm{e}^{-\epsilon^2/(2\sigma_1^2)}}{\left(1 - \mathrm{e}^{-\epsilon^2/(2\sigma_1^2)}\right)^2} + \frac{2\left(1 + \frac{\epsilon}{\sigma_k^2}\right)^{-1}}{1 - \mathrm{e}^{-h(1)}} + \frac{8\sigma_1^2}{\epsilon^2} \\
&= \mathrm{O}(\epsilon^{-2}) + \mathrm{O}(\epsilon^{-6}) + \mathrm{O}(1) + \mathrm{O}(\epsilon^{-2}) = \mathrm{O}(\epsilon^{-6}). \qquad (46)
\end{aligned}
$$

■