Supplementary Document for "Bandit Online Learning with Unknown Delays"

A Real to virtual slot mapping

For the analysis, let $t(\tau)$ denote the real slot when the real loss $\boldsymbol{l}_{t(\tau)}$ corresponding to $\tilde{\boldsymbol{l}}_\tau$ was incurred, i.e., $\tilde{\boldsymbol{l}}_\tau = \hat{\boldsymbol{l}}_{t(\tau)|t(\tau)+d_{t(\tau)}}$. Also define an auxiliary variable $\tilde{s}_\tau = \tau - 1 - L_{t(\tau)-1}$. See an example in Fig. 6 and Table 1.

Lemma 6. *The following relations hold: i) $\tilde{s}_\tau \geq 0,\ \forall \tau$; ii) $\sum_{\tau=1}^T \tilde{s}_\tau = \sum_{t=1}^T d_t$; and, iii) if $\max_t d_t \leq \bar{d}$, we have $\tilde{s}_\tau \leq 2\bar{d},\ \forall \tau$.*

Proof. We first prove the property i) $\tilde{s}_\tau \geq 0,\ \forall t$. Consider at virtual slot τ, the observed loss is $l_{t(\tau)}(a_{t(\tau)})$ with corresponding $\tilde{s}_\tau = \tau - 1 - L_{t(\tau)-1}$. Suppose that $L_{t(\tau)-1} = m$, where $0 \leq m \leq t(\tau) - 1$ (by definition of $L_{t(\tau)-1}$). The history $L_{t(\tau)-1} = m$ suggests that at the beginning of $t_1 = t(\tau)$, the number of received feedback is m. On the other hand, the loss $l_{t(\tau)}(a_{t(\tau)})$ is observed at the end of slot $t_2 = t(\tau) + d_{t(\tau)} \geq t_1$, thus at the beginning of t_2, there are at least m observations. Hence we must have $\tau \geq m+1$. Then by the definition, $\tilde{s}_\tau \geq m + 1 - 1 - m = 0$.

Then for the property ii) $\sum_{\tau=1}^T \tilde{s}_\tau = \sum_{t=1}^T d_t$, the proof follows from the definition of $\tilde{s}_\tau$, i.e.,

$$\begin{aligned}\sum_{\tau=1}^T \tilde{s}_\tau &= \sum_{\tau=1}^T \Big(\tau - 1 - L_{t(\tau)-1}\Big) = \sum_{t=1}^T (t-1) - \sum_{\tau=1}^T L_{t(\tau)-1} \\ &\overset{(a)}{=} \sum_{t=1}^T \big(t - 1 - L_{t-1}\big) \overset{(b)}{=} \sum_{t=1}^T d_t \end{aligned} \tag{20}$$

where (a) is due to the fact that $\{t(\tau)\}_{\tau=1}^T$ is a permutation of $\{1, \cdots, T\}$; and (b) follows from the definition of L_{t-1}.

Finally, for property iii), notice that $L_{t(\tau)-1} \geq t(\tau) - 1 - \bar{d}$, which follows that at the beginning of $t = t(\tau)$, the losses of slots $t \leq t(\tau) - 1 - \bar{d}$ must have been received. Therefore, we have

$$\tilde{s}_\tau = \tau - 1 - L_{t(\tau)-1} \leq \tau - 1 - t(\tau) + 1 + \bar{d} \overset{(c)}{\leq} 2\bar{d} \tag{21}$$

where (c) follows from that $l_{t(\tau)}(a_{t(\tau)})$ is observed at the end of $t = t(\tau) + d_{t(\tau)}$, and $L_{t(\tau)+d_{t(\tau)}-1}$ is at most $t(\tau) + d_{t(\tau)} - 2$ (since $l_{t(\tau)}(a_{t(\tau)})$ is not observed), leading to the fact that τ is at most $t(\tau) + d_{t(\tau)}$, and thus $\tau - t(\tau) \leq d_{t(\tau)} \leq \bar{d}$. ☐

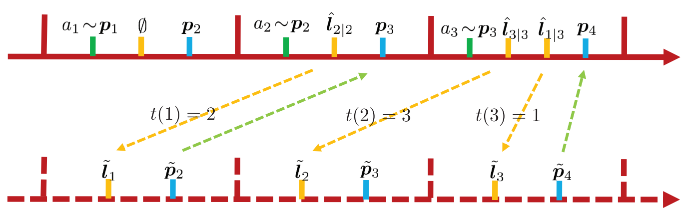

Figure 6: An example of mapping from real slots (solid line) to virtual slots (dotted line). The value of $t(\tau)$ is marked beside the corresponding yellow arrow. In the example, we consider $T = 3$ with delay $d_1 = 2$, $d_2 = 0$, and $d_3 = 0$.

Table 1: The Value of $t(\tau)$, $L_{t(\tau)-1}$, and $\tilde{s}_\tau$ in Fig. 6.

Virtual slot	$\tau = 1$	$\tau = 2$	$\tau = 3$
$t(\tau)$	2	3	1
$L_{t(\tau)-1}$	0	1	0
$\tilde{s}_\tau$	0	0	2

B Proofs for DEXP3

Before diving into the proofs, we first show some useful yet simple bounds for different parameters of the DEXP3's (in virtual slots). In virtual slot τ, the update is carried out the same as (6), (7) and (8), given by

$$\tilde{w}_{\tau+1}(k) = \tilde{p}_\tau(k) \exp\Big[-\eta \min\big\{\delta_1, \tilde{l}_\tau(k)\big\}\Big],\ \forall k, \tag{22}$$

$$w_{\tau+1}(k) = \max\left\{\frac{\tilde{w}_{\tau+1}(k)}{\sum_{j=1}^K \tilde{w}_{\tau+1}(j)}, \frac{\delta_2}{K}\right\},\ \forall k, \tag{23}$$

$$\tilde{p}_{\tau+1}(k) = \frac{w_{\tau+1}(k)}{\sum_{j=1}^K w_{\tau+1}(j)},\ \forall k. \tag{24}$$

Since $\tilde{l}_\tau(k) \geq 0, \forall k, \tau$, we have

$$\sum_{j=1}^{K} \tilde{w}_\tau(j) \leq \sum_{j=1}^{K} \tilde{p}_{\tau-1}(j) = 1. \tag{25}$$

And $\sum_{k=1}^{K} w_\tau(k)$ is bounded by

$$\sum_{k=1}^{K} w_\tau(k) \geq \sum_{k=1}^{K} \frac{\tilde{w}_\tau(k)}{\sum_{j=1}^{K} \tilde{w}_\tau(j)} = 1; \tag{26}$$

$$\sum_{k=1}^{K} w_\tau(k) \leq \sum_{k=1}^{K} \frac{\tilde{w}_\tau(k)}{\sum_{j=1}^{K} \tilde{w}_\tau(j)} + \delta_2 = 1 + \delta_2. \tag{27}$$

Finally, $\tilde{p}_\tau(k)$ is bounded by

$$\frac{\delta_2}{K(1+\delta_2)} \leq \frac{w_\tau(k)}{1+\delta_2} \leq \tilde{p}_\tau(k) \leq w_\tau(k). \tag{28}$$

B.1 Proof of Lemma 1

Lemma 7. *In consecutive virtual slots $\tau - 1$ and τ, the following inequality holds for any k.*

$$\tilde{p}_{\tau-1}(k) - \tilde{p}_\tau(k) \leq \tilde{p}_{\tau-1}(k) \frac{\delta_2 + \eta \min\left\{\delta_1, \tilde{l}_{\tau-1}(k)\right\}}{1+\delta_2}. \tag{29}$$

Proof. First, we have

$$\tilde{p}_\tau(k) \overset{(a)}{\geq} \frac{w_\tau(k)}{1+\delta_2} \geq \frac{\tilde{w}_\tau(k)}{\sum_{j=1}^{K} \tilde{w}_\tau(j)(1+\delta_2)} \overset{(b)}{\geq} \frac{\tilde{w}_\tau(k)}{1+\delta_2} = \frac{\tilde{p}_{\tau-1}(k) \exp\left[-\eta \min\left\{\delta_1, \tilde{l}_{\tau-1}(k)\right\}\right]}{1+\delta_2} \tag{30}$$

where (a) is the result of (28); (b) is due to (25). Hence, we have

$$\begin{aligned}
\tilde{p}_\tau(k) - \tilde{p}_{\tau-1}(k) &\geq \frac{\tilde{p}_{\tau-1}(k) \exp\left[-\eta \min\left\{\delta_1, \tilde{l}_{\tau-1}(k)\right\}\right]}{1+\delta_2} - \tilde{p}_{\tau-1}(k) \\
&\overset{(c)}{\geq} \frac{\tilde{p}_{\tau-1}(k)}{1+\delta_2}\left[1 - \eta \min\left\{\delta_1, \tilde{l}_{\tau-1}(k)\right\}\right] - \tilde{p}_{\tau-1}(k) \\
&= \tilde{p}_{\tau-1}(k) \frac{-\delta_2 - \eta \min\left\{\delta_1, \tilde{l}_{\tau-1}(k)\right\}}{1+\delta_2}
\end{aligned} \tag{31}$$

where (c) follows from $e^{-x} \geq 1 - x$ and the proof is completed by multiplying -1 on both sides of (31). □

From Lemma 7, we have

$$\tilde{p}_{\tau-1}(k) - \tilde{p}_\tau(k) \leq \tilde{p}_{\tau-1}(k) \frac{\delta_2 + \eta \min\left\{\delta_1, \tilde{l}_{\tau-1}(k)\right\}}{1+\delta_2} \leq \tilde{p}_{\tau-1}(k)\big(\delta_2 + \eta\delta_1\big). \tag{32}$$

Hence, as long as $1 - \delta_2 - \eta\delta_1 \geq 0$, we can guarantee that (13) is satisfied.

B.2 Proof of Lemma 2

Lemma 8. *The following inequality holds for any τ and any k*

$$\tilde{p}_\tau(k) - \tilde{p}_{\tau-1}(k) \leq \tilde{p}_\tau(k)\left[1 - I_\tau(k) \sum_{j=1}^{K} \tilde{p}_{\tau-1}(j)\Big(1 - \eta \min\left\{\delta_1, \tilde{l}_{\tau-1}(j)\right\}\Big)\right] \tag{33}$$

where $I_\tau(k) := \mathbb{1}\big(w_\tau(k) > \frac{\delta_2}{K}\big)$.

Proof. We first show that

$$\tilde{w}_\tau(k) \geq \tilde{p}_\tau(k) I_\tau(k) \sum_{j=1}^{K} \tilde{w}_\tau(j). \tag{34}$$

It is easy to see that inequality (34) holds when $I_\tau(k)=0$. When $I_\tau(k)=1$, we have $w_\tau(k)=\tilde{w}_\tau(k)/\big(\sum_{j=1}^K \tilde{w}_\tau(j)\big)$. By (28), we have $\tilde{p}_\tau(k)\le w_\tau(k)=\tilde{w}_\tau(k)/\big(\sum_{j=1}^K \tilde{w}_\tau(j)\big)$, from which (34) holds. Then we have

$$\begin{aligned}\tilde{p}_\tau(k)-\tilde{p}_{\tau-1}(k) &\le \tilde{p}_\tau(k)-\tilde{w}_\tau(k)\le \tilde{p}_\tau(k)-\tilde{p}_\tau(k)I_\tau(k)\sum_{j=1}^K\tilde{w}_\tau(j)\\ &= \tilde{p}_\tau(k)\Big[1-I_\tau(k)\sum_{j=1}^K\tilde{w}_\tau(j)\Big]=\tilde{p}_\tau(k)\Big\{1-I_\tau(k)\sum_{j=1}^K\tilde{p}_{\tau-1}(j)\exp\Big[-\eta\min\big\{\delta_1,\tilde{l}_{\tau-1}(j)\big\}\Big]\Big\}\\ &\overset{(a)}{\le} \tilde{p}_\tau(k)\Big[1-I_\tau(k)\sum_{j=1}^K\tilde{p}_{\tau-1}(j)\Big(1-\eta\min\big\{\delta_1,\tilde{l}_{\tau-1}(j)\big\}\Big)\Big]\end{aligned}\tag{35}$$

where in (a) we used $e^{-x}\ge 1-x$. □

The proof of Lemma 2 builds on Lemma 8. First consider the case of $I_\tau(k)=0$. In this case Lemma 8 becomes $\tilde{p}_\tau(k)-\tilde{p}_{\tau-1}(k)\le\tilde{p}_\tau(k)$, which is trivial. On the other hand, since $I_\tau(k)=0$, we have $w_\tau(k)=\frac{\delta_2}{K}$. Then leveraging (28), we have $\tilde{p}_\tau(k)\le w_\tau(k)=\frac{\delta_2}{K}$. Plugging the lower bound of $\tilde{p}_{\tau-1}(k)$ into (28), we have

$$\frac{\tilde{p}_\tau(k)}{\tilde{p}_{\tau-1}(k)}\le\frac{\delta_2}{K}\frac{1}{\tilde{p}_{\tau-1}(k)}\le\frac{\delta_2}{K}\frac{K(1+\delta_2)}{\delta_2}=1+\delta_2.\tag{36}$$

Considering the case of $I_\tau(k)=1$, Lemma 8 becomes

$$\begin{aligned}\tilde{p}_\tau(k)-\tilde{p}_{\tau-1}(k)&\le\tilde{p}_\tau(k)\Big[1-\sum_{j=1}^K\tilde{p}_{\tau-1}(j)\Big(1-\eta\min\big\{\delta_1,\tilde{l}_{\tau-1}(j)\big\}\Big)\Big]\\ &=\eta\tilde{p}_\tau(k)\sum_{j=1}^K\tilde{p}_{\tau-1}(j)\min\big\{\delta_1,\tilde{l}_{\tau-1}(k)\big\}\le\eta\tilde{p}_\tau(k)\delta_1.\end{aligned}\tag{37}$$

Rearranging (37) and combining it with (36), we complete the proof.

B.3 Proof of Lemma 3

For conciseness, define $\tilde{\boldsymbol{c}}_\tau:=\min\big\{\tilde{\boldsymbol{l}}_\tau,\delta_1\cdot\mathbf{1}\big\}$, and correspondingly $\tilde{c}_\tau(k):=\min\{\tilde{l}_\tau(k),\delta_1\}$. We further define $\tilde{W}_\tau:=\sum_{k=1}^K\tilde{w}_\tau(k)$, and $W_\tau:=\sum_{k=1}^K w_\tau(k)$. Leveraging these auxiliary variables, we have

$$\begin{aligned}\tilde{W}_{T+1}&=\sum_{k=1}^K\tilde{w}_{T+1}(k)=\sum_{k=1}^K\tilde{p}_T(k)\exp\big[-\eta\tilde{c}_T(k)\big]=\sum_{k=1}^K\frac{w_T(k)}{W_T}\exp\big[-\eta\tilde{c}_T(k)\big]\\ &\ge\sum_{k=1}^K\frac{\tilde{w}_T(k)}{\tilde{W}_T}\frac{\exp\big[-\eta\tilde{c}_T(k)\big]}{W_T}=\sum_{k=1}^K\tilde{p}_{T-1}(k)\frac{\exp\big[-\eta\tilde{c}_T(k)-\eta\tilde{c}_{T-1}(k)\big]}{\tilde{W}_TW_T}\\ &=\sum_{k=1}^K\frac{w_{T-1}(k)}{W_{T-1}}\frac{\exp\big[-\eta\tilde{c}_T(k)-\eta\tilde{c}_{T-1}(k)\big]}{\tilde{W}_TW_T}\ge\cdots\ge\sum_{k=1}^K\frac{\tilde{w}_1(k)\exp\Big[-\eta\sum_{\tau=1}^T\tilde{c}_\tau(k)\Big]}{\prod_{\tau=1}^T\big(W_\tau\tilde{W}_\tau\big)}.\end{aligned}\tag{38}$$

Then, for any probability distribution $\boldsymbol{p}\in\Delta_K$ noticing that the initialization of $\tilde{w}_1(k)=1,\forall k$ and hence $\tilde{W}_1=K$, inequality (38) implies that

$$\sum_{k=1}^K p(k)\exp\Big[-\eta\sum_{\tau=1}^T\tilde{c}_\tau(k)\Big]\le\sum_{k=1}^K\exp\Big[-\eta\sum_{\tau=1}^T\tilde{c}_\tau(k)\Big]\le\tilde{W}_1\prod_{\tau=1}^T\big(W_\tau\tilde{W}_{\tau+1}\big)\overset{(a)}{\le}K(1+\delta_2)^T\prod_{\tau=2}^{T+1}\tilde{W}_\tau,\tag{39}$$

where in (a) we used the fact that $W_\tau\le 1+\delta_2$. Then, using the the Jensen's inequality on e^{-x}, we have

$$\sum_{k=1}^K p(k)\exp\Big[-\eta\sum_{\tau=1}^T\tilde{c}_\tau(k)\Big]\ge\exp\Big[-\eta\sum_{k=1}^K\sum_{\tau=1}^T p(k)\tilde{c}_\tau(k)\Big].\tag{40}$$

Plugging (40) into (39), we arrive at

$$\exp\Big[-\eta\sum_{k=1}^K\sum_{\tau=1}^T p(k)\tilde{c}_\tau(k)\Big]\le K(1+\delta_2)^T\prod_{\tau=2}^{T+1}\tilde{W}_\tau.\tag{41}$$

On the other hand, $\tilde{W}_\tau$ can be upper bounded by

$$\begin{aligned}\tilde{W}_\tau &= \sum_{k=1}^{K} \tilde{w}_\tau = \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \exp\big[-\eta \tilde{c}_{\tau-1}(k)\big] \\ &\overset{(b)}{\leq} \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \bigg(1 - \eta \tilde{c}_{\tau-1}(k) + \frac{\eta^2}{2}\big[\tilde{c}_{\tau-1}(k)\big]^2\bigg) \\ &= 1 - \eta \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \tilde{c}_{\tau-1}(k) + \frac{\eta^2}{2} \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \big[\tilde{c}_{\tau-1}(k)\big]^2 \end{aligned} \tag{42}$$

where (b) follows from $e^{-x} \leq 1 - x + x^2/2,\ \forall x \geq 0$. Taking logarithm on both sides of (42), we arrive at

$$\begin{aligned}\ln \tilde{W}_\tau &\leq \ln\left(1 - \eta \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \tilde{c}_{\tau-1}(k) + \frac{\eta^2}{2} \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \big[\tilde{c}_{\tau-1}(k)\big]^2\right) \\ &\overset{(c)}{\leq} -\eta \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \tilde{c}_{\tau-1}(k) + \frac{\eta^2}{2} \sum_{k=1}^{K} \tilde{p}_{\tau-1}(k) \big[\tilde{c}_{\tau-1}(k)\big]^2 \end{aligned} \tag{43}$$

where (c) follows from $\ln(1+x) \leq x$. Then taking logarithm on both sides of (41) and plugging (43) in, we arrive at

$$-\eta \sum_{k=1}^{K} \sum_{\tau=1}^{T} p(k) \tilde{c}_\tau(k) \leq T \ln(1+\delta_2) + \ln K - \eta \sum_{\tau=1}^{T} \sum_{k=1}^{K} \tilde{p}_\tau(k) \tilde{c}_\tau(k) + \frac{\eta^2}{2} \sum_{\tau=1}^{T} \sum_{k=1}^{K} \tilde{p}_\tau(k) \big[\tilde{c}_\tau(k)\big]^2. \tag{44}$$

Rearranging the terms of (44) and writing it compactly, we obtain

$$\begin{aligned}\sum_{\tau=1}^{T} \big(\tilde{\boldsymbol{p}}_\tau - \boldsymbol{p}\big)^\top \tilde{\boldsymbol{c}}_\tau &\leq \frac{T \ln(1+\delta_2) + \ln K}{\eta} + \frac{\eta}{2} \sum_{\tau=1}^{T} \sum_{k=1}^{K} \tilde{p}_\tau(k) \big[\tilde{c}_\tau(k)\big]^2 \\ &\leq \frac{T \ln(1+\delta_2) + \ln K}{\eta} + \frac{\eta}{2} \sum_{\tau=1}^{T} \sum_{k=1}^{K} \tilde{p}_\tau(k) \big[\tilde{l}_\tau(k)\big]^2. \end{aligned} \tag{45}$$

B.4 Proof of Theorem 1

To begin with, the instantaneous regret can be written as

$$\begin{aligned}\boldsymbol{p}_t^\top \boldsymbol{l}_t - \boldsymbol{p}^\top \boldsymbol{l}_t &= \sum_{k=1}^{K} p_t(k) l_t(k) - \sum_{k=1}^{K} p(k) l_t(k) \\ &\overset{(a)}{=} \sum_{k=1}^{K} p_t(k) \mathbb{E}_{a_t}\left[\frac{l_t(k) \mathbb{1}(a_t = k)}{p_t(k)}\right] - \sum_{k=1}^{K} p(k) \mathbb{E}_{a_t}\left[\frac{l_t(k) \mathbb{1}(a_t = k)}{p_t(k)}\right] \\ &= \sum_{k=1}^{K} \big(p_t(k) - p(k)\big) \mathbb{E}_{a_t}\left[\frac{l_t(k) \mathbb{1}(a_t = k)}{p_{t+d_t}(k)} \frac{p_{t+d_t}(k)}{p_t(k)}\right] \\ &\leq \max_k \frac{p_{t+d_t}(k)}{p_t(k)} \sum_{k=1}^{K} \big(p_t(k) - p(k)\big) \mathbb{E}_{a_t}\left[\frac{l_t(k) \mathbb{1}(a_t = k)}{p_{t+d_t}(k)}\right] \\ &\overset{(b)}{=} \left(\max_k \frac{p_{t+d_t}(k)}{p_t(k)}\right) \mathbb{E}_{a_t}\Big[\boldsymbol{p}_t^\top \hat{\boldsymbol{l}}_{t|t+d_t} - \boldsymbol{p}^\top \hat{\boldsymbol{l}}_{t|t+d_t}\Big] \end{aligned} \tag{46}$$

where (a) is due to $\mathbb{E}_{a_t}\left[\frac{l_t(k) \mathbb{1}(a_t=k)}{p_t(k)}\right] = l_t(k)$, and (b) follows from $\hat{l}_{t|t+d_t}(k) = \frac{l_t(k) \mathbb{1}(a_t=k)}{p_{t+d_t}(k)}$.

Then the overall regret of T slots is given by

$$\begin{aligned}
\text{Reg}_T &= \mathbb{E}\Big[\sum_{t=1}^{T}\boldsymbol{p}_t^\top \boldsymbol{l}_t\Big] - \boldsymbol{p}^\top \boldsymbol{l}_t \leq \mathbb{E}\Big[\sum_{t=1}^{T}\Big(\max_k \frac{p_{t+d_t}(k)}{p_t(k)}\Big)\mathbb{E}_{a_t}\big[\boldsymbol{p}_t^\top \hat{\boldsymbol{l}}_{t|t+d_t} - \boldsymbol{p}^\top \hat{\boldsymbol{l}}_{t|t+d_t}\big]\Big] \\
&\overset{(c)}{=} \mathbb{E}\Big[\sum_{\tau=1}^{T}\Big(\max_k \frac{p_{t(\tau)+d_{t(\tau)}}(k)}{p_{t(\tau)}(k)}\Big)\mathbb{E}_{a_{t(\tau)}}\Big[\boldsymbol{p}_{t(\tau)}^\top \hat{\boldsymbol{l}}_{t(\tau)|t(\tau)+d_t(\tau)} - \boldsymbol{p}^\top \hat{\boldsymbol{l}}_{t(\tau)|t(\tau)+d_t(\tau)}\Big]\Big] \\
&\overset{(d)}{=} \mathbb{E}\Big[\sum_{\tau=1}^{T}\Big(\max_k \frac{p_{t(\tau)+d_{t(\tau)}}(k)}{p_{t(\tau)}(k)}\Big)\mathbb{E}_{a_{t(\tau)}}\Big[\boldsymbol{p}_{t(\tau)}^\top \tilde{\boldsymbol{l}}_\tau - \boldsymbol{p}^\top \tilde{\boldsymbol{l}}_\tau\Big]\Big] \\
&\overset{(e)}{=} \mathbb{E}\Big[\sum_{\tau=1}^{T}\Big(\max_k \frac{p_{t(\tau)+d_{t(\tau)}}(k)}{p_{t(\tau)}(k)}\Big)\mathbb{E}_{a_{t(\tau)}}\Big[\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}^\top \tilde{\boldsymbol{l}}_\tau - \boldsymbol{p}^\top \tilde{\boldsymbol{l}}_\tau\Big]\Big] \\
&= \mathbb{E}\Big[\sum_{\tau=1}^{T}\Big(\max_k \frac{p_{t(\tau)+d_{t(\tau)}}(k)}{p_{t(\tau)}(k)}\Big)\Big(\mathbb{E}_{a_{t(\tau)}}\Big[\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}^\top \tilde{\boldsymbol{l}}_\tau - \tilde{\boldsymbol{p}}_\tau^\top \tilde{\boldsymbol{l}}_\tau\Big] + \mathbb{E}_{a_{t(\tau)}}\Big[\tilde{\boldsymbol{p}}_\tau^\top \tilde{\boldsymbol{l}}_\tau - \boldsymbol{p}^\top \tilde{\boldsymbol{l}}_\tau\Big]\Big)\Big] \qquad (47)
\end{aligned}$$

where (c) is due to the fact that $\{t(1), t(2), \dots, t(T)\}$ is a permutation of $\{1, 2, \dots, T\}$; (d) follows from $\tilde{\boldsymbol{l}}_\tau = \hat{\boldsymbol{l}}_{t(\tau)|t(\tau)+d_t(\tau)}$; (e) uses the fact $\boldsymbol{p}_t = \tilde{\boldsymbol{p}}_{L_{t-1}+1}$ and $\boldsymbol{p}_{t(\tau)} = \tilde{\boldsymbol{p}}_{L_{t(\tau)-1}+1} = \tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}$.

First note that between real time slot $t(\tau)$ and $t(\tau)+d_{t(\tau)}$, there is at most $\bar{d}+d_{t(\tau)} \leq 2\bar{d}$ feedback received. Hence the corresponding virtual slots will not differ larger than $2\bar{d}$. Note also that the index of virtual slot corresponding to $t(\tau)$ must be no larger than that of $t(\tau)+d_{t(\tau)}$. Hence we have for all $\tau \in [1, T]$,

$$\max_k \frac{p_{t(\tau)+d_{t(\tau)}}(k)}{p_{t(\tau)}(k)} \leq \Big(\max_k \frac{\tilde{p}_{\tau+1}(k)}{\tilde{p}_\tau(k)}\Big)^{2\bar{d}} \overset{(f)}{\leq} \max\Big\{(1+\delta_2)^{2\bar{d}}, \frac{1}{(1-\eta\delta_1)^{2\bar{d}}}\Big\} \qquad (48)$$

where (f) is the result of Lemma 2.

Then, to bound the terms in the second brackets of (47), again we denote $\tilde{\boldsymbol{c}}_\tau := \min\{\tilde{\boldsymbol{l}}_\tau, \delta_1 \cdot \mathbf{1}\}$, and correspondingly $\tilde{c}_\tau(k) := \min\{\tilde{l}_\tau(k), \delta_1\}$ for conciseness. Then we have

$$\begin{aligned}
\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}^\top \tilde{\boldsymbol{c}}_\tau - \tilde{\boldsymbol{p}}_\tau^\top \tilde{\boldsymbol{c}}_\tau &= \tilde{\boldsymbol{c}}_\tau^\top(\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau} - \tilde{\boldsymbol{p}}_\tau) \overset{(g)}{=} \tilde{c}_\tau(m)\sum_{j=0}^{\tilde{s}_\tau-1}\big(\tilde{p}_{\tau-\tilde{s}_\tau+j}(m) - \tilde{p}_{\tau-\tilde{s}_\tau+j+1}(m)\big) \\
&\overset{(h)}{\leq} \tilde{c}_\tau(m)\sum_{j=0}^{\tilde{s}_\tau-1}\tilde{p}_{\tau-\tilde{s}_\tau+j}(m)\frac{\delta_2+\eta\tilde{c}_{\tau-\tilde{s}_\tau+j}(m)}{1+\delta_2} \leq \tilde{c}_\tau(m)\sum_{j=0}^{\tilde{s}_\tau-1}\big(\eta\tilde{p}_{\tau-\tilde{s}_\tau+j}(m)\tilde{c}_{\tau-\tilde{s}_\tau+j}(m)+\delta_2\big) \\
&\leq \tilde{l}_\tau(m)\sum_{j=0}^{\tilde{s}_\tau-1}\big(\eta\tilde{p}_{\tau-\tilde{s}_\tau+j}(m)\tilde{l}_{\tau-\tilde{s}_\tau+j}(m)+\delta_2\big) \qquad (49)
\end{aligned}$$

where (g) follows from the facts that $\tilde{\boldsymbol{l}}_\tau$ has at most one entry (with index m) being non-zero [cf. (59)] and $\tilde{s}_\tau \geq 0$ [cf. Lemma 6]; and (h) is the result of Lemma 7. Then notice that

$$\tilde{l}_\tau(k)\tilde{p}_\tau(k) = \frac{l_{t(\tau)}(k)}{p_{t(\tau)+d_t(\tau)}(k)}\tilde{p}_\tau(k) \overset{(i)}{\leq} \Big(\max_k \frac{\tilde{p}_\tau(k)}{\tilde{p}_{\tau+1}(k)}\Big)^{2\bar{d}} \leq \frac{1}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}} \qquad (50)$$

where (i) uses the fact that between $t(\tau)$ and $t(\tau)+d_{t(\tau)}$ there is at most $2\bar{d}$ feedback; then further applying the result of Lemma 1, inequality (50) can be obtained. Plugging (50) back in to (49) and taking expectation w.r.t. $a_{t(\tau)}$, we arrive at

$$\begin{aligned}
\mathbb{E}_{a_{t(\tau)}}\big[\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}^\top \tilde{\boldsymbol{c}}_\tau - \tilde{\boldsymbol{p}}_\tau^\top \tilde{\boldsymbol{c}}_\tau\big] &\leq \Big(\frac{\eta\tilde{s}_\tau}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}} + \delta_2\tilde{s}_\tau\Big)\sum_{k=1}^{K} p_{t(\tau)}(k)\tilde{l}_\tau(k) \\
&\overset{(j)}{\leq} K\frac{1}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}}\Big(\frac{\eta\tilde{s}_\tau}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}} + \delta_2\tilde{s}_\tau\Big) \qquad (51)
\end{aligned}$$

where (j) follows a similar reason of (50). Then, noticing $\sum_{\tau=1}^T \tilde{s}_\tau = \sum_{t=1}^T d_t = D$, we have

$$\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\big[\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}^\top \tilde{\boldsymbol{c}}_\tau - \tilde{\boldsymbol{p}}_\tau^\top \tilde{\boldsymbol{c}}_\tau\big] \leq \frac{KD}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}}\Big(\frac{\eta}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}} + \delta_2\Big). \qquad (52)$$

Using a similar argument of (50), we can obtain

$$\mathbb{E}_{a_{t(\tau)}}\Big[\tilde{p}_\tau(k)\big[\tilde{l}_\tau(k)\big]^2\Big] = \tilde{p}_\tau(k)\frac{l^2_{t(\tau)}(k)}{p^2_{t(\tau)+d_{t(\tau)}}(k)}p_{t(\tau)}(k) \leq \frac{1}{(1-\delta_2-\eta\delta_1)^{4\bar{d}}} \tag{53}$$

Then leveraging Lemma 3, we arrive at

$$\begin{aligned}\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{c}}_\tau\big] &\leq \frac{T\ln(1+\delta_2)+\ln K}{\eta}+\frac{\eta}{2}\sum_{\tau=1}^{T}\sum_{k=1}^{K}\mathbb{E}_{a_{t(\tau)}}\Big[\tilde{p}_\tau(k)\big[\tilde{l}_\tau(k)\big]^2\Big]\\ &\leq \frac{T\ln(1+\delta_2)+\ln K}{\eta}+\frac{\eta KT}{2(1-\delta_2-\eta\delta_1)^{4\bar{d}}}.\end{aligned} \tag{54}$$

The last step is to show that introducing δ_1 will not incur too much extra regret. Note that both $\tilde{\boldsymbol{c}}_\tau$ and $\tilde{\boldsymbol{l}}_\tau$ have only one entry being non-zero, whose index is denoted by m_τ. Notice that $\tilde{l}_\tau(m_\tau) > \tilde{c}_\tau(m_\tau)$ only when $\tilde{l}_\tau(m_\tau) = \frac{l_{t(\tau)}(m_\tau)}{p_{t(\tau)+d_{t(\tau)}}(m_\tau)} > \delta_1$, which is equivalent to $p_{t(\tau)+d_{t(\tau)}}(m_\tau) < l_{t(\tau)}(m_\tau)/\delta_1 \leq 1/\delta_1$. Hence, we have

$$\begin{aligned}\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{l}}_\tau\big] &= \sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{c}}_\tau\big]+\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\big(\tilde{\boldsymbol{l}}_\tau-\tilde{\boldsymbol{c}}_\tau\big)\big]\\ &\overset{(h)}{\leq} \sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{c}}_\tau\big]+\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[\tilde{p}_\tau(m_\tau)\big(\tilde{l}_\tau(m_\tau)-\tilde{c}_\tau(m_\tau)\big)\mathbb{1}\big(p_{t(\tau)+d_{t(\tau)}}(m_\tau)<1/\delta_1\big)\Big]\\ &\leq \sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{c}}_\tau\big]+\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[\tilde{p}_\tau(m_\tau)\tilde{l}_\tau(m_\tau)\mathbb{1}\big(p_{t(\tau)+d_{t(\tau)}}(m_\tau)<1/\delta_1\big)\Big]\end{aligned} \tag{55}$$

where in (h), m_τ denotes the index of the only one none-zero entry of $\tilde{\boldsymbol{l}}_\tau$, and $\tilde{\boldsymbol{p}}$ is dropped due to the appearance of the indicator function. To proceed, notice that

$$\begin{aligned}&\mathbb{E}_{a_{t(\tau)}}\Big[\tilde{l}_\tau(m_\tau)\tilde{p}_\tau(m_\tau)\mathbb{1}\big(p_{t(\tau)+d_{t(\tau)}}(m_\tau)<1/\delta_1\big)\Big] = \sum_{k=1}^{K}\frac{p_{t(\tau)}(k)l_{t(\tau)}(k)}{p_{t(\tau)+d_{t(\tau)}}(k)}\tilde{p}_\tau(k)\mathbb{1}\big(p_{t(\tau)+d_{t(\tau)}}(k)<1/\delta_1\big)\\ &\overset{(i)}{\leq} \frac{\sum_{k=1}^{K}\tilde{p}_\tau(k)\mathbb{1}\big(p_{t(\tau)+d_{t(\tau)}}(k)<1/\delta_1\big)}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}} = \sum_{k=1}^{K}\frac{\tilde{p}_\tau(k)}{p_{t(\tau)+d_{t(\tau)}}(k)}\frac{p_{t(\tau)+d_{t(\tau)}}(k)\mathbb{1}\big(p_{t(\tau)+d_{t(\tau)}}(k)<1/\delta_1\big)}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}}\\ &\overset{(j)}{\leq} \frac{K}{\delta_1(1-\delta_2-\eta\delta_1)^{4\bar{d}}}\end{aligned} \tag{56}$$

where in (i) we used the a similar argument of (50); and in (j) we used the fact $x\mathbb{1}(x<a)\leq a$.

Plugging (56) back into (55), we arrive at

$$\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{l}}_\tau\Big] \leq \sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{c}}_\tau\Big]+\frac{KT}{\delta_1(1-\delta_2-\eta\delta_1)^{4\bar{d}}} \tag{57}$$

Applying similar arguments as (55) and (56), we can also show that

$$\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[\big(\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}-\tilde{\boldsymbol{p}}_\tau\big)^\top\tilde{\boldsymbol{l}}_\tau\Big] \leq \sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[\big(\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}-\tilde{\boldsymbol{p}}_\tau\big)^\top\tilde{\boldsymbol{c}}_\tau\Big]+\frac{KD}{\delta_1(1-\delta_2-\eta\delta_1)^{6\bar{d}}}. \tag{58}$$

For the parameter selection, we have $T\ln(1+\delta_2) = T\ln(1+\frac{1}{T+D}) \leq \ln e = 1$. Leveraging the inequality that $e \leq (1-2x)^{-2x} \leq 4, \forall x\in\mathbb{N}^+$, we have that

$$\frac{1}{(1-\eta\delta_1)^{2\bar{d}}} \leq \frac{1}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}} = \mathcal{O}(1). \tag{59}$$

From (59) it is not hard to see the bound on (48), which is

$$\max_k\frac{p_{t(\tau)+d_{t(\tau)}}(k)}{p_{t(\tau)}(k)} \leq \max\left\{(1+\delta_2)^{2\bar{d}}, \frac{1}{(1-\eta\delta_1)^{2\bar{d}}}\right\} = \mathcal{O}(1). \tag{60}$$

Then for (52), we have

$$\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\left[\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}^\top\tilde{\boldsymbol{c}}_\tau-\tilde{\boldsymbol{p}}_\tau^\top\tilde{\boldsymbol{c}}_\tau\right]\leq\frac{KD}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}}\left(\frac{\eta}{(1-\delta_2-\eta\delta_1)^{2\bar{d}}}+\delta_2\right)=\mathcal{O}(\eta KD+\delta_2 KD). \tag{61}$$

For (54), we have

$$\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\left[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{c}}_\tau\right]\leq\frac{T\ln(1+\delta_2)+\ln K}{\eta}+\frac{\eta KT}{2(1-\delta_2-\eta\delta_1)^{4\bar{d}}}=\mathcal{O}\bigg(\eta KT+\frac{1+\ln K}{\eta}\bigg). \tag{62}$$

Using (62) and the selection of δ_1, we can bound (57) by

$$\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{l}}_\tau\Big]\leq\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[(\tilde{\boldsymbol{p}}_\tau-\tilde{\boldsymbol{p}})^\top\tilde{\boldsymbol{c}}_\tau\Big]+\frac{KT}{\delta_1(1-\delta_2-\eta\delta_1)^{4\bar{d}}}=\mathcal{O}\bigg(\eta KT+\frac{1+\ln K}{\eta}\bigg). \tag{63}$$

Using (61) and the selection of δ_1, we have

$$\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[\big(\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}-\tilde{\boldsymbol{p}}_\tau\big)^\top\tilde{\boldsymbol{l}}_\tau\Big]\leq\sum_{\tau=1}^{T}\mathbb{E}_{a_{t(\tau)}}\Big[\big(\tilde{\boldsymbol{p}}_{\tau-\tilde{s}_\tau}-\tilde{\boldsymbol{p}}_\tau\big)^\top\tilde{\boldsymbol{c}}_\tau\Big]+\frac{KD}{\delta_1(1-\delta_2-\eta\delta_1)^{6\bar{d}}}=\mathcal{O}\big(\eta KD+\delta_2 KD\big). \tag{64}$$

Plugging (60), (63) , and (64) into (47), the regret is bounded by

$$\text{Reg}_T=\sum_{t=1}^{T}\mathbb{E}\big[\boldsymbol{p}_t^\top\boldsymbol{l}_t\big]-\sum_{t=1}^{T}\boldsymbol{p}^{*\top}\boldsymbol{l}_t=\mathcal{O}\big(\sqrt{(T+D)K(1+\ln K)}\big). \tag{65}$$

C Proofs for DBGD

C.1 Proof of Lemma 4

Since $f_{s|t}(\cdot)$ is L-Lipschitz, we have $g_{s|t}(k)\leq\frac{1}{\delta}L\|\delta\boldsymbol{e}_k\|=L$, and thus $\|\boldsymbol{g}_{s|t}\|\leq\sqrt{K}L$. On the other hand, let $\boldsymbol{\nabla}_{s|t}:=\nabla f_{s|t}(\boldsymbol{x}_{s|t})$, and $\nabla_{s|t}(k)$ being the k-th entry of $\boldsymbol{\nabla}_{s|t}$. Due to the β-smoothness of $f_{s|t}(\cdot)$, we have

$$g_{s|t}(k)-\nabla_{s|t}(k)\leq\frac{1}{\delta}\big(\delta\boldsymbol{\nabla}_{s|t}^\top\boldsymbol{e}_k+\frac{\beta}{2}\delta^2\big)-\nabla_{s|t}(k)=\frac{\beta\delta}{2} \tag{66}$$

suggesting that $\|\boldsymbol{g}_{s|t}-\nabla f_{s|t}(\boldsymbol{x}_{s|t})\|\leq\frac{\beta\delta}{2}\sqrt{K}$.

C.2 Proof of Lemma 5

Lemma 5 (Restate). *In virtual slots, it is guaranteed to have*

$$\|\tilde{\boldsymbol{x}}_\tau-\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}\|\leq\eta\tilde{s}_\tau\sqrt{K}L \tag{67}$$

and for any $\boldsymbol{x}\in\mathcal{X}_\delta$, we have

$$\eta\tilde{\boldsymbol{g}}_\tau^\top\big(\tilde{\boldsymbol{x}}_\tau-\boldsymbol{x}\big)\leq\frac{\eta^2}{2}KL^2+\frac{\big\|\tilde{\boldsymbol{x}}_\tau-\boldsymbol{x}\big\|^2-\big\|\tilde{\boldsymbol{x}}_{\tau+1}-\boldsymbol{x}\big\|^2}{2}. \tag{68}$$

Proof. The proof begins with

$$\|\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}-\tilde{\boldsymbol{x}}_\tau\|\leq\sum_{j=0}^{\tilde{s}_\tau-1}\|\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau+j}-\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau+j+1}\|\overset{(a)}{\leq}\eta\tilde{s}_\tau\sqrt{K}L \tag{69}$$

where (a) uses the fact that $\|\tilde{\boldsymbol{x}}_\tau-\tilde{\boldsymbol{x}}_{\tau+1}\|=\big\|\tilde{\boldsymbol{x}}_\tau-\Pi_{\mathcal{X}_\delta}[\tilde{\boldsymbol{x}}_\tau-\eta\tilde{\boldsymbol{g}}_\tau]\big\|\leq\eta\|\tilde{\boldsymbol{g}}_\tau\|$. The first inequality is thus proved

Then, notice that

$$\begin{aligned}\big\|\tilde{\boldsymbol{x}}_{\tau+1}-\boldsymbol{x}\big\|^2-\big\|\tilde{\boldsymbol{x}}_\tau-\boldsymbol{x}\big\|^2&=\big\|\Pi_{\mathcal{X}_\delta}[\tilde{\boldsymbol{x}}_\tau-\eta\tilde{\boldsymbol{g}}_\tau]-\boldsymbol{x}\big\|^2-\big\|\tilde{\boldsymbol{x}}_\tau-\boldsymbol{x}\big\|^2\\&\overset{(b)}{\leq}\big\|\tilde{\boldsymbol{x}}_\tau-\boldsymbol{x}-\eta\tilde{\boldsymbol{g}}_\tau\big\|^2-\big\|\tilde{\boldsymbol{x}}_\tau-\boldsymbol{x}\big\|^2=-2\eta\tilde{\boldsymbol{g}}_\tau^\top\big(\tilde{\boldsymbol{x}}_\tau-\boldsymbol{x}\big)+\eta^2\big\|\tilde{\boldsymbol{g}}_\tau\big\|^2\end{aligned} \tag{70}$$

where inequality (b) uses the non-expansion property of projection. Rearranging the terms of (70) completes the proof. □

C.3 Proof of Theorem 2

Lemma 9. *Let* $h_t(\boldsymbol{x}) := f_t(\boldsymbol{x}) + \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top \boldsymbol{x}$*, where* $\boldsymbol{g}_t := \boldsymbol{g}_{t|t+d_t}$*. Then* $h_t(\boldsymbol{x})$ *has the following properties: i)* $h_t(\boldsymbol{x})$ *is* $\big(L + \frac{\beta\delta\sqrt{K}}{2}\big)$*-Lipschitz; and ii)* $h_t(\boldsymbol{x})$ *is* β *smooth and convex.*

Proof. Starting with the first property, consider that

$$\|h_t(\boldsymbol{x}) - h_t(\boldsymbol{y})\| = \big\|f_t(\boldsymbol{x}) + \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top \boldsymbol{x} - f_t(\boldsymbol{y}) - \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top \boldsymbol{y}\big\|$$
$$\leq \|f_t(\boldsymbol{x}) - f_t(\boldsymbol{y})\| + \big\|\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big\|\|\boldsymbol{x} - \boldsymbol{y}\| \overset{(a)}{\leq} \bigg(L + \frac{\beta\delta\sqrt{K}}{2}\bigg)\|\boldsymbol{x} - \boldsymbol{y}\| \tag{71}$$

where in (a) we used the results in Lemma 4. For the second property, the convexity of $h_t(\boldsymbol{x})$ is obvious. Then noticing that $\nabla h_t(\boldsymbol{x}) = \nabla f_t(\boldsymbol{x}) + \boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)$, we have

$$h_t(\boldsymbol{y}) - h_t(\boldsymbol{x}) = f_t(\boldsymbol{y}) - f_t(\boldsymbol{x}) + \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top (\boldsymbol{y} - \boldsymbol{x})$$
$$\leq \big(\nabla f_t(\boldsymbol{x})\big)^\top (\boldsymbol{y} - \boldsymbol{x}) + \frac{\beta}{2}\|\boldsymbol{y} - \boldsymbol{x}\|^2 + \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top (\boldsymbol{y} - \boldsymbol{x})$$
$$= \big(\nabla h_t(\boldsymbol{x})\big)^\top (\boldsymbol{y} - \boldsymbol{x}) + \frac{\beta}{2}\|\boldsymbol{y} - \boldsymbol{x}\|^2 \tag{72}$$

which implies that $h_t(\boldsymbol{x})$ is β smooth. □

Then we are ready to prove Theorem 2. Let $h_t(\boldsymbol{x}) := f_t(\boldsymbol{x}) + \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top \boldsymbol{x}$, where $\boldsymbol{g}_t := \boldsymbol{g}_{t|t+d_t}$. Using the property of $h_t(\boldsymbol{x})$ in Lemma 9 as well as the fact $\nabla h_t(\boldsymbol{x}_t) = \boldsymbol{g}_t$, we have

$$\text{Reg}_T = \sum_{t=1}^T f_t(\boldsymbol{x}_t) - \sum_{t=1}^T f_t(\boldsymbol{x}^*)$$
$$= \sum_{t=1}^T \Big(h_t(\boldsymbol{x}_t) - \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top \boldsymbol{x}_t\Big) - \sum_{t=1}^T \Big(h_t(\boldsymbol{x}^*) - \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top \boldsymbol{x}^*\Big)$$
$$= \sum_{t=1}^T \Big(h_t(\boldsymbol{x}_t) - h_t(\boldsymbol{x}^*)\Big) + \sum_{t=1}^T \big(\boldsymbol{g}_t - \nabla f_t(\boldsymbol{x}_t)\big)^\top \big(\boldsymbol{x}^* - \boldsymbol{x}_t\big)$$
$$\overset{(a)}{\leq} \sum_{t=1}^T \Big(h_t(\boldsymbol{x}_t) - h_t(\boldsymbol{x}_\delta)\Big) + \sum_{t=1}^T \Big(h_t(\boldsymbol{x}_\delta) - h_t(\boldsymbol{x})\Big) + \frac{RT\beta\delta\sqrt{K}}{2}$$
$$\overset{(b)}{\leq} \sum_{t=1}^T \Big(h_t(\boldsymbol{x}_t) - h_t(\boldsymbol{x}_\delta)\Big) + \delta RT\Big(L + \frac{\beta\delta\sqrt{K}}{2}\Big) + \frac{RT\beta\delta\sqrt{K}}{2} \tag{73}$$

where in (a) $\boldsymbol{x}_\delta := \Pi_{\mathcal{X}_\delta}(\boldsymbol{x}^*)$, and the inequality follows from the results in Lemma 4; (b) follows from the fact that $h_t(\cdot)$ is $(L + \frac{\beta\delta\sqrt{K}}{2})$-Lipschitz, as well as $\|\boldsymbol{x}_\delta - \boldsymbol{x}\| \leq \delta R$.

Hence, at virtual slots, it is like learning according to $h_t(\boldsymbol{x}_t)$, with $\nabla h_t(\boldsymbol{x}_t)$ being revealed. With the short-hand notation $\tilde{h}_\tau(\cdot) := h_{t(\tau)}(\cdot)$, we have (using similar arguments like the proof of Theorem 1)

$$\sum_{t=1}^T h_t(\boldsymbol{x}_t) - \sum_{t=1}^T h_t(\boldsymbol{x}_\delta) = \sum_{\tau=1}^T h_{t(\tau)}(\boldsymbol{x}_{t(\tau)}) - \sum_{\tau=1}^T h_{t(\tau)}(\boldsymbol{x}_\delta) = \sum_{\tau=1}^T \tilde{h}_\tau(\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}) - \sum_{\tau=1}^T \tilde{h}_\tau(\boldsymbol{x}_\delta)$$
$$= \sum_{\tau=1}^T \tilde{h}_\tau(\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}) - \sum_{\tau=1}^T \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) + \sum_{\tau=1}^T \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) - \sum_{\tau=1}^T \tilde{h}_\tau(\boldsymbol{x}_\delta). \tag{74}$$

The first term in the RHS of (74) can be bounded as

$$\tilde{h}_\tau(\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}) - \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) \leq \big\|\tilde{h}_\tau(\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}) - \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau)\big\| \overset{(c)}{\leq} \bigg(L + \frac{\beta\delta\sqrt{K}}{2}\bigg)\big\|\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau} - \tilde{\boldsymbol{x}}_\tau\big\| \overset{(d)}{\leq} \eta\tilde{s}_\tau\sqrt{K}L\bigg(L + \frac{\beta\delta\sqrt{K}}{2}\bigg) \tag{75}$$

where (c) follows from Lemma 9; and (d) is the result of Lemma 5. Hence, using $\sum_{\tau=1}^T \tilde{s}_\tau = D$ in Lemma 6, we obtain

$$\sum_{\tau=1}^T \tilde{h}_\tau(\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}) - \sum_{\tau=1}^T \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) \leq \eta D\sqrt{K}L\bigg(L + \frac{\beta\delta\sqrt{K}}{2}\bigg). \tag{76}$$

On the other hand, by the convexity of $\tilde{h}_\tau(\cdot)$, we have

$$\tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) - \tilde{h}_\tau(\boldsymbol{x}_\delta) \leq \big(\nabla \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau)\big)^\top \big(\tilde{\boldsymbol{x}}_\tau - \boldsymbol{x}_\delta\big) = \big[\nabla \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) - \tilde{\boldsymbol{g}}_\tau\big]^\top \big(\tilde{\boldsymbol{x}}_\tau - \boldsymbol{x}_\delta\big) + \tilde{\boldsymbol{g}}_\tau^\top \big(\tilde{\boldsymbol{x}}_\tau - \boldsymbol{x}_\delta\big)$$
$$\stackrel{(e)}{\leq} \beta\big\|\tilde{\boldsymbol{x}}_\tau - \tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}\big\|\big\|\tilde{\boldsymbol{x}}_\tau - \boldsymbol{x}_\delta\big\| + \tilde{\boldsymbol{g}}_\tau^\top \big(\tilde{\boldsymbol{x}}_\tau - \boldsymbol{x}_\delta\big) \leq \beta R \big\|\tilde{\boldsymbol{x}}_\tau - \tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}\big\| + \tilde{\boldsymbol{g}}_\tau^\top \big(\tilde{\boldsymbol{x}}_\tau - \boldsymbol{x}_\delta\big) \tag{77}$$

where (e) is because $\tilde{h}_\tau(\cdot)$ is β-smoothness [cf. (Nesterov, 2013, Thm 2.1.5)]. Taking summation over τ and leveraging the results in Lemma 5, we have

$$\sum_{\tau=1}^{T} \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) - \tilde{h}_\tau(\boldsymbol{x}_\delta) \leq \sum_{\tau=1}^{T} \eta \tilde{s}_\tau \sqrt{K} L \beta R + \sum_{\tau=1}^{T} \frac{\eta}{2} \big\|\tilde{\boldsymbol{g}}_\tau\big\|^2 + \frac{R^2}{\eta} \leq \eta D \sqrt{K} L \beta R + \frac{\eta T}{2} K L^2 + \frac{R^2}{\eta}. \tag{78}$$

Selecting $\delta = \mathcal{O}\big(1/(T+D)\big)$, (76) implies

$$\sum_{\tau=1}^{T} \tilde{h}_\tau(\tilde{\boldsymbol{x}}_{\tau-\tilde{s}_\tau}) - \sum_{\tau=1}^{T} \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) \leq \eta D \sqrt{K} L \bigg(L + \frac{\beta\delta\sqrt{K}}{2}\bigg) = \mathcal{O}\big(\eta\sqrt{K}D\big). \tag{79}$$

Inequality (78) then becomes

$$\sum_{\tau=1}^{T} \tilde{h}_\tau(\tilde{\boldsymbol{x}}_\tau) - \tilde{h}_\tau(\boldsymbol{x}_\delta) \leq \eta D \sqrt{K} L \beta R + \frac{\eta T}{2} K L^2 + \frac{R^2}{\eta} = \mathcal{O}\bigg(\eta K T + \eta\sqrt{K} D + \frac{1}{\eta}\bigg). \tag{80}$$

Plugging (74), (76), and (78) into (73), and choosing $\eta = \mathcal{O}(1/\sqrt{K(T+D)})$, the proof is complete.

C.4 Proof of Corollary 1

To prove Corollary 1, we will show that

$$\frac{1}{K+1} \sum_{t=1}^{T} \sum_{k=0}^{K} f_t(\boldsymbol{x}_{t,k}) - \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) = \mathcal{O}(\sqrt{K}). \tag{81}$$

Using the β-smoothness in Assumption 4, we have for any $k \neq 0$

$$f_t(\boldsymbol{x}_{t,k}) - f_t(\boldsymbol{x}_t) \leq \big(\nabla f_t(\boldsymbol{x}_t)\big)^\top (\boldsymbol{x}_{t,k} - \boldsymbol{x}_t) + \frac{\beta\delta^2}{2} \leq \delta \|\nabla f_t(\boldsymbol{x}_t)\| + \frac{\beta\delta^2}{2}. \tag{82}$$

Then leveraging the result of Lemma 4, we have

$$\|\nabla f_t(\boldsymbol{x}_t)\| = \|\nabla f_{t|t+d_t}(\boldsymbol{x}_{t|t+d_t})\| = \|\nabla f_{t|t+d_t}(\boldsymbol{x}_{t|t+d_t}) + \boldsymbol{g}_{t|t+d_t} - \boldsymbol{g}_{t|t+d_t}\|$$
$$\leq \|\boldsymbol{g}_{t|t+d_t}\| + \|\nabla f_{t|t+d_t}(\boldsymbol{x}_{t|t+d_t}) - \boldsymbol{g}_{t|t+d_t}\| \leq \sqrt{K} L + \frac{\beta\delta\sqrt{K}}{2}. \tag{83}$$

Plugging (83) back to (82), we have

$$f_t(\boldsymbol{x}_{t,k}) - f_t(\boldsymbol{x}_t) \leq \delta\sqrt{K} L + \frac{\beta\delta^2\sqrt{K}}{2} + \frac{\beta\delta^2}{2} \stackrel{(a)}{=} \mathcal{O}\Big(\frac{\sqrt{K}}{T+D}\Big) \tag{84}$$

where (a) follows from $\delta = \mathcal{O}\big((T+D)^{-1}\big)$. Summing over k and t readily implies (81).