$B^0 - \bar{B}^0$ Mixing and Decay Constants from Lattice QCD*

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ABSTRACT: We present updates of our results for neutral B-meson mixing and leptonic decay constants obtained in the quenched approximation from a mean-field-improved Sheikholeslami-Wohlert action at two values of lattice spacing. We consider quantities such as $B_{B_{d(s)}}$, $f_{D_{(s)}}$, $f_{B_{(s)}}$ and the full $\Delta B = 2$ matrix-elements, as well as the corresponding $SU(3)$-breaking ratios.

1. Introduction

The study of $B_d - \bar{B}_d$ oscillations enables measurement of the poorly known CKM matrix element $|V_{td}|$. The frequency of these oscillations is determined by

$$\Delta m_d \equiv M^H_{B_d} - M^L_{B_d} \ , \tag{1.1}$$

where $M^H_{B_d}$ and $M^L_{B_d}$ are the heavy and light mass eigenvalues of the mixing system. Δm_d is experimentally measurable from tagged B_d meson samples, and is also calculable in the Standard Model. To leading order in $1/M_W$, the Standard Model prediction for Δm_d is

$$\begin{aligned}\Delta m_d = & \frac{G_F^2}{8\pi^2} M_W^2 \, |V_{td}V_{tb}^*|^2 \, C\left(\{S_i(x_t)\}, M_W, \mu\right) \\ & \times |\langle \bar{B}_d|\mathcal{O}_d^{\Delta B=2}(\mu)|B_d\rangle| \ , \end{aligned} \tag{1.2}$$

where $x_t = m_t^2/M_W^2$, $S_i(x_t)$ are the relevant Inami-Lim functions [1], μ is the renormalisation scale, $\mathcal{O}_d^{\Delta B=2}$ is the four-quark operator $\left[\bar{b}\gamma^\mu(1-\gamma^5)d\right]\left[\bar{b}\gamma_\mu(1-\gamma^5)d\right]$ and C is the relevant Wilson coefficient. Since $|V_{tb}|$ is equal to unity to very good accuracy, a measurement of Δm_d enables the determination of $|V_{td}|$. The accuracy of this determination is currently limited by the theoretical uncertainty in the calculation of the non-perturbative strong-interaction effects in the matrix-element $\langle \bar{B}_d|\mathcal{O}_d^{\Delta B=2}|B_d\rangle$. An alternative approach [2], in which many theoretical uncertainties cancel, is to consider the ratio, $\Delta m_s/\Delta m_d$, where Δm_s is the mass difference in the neutral $B_s - \bar{B}_s$ system. In the Standard Model, one has

$$\begin{aligned}\frac{\Delta m_s}{\Delta m_d} &= \left|\frac{V_{ts}}{V_{td}}\right|^2 \left(\frac{M_{B_s}}{M_{B_d}}\right)^2 \xi^2 = \left|\frac{V_{ts}}{V_{td}}\right|^2 r_{sd} \\ &\equiv \left|\frac{V_{ts}}{V_{td}}\right|^2 \left|\frac{\langle \bar{B}_s|\mathcal{O}_s^{\Delta B=2}|B_s\rangle}{\langle \bar{B}_d|\mathcal{O}_d^{\Delta B=2}|B_d\rangle}\right| \ , \end{aligned} \tag{1.3}$$

where $\mathcal{O}_s^{\Delta B=2}$ is the same operator as $\mathcal{O}_d^{\Delta B=2}$ with d replaced by s and where we have omitted the renormalisation-scale dependence of these operators as it cancels in the ratio. Because the unitarity of the CKM matrix implies $|V_{ts}| \simeq |V_{cb}|$ and because $|V_{cb}|$ can be accurately obtained from semileptonic B to charm decays, a measurement of $\Delta m_s/\Delta m_d$ determines $|V_{td}|$. This is a challenging measurement and, at present, only a lower bound on $\Delta m_s/\Delta m_d$ exists [3].

The matrix elements which appear in Eq.

*CERN-TH/99-344, CPT-99/P.3909, Edinburgh Preprint 1999/19, LAPTH-Conf-762/99, SHEP 99/20, UK/99-16.

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(1.3) are traditionally parameterised by

$$\mathcal{M}_{B_q}(\mu) = \langle \bar{B}_q | \mathcal{O}_q^{\Delta B=2}(\mu) | B_q \rangle = \frac{8}{3} M_{B_q}^2 f_{B_q}^2 B_{B_q}(\mu) \ , \qquad (1.4)$$

where $q = d$ or s, where the B-parameter, B_{B_q}, measures deviations from vacuum saturation, corresponding to $B_{B_q} = 1$, and f_{B_q} is the leptonic decay constant:

$$\langle 0 | \bar{b}\gamma_\mu\gamma_5 q | B_q(p) \rangle = i p_\mu f_{B_q} \ . \qquad (1.5)$$

With this parameterisation, the quantity ξ defined in Eq. (1.3) is given by

$$\xi = \frac{f_{B_s}}{f_{B_d}} \sqrt{\frac{B_{B_s}}{B_{B_d}}} \ . \qquad (1.6)$$

Because M_{B_s} and M_{B_d} are measured experimentally, ξ^2 is the quantity in Eq. (1.3) which requires a non-perturbative determination.

We report on $B_{B_{d(s)}}$, $f_{B_{(s)}}$, $B_{B_s}/\ B_{B_d}$, $f_{B_s}/\ f_B$, r_{sd} and ξ. Results for D-meson decay constants $f_{D_{(s)}}$ and the $SU(3)$ breaking ratio $f_{D_s}/\ f_D$ are also given. These results are updates of those we presented in [4, 5].

2. Main features of the lattice calculation

Numerical calculations are performed in the quenched approximation at two values of the coupling, $\beta = 6.2$ and $\beta = 6.0$, corresponding to an inverse lattice spacing $1/a \sim 2.5$ GeV (finer) and $1/a \sim 2.0$ GeV (coarser), respectively. We use a mean-field-improved Sheikholeslami-Wohlert (SW) action [6] to describe the quarks. With this action, discretisation errors are formally reduced from $\mathcal{O}(a)$ to $\mathcal{O}(\alpha_s a)$ and may be numerically smaller because of the mean-field improvement. This reduction of discretisation errors is important in lattice calculations involving heavy quarks, because these quarks have small Compton wavelengths. For details on the parameters used in the numerical calculations, please refer to table 2 in Appendix A.

At each value of the lattice spacing, we have three light quarks with masses in a range between $\sim m_s/2$ and $\sim m_s$, which allows us to linearly extrapolate the quantities we are after to vanishing quark mass and interpolate them to the strange-quark mass. To obtain results for the b quark, while keeping discretisation errors under control, we work with five heavy-quark masses straddling the charm mass[1] and extrapolate up to the b mass, guided by HQET.

We use two methods to calculate r_{sd}:

- **Direct method**: r_{sd} is obtained from the direct calculations of $\mathcal{M}_{B_s}$ and $\mathcal{M}_{B_d}$.
- **Indirect method**: r_{sd} is obtained by calculating f_{B_s}/f_{B_d} and B_{B_s}/B_{B_d}, and then combining them with the experimental $M_{B_s}/\ M_{B_d}$.

We find that both heavy-quark-mass and light-quark-mass extrapolations are under better control for the indirect method than they are for the direct method.

3. Matching to the continuum and running in the $\overline{\mathrm{MS}}$ scheme

Results of lattice-regularised calculations have to be matched to the continuum renormalisation scheme in which Wilson coefficients are calculated. We perform this matching at one loop [7, 8, 9, 10] with mean-field improvement [11]. At this order, it is consistent to use the tree-level value for improvement coefficient $c_{\rm SW}$ (see Appendix A). This is the procedure we use to obtain the central values for our results.

Moreover, because chiral symmetry is explicitly broken by Wilson fermions, the axial vector current A_μ requires a (multiplicative) renormalisation and is related to its continuum counterpart via

$$A_\mu^{\rm cont} = Z_A(\alpha_s) A_\mu^{\rm latt}, \qquad (3.1)$$

where Z_A is finite.

For the four-quark operators, to subtract the contributions arising from the explicit breaking

[1] However, only three of these are used on the finer lattice ($\beta = 6.2$) when calculating matrix elements and B-parameters of the four-quark operators.

of chiral symmetry, it is sufficient to consider the following basis of parity-conserving operators

$$\begin{aligned} \mathcal{O}_{1,2} &= \gamma_\mu \times \gamma_\mu \pm \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5, \\ \mathcal{O}_{3,4} &= I \times I \pm \gamma_5 \times \gamma_5, \\ \mathcal{O}_5 &= \sigma_{\mu\nu} \times \sigma_{\mu\nu}, \end{aligned} \tag{3.2}$$

where we only show their Dirac structure for simplicity. $\mathcal{O}_1$ is the parity-even part of $\mathcal{O}_q^{\Delta B=2}$. This operator, in the $\overline{\mathrm{MS}}$ scheme at the renormalisation scale μ, is related to the above lattice operators by

$$\mathcal{O}_1^{\overline{\mathrm{MS}}}(\mu) = Z_{11}(\alpha_s, a\mu)\hat{\mathcal{O}}_1^{\mathrm{latt}}(a)\ , \tag{3.3}$$

where

$$\hat{\mathcal{O}}_1^{\mathrm{latt}}(a) = \mathcal{O}_1^{\mathrm{latt}}(a) + \sum_{i=2}^{5} Z_{1i}(\alpha_s)\mathcal{O}_i^{\mathrm{latt}}(a)\ . \tag{3.4}$$

Z_{11} depends logarithmically on $a\mu$. The Z_{1i}, $i \neq 1$, account for the operator mixing due to the explicit chiral symmetry, and do not depend on $a\mu$.

The scheme of α_s is not fixed at one loop. We choose $\alpha_s = \alpha_{\overline{\mathrm{MS}}}$ obtained from the procedure described in [11], with $n_f = 0$ (quenched approximation), which was shown to lead to particularly convergent perturbative expansions [11]. Central values are obtained by identifying the scale of the coupling with the matching scale μ and matching at $\mu = 2/a$–a typical lattice ultraviolet scale. Running in the $\overline{\mathrm{MS}}$ scheme is performed at two loops with the same coupling constant as for the matching, and $n_f = 0$.

4. Scaling with heavy-quark mass

To study the behaviour of the various quantities with heavy-quark mass, we define

$$\Phi_f(M_P) \equiv \frac{af_P}{Z_A}\sqrt{aM_P}\left(\frac{\alpha_s(M_P)}{\alpha_s(M_B)}\right)^{2/\beta_0} \tag{4.1}$$

$$\Phi_B(\mu, M_P) \equiv B_P(\mu)\left(\frac{\alpha_s(M_P)}{\alpha_s(M_B)}\right)^{2/\beta_0} \tag{4.2}$$

$$\Phi_{\Delta F=2}(\mu, M_P) \equiv \frac{a^4\mathcal{M}_P(\mu)}{aM_P}\left(\frac{\alpha_s(M_P)}{\alpha_s(M_B)}\right)^{6/\beta_0} \tag{4.3}$$

where M_P is the heavy-meson mass and $\mathcal{M}_P$ is the $\Delta F = 2$ matrix element calculated at that mass. β_0 is the one-loop coefficient of the QCD β-function, with $n_f = 0$. In Φ_f, Φ_B and $\Phi_{\Delta F=2}$, we have cancelled the logarithmic dependence of f_P, B_P and $\mathcal{M}_P$ on M_P at leading-log order [12].

For $X(M_P) = \Phi_f$, Φ_B, $\Phi_{\Delta F=2}$ and the $SU(3)$-breaking ratios, we use the HQET-inspired relation,

$$X(M_P) = a_X + b_X\left(\frac{1}{aM_P}\right) + c_X\left(\frac{1}{aM_P}\right)^2 + \cdots, \tag{4.4}$$

to investigate the heavy-quark-mass scaling behaviour of these quantities, as shown in figures 1, 2 and 3.

5. Systematic uncertainties

Our main results at the two values of lattice spacing are summarised in table 1. In this table, the first error bar for each quantity is statistical. The other errors are systematic and we discuss them now.

5.1 Discretisation errors

In table 1, results for the decay constants display significant variation with lattice spacing[2]. This suggests that discretisation errors for these quantities may be important. To quantify these errors we estimate residual, $\mathcal{O}(am_Q\alpha_s)$ discretisation effects, associated with the mass m_Q of the heavy quark, as described in Appendix B. This is the second error bar on the decays constants and their $SU(3)$-breaking ratios, f_{D_s}/f_D and f_{B_s}/f_B.

Such an estimate could, in principle, be carried out for the B-parameters and the $SU(3)$-breaking ratios r_{sd} and ξ. However, many of

[2] This poor scaling is not fully understood. It could be improved by using f_π instead of m_ρ to set the scale (see table 2). For instance, f_B at $\beta = 6.0$ would be ~ 191 MeV instead of 201 MeV while its value at $\beta = 6.2$ would be 160 MeV. However, m_ρ is a valid means of setting the scale in quenched calculations and, as discussed below, we include a systematic associated with the uncertainty in the lattice spacing. Furthermore, this poor scaling is not present in the B-parameters and $SU(3)$-breaking ratios which are the main thrust of the present work.

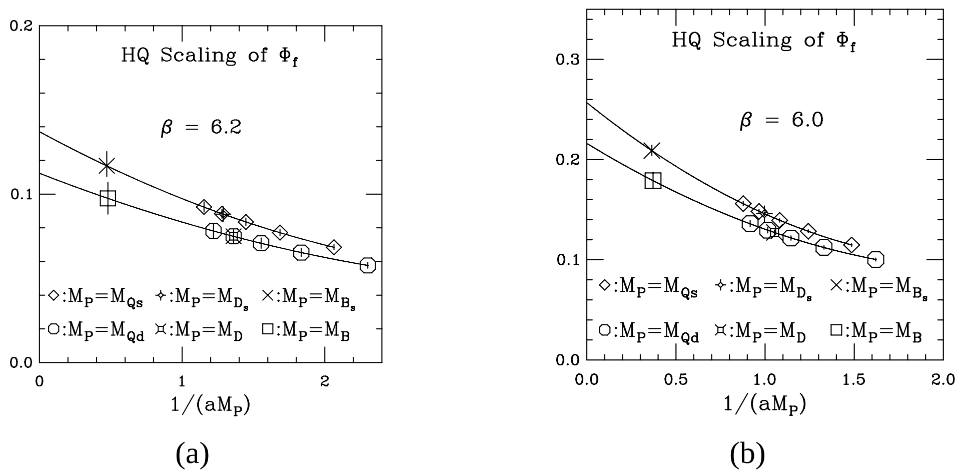

Figure 1: *Scaling of Φ_f with heavy-quark mass on (a): the finer lattice and (b): the coarser lattice. The points labelled M_{Qs} and M_{Qd} correspond to the heavy quarks, Q, used in our simulation. The curves are fits to the RHS of Eq. (4.4). The other points are the result of interpolation to $Q = c$ or extrapolation to $Q = b$.*

these discretisation effects cancel trivially in the ratios of matrix elements defining these quantities. Furthermore, a full quantification of $\mathcal{O}(a\, m_Q \alpha_s)$ effects for $\mathcal{M}_{B_q}$ and their B-parameters would require one to consider the mixing of the four-quark operators in Eq. (3.3) with operators of dimension seven, which is beyond the scope of the present work. Finally, in table 1, results for B-parameters and their $SU(3)$-breaking ratios exhibit very little lattice-spacing dependence, supporting the idea that discretisation errors for these quantities are small. Thus, we assume that the statistical error for these quantities encompasses possible residual discretisation errors. For $r_{sd}^{\rm indirect}$ and ξ, however, which are obtained using f_{B_s}/f_B, we take into account the discretisation error on this quantity.

5.2 Matching uncertainties

To estimate the systematic errors arising from the perturbative matching in the B-parameters, we match at different μ in the range between $1/a$ and π/a [3], then run the resultant B-parameters to $2/a$ to compare them with the ones matched "directly" at $2/a$. The range $[1/a, \pi/a]$ covers typical lattice ultraviolet scales and is vindicated by our study of B_K [13], where we find that continuum chiral behaviour is restored for these scales. We also consider the variation coming from computing $Z_{11}(\mu = 2/a)$ and Z_{1i} with the constant $c_{\rm SW}$ set to its mean-field-improved value instead of 1. All of these variations, which affect B_B and B_{B_s} significantly, but not $B_{B_s}/\, B_B$, are reflected in the second error bars on these B-parameters.

Decay constants independent of renormalisation scale. However, the above procedure results in a $\sim$ 4% change in Z_A through the μ-dependence of $\alpha_{\overline{\rm MS}}(\mu)$ and the change in the value of $c_{\rm SW}$. This is reflected in the decay constants' third error bar but does not affect the corresponding $SU(3)$-breaking ratios.

5.3 Heavy-quark-mass extrapolations

As shown in figure 1, the decay constants have a pronounced extrapolation in heavy-quark mass, and the term quadratic in $1/M_P$ on the RHS of Eq. (4.4) contributes significantly. To quantify the systematic error associated with this extrapolation–the fourth error bar on the decay constants–we perform a fit of the heaviest three points in figure 1 to the RHS of Eq. (4.4), without the quadratic term.

Figure 2 indicates that the linear heavy-quark-mass extrapolation of the B-parameters works

[3] We always identify the scale of $\alpha_{\overline{\rm MS}}$ with the matching scale.

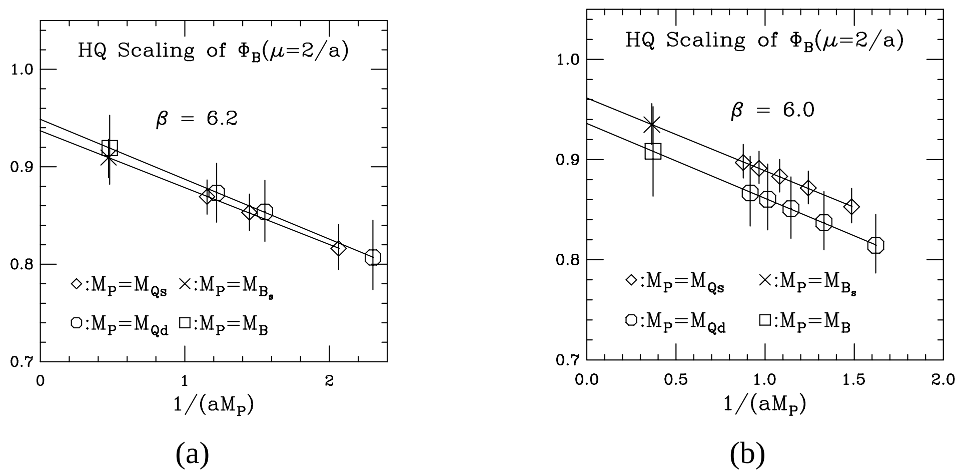

Figure 2: *Scaling of Φ_B with heavy-quark mass on (a): the finer lattice and (b): the coarser lattice. The points labelled M_{Qs} and M_{Qd} correspond to the heavy quarks, Q, used in our simulation. The curves are fits to the RHS of Eq. (4.4) without the quadratic term in $1/M_P$. The other points are the result of extrapolation to $Q = b$.*

well and is mild: the associated uncertainty should be covered by the statistical error.

The $\Delta F = 2$ matrix elements have a very pronounced dependence on heavy-quark mass, as seen in figure 3. Since we are not reporting results for the individual $\mathcal{M}_{B_{d(s)}}$, we do not quantify the systematic errors associated with their determination. However, this strong mass-dependence is one of the elements which make a reliable determination of r_{sd}, from the ratio of individually calculated $\mathcal{M}_{B_{d(s)}}$, difficult [14].

5.4 Uncertainties in the determination of the lattice spacing

In quenched calculations, the value of the lattice spacing varies significantly with the quantity used to set the scale. This variation is due to quenching effects, as well as any other systematic uncertainty which may affect the quantity used to set the scale. In this work, we determine the lattice spacing from the ρ-meson mass[4]. We then vary the inverse lattice spacing, $1/a$, increasing it by 10% and decreasing it by 5%. This range covers the typical variations observed in the determination of the scale from gluonic or light-hadron spectral quantities, for the action and parameters we use [15].

Uncertainties in the lattice spacing will obviously affect the determination of all the decay constants, as they are dimensionful. They will also slightly change the curves in the heavy-quark-mass extrapolations (figures 1, 2 and 3). Furthermore, they induce a variation in the strange-quark mass, which we obtain from the mass of the kaon, and therefore affect all quantities which depend on this mass.

In practice, we find that the variation of the lattice spacing discussed above does not induce a significant change in the B-parameters. However, it does affect all the decay constants and $SU(3)$-breaking ratios. This is reflected in the last error bar on these quantities.

5.5 Quenching errors

Quenching errors have been studied using the quenched Chiral Perturbation Theory (qχPT) and have been found to be small for the B-parameters [16]. Moreover, numerical simulation [17] with two light flavours of dynamical quarks indicate that those in f_{B_s}/f_B are also small. Thus, they ought to be small for r_{sd} and ξ. Contrary to this, quenching errors may be significant for the decay

[4] The scale determined from f_π gives compatible results at $\beta = 6.2$, as shown in table 2.

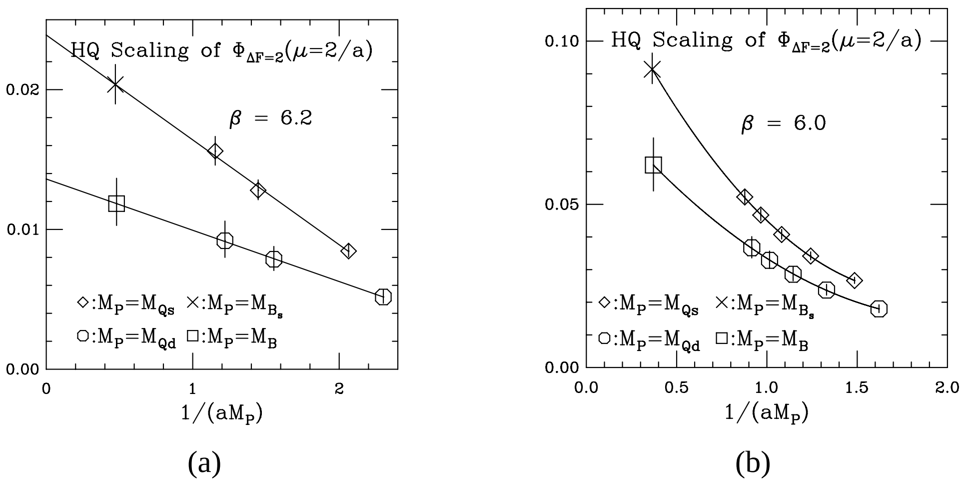

Figure 3: *Scaling of $\Phi_{\Delta F=2}$ with heavy-quark mass on (a) the finer lattice and (b) the coarser lattice. The points labelled M_{Qs} and M_{Qd} correspond to the heavy quarks, Q, used in our simulation. The curves are fits to the RHS of Eq. (4.4) (without the quadratic term in $1/M_P$ for the finer lattice). The other points are the result of extrapolation to $Q = b$.*

constants themselves, as indicated by both qχPT [16] and numerical simulation [17].

As we mentionned in the previous section, the uncertainty in the lattice scale is, in part, a quenching effect. Thus, to the extent that it is, we have already accounted for some quenching errors. A more thorough estimate of these effects, however, would require a dedicated unquenched simulation which is beyond the scope of this work. Therefore, we do not attempt to quantify quenching errors any further.

6. Final results

Since two lattice spacings are not sufficient for an extrapolation to the continuum limit ($a = 0$), we quote the results obtained on the finer lattice ($\beta = 6.2$) as our best estimates. And because it appears to be more reliable, we quote r_{sd}^{indirect} for r_{sd}.

Our main preliminary results are thus

$$\xi = 1.15(6)^{+2}_{-3} \ ,$$
$$r_{sd} = 1.37(14)^{+4}_{-6} \ ,$$
$$\frac{B_{B_s}}{B_B} = 0.98(3) \ ,$$
$$\frac{f_{B_s}}{f_B} = 1.16(6)^{+2}_{-3} \ ,$$
$$B_B(5 \text{ GeV}) = 0.92(4)^{+3}_{-0} \ ,$$
$$B_{B_s}(5 \text{ GeV}) = 0.91(2)^{+3}_{-0} \ ,$$
$$f_B = 161(16)^{+24}_{-13} \text{ MeV} \ ,$$
$$f_{B_s} = 192(14)^{+24}_{-13} \text{ MeV} \ ,$$
$$f_D = 195(10)^{+22}_{-10} \text{ MeV} \ ,$$
$$f_{D_s} = 224(7)^{+21}_{-9} \text{ MeV} \ ,$$
$$\frac{f_{D_s}}{f_D} = 1.15(4)^{+2}_{-3} \ ,$$

where the first error bar is statistical and the second is systematic, the result of adding our long list of systematic errors in quadrature. By choosing the results obtained on the finer lattice, we are also being conservative in our estimate of statistical errors as they are larger than on the coarser lattice for which we have higher statistics.

Note that our results, obtained with large statistics and a highly improved action are compatible with recent world averages [5, 18, 19, 20].

Acknowledgments

This work is supported in part by EPSRC and PPARC under grants GR/K41663 and GR/L29-927. We thank Jonathan Flynn and Guido Martinelli for inspiring discussions. We are grateful to the UKQCD Collaboration, especially Peter Boyle, Pablo Martinez, Chris Maynard and

lattice	coarser	finer
β	6.0	6.2
f_{D_s}(MeV)	$251(3)^{+16+11+2+17}_{-\ 0-\ 4-0-\ 8}$	$224(7)^{+9+8+0+16}_{-0-3-0-\ 8}$
f_D(MeV)	$224(4)^{+12+10+1+20}_{-\ 0-\ 4-0-10}$	$195(10)^{+7+7+0+19}_{-0-3-0-10}$
f_{B_s}(MeV)	$232(6)^{+27+10+\ 0+22}_{-\ 0-\ 4-16-11}$	$192(14)^{+14+7+\ 0+19}_{-\ 0-3-10-\ 9}$
f_B(MeV)	$201(9)^{+20+9+\ 0+25}_{-\ 0-4-13-13}$	$161(16)^{+11+6+0+21}_{-\ 0-2-8-10}$
f_{D_s}/f_D	$1.12(1)^{+1+\ +\ +2}_{-0-\ -\ -2}$	$1.15(4)^{+1+\ +\ +2}_{-0-\ -\ -3}$
f_{B_s}/f_B	$1.14(2)^{+1+\ +\ +2}_{-0-\ -\ -3}$	$1.16(6)^{+1+\ +\ +2}_{-0-\ -\ -3}$
B_{B_s}(5 GeV)	$0.92(2)^{+\ +4+\ +}_{-\ -0-\ -}$	$0.91(2)^{+\ +3+\ +}_{-\ -0-\ -}$
B_B(5 GeV)	$0.90(4)^{+\ +4+\ +}_{-\ -0-\ -}$	$0.92(4)^{+\ +3+\ +}_{-\ -0-\ -}$
B_{B_s}/B_{B_d}	$1.02(3)^{+\ +0+\ +0}_{-\ -0-\ -0}$	$0.98(3)^{+\ +0+\ +0}_{-\ -0-\ -0}$
r_{sd}^{indirect}	$1.38(7)^{+2+\ +\ +4}_{-0-\ -\ -6}$	$1.37(14)^{+1+\ +\ +4}_{-0-\ -\ -6}$
r_{sd}^{direct}	$1.52(18)^{+\ +\ +\ +6}_{-\ -\ -\ -9}$	$1.71(28)^{+\ +\ +\ +\ 8}_{-\ -\ -\ -11}$
ξ	$1.15(3)^{+1+\ +\ +2}_{-0-\ -\ -3}$	$1.15(6)^{+1+\ +\ +2}_{-0-\ -\ -3}$

Table 1: *Results at the two values of lattice spacing.* $r_{sd}^{\text{indirect}} = (\frac{M_{B_s}}{M_B}\frac{f_{B_s}}{f_B})^2\frac{B_{B_s}}{B_B}$ *and* $r_{sd}^{\text{direct}} = (\mathcal{M}_{B_s}/\mathcal{M}_{B_d})$. *The first error bar on each quantity is statistical while the others are systematic, as described in Section 5. Blank error bars are put in to help keep track of which systematic effect each error corresponds to.*

David Richards, for their help. L.L. wishes to thank the Department of Physics & Astronomy of the University of Edinburgh and C.-J.D.L. wishes to thank CPT Marseille and the CERN Theory Division, for hospitality. Financial support from the DOE EPSCoR grant DE-FG05-84ER40154 (PI Liu, CO-PI Draper) is also acknowledged by C.-J.D.L..

A. Simulation Details

The quarks in the simulation are described by the SW action

$$S_{\mathrm{F}}^{\mathrm{SW}} = S_{\mathrm{F}}^{\mathrm{W}} - ig_0\, c_{\mathrm{SW}} \frac{\kappa_q}{2} \sum_{x,\mu,\nu} \bar{q}\, F_{\mu\nu}\sigma_{\mu\nu}\, q(x) \ , \tag{A.1}$$

where $S_{\mathrm{F}}^{\mathrm{W}}$ is the standard Wilson action, g_0 the bare gauge coupling, $F_{\mu\nu}$ a lattice realisation of the Yang-Mills field strength tensor, κ_q the appropriate quark hopping parameter and c_{SW}, the so-called clover coefficient. $c_{\mathrm{SW}} = 1$ corresponds to tree-level improvement. With this value of c_{SW}, leading discretisation errors are $\mathcal{O}(\alpha_s a)$ instead of $\mathcal{O}(a)$ as they are with the standard Wilson action, corresponding to $c_{\mathrm{SW}} = 0$. We actually use a mean-field-improved SW action with values of c_{SW} given in table 2, where the parameters used in our simulations are summarised.

lattice	coarser	finer
β	6.0	6.2
lattice size	$16^3 \times 48$	$24^3 \times 48$
c_{SW}	1.47852	1.44239
# of cfs.	498	188
$a^{-1}(M_\rho)$ (GeV)	1.96(5)	2.54(8)
$a^{-1}(f_\pi)$ (GeV)	1.87(4)	2.52(8)

Table 2: *Simulation parameters.* $a^{-1}(M_\rho)$ *and* $a^{-1}(f_\pi)$ *are the values of the inverse lattice spacing determined from calculations of the* ρ*-meson mass and the pion decay constant, respectively. The latter is given for* $\mu = 2/a$.

B. $\mathcal{O}(a)$-improvement of the axial current

The leading discretisation errors with the mean-field-improved SW action are *formally* of $\mathcal{O}(\alpha_s a)$, as they are for the tree-level improved SW action. To estimate these errors, we consider the following variation in our procedure.

$\mathcal{O}(\alpha_s a)$-improvement of the axial current requires one to include the effect of the $a\partial_\mu P$ (P the pseudoscalar density) counterterm through the replacement

$$A_\mu \to A_\mu + c_A a\partial_\mu P \ , \tag{B.1}$$

as well as to rescale the quark fields as

$$q \to (1 + \frac{b_A}{2} am_q) q \ , \qquad \text{(B.2)}$$

with both c_A and b_A evaluated at one loop [21, 22]. Thus, from a comparison of results obtained with c_A and b_A set to their tree-level values ($c_A = 0$ and $b_A = 1$) to those obtained with c_A and b_A evaluated at one loop, we can get an estimate of the effect of $\mathcal{O}(\alpha_s a)$ discretisation errors. We do not use the one-loop results as central values for the decay constants, for consistency with our determination of the B-parameters. Indeed, $\mathcal{O}(\alpha_s a)$-improvement of the four-quark operators would require one to consider the mixing of these operators with operators of dimension seven, which is beyond the scope of the present work.

To correct for some higher-order discretisation effects, we actually use KLM normalisation [23] for the quark fields. Thus, our central values are obtained with the normalisation

$$q \to \sqrt{1 + am_q}\, q \qquad \text{(B.3)}$$

and the one-loop variation with

$$q \to \frac{\sqrt{1 + am_q}}{1 + am_q/2} (1 + \frac{b_A^{1-\text{loop}}}{2} am_q) q \ . \qquad \text{(B.4)}$$

We also check that results obtained with tree-level normalisation ($b_A = 1$ in Eq. (B.2)) [24] lie within the discretisation error we quote.

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