A Population CATE sensitivity bounds

Lemma 1. *The sensitivity bounds for the conditional expected potential outcomes* $\overline{\mu}_t(x)$ *and* $\underline{\mu}_t(x)$ *defined in* (5)(6) *have the following equivalent characterization:*

$$\overline{\mu}_t(x) = \sup_{u \in \mathcal{U}^{nd}} \frac{\alpha_t(x)\int y f_t(y \mid x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) y f_t(y \mid x)dy}{\alpha_t(x)\int f_t(y \mid x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) f_t(y \mid x)dy}$$
$$\underline{\mu}_t(x) = \inf_{u \in \mathcal{U}^{ni}} \frac{\alpha_t(x)\int y f_t(y \mid x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) y f_t(y \mid x)dy}{\alpha_t(x)\int f_t(y \mid x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) f_t(y \mid x)dy}$$

where

$$\mathcal{U}^{nd} = \{u : \mathcal{Y} \to [0,1] \mid u(y) \text{is nondecreasing}\},$$
$$\mathcal{U}^{ni} = \{u : \mathcal{Y} \to [0,1] \mid u(y) \text{is nonincreasing}\},$$

and $\alpha_t(x)$ *and* $\beta(x)$ *defined in* (4).

Proof. Recall that

$$\overline{\mu}_t(x) = \sup_{w_t(y|x) \in [\alpha_t(x), \beta_t(x)]} \frac{\int y w_t(y \mid x) f_t(y \mid x)dy}{\int w_t(y \mid x) f_t(y \mid x)dy}, \tag{25}$$
$$\underline{\mu}_t(x) = \inf_{w_t(y|x) \in [\alpha_t(x), \beta_t(x)]} \frac{\int y w_t(y \mid x) f_t(y \mid x)dy}{\int w_t(y \mid x) f_t(y \mid x)dy}. \tag{26}$$

By one-to-one change of variables $w_t(y \mid x) = \alpha_t(x) + u(y)(\beta_t(x) - \alpha_t(x))$ with $u : \mathcal{Y} \to [0,1]$,

$$\overline{\mu}_t(x) = \sup_{u:\mathcal{Y} \to [0,1]} \frac{\alpha_t(x)\int y f_t(y, x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) y f_t(y \mid x)dy}{\alpha_t(x)\int f_t(y, x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) f_t(y \mid x)dy} \tag{27}$$
$$\underline{\mu}_t(x) = \inf_{u:\mathcal{Y} \to [0,1]} \frac{\alpha_t(x)\int y f_t(y, x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) y f_t(y \mid x)dy}{\alpha_t(x)\int f_t(y, x)dy + (\beta_t(x) - \alpha_t(x))\int u(y) f_t(y \mid x)dy} \tag{28}$$

We next use duality to prove that the $u^*(y)$ that achieves the supremum in (27) belongs to $\mathcal{U}^{nd}$. Similar result can be proved analogously for the infimum in (28).

Denote that $a(x) = (\beta_t(x) - \alpha_t(x))$, $b(x) = (\beta_t(x) - \alpha_t(x))$, $c(x) = \alpha_t(x)\int y f_t(y \mid x)dy$, $d(x) = \alpha_t(x)\int f_t(y \mid x)dy$. Then the optimization problem in (27) can be written as:

$$\max_{u:\mathcal{Y} \to [0,1]} \quad \frac{a(x)\langle y, u\rangle_{f_t(y|x)} + c(x)}{b(x)\langle 1, u\rangle_{f_t(y|x)} + d(x)}$$

where $\langle \cdot, \cdot \rangle_{f_t(y|x)}$ is the inner product with respect to measure $f_t(y \mid x)$.

By Charnes-Cooper transformation with $\tilde{u} = \frac{u}{b(x)\langle 1, u\rangle_{f_t(y|x)} + d(x)}$ and $\tilde{v}(x) = \frac{1}{b(x)\langle 1, u\rangle_{f_t(y|x)} + d(x)}$, the optimization problem in (27) is equivalent to the following linear program:

$$\begin{aligned} \max_{\substack{\tilde{u}:\mathcal{Y} \to [0,1] \\ \tilde{v}(x)}} \quad & a(x)\langle y, \tilde{u}\rangle_{f_t(y|x)} + c(x)\tilde{v}(x) \\ \text{s.t.} \quad & \tilde{u}(y) \le \tilde{v}(x), -\tilde{u}(y) \le 0, \text{ for } \forall y \in \mathcal{Y} \\ & b(x)\langle 1, \tilde{u}\rangle_{f_t(y|x)} + d(x)\tilde{v}(x) = 1, \tilde{v}(x) \ge 0 \end{aligned}$$

Let the dual function $p(y)$ be associated with the primal constraint $\tilde{u}(y) \le \tilde{v}(x)$ $(u(y) \le 1)$, and $q(y)$ be the dual function associated with $-\tilde{u}(y) \le 0$ $(u(y) \ge 0)$, and λ be the dual variable associated with the constraint

$b(x)\left\langle 1,\tilde{u}\right\rangle_{f_t(y|x)} + d(x)\tilde{v} = 1$. The dual program is

$$\begin{aligned} \min_{\lambda, p\succeq 0, q\succeq 0} \quad & \lambda \\ \text{s.t.} \quad & p - q + \lambda b(x) f_t(y \mid x) = a(x) y f_t(y \mid x) \\ & -\langle 1, p\rangle + \lambda d(x) \geq c(x) \end{aligned}$$

By complementary slackness, at most one of z_i or ρ_i is nonzero. The first dual constraint implies that

$$\begin{aligned} p &= (\beta_t(x) - \alpha_t(x)) f_t(y \mid x) \max\{y - \lambda, 0\}, \\ q &= (\beta_t(x) - \alpha_t(x)) f_t(y \mid x) \max\{\lambda - y, 0\}. \end{aligned}$$

Moreover, the constraint that $-\langle 1, p\rangle + \lambda d(x) \geq c(x)$ should be tight at optimality. (otherwise there exists smaller yet feasible λ that achives lower objective of the dual program.) This implies that

$$(\beta_t(x) - \alpha_t(x)) \int f_t(y \mid x) \max\{y - \lambda, 0\} dy = \alpha_t(x) \int (\lambda - y) f_t(y \mid x) dy$$

This rules out the possibility that $\lambda > C_Y$ or $\lambda < -C_Y$ where $C_Y > 0$ such that $|Y| \leq C_Y$. Thus $\exists y^H \in [-C_Y, C_y]$ such that when $y < y^H$, $q > 0$ so $u = 0$ and when $y \geq y^H$, $p > 0$ so $u = 1$. Therefore, the optimal $u^*(y)$ that achieves the supremum in (27) belongs to $\mathcal{U}^{nd}$.

□

B CATE sensitivity bounds estimators

Lemma 2. *The kernel-regression based sensitivity bound estimators* $\hat{\overline{\mu}}_t(x), \hat{\underline{\mu}}_t(x)$ *given in* (12)(13) *have the following equivalent characterization: for* $t \in \{0, 1\}$

$$\begin{aligned} \hat{\overline{\mu}}_t(x) &= \sup_{u\in\mathcal{U}^{nd}} \frac{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h})Y_i u(Y_i)}{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h}) u(Y_i)} \\ \hat{\underline{\mu}}_t(x) &= \inf_{u\in\mathcal{U}^{ni}} \frac{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h})Y_i u(Y_i)}{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h}) u(Y_i)} \end{aligned}$$

where $\mathcal{U}^{nd}$ *and* $\mathcal{U}^{ni}$ *are defined in Lemma 1.*

Proof. We prove the result for $\hat{\overline{\mu}}_t(x)$ and the result for $\hat{\underline{\mu}}_t(x)$ can be proved analogously. Given (12), by one-to-one change of variable $W_i = \alpha(X_i) + (\beta(X_i) - \alpha(X_i))U_i$ where $U_i \in [0, 1]$,

$$\hat{\overline{\mu}}_t(x) = \sup_{U_i\in[0,1]} \frac{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h})Y_i U_i}{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h}) U_i}, \tag{29}$$

$$\hat{\underline{\mu}}_t(x) = \inf_{U_i\in[0,1]} \frac{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h})Y_i U_i}{\sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i:T_i=t}^n (\beta(X_i)-\alpha(X_i))\mathbf{K}(\frac{X_i-x}{h}) U_i}. \tag{30}$$

Now we use duality to prove that the optimal weights U_i^* that attains the supremum in (29) satisfies that $U_i^* = u(Y_i)$ for some function $u : \mathcal{Y} \to [0, 1]$ such that $u(y)$ is nondecreasing in y. The analogous result for (30) can be proved similarly.

Essentially, (29) gives the following fractional linear program:

$$\begin{aligned} &\max_U \frac{A^T U + C}{B^T U + D} \\ &\text{s.t.} \quad \begin{bmatrix} I_N \\ -I_N \end{bmatrix} U \leq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned}$$

where $U = [U_1, U_2, \dots, U_{n-1}, U_n]^\top$, $A = [a_1, a_2, \dots, a_n]^\top$ with $a_i = \mathbb{I}[T_i = t](\beta(X_i) - \alpha(X_i))\mathbf{K}(\frac{X_i - x}{h})Y_i$, $B = [b_1, b_2, \dots, b_n]^\top$ with $b_i = \mathbb{I}[T_i = t](\beta(X_i) - \alpha(X_i))\mathbf{K}(\frac{X_i - x}{h})$, $C = \sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i$, and $D = \sum_{i:T_i=t}^n \alpha(X_i)\mathbf{K}(\frac{X_i-x}{h})$.

By Charnes-Cooper transformation with $\tilde{U} = \frac{U}{B^\top U + D}$ and $\tilde{V} = \frac{1}{B^\top U + D}$, the linear-fractional program above is equivalent to the following linear program:

$$
\begin{aligned}
\max_{\tilde{U}, v} \; & A^\top \tilde{U} + C\tilde{V} \\
\text{s.t.} \; & \begin{bmatrix} I_n \\ -I_n \end{bmatrix} \tilde{U} \leq \tilde{V} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
& B^\top \tilde{U} + \tilde{V} D = 1, \tilde{V} \geq 0
\end{aligned}
$$

where the solution for $\tilde{U}, \tilde{V}$ yields a solution for the original program, $U = \frac{\tilde{U}}{\tilde{V}}$.

Let the dual variables $p_i \geq 0$ be associated with the primal constraints $\tilde{U}_i \leq \tilde{V}$ (corresponding to $U_i \leq 1$), $q_i \geq 0$ associated with $\tilde{U}_i \geq 0$ (corresponding to $U_i \geq 0$), and λ associated with the constraint $B^\top \tilde{U} + D\tilde{V} = 1$. Denote $P = [p_1, \dots, p_n]^\top$ and $Q = [q_1, \dots, q_n]^\top$.

The dual problem is:

$$
\begin{aligned}
\min_{\lambda, z, \rho} \; & \lambda \\
\text{s.t.} \; & P - Q + \lambda B = A, p_i \geq 0, q_i \geq 0 \\
& -1^T P + \lambda D \geq C
\end{aligned}
$$

By complementary slackness, at most one of p_i or q_i is nonzero. Rearranging the first set of equality constraints gives $p_i - q_i = \mathbb{I}(T_i = t)(\beta(X_i) - \alpha(X_i))\mathbf{K}(\frac{X_i - x}{h})(Y_i - \lambda)$, which implies that

$$
\begin{aligned}
p_i &= \mathbb{I}[T_i = t](\beta(X_i) - \alpha(X_i))\mathbf{K}(\frac{X_i - x}{h}) \max(Y_i - \lambda, 0) \\
q_i &= \mathbb{I}[T_i = t](\beta(X_i) - \alpha(X_i))\mathbf{K}(\frac{X_i - x}{h}) \max(\lambda - Y_i, 0)
\end{aligned}
$$

Since the constraint $-1^T P + \lambda D \geq c$ is tight at optimality (otherwise there exists smaller yet feasible λ that achives lower objective of the dual program),

$$
\sum_{i=1}^n \mathbb{I}[T_i = t]\alpha(X_i)\mathbf{K}(\frac{X_i - x}{h})(\lambda - Y_i) = \sum_{i=1}^n \mathbb{I}[T_i = t](\beta(X_i) - \alpha(X_i))\mathbf{K}(\frac{X_i - x}{h}) \max(Y_i - \lambda, 0) \tag{31}
$$

This rules out both $\lambda > \max_i Y_i$ and $\lambda < \min_i Y_i$, thus $Y_{(k)} < \lambda \leq Y_{(k+1)}$ for some k where $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ are the order statistics of the sample outcomes . This means that $q_i > 0$ can happen only when $Y_i \leq Y_{(k)}$, *i.e.*, $U_i = 0$; and $p_i > 0$ can happen only when $i > k+1$, *i.e.*, $U_i = 1$. This proves there exist a nondecreasing function $u : \mathcal{Y} \to [0, 1]$ such that $U_i = u(Y_i)$ attains the upper bound in (29). □

Proof for Proposition 1. We prove the result for $\hat{\overline{\mu}}_t(x)$ and the result for $\underline{\hat{\mu}}_t(x)$ can be proved analogously. In the proof of Lemma 2, (31) implies that $\exists k^H$ such that the optimal $Y_{k^H} < \lambda^* \leq Y_{k^H+1}$ and

$$
\sum_{i \leq k^H} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i - x}{h})(\lambda^* - Y_i) = \sum_{i \geq k^H + 1} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i - x}{h})(Y_i - \lambda^*).
$$

Thus

$$
\lambda^* = \frac{\sum_{i \leq k^H} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i - x}{h})Y_i + \sum_{i \geq k^H+1} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i - x}{h})Y_i}{\sum_{i \leq k^H} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i - x}{h}) + \sum_{i \geq k^H+1} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i - x}{h})}. \tag{32}
$$

Now we prove that if $\lambda(k) \geq \lambda(k+1)$, then $\lambda(k+1) \geq \lambda(k+2)$, so $k^H = \inf\{k : \lambda(k) \geq \lambda(k+1)\}$. Note that $\lambda(k) \geq \lambda(k+1)$ is equivalent to

$$\frac{\sum_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + \sum_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i}{\sum_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h})} \leq Y_{k+1}.$$

Thus, if we denote $(\beta_t(x) - \alpha_t(x))$ as $\Delta_{\beta,\alpha}(x)$ for short,

$$\begin{aligned}
&\frac{\sum_{i\leq k+2} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + \sum_{i\geq k+3} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i}{\sum_{i\leq k+2} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i\geq k+3} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h})} \\
\leq &\frac{\sum_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + \sum_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h})Y_i + (\beta_t(x) - \alpha_t(x))\mathbb{I}[T_i = t]\mathbf{K}(\frac{X_i-x}{h})Y_i}{\sum_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \Delta_{\beta,\alpha}(x)\mathbb{I}[T_i = t]\mathbf{K}(\frac{X_i-x}{h})} \\
\overset{(*)}{\leq} &\frac{\left(\sum\limits_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum\limits_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h})\right)Y_{k+1} + \Delta_{\beta,\alpha}(x)\mathbb{I}[T_{k+2} = t]\mathbf{K}(\frac{X_{k+2}-x}{h})Y_{k+2}}{\sum_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \Delta_{\beta,\alpha}(x)\mathbb{I}[T_i = t]\mathbf{K}(\frac{X_i-x}{h})} \\
\leq &\frac{\left(\sum\limits_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h})\right)Y_{k+2} + \Delta_{\beta,\alpha}(x)\mathbb{I}[T_{k+2} = t]\mathbf{K}(\frac{X_{k+2}-x}{h})Y_{k+2}}{\sum_{i\leq k+1} \mathbb{I}[T_i = t]\alpha_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \sum_{i\geq k+2} \mathbb{I}[T_i = t]\beta_t(X_i)\mathbf{K}(\frac{X_i-x}{h}) + \Delta_{\beta,\alpha}(x)\mathbb{I}[T_i = t]\mathbf{K}(\frac{X_i-x}{h})} \\
= &Y_{k+2},
\end{aligned}$$

where (*) holds due to (32).

This implies that $\lambda(k+1) \geq \lambda(k+2)$. By strong duality, we know that $\hat{\overline{\mu}}_t(x) = \lambda^* = \overline{\lambda}(k^H; x)$ thus we prove the result for $\hat{\overline{\mu}}_t(x)$. We can analogously prove the result for $\hat{\underline{\mu}}_t(x)$.

□

Proof for Theorem 1. Here we prove that $\hat{\overline{\mu}}_t(x) \to \overline{\mu}_t(x)$. $\hat{\underline{\mu}}_t(x) \to \underline{\mu}_t(x)$ can be proved analogously.

Since $\int \mathcal{K}(u)du < \infty$, without loss of generality we assume $\int \mathcal{K}(u) = 1$.

Define the following quantities:

$$\begin{aligned}
\kappa^y_\alpha(t, x; n, h) &= \frac{1}{nh^d} \sum_{i:T_i=t}^n \alpha_t(X_i)\mathbf{K}(\frac{X_i - x}{h})Y_i, \quad I^y_\alpha(t, x) = \alpha_t(x) \int y f_t(y \mid x)dy, \\
\kappa^{u,y}_{\beta-\alpha}(t, x; n, h) &= \frac{1}{nh^d} \sum_{i:T_i=t}^n (\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})Y_i u(Y_i), \quad I^{u,y}_{\beta-\alpha}(t, x) = (\beta_t(x) - \alpha_t(x)) \int u(y)y f_t(y \mid x)dy, \\
\kappa_\alpha(t, x; n, h) &= \frac{1}{nh^d} \sum_{i:T_i=t}^n \alpha_t(X_i)\mathbf{K}(\frac{X_i - x}{h}), \quad I_\alpha(t, x) = \alpha_t(x) \int f_t(y \mid x)dy, \\
\kappa^u_{\beta-\alpha}(t, x; n, h) &= \frac{1}{nh^d} \sum_{i:T_i=t}^n (\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})u(Y_i), \quad I^u_{\beta-\alpha}(t, x) = (\beta_t(x) - \alpha_t(x)) \int u(y) f_t(y \mid x)dy.
\end{aligned}$$

Then

$$\begin{aligned}
\hat{\overline{\mu}}_t(x) &= \sup_{u\in\mathcal{U}^{nd}} \frac{\kappa^y_\alpha(t, x; n, h) + \kappa^{u,y}_{\beta-\alpha}(t, x; n, h)}{\kappa_\alpha(t, x; n, h) + \kappa^u_{\beta-\alpha}(t, x; n, h)} \\
\overline{\mu}_t(x) &= \sup_{u\in\mathcal{U}^{nd}} \frac{I^y_\alpha(t, x) + I^{u,y}_{\beta-\alpha}(t, x)}{I_\alpha(t, x) + I^u_{\beta-\alpha}(t, x)}
\end{aligned}$$

According to Lemma 3,

$$
\begin{aligned}
\left|\hat{\overline{\mu}}_t(x) - \overline{\mu}_t(x)\right| &\leq \sup_{u\in\mathcal{U}^{nd}} \left|\frac{\kappa_\alpha^y(t,x;n,h) + \kappa_{\beta-\alpha}^{u,y}(t,x;n,h)}{\kappa_\alpha(t,x;n,h) + \kappa_{\beta-\alpha}^{u}(t,x;n,h)} - \frac{I_\alpha^y(t,x) + I_{\beta-\alpha}^{u,y}(t,x)}{I_\alpha(t,x) + I_{\beta-\alpha}^{u}(t,x)}\right| \\
&\leq \sup_{u\in\mathcal{U}^{nd}} \left\{ \left|\kappa_\alpha^y + \kappa_{\beta-\alpha}^{u,y}\right| \frac{\left|\kappa_\alpha + \kappa_{\beta-\alpha}^u - (I_\alpha + I_{\beta-\alpha}^u)\right|}{\left|\kappa_\alpha + \kappa_{\beta-\alpha}^u\right|\left|I_\alpha + I_{\beta-\alpha}^u\right|} + \frac{1}{\left|I_\alpha + I_{\beta-\alpha}^u\right|}\left|\kappa_\alpha^y + \kappa_{\beta-\alpha}^{u,y} - (I_\alpha^y + I_{\beta-\alpha}^{u,y})\right| \right\} \\
&\leq \frac{(\Delta_1(t,x;n,h) + \left|I_\alpha^y + I_{\beta-\alpha}^{u,y}\right|)\Delta_2(t,x;n,h)}{\left|I_\alpha + I_{\beta-\alpha}^u\right|(\left|I_\alpha + I_{\beta-\alpha}^u\right| - \Delta_2(t,x;n,h))} + \frac{\Delta_1(t,x;n,h)}{\left|I_\alpha + I_{\beta-\alpha}^u\right|} \qquad (33)
\end{aligned}
$$

where

$$
\Delta_1(t,x;n,h) = \sup_{u\in\mathcal{U}^{nd}} \left|[\kappa_\alpha^y(t,x;n,h) + \kappa_{\beta-\alpha}^{u,y}(t,x;n,h)] - [I_\alpha^y(t,x) + I_{\beta-\alpha}^{u,y}(t,x)]\right|, \qquad (34)
$$

$$
\Delta_2(t,x;n,h) = \sup_{u\in\mathcal{U}^{nd}} \left|[\kappa_\alpha(t,x;n,h) + \kappa_{\beta-\alpha}^{u}(t,x;n,h)] - [I_\alpha(t,x) + I_{\beta-\alpha}^{u}(t,x)]\right|. \qquad (35)
$$

Therefore, we only need to prove that when $n \to \infty$, $h \to 0$, and $nh^{2d} \to \infty$, $\Delta_1(t,x;n,h) \overset{\text{p}}{\to} 0$ and $\Delta_2(t,x;n,h) \overset{\text{p}}{\to} 0$ for $t \in \{0,1\}$ and $x \in \mathcal{X}$. We prove $\Delta_1(t,x;n,h) \overset{\text{p}}{\to} 0$ in this proof. $\Delta_2(t,x;n,h) \overset{\text{p}}{\to} 0$ can be proved analogously.

Note that

$$
\Delta_1(t,x;n,h) \leq \left|\kappa_\alpha^y(t,x;n,h) - I_\alpha^y(t,x)\right| + \sup_{u\in\mathcal{U}^{nd}} \left|\kappa_{\beta-\alpha}^{u,y}(t,x;n,h) - I_{\beta-\alpha}^{u,y}(t,x)\right|.
$$

Step 1: prove that $\sup_{u\in\mathcal{U}^{nd}} \left|\kappa_{\beta-\alpha}^{u,y}(t,x;n,h) - I_{\beta-\alpha}^{u,y}(t,x)\right| \to 0$.

Obviously

$$
\begin{aligned}
&\sup_{u\in\mathcal{U}^{nd}} \left|\kappa_{\beta-\alpha}^{u,y}(t,x;n,h) - I_{\beta-\alpha}^{u,y}(t,x)\right| \\
&\leq \sup_{u\in\mathcal{U}^{nd}} \left|\kappa_{\beta-\alpha}^{u,y}(t,x;n,h) - \mathbb{E}\kappa_{\beta-\alpha}^{u,y}(t,x;n,h)\right| + \sup_{u\in\mathcal{U}^{nd}} \left|\mathbb{E}\kappa_{\beta-\alpha}^{u,y}(t,x;n,h) - I_{\beta-\alpha}^{u,y}(t,x)\right| \\
&:= \Lambda_1 + \Lambda_2
\end{aligned}
$$

Step 1.1: prove $\Lambda_1 \overset{\text{p}}{\to} 0$.

By assumption, there exists $\delta > 0$ with $e_t(x,y) \in [\delta, 1-\delta]$. Hence, $\alpha_t(x) \leq C_{\delta,\Gamma}(\alpha) = \frac{1}{\Gamma}(\frac{1}{\delta} - 1) + 1$ and $\beta_t(x) - \alpha_t(x) \leq C_{\delta,\Gamma}(\beta-\alpha) = (\Gamma - \frac{1}{\Gamma})(\frac{1}{\delta} - 1)$. Under the assumptions that $|K(x)| \leq C_K$ and $|Y| \leq C_Y$, there exists a constant $c > 0$ such that for any two different observations (X_i, T_i, Y_i) and (X_i', T_i', Y_i'),

$$
\begin{aligned}
&\Big|\frac{1}{nh^d}(\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})\mathbb{I}(T_i = t)u(Y_i)Y_i \\
&- \frac{1}{nh^d}(\beta_t(X_i') - \alpha_t(X_i'))\mathbf{K}(\frac{X_i' - x}{h})\mathbb{I}(T_i' = t)u(Y_i')Y_i'\Big| \\
&\leq \frac{cC_K^d C_Y C_{\delta,\Gamma}(\beta-\alpha)}{nh^d}.
\end{aligned}
$$

Then Lemma 4 and Mcdiarmid inequality implies that with high probability at least $1 - \exp(-\frac{2nh^{2d}\epsilon^2}{c^2 C_Y^2 C_K^{2d} C_{\delta,\Gamma}^2(\beta-\alpha)})$,

$$
\Lambda_1 \leq \mathbb{E}\Lambda_1 + \epsilon.
$$

Moreover, we can bound $\mathbb{E}\Lambda_1$ by Rademacher complexity: for i.i.d Rademacher random variables $\sigma_1, \ldots, \sigma_n$,

$$\mathbb{E}\Lambda_1 \le 2\mathbb{E} \sup_{u \in \mathcal{U}^{nd}} \left| \frac{1}{nh^d} \sum_{i=1}^n \sigma_i(\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})\mathbb{I}(T_i = t)u(Y_i)Y_i \right|. \tag{36}$$

Furthermore, we can bound the Rademacher complexity given the monotonicity structure of $\mathcal{U}^{nd}$. Suppose we reorder the data so that $Y_1 \le Y_2 \le \cdots \le Y_n$. Denote the whole sample by $\mathcal{S} = \{(X_i, T_i, Y_i)\}_{i=1}^n$ and $\kappa_i = (\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})\mathbb{I}(T_i = t)u(Y_i)Y_i$. Since (36) is a linear programming problem, we only need to consider the vertex solutions, *i.e.*, $u \in \{0, 1\}$. Therefore,

$$\mathbb{E}\Lambda_1 \le 2\mathbb{E} \sup_{u \in \mathcal{U}^{nd}, u \in \{0,1\}} \left| \frac{1}{nh^d} \sum_{i=1}^n \sigma_i(\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})\mathbb{I}(T_i = t)u(Y_i)Y_i \right|. \tag{37}$$

Conditionally on $\mathcal{S}$, $(u(Y_1), \ldots, u(Y_n))$ thus can only have $n + 1$ possible values:

$$(0, 0, \ldots, 0, 0), (0, 0, \ldots, 0, 1), \ldots,$$
$$(0, 1, \ldots, 1, 1), (1, 1, \ldots, 1, 1).$$

This means that conditionally on $\mathcal{S}$, $(\kappa_1, \kappa_2, \ldots, \kappa_N)$ can have at most $N + 1$ possible values. Plus, $|\kappa_i| \le C_K^d C_Y C_{\delta,\Gamma}(\beta - \alpha)$. So by Massart's finite class lemma,

$$\begin{aligned} &\mathbb{E} \sup_{u \in \mathcal{U}^{nd}} \left| \frac{1}{nh^d} \sum_{i=1}^N \sigma_i(\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})\mathbb{I}(T_i = t)u(Y_i)Y_i \right| \\ &\le \sqrt{\frac{2C_Y^2 C_K^{2d} C_{\delta,\Gamma}^2(\beta - \alpha)\log(n+1)}{nh^{2d}}}. \end{aligned}$$

Therefore, with high probability at least $1 - \exp(-\frac{2nh^{2d}\epsilon^2}{c^2 C_Y^2 C_K^{2d} C_{\delta,\Gamma}^2(\beta - \alpha)})$,

$$\Lambda_1 \le 2\sqrt{\frac{2C_Y^2 C_K^{2d} C_{\delta,\Gamma}^2(\beta - \alpha)\log(n+1)}{nh^{2d}}} + \epsilon,$$

which means that $\Lambda_1 \xrightarrow{\text{p}} 0$ when $nh^{2d} \to \infty$.

Step 1.2: prove $\Lambda_2 \xrightarrow{\text{p}} 0$.

$$\begin{aligned} &\mathbb{E}\frac{1}{nh^d} \sum_{i=1}^n (\beta(X_i) - \alpha(X_i))\mathbf{K}(\frac{X_i - x}{h})\mathbb{I}(T_i = t)u(Y_i)Y_i \\ &= \frac{1}{h^d}\mathbb{E}[(\beta_t(X_i) - \alpha_t(X_i))\mathbf{K}(\frac{X_i - x}{h})\mathbb{I}(T_i = t)u(Y_i)Y_i] \\ &= \frac{1}{h^d} \int u(y)y \left[\int (\beta_t(z') - \alpha_t(z'))\mathbf{K}(\frac{z' - x}{h}) f_t(y \mid x) dz' \right] dy \\ &\overset{(a)}{=} \int u(y)y \left[\int (\beta_t - \alpha_t)(x + zh)\mathbf{K}(z) f_t(y \mid x + zh) dz \right] dy \end{aligned}$$

where in (a) we use change-of-variable $z = \frac{z' - x}{h}$.

Since $\beta_t(x)$, $\alpha_t(x)$, and $f_t(y \mid x)$ are twice continuously differentiable with respect to x at any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Apply Taylor expansion to $(\beta_t - \alpha_t)(x + zh)$ and $f_t(y \mid x + zh)$ around x:

$$(\beta_t - \alpha_t)(x + zh) = (\beta_t - \alpha_t)(x) + hz^\top \frac{d}{dx}(\beta_t - \alpha_t)(x) + \frac{1}{2}h^2 z^\top \frac{d^2}{dx^2}(\beta_t - \alpha_t)(x)z + o(h^2)$$
$$f_t(y \mid x + zh) = f_t(y \mid x) + hz^\top \frac{\partial}{\partial x} f_t(y \mid x) + \frac{1}{2}h^2 z^\top \frac{\partial^2}{\partial x^2} f_t(y \mid x)z + o(h^2)$$

Then

$$
\begin{aligned}
&\mathbb{E}\frac{1}{nh^d}\sum_{i=1}^{n}(\beta_t(X_i)-\alpha_t(X_i))\mathbf{K}(\frac{X_i-x}{h})\mathbb{I}(T_i=t)u(Y_i)Y_i \\
&=\int(\beta_t-\alpha_t)(x)u(y)yf_t(y\mid x)dy+\frac{h^2}{2}(\int \mathbf{K}(z)z^2dz)\int u(y)y\Big(f_t(y\mid x)\frac{d^2}{dx^2}(\beta_t-\alpha_t)(x) \\
&\qquad +(\beta_t-\alpha_t)(x)\frac{\partial^2}{\partial x^2}f_t(y\mid x)+2\frac{d}{dx}(\beta_t(x)-\alpha_t(x))\frac{\partial}{\partial x}f_t(y\mid x)\Big)dy+o(h^2) \\
&=\frac{h^2}{2}(\int \mathbf{K}(z)z^2dz)\int u(y)y\Big(\frac{\partial^2}{\partial x^2}\big((\beta_t-\alpha_t)(x)f_t(y\mid x)\big)\Big)dy+I^u_{\beta-\alpha}(t,x)+o(h^2)
\end{aligned}
$$

Since the first order and second order derivatives of $\beta_t(x)$, $\alpha_t(x)$, and $f_t(y\mid x)$ with respect to x are bounded, obviously,

$$
\left|\int u(y)y\Big(\frac{\partial^2}{\partial x^2}\big((\beta_t-\alpha_t)(x)f_t(y\mid x)\big)\Big)dy\right|<\infty.
$$

Thus as $h\to 0$,

$$
\begin{aligned}
\Lambda_2=&\sup_{u\in\mathcal{U}^{nd}}\left|\frac{h^2}{2}(\int \mathbf{K}(z)z^2dz)\int u(y)y\Big(\frac{\partial^2}{\partial x^2}\big((\beta_t-\alpha_t)(x)f_t(y\mid x)\big)\Big)dy+o(h^2)\right| \\
&\to 0.
\end{aligned}
$$

Step 2: prove that $\left|\kappa^y_\alpha(t,x;n,h)-I^y_\alpha(t,x)\right|\xrightarrow{\text{p}}0$. Obviously

$$
\begin{aligned}
&\left|\kappa^y_\alpha(t,x;n,h)-I^y_\alpha(t,x)\right| \\
&\le\left|\kappa^y_\alpha(t,x;n,h)-\mathbb{E}\kappa^y_\alpha(t,x;n,h)\right|+\left|\mathbb{E}\kappa^y_\alpha(t,x;n,h)-I^y_\alpha(t,x)\right| \\
&:=\Lambda_3+\Lambda_4
\end{aligned}
$$

Step 2.1: prove $\Lambda_3\xrightarrow{\text{p}}0$. By Mcdiarmid inequality, with high probability at least $1-2\exp(-\frac{2nh^{2d}\epsilon^2}{c^2C_Y^2C_K^2C^2_{\delta,\Gamma}(\alpha)})$,

$$
\Lambda_3\le\epsilon.
$$

Thus $\Lambda_3\xrightarrow{\text{p}}0$ when $nh^{2d}\to\infty$.

Step 2.2: prove $\Lambda_4\xrightarrow{\text{p}}0$. Similarly to Step 1.2, we can prove that

$$
\begin{aligned}
\Lambda_4=&\left|\frac{h^2}{2}(\int \mathbf{K}(z)z^2dz)\int u(y)y\frac{\partial^2}{\partial x^2}\big(\alpha_t(x)f_t(y\mid x)\big)dy+o(h^2)\right| \\
&\to 0.
\end{aligned}
$$

So far, we have proved that $\Delta_1(t,x;n,h)\xrightarrow{\text{p}}0$. Analogously we can prove that $\Delta_2(t,x;n,h)\xrightarrow{\text{p}}0$. Thus when $n\to\infty$, $h\to 0$, and $nh^{2d}\to\infty$,

$$
\left|\hat{\overline{\mu}}_t(x)-\overline{\mu}_t(x)\right|\xrightarrow{\text{p}}0.
$$

Analogously,

$$
\left|\hat{\underline{\mu}}_t(x)-\underline{\mu}_t(x)\right|\xrightarrow{\text{p}}0.
$$

Therefore,

$$
\hat{\overline{\tau}}(x)\xrightarrow{\text{p}}\overline{\tau}(x),\quad \hat{\underline{\tau}}(x)\xrightarrow{\text{p}}\underline{\tau}(x).
$$

□

Lemma 3. *For functions $J : S \to \mathbb{R}$ and $\tilde{J} : S \to \mathbb{R}$ where S is some subset in Euclidean space,*

$$\big|\sup_{x\in S} J(x) - \sup_{x\in S}\tilde{J}(x)\big| \le \sup_{x\in S}|J(x)-\tilde{J}(x)|$$
$$\big|\inf_{x\in S} J(x) - \inf_{x\in S}\tilde{J}(x)\big| \le \sup_{x\in S}|J(x)-\tilde{J}(x)|$$

Proof. Obviously,

$$\sup_{x\in S} J(x) \le \sup_{x\in S}\tilde{J}(x) + \sup_{x\in S}\{J(x)-\tilde{J}(x)\}$$
$$\inf_{x\in S} -\tilde{J}(x) \ge \inf_{x\in S} -J(x) + \inf_{x\in S}\{\tilde{J}(x)-J(x)\}.$$

This implies that

$$\begin{aligned}
-\sup_{x\in S}|J(x)-\tilde{J}(x)| &\le \inf_{x\in S}\{\tilde{J}(x)-J(x)\}\\
&\le \sup_{x\in S} J(x) - \sup_{x\in S}\tilde{J}(x)\\
&\le \sup_{x\in S}\{J(x)-\tilde{J}(x)\} \le \sup_{x\in S}|J(x)-\tilde{J}(x)|,
\end{aligned}$$

i.e., $|\sup_{x\in S} J(x) - \sup_{x\in S}\tilde{J}(x)| \le \sup_{x\in S}|J(x)-\tilde{J}(x)|$.

On the other hand,

$$\sup_{x\in S} -\tilde{J}(x) \le \sup_{x\in S} -J(x) + \sup_{x\in S}\{J(x)-\tilde{J}(x)\}$$
$$\inf_{x\in S} J(x) \ge \inf_{x\in S}\tilde{J}(x) + \inf_{x\in S}\{J(x)-\tilde{J}(x)\}$$

which implies that

$$\begin{aligned}
-\sup_{x\in S}|J(x)-\tilde{J}(x)| &\le -\sup_{x\in S}\{\tilde{J}(x)-J(x)\}\\
&\le \inf_{x\in S} J(x) - \inf_{x\in S}\tilde{J}(x)\\
&\le \sup_{x\in S}\{J(x)-\tilde{J}(x)\} \le \sup_{x\in S}|J(x)-\tilde{J}(x)|
\end{aligned}$$

Namely $|\inf_{x\in S} J(x) - \inf_{x\in S}\tilde{J}(x)| \le \sup_{x\in S}|J(x)-\tilde{J}(x)|$. □

Lemma 4. *For functions $J : S \to \mathbb{R}$ and $\tilde{J} : S \to \mathbb{R}$ where S is some subset in Euclidean space,*

$$\big|sup_{x\in S}|J(x)| - sup_{x\in S}|\tilde{J}(x)|\big| \le \sup_{x\in S}|J(x)-\tilde{J}(x)|$$

Proof. On the one hand,

$$\sup_{x\in S}|J(x)| = \sup_{x\in S}|J(x)-\tilde{J}(x)+\tilde{J}(x)| \le \sup_{x\in S}|J(x)-\tilde{J}(x)| + \sup_{x\in S}|\tilde{J}(x)|,$$

which implies that

$$\sup_{x\in S}|J(x)| - \sup_{x\in S}|\tilde{J}(x)| \le \sup_{x\in S}|J(x)-\tilde{J}(x)|.$$

On the other hand,

$$\inf_{x\in S} -|\tilde{J}(x)| = \inf_{x\in S} -|\tilde{J}(x)-J(x)+J(x)| \ge \inf_{x\in S}(-|\tilde{J}(x)-J(x)| - |J(x)|) \ge \inf_{x\in S}(-|\tilde{J}(x)-J(x)|) + \inf_{x\in S}(-|J(x)|),$$

which implies that

$$\sup_{x\in S}|J(x)| - \sup_{x\in S}|\tilde{J}(x)| \ge -\sup_{x\in S}|J(x)-\tilde{J}(x)|.$$

Therefore, the conclusion follows. □

C Policy Learning

Proof for Proposition 2. The optimal policy $\pi^*(\cdot;\Gamma)$ solves the following optimization problem:

$$\inf_{\pi:\mathcal{X}\to[0,1]} \sup_{\tau(x)\in\mathcal{T}(x;\Gamma)\ \forall x\in\mathcal{X}} \mathbb{E}[(\pi(X)-\pi_0(X))\tau(X)]. \tag{38}$$

Since both π and τ are bounded, according to Von Neumann theorem, the optimaization problem (38) is equivalent to

$$\sup_{\tau\in\mathcal{T}} \inf_{\pi:\mathcal{X}\to[0,1]} \mathbb{E}[(\pi(X)-\pi_0(X))\tau(X)], \tag{39}$$

which means that there exist optimal $\tau^*\in\mathcal{T}$ and π^* such that: (a) τ^* is pessimal for π^* in that $\mathbb{E}[(\pi^*(X)-\pi_0(X))\tau^*(X)] \geq \mathbb{E}[(\pi^*(X)-\pi_0(X))\tau(X)]$ for $\forall\tau\in\mathcal{T}$ and (b) π^* is optimal for τ^* in that $\mathbb{E}[(\pi^*(X)-\pi_0(X))\tau^*(X)] \leq \mathbb{E}[(\pi(X)-\pi_0(X))\tau^*(X)]$, $\forall\pi:\mathcal{X}\to[0,1]$. Obviously (b) implies that $\pi^* = \mathbb{I}\left[\tau^*(x)<0\right]+\pi_0(x)\mathbb{I}\left[\tau^*(x)=0\right]$ can make an optimal policy. Plugging $\pi^* = \mathbb{I}\left[\tau^*(x)<0\right]+\pi_0(x)\mathbb{I}\left[\tau^*(x)=0\right]$ into (39) gives

$$\tau^* = \operatorname*{argmax}_{\tau\in\mathcal{T}} \mathbb{E}\min((1-\pi_0(X))\tau(X), -\pi_0(X)\tau(X)). \tag{40}$$

Actually τ^* has closed form solution:

- When $\overline{\tau}(x)\leq 0$, obviously $\tau(x)\leq 0$ so $\min((1-\pi_0(x))\tau(x), -\pi_0(x)\tau(x)) = (1-\pi_0(x))\tau(x)$.
  - $\tau^*(x) = \overline{\tau}(x)$ if $\pi_0(x)<1$;
  - $\tau^*(x)$ can be anything between $\underline{\tau}(x)$ and $\overline{\tau}(x)$ if $\pi_0(x)=1$.
- When $\underline{\tau}(x)\geq 0$, obviously $\tau(x)\geq 0$ so $\min((1-\pi_0(x))\tau(x), -\pi_0(x)\tau(x)) = -\pi_0(x)\tau(x)$.
  - $\tau^*(x) = \underline{\tau}(x)$ if $\pi_0(x)>0$;
  - $\tau^*(x)$ can be anything between $\underline{\tau}(x)$ and $\overline{\tau}(x)$ if $\pi_0(x)=0$.
- When $\underline{\tau}(x)<0<\overline{\tau}(x)$,
  - If $0<\pi_0(x)<1$, when choosing $\tau^*(x)\geq 0$, $\min((1-\pi_0(x))\tau^*(x), -\pi_0(x)\tau^*(x)) = -\pi_0(x)\tau^*(x))\leq 0$, so $\tau^*(x)$ must be 0; similarly, when choosing $\tau^*(x)\leq 0$, $\tau^*(x)$ must be 0. This means that $\tau^*(x)=0$.
  - When $\pi_0(x)=0$, $\tau^*(x)$ can be anything between 0 and $\overline{\tau}(x)$.
  - When $\pi_0(x)=1$, $\tau^*(x)$ can be anything between $\underline{\tau}(x)$ and 0.

In summary, the following τ^* always solves the optimization problem in (40):

$$\tau^*(x) = \overline{\tau}(x)\mathbb{I}(\overline{\tau}(x)\leq 0) + \underline{\tau}(x)\mathbb{I}(\underline{\tau}(x)\geq 0).$$

Therefore, the following policy is a minimax-optimal policy that optimizes (39):

$$\pi^*(x) = \mathbb{I}\left[\tau^*(x)<0\right] + \pi_0(x)\mathbb{I}\left[\tau^*(x)=0\right],$$

with

$$\tau^*(x) = \overline{\tau}(x)\mathbb{I}(\overline{\tau}(x)\leq 0) + \underline{\tau}(x)\mathbb{I}(\underline{\tau}(x)\geq 0).$$

Namely,

$$\pi^*(x) = \mathbb{I}(\overline{\tau}(x)\leq 0) + \pi_0(x)\mathbb{I}(\underline{\tau}(x)<0\leq\overline{\tau}(x)).$$

□

Proof for Theorem 2. According to the proof for Proposition 2,

$$
\begin{aligned}
\sup_{\tau\in\mathcal{T}} \overline{R}_{\pi_0}(\pi^*(\cdot;\Gamma);\Gamma) &= \mathbb{E}\min((1-\pi_0(X))\tau^*(X), -\pi_0(X)\tau^*(X)) \\
&= \mathbb{E}(1-\pi_0(X))\overline{\tau}(X)\mathbb{I}(\overline{\tau}(X)<0) + \mathbb{E}(-\pi_0(X))\underline{\tau}(X)\mathbb{I}(\underline{\tau}(X)>0)
\end{aligned}
$$

In contrast,

$$
\begin{aligned}
\sup_{\tau\in\mathcal{T}} \overline{R}_{\pi_0}(\hat{\pi}^*(\cdot;\Gamma);\Gamma) &= \max_{\tau\in\mathcal{T}} \mathbb{E}[(1-\pi_0(X))\tau(X)\mathbb{I}(\hat{\overline{\tau}}(X)<0) + (-\pi_0(X))\tau(X)\mathbb{I}(\hat{\underline{\tau}}(X)>0)] \\
&= \mathbb{E}[(1-\pi_0(X))\overline{\tau}(X)\mathbb{I}(\hat{\overline{\tau}}(X)<0) + (-\pi_0(X))\underline{\tau}(X)\mathbb{I}(\hat{\underline{\tau}}(X)>0)]
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sup_{\tau\in\mathcal{T}} \overline{R}_{\pi_0}(\hat{\pi}^*(\cdot;\Gamma);\Gamma) - \sup_{\tau\in\mathcal{T}} \overline{R}_{\pi_0}(\pi^*(\cdot;\Gamma);\Gamma) &= \mathbb{E}\Bigg[\Bigg((1-\pi_0(X))\overline{\tau}(X)\mathbb{I}(\overline{\tau}(X)<0) + (-\pi_0(X))\underline{\tau}(X)\mathbb{I}(\underline{\tau}(X)>0)\Bigg) \\
&\quad - \Bigg((1-\pi_0(X))\overline{\tau}(X)\mathbb{I}(\hat{\overline{\tau}}(X)<0) + (-\pi_0(X))\underline{\tau}(X)\mathbb{I}(\hat{\underline{\tau}}(X)>0)\Bigg)\Bigg] \\
&= \mathbb{E}\Bigg[(1-\pi_0(X))\overline{\tau}(X)\Bigg(\mathbb{I}(\overline{\tau}(X)<0) - \mathbb{I}(\hat{\overline{\tau}}(X)<0)\Bigg) \\
&\quad + (-\pi_0(X))\underline{\tau}(X)\Bigg(\mathbb{I}(\underline{\tau}(X)>0) - \mathbb{I}(\hat{\underline{\tau}}(X)>0)\Bigg)\Bigg] \\
&= -\mathbb{E}\Bigg[\Bigg((1-\pi_0(X))\,|\overline{\tau}(X)|\,\mathbb{I}(\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X)))\Bigg) \\
&\quad + \Bigg((-\pi_0(X))\,|\underline{\tau}(X)|\,\mathbb{I}(\mathrm{sign}(\underline{\tau}(X)) \neq \mathrm{sign}(\hat{\underline{\tau}}(X)))\Bigg)\Bigg]
\end{aligned}
$$

Next, we prove that under the assumptions in Theorem 1, when $n\to\infty$, $h\to 0$, and $nh^2\to\infty$,

$$
\mathbb{E}\Bigg((1-\pi_0(X))\,|\overline{\tau}(X)|\,\mathbb{I}(\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X)))\Bigg) \to 0.
$$

Given that $|Y|\le C_Y$, $|\overline{\tau}(x)|\le 2C_Y$ and $|\underline{\tau}(x)|\le 2C_Y$. For any $\eta>0$,

$$
\begin{aligned}
&\mathbb{E}\left[(1-\pi_0(X))\,|\overline{\tau}(X)|\,\mathbb{I}(\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X)))\right] \\
&\le 2C_Y\mathbb{P}\Bigg(\mathbb{I}(\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X)))\mathbb{I}(|\overline{\tau}(X)|>\eta)\Bigg) + \eta\mathbb{P}\Bigg(\mathbb{I}(\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X)))\mathbb{I}(|\overline{\tau}(X)|\le\eta)\Bigg) \\
&\overset{(b)}{\le} 2C_Y\mathbb{P}\Bigg(\mathbb{I}(\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X)))\mathbb{I}(|\overline{\tau}(X)-\hat{\overline{\tau}}(X)|>\eta)\Bigg) + \eta \\
&\le 2C_Y\mathbb{P}(|\overline{\tau}(X)-\hat{\overline{\tau}}(X)|>\eta) + \eta \\
&= 2C_Y\mathbb{E}\Bigg[\mathbb{P}(|\overline{\tau}(X)-\hat{\overline{\tau}}(X)|>\eta \mid X)\Bigg] + \eta \overset{(c)}{\to} \eta
\end{aligned}
$$

Here (b) holds because when $\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X))$, $|\overline{\tau}(X)-\hat{\overline{\tau}}(X)| > |\overline{\tau}(X)|$; (c) holds because Theorem 1 proves that $\mathbb{P}(|\overline{\tau}(X)-\hat{\overline{\tau}}(X)|>\eta \mid X)\to 0$, which implies $\mathbb{E}\Bigg[\mathbb{P}(|\overline{\tau}(X)-\hat{\overline{\tau}}(X)|>\eta \mid X)\Bigg]\to 0$ according to bounded convergence theorem considering that $\mathbb{P}(|\overline{\tau}(X)-\hat{\overline{\tau}}(X)|>\eta \mid X)\le 1$.

Therefore, when $n\to\infty$, $h\to 0$ and $nh^{2d}\to\infty$,

$$
\mathbb{E}\left[(1-\pi_0(X))\,|\overline{\tau}(X)|\,\mathbb{I}(\mathrm{sign}(\overline{\tau}(X)) \neq \mathrm{sign}(\hat{\overline{\tau}}(X)))\right] \to 0.
$$

Analogously, we can prove that, when $n \to \infty$, $h \to 0$ and $nh^{2d} \to \infty$,

$$\mathbb{E}\left[\pi_0(X) \left|\overline{\tau}(X)\right| \mathbb{I}(\text{sign}(\overline{\tau}(X)) \neq \text{sign}(\hat{\overline{\tau}}(X)))\right] \to 0.$$

As a result, when $n \to \infty$, $h \to 0$ and $nh^{2d} \to \infty$, $\sup_{\tau \in \mathcal{T}} \overline{R}_{\pi_0}(\hat{\pi}^*(\cdot;\Gamma);\Gamma) - \sup_{\tau \in \mathcal{T}} \overline{R}_{\pi_0}(\pi^*(\cdot;\Gamma);\Gamma) \to 0.$

□

D PCATE sensitivity bounds

Analogously, the corresponding sensitivity bounds for partial conditional average treatment effect are:

$$\overline{\tau}(x_S;\Gamma) = \overline{\mu}_1(x_S;\Gamma) - \underline{\mu}_0(x_S;\Gamma), \tag{41}$$
$$\underline{\tau}(x_S;\Gamma) = \underline{\mu}_1(x_S;\Gamma) - \overline{\mu}_0(x_S;\Gamma), \tag{42}$$

where $\overline{\mu}_t(x_S;\Gamma)$ and $\underline{\mu}_t(x_S;\Gamma)$ for $t \in \{0,1\}$ are given in (17)(18). The corresponding PCATE bounds estimators are:

$$\hat{\overline{\tau}}(x_S;\Gamma) = \hat{\overline{\mu}}_1(x_S;\Gamma) - \hat{\underline{\mu}}_0(x_S;\Gamma), \tag{43}$$
$$\hat{\underline{\tau}}(x_S;\Gamma) = \hat{\underline{\mu}}_1(x_S;\Gamma) - \hat{\overline{\mu}}_0(x_S;\Gamma), \tag{44}$$

where $\hat{\overline{\mu}}_t(x_S;\Gamma)$ and $\hat{\underline{\mu}}_t(x_S;\Gamma)$ for $t \in \{0,1\}$ are given in (22)(23).

For any $\pi_S : \mathcal{X}_S \to [0,1]$, the policy value and the worst-case policy regret are:

$$V(\pi_S;\tau) = \mathbb{E}[\pi(X_S)Y(1) + (1-\pi(X_S))Y(0)]$$
$$\overline{R}^S_{\pi_0}(\pi;\Gamma) = \sup_{\tau(x_S) \in \mathcal{T}(x_S;\Gamma)\ \forall x_S \in \mathcal{X}_S} (V(\pi_S;\tau) - V(\pi_0;\tau)) \tag{45}$$

Corollary 2.1. *Consider the partial conditional expected potential outcome*

$$\mu_t(x_S) = \mathbb{E}[Y(t) \mid X_S = x_S],$$

where $t \in \{0,1\}$, X_S is a subset of the observed covariates X, and $x_S \in \mathcal{X}_S$. The corresponding population PCAT sensitivity bounds (17)(18) *have the following equivalent characterization:*

$$\overline{\mu}_t(x_S;\Gamma) = \sup_{u \in \mathcal{U}^{nd}} \frac{\iint \alpha_t(x_S, x_{S^c}) y f_t(y, x_{S^c} \mid x_S) dy dx_{S^c} + \iint (\beta_t(x_S, x_{S^c}) - \alpha_t(x_S, x_{S^c})) u(y) y f_t(y, x_{S^c} \mid x_S) dy dx_{S^c}}{\iint \alpha_t(x_S, x_{S^c}) f_t(y, x_{S^c} \mid x_S) dy dx_{S^c} + \iint (\beta_t(x_S, x_{S^c}) - \alpha_t(x_S, x_{S^c})) u(y) f_t(y, x_{S^c} \mid x_S) dy dx_{S^c}}$$
$$\underline{\mu}_t(x_S;\Gamma) = \inf_{u \in \mathcal{U}^{ni}} \frac{\iint \alpha_t(x_S, x_{S^c}) y f_t(y, x_{S^c} \mid x_S) dy dx_{S^c} + \iint (\beta_t(x_S, x_{S^c}) - \alpha_t(x_S, x_{S^c})) u(y) y f_t(y, x_{S^c} \mid x_S) dy dx_{S^c}}{\iint \alpha_t(x_S, x_{S^c}) f_t(y, x_{S^c} \mid x_S) dy dx_{S^c} + \iint (\beta_t(x_S, x_{S^c}) - \alpha_t(x_S, x_{S^c})) u(y) f_t(y, x_{S^c} \mid x_S) dy dx_{S^c}}$$

where $f_t(y, x_{S^c} \mid x_S)$ is the conditional joint density function for $\{T = t, Y(t), X_{S^c}\}$ given $X_S = x_S$ with X_{S^c} as the complementary subset of X with respect to X_S, $\alpha_t(\cdot)$ and $\beta_t(\cdot)$ are defined in (4)*, and $\mathcal{U}^{nd}$ and $\mathcal{U}^{ni}$ are defined in Lemma 1.*

Proof. By analogous arguments of change of variable and duality in the proof for Lemma 1, we can prove the conclusions in Corollary 2.1. □

Corollary 2.2. *Consider the following estimators:*

$$\hat{\overline{\mu}}_t(x_S;\Gamma) = \sup_{W_{ti} \in [\alpha_t(X_i;\Gamma), \beta_t(X_i;\Gamma)]} \frac{\sum_{i=1}^n \mathbb{I}(T_i = t) K(\frac{X_{i,S} - x_S}{h}) W_{ti} Y_i}{\sum_{i=1}^n \mathbb{I}(T_i = t) K(\frac{X_{i,S} - x_S}{h}) W_{ti}},$$
$$\hat{\underline{\mu}}_t(x_S;\Gamma) = \inf_{W_{ti} \in [\alpha_t(X_i;\Gamma), \beta_t(X_i;\Gamma)]} \frac{\sum_{i=1}^n \mathbb{I}(T_i = t) K(\frac{X_{i,S} - x_S}{h}) W_{ti} Y_i}{\sum_{i=1}^n \mathbb{I}(T_i = t) K(\frac{X_{i,S} - x_S}{h}) W_{ti}}.$$

where $\alpha_t(\cdot)$ *and* $\beta_t(\cdot)$ *are defined in* (4).

Assume that $e_t(x_S, x_{S^c})$ *and* $f_t(y, x_{S^c} \mid x_S)$ *are twice continuously differentiable with respect to* x_S *for any* $y \in \mathcal{Y}$ *and* $x_{S^c} \in \mathcal{X}_S$ *with bounded first and second derivatives. Under the other assumptions in Theorem 1, when* $n \to \infty$*,* $h \to 0$*, and* $nh^{2|S|} \to \infty$*,* $\hat{\overline{\mu}}_t(x_S) \xrightarrow{p} \overline{\mu}_t(x_S)$ *and* $\hat{\underline{\mu}}_t(x_S) \xrightarrow{p} \underline{\mu}_t(x_S)$*.*

Proof. Following the proof for Theorem 1, we can analogously prove that when $n \to \infty$ and $nh^{2|S|} \to \infty$, with $\Delta_{\beta,\alpha}(X_{i,S}, X_{i,S^c}) = (\beta_t(X_{i,S}, X_{i,S^c}) - \alpha_t(X_{i,S}, X_{i,S^c}))$ for short,

$$\hat{\overline{\mu}}_t(x_S; \Gamma) \xrightarrow{\text{p}} \tag{46}$$

$$\sup_{u \in \mathcal{U}^{nd}} \frac{\mathbb{E}\Big[\mathbb{I}(T_i = t)\alpha_t(X_{i,S}, X_{i,S^c})K(\frac{X_{i,S}-x_S}{h})Y_i + \mathbb{I}(T_i = t)u(Y_i)(\Delta_{\beta,\alpha}(X_{i,S}, X_{i,S^c}))K(\frac{X_{i,S}-x_S}{h})Y_i\Big]}{\mathbb{E}\Big[\mathbb{I}(T_i = t)\alpha_t(X_{i,S}, X_{i,S^c})K(\frac{X_{i,S}-x_S}{h}) + \mathbb{I}(T_i = t)u(Y_i)K(\frac{X_{i,S}-x_S}{h})\Big]} \tag{47}$$

Note that

$$\mathbb{E}\Big[\mathbb{I}(T_i = t)\alpha_t(X_{i,S}, X_{i,S^c})K(\frac{X_{i,S} - x_S}{h})Y_i\Big] = \iiint \alpha_t(x'_S, x_{S^c})f_t(y, x'_S, x_{S^c})K(\frac{x'_S - x_S}{h})ydydx'_Sdx_{S^c}.$$

By the similar Taylor expansion argument in the proof for Theorem 1, when $h \to 0$,

$$\mathbb{E}\Big[\mathbb{I}(T_i = t)\alpha_t(X_{i,S}, X_{i,S^c})K(\frac{X_{i,S} - x_S}{h})Y_i\Big] \to \iint \alpha_t(x_S, x_{S^c})f_t(y, x_S, x_{S^c})ydydx_{S^c}.$$

Similarly, we can prove the convergence of other components in Corollary 2.2. Given the characterization in Corollary 2.1, we can decompose the estimation bias in a way similar to (33), which leads to the final conclusions. □

Corollary 2.3. *Define the following policies based on the subset observed covariates* X_S*: for any* $x_S \in \mathcal{X}$*,*

$$\pi^P(x_S; \Gamma) = \mathbb{I}(\overline{\tau}(x_S; \Gamma) \leq 0) + \pi_0(x_S)\mathbb{I}(\underline{\tau}(x_S; \Gamma) \leq 0 < \overline{\tau}(x_S; \Gamma))$$
$$\hat{\pi}^P(x_S; \Gamma) = \mathbb{I}(\hat{\overline{\tau}}(x_S; \Gamma) \leq 0) + \pi_0(x_S)\mathbb{I}(\hat{\underline{\tau}}(x_S; \Gamma) \leq 0 < \hat{\overline{\tau}}(x_S; \Gamma)).$$

where $\overline{\tau}(x_S; \Gamma)$ *and* $\underline{\tau}(x_S; \Gamma)$ *are the population PCATE sentivity bounds defined in* (41)(42)*, and* $\hat{\overline{\tau}}(x_S; \Gamma)$ *and* $\hat{\underline{\tau}}(x_S; \Gamma)$ *are the PCATE sensitivity bounds estimators given in* (43) (44).

Then $\pi^P(\cdot; \Gamma)$ *is the population minimax-optimal policies. Namely,*

$$\pi^P(\cdot; \Gamma) \in \operatorname*{argmin}_{\pi_S : \mathcal{X}_S \to [0,1]} [\sup_{\tau \in \mathcal{T}_S} \overline{R}^S_{\pi_0}(\pi; \Gamma)]$$

where $\mathcal{T}_S = \{\tau : \tau(x_S) \in [\underline{\tau}(x_S), \overline{\tau}(x_S)], \forall x_S \in \mathcal{X}_S\}$*. Furthermore, the sample policy* $\hat{\pi}^P$ *is asymptotically minimax-optimal:*

$$\overline{R}^S_{\pi_0}(\hat{\pi}^P(\cdot; \Gamma); \Gamma) \xrightarrow{p} \overline{R}^S_{\pi_0}(\pi^P(\cdot; \Gamma); \Gamma).$$

Proof. The conclusions can be proved analougously to the proofs for Proposition 2 and Theorem 2. □

E Additional figures

E.1 Bounds on Partial CATE

We next illustrate the case of learning the PCATE using our interval estimators in eqs. (21) and (23). In Fig. 4, we show the same CATE specification used in Fig. 1, but introduce additional confounders which impact selection to

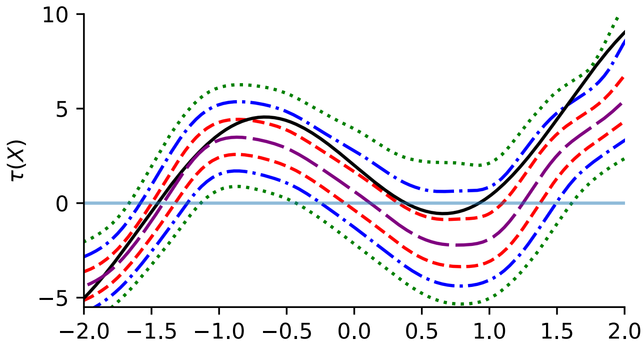

Figure 4: Bounds on PCATE. Legend same as in Fig 1.

illustrate the use of this approach with higher-dimensional observed covariates. We consider observed covariates $X \in \mathbb{R}^3$, uniformly generated on $[-1,1]^3$, where heterogeneity in treatment effect is only due to x_S, $S = \{1\}$, the first dimension. That is, we specify the outcome model for $t \in \{0,1\}$ as:

$$\begin{aligned} Y(t) = (2t-1)X_S + (2t-1) - 2\sin(2(2t-1)X_S) \\ - 2(2u-1)(1+0.5X_S) + \beta_x^\top X + \epsilon \end{aligned}$$

We fix the nominal propensities as $e(x) = \sigma(\theta^\top x + 0.5)$, with $\theta = [0.75, -0.5, 0.5]$ and the outcome coefficient vector $\beta_x = [0.5, 0.5, 0.5]$. Again, we set the propensity scores such that the complete propensities achieve the extremal bounds. Additional confounding dimensions tend to increase the outcome variation for any given x_S value, so the bounds are wider in Fig. 4.

E.2 Estimated vs. fixed marginal propensity scores

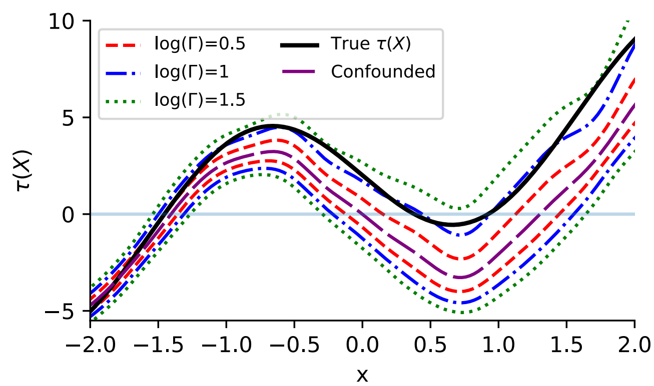

Figure 5: Bounds on CATE for differing values of Γ.

In the main text, for the sake of illustration we presented an example where we fix the nominal propensities $e(X)$ and set the true propensities $e(X,U)$ such that the maximal odds-ratio bounds $\alpha(X), \beta(X)$ are "achieved" by the true propensities. We computed bounds on the pre-specified $e(X)$. We now consider a setting with a binary confounder where $\log(\Gamma) = 1$ is not a uniform bound, but bounds most of the observed odds ratios, and instead we learn the marginal propensities $\Pr[T = 1 \mid X = x]$ from data using logistic regression. The results (Fig. 5) are materially the same as in the main text.

We consider the same setting as in Fig. 1 with a binary confounder $u \sim \text{Bern}(1/2)$ generated independently, and $X \sim \text{Unif}[-2,2]$. Then we set the true propensity score as

$$e^*(x,u) = \sigma(\theta x + 2(u - 0.5) + 0.5)$$

We learn the nominal propensity scores $e(x)$ by predicting them from data with logistic regression, which essentially learns the marginalized propensity scores $e(x) = \Pr[T = 1 \mid X = x]$. The outcome model yields a nonlinear

CATE, with linear confounding and with randomly generated mean-zero noise, $\epsilon \sim N(0,1)$:

$$Y(t) = (2t-1)x + (2t-1) - 2\sin(2(2t-1)X) - 2(u-1)(1+0.5X) + \epsilon$$

This outcome model specification yields a confounded CATE estimate of

$$\begin{aligned}&\mathbb{E}[Y \mid X=x, T=1] - \mathbb{E}[Y \mid X=x, T=0]\\&= 2 - 2x + 2(\sin(-2x) - \sin(2x)) +\\&2(2+x)(\Pr[u=1 \mid {\textstyle\genfrac{}{}{0pt}{}{X=x}{T=1}}] - \Pr[u=1 \mid {\textstyle\genfrac{}{}{0pt}{}{X=x}{T=0}}])\end{aligned}$$

By Bayes' rule,

$$\Pr[u=1 \mid X=x, T=1] = \frac{\Pr[T=1 \mid X=x, u=1]\Pr[u=1 \mid X=x]}{\Pr[T=1 \mid X=x]}$$

In Fig. 5, we compute the bounds using our approach for $\log(\Gamma) = 0.5, 1, 1.5$ on a dataset with $n = 2000$. The purple long-dashed line corresponds to a confounded kernel regression. (Bandwidths are estimated by leave-one-out cross-validation for each treatment arm regression). The confounding is greatest for large, positive x. The true, unconfounded CATE is plotted in black. While the confounded estimation suggests a large region, $x \in [0, 1.25]$, where $T = 1$ is beneficial, the true CATE suggests a much smaller region where $T = 1$ is optimal.