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The Number of Convex Topologies on a Finite Totally Ordered Set

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Abstract

We give an algorithm to find the number $T_{cvx}(n)$ of convex topologies on a totally ordered set X with n elements, and present these numbers for $n \leq 10$.

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1 Introduction

A subset B of poset $(X, \leq)$ is *increasing* if $x \in B$ and $y \geq x$ imply $y \in B$, and is *convex* if $x, z \in B$ and $x \leq y \leq z$ imply $y \in B$. An n-point totally ordered set X may be labeled $X = \{1, 2, \ldots, n\}$ where $1 < 2 < \cdots < n$. This set will be denoted $[1, n]$, and in general, $[a, b]$ will denote $\{a, a+1, \ldots, b\} \subset \mathbb{N}$ with the natural order from $\mathbb{N}$. A topology on $(X, \leq)$ is *convex* if it has a base of convex sets, or equivalently, if each point has a neighborhood base of convex sets. Because of these equivalent characterizations, convex topologies are often called *locally convex* topologies. (See Nachbin [5]). For finite sets, every point j has a minimal neighborhood $MN(j)$ which is the intersection of all neighborhoods of j. It is convenient to identify a topology on $[1, n]$ with its base $\{MN(j) : j \in [1, n]\}$ of minimal neighborhoods of each point. Finite topological spaces are used in computer graphics, where the Euclidean plane is modeled by a topology on a finite set of pixels. If $a < b < c$ in a finite poset with a topology, if c is "near" a and there is any compatibility between the topology and order, we would expect b to also be near a. This is the convexity condition, which is a natural, weak compatibility condition between a topology and order assumed in most applications. We will consider the number of convex topologies on a finite totally ordered set $[1, n]$.

An excellent reference on finding the number $T(n)$ of topologies on an n-element set is Erné and Stege [3]. Currently, $T(n)$ is known for $n \leq 18$. A standard approach to counting topologies on a finite set X is to employ the one-to-one correspondence between a topology τ on X and the associated specialization quasiorder defined by $x \leq y$ if and only if x is in the closure of y. This correspondence dates back to Alexandroff [1]. (See Richmond [6] for a survey of this connection.) One approach to counting the convex topologies would be to find a (bi-ordered) characterization of convex topologies using some compatibility between the specialization order and the given total order. Fruitful results in this direction have not been found.

For $j \in [1, n]$, a convex subset $N(j)$ of $[1, n]$ containing j has form $[a, b]$ where $1 \leq a \leq j \leq b \leq n$. There are j choices for a and $n-j+1$ choices for b, and thus $j(n+j-1)$ choices for $N(j)$. Since a base of minimal neighborhoods for a locally convex topology on $[1, n]$ consists of one convex subset $N(j)$ for each $j \in [1, n]$, we see that

$$\prod_{j=1}^{n} (j)(n + j - 1) = (n!)^2$$

gives an upper bound on $T_{cvx}(n)$. Of course, arbitrarily selecting a convex set $N(j)$ containing j for each $j \in [1, n]$ is unlikely to give a base for a topology, so this upper bound is not sharp.

2 Nested Convex Topologies

Stephen [8] gives a recursive formula for the number of nested topologies (or equivalently, ordered partitions) on an n-point set X, generating the sequence $1, 3, 13, 75, 541, 4683, 47293, \ldots$, which is A000670 in Sloane [7]. If $X = [1, n]$ is totally ordered set with n-elements, let $T_{Nest}(n)$ be the number of nested convex topologies on X, and let $T_{Nest}(n, k)$ be the number of those convex topologies consisting of k nested non-empty open sets $U_1, U_2, \ldots, U_k$ where

$X = U_1 \supset U_2 \supset \cdots \supset U_k \neq \emptyset$. Since the indiscrete topology is the only nested topology with one non-empty open set, $T_{Nest}(n, 1) = 1$. Suppose we have found $T_{Nest}(m, j)$ for all $m \leq n$ and $j \leq k$. To find $T_{Nest}(n, k+1)$, note that $X = U_1 \supset U_2 \supset U_3 \supset \cdots \supset U_{k+1} \neq \emptyset$ implies that U_2 must contain at least k elements and at most $n-1$ elements. If $|U_2| = j$, there are $n-j+1$ ways to choose U_2 as a convex subset of X, and $T_{Nest}(j, k)$ ways to complete the nested convex topology $\{U_2, \ldots, U_{k+1}\}$ on the j-point totally ordered set U_2. Thus, we have

$$T_{Nest}(n, k+1) = \sum_{j=k}^{n-1} (n-j+1) T_{Nest}(j, k) = \sum_{m=2}^{n-k+1} m \cdot T_{Nest}(n-m+1, k),$$

where the second equality follows from the substitution $m = n - j + 1$. In Table 1 we tabulate the values of $T_{Nest}(n, k)$ for $n, k \leq 10$.

n \ k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1	5	4							
4	1	9	16	8						
5	1	14	41	44	16					
6	1	20	85	146	112	32				
7	1	27	155	377	456	272	64			
8	1	35	259	833	1,408	1,312	640	128		
9	1	44	406	1,652	3,649	4,712	3,568	1,472	256	
10	1	54	606	3,024	8,361	14,002	14,608	9,312	3,328	512

Table 1: $T_{Nest}(n, k)$, the number of topologies on a totally ordered n-point set consisting of k nested convex sets.

This table (sequence A056242 in Sloane [7]) is also used by Hwang and Mallow [4] to count the number of *order consecutive partitions* of $X = \{1, 2, \ldots, n\}$, which they define as follows: An ordered list $S_1, S_2, \ldots, S_m$ of subsets of X is an order consecutive partition of X if $\{S_1, \ldots, S_m\}$ is a partition of X and each of the sets $\bigcup_{j=1}^{k} S_j$ $(1 \leq k \leq m)$ is a consecutive set of integers. If $\{S_1, \ldots, S_m\}$ is an order consecutive partition, clearly

$\{S_1, S_1 \cup S_2, S_1 \cup S_2 \cup S_3, \ldots, X\}$ is a nested convex topology on X. Conversely, any nested convex topology $\tau = \{U_1, U_2, \ldots, U_k\}$ on $X = \{1, 2, \ldots n\}$ generates the order consecutive partition U_1, $U_2 \setminus U_1$, $U_3 \setminus U_2, \ldots, U_k \setminus U_{k-1}$.

It is easy to confirm from our formula for $T_{Nest}(n, k)$ that $T_{Nest}(n, n) = 2^{n-1}$ and $T_{Nest}(n, 2) = \Delta_n - 1$ where Δ_n is the n^{th} triangular number.

Now, we note that

$$T_{Nest}(n) = \sum_{k=1}^{n} T_{Nest}(n, k).$$

This sequence $(T_{Nest}(n))_{n=1}^{10} = (1, 3, 10, 34, 116, 396, 1352, 4616, 15760, 53808)$ appears as A007052 in Sloane [7], where it is noted that $T_{nest}(n) = 4\,T_{nest}(n-1) - 2\,T_{nest}(n-2)$ for $n > 2$. Solving this recurrence relation by standard techniques gives

$$T_{nest}(n) = \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{4}.$$

Nested convex topologies have as much inclusion as possible. Not only are they totally ordered by inclusion, but they maximize "overlap." The other extreme would be to have as little inclusion and overlap as possible. This suggests considering mutually disjoint collections. A collection $\mathcal{D}$ of mutually disjoint convex subsets of X is not a basis for a topology if $\bigcup \mathcal{D} \neq X$, but $\mathcal{D} \cup \{X\}$ is always a basis for a convex topology on X. The authors [2] have shown that the number of topologies on an n-element totally ordered set having a base consisting of a mutually disjoint collection $\mathcal{D}$ of convex sets, or such a collection $\mathcal{D}$ together with X, is $F_{2n+1} - 1$, where F_k is the k^{th} Fibonacci number.

3 An Algorithm for $T_{cvx}(n)$

We now present a recursive algorithm to find the number $T_{cvx}(n)$ of convex topologies on a totally ordered set $[1, n]$. It is easy to check that $T_{cvx}(1) = 1 = T(1)$ and $T_{cvx}(2) = 4 = T(2)$. That is, the only topology on a 1-point set is convex, as are all four topologies on a 2-point set.

Suppose $T_{cvx}(n)$ is known. To find $T_{cvx}(n+1)$, note that each convex topology on $[1, n+1]$, when restricted to $[1, n]$, gives a unique convex topology on $[1, n]$. Thus, we may count $T_{cvx}(n)$ by looping through each topology τ counted in $T_{cvx}(n)$, adding $n+1$ as the greatest point, adjusting the minimal neighborhoods of $j \in [1, n]$, and defining the minimal neighborhood of $n+1$ so that the subspace topology on $[1, n]$ is still τ. That is, considering how each topology on $[1, n]$ may be appropriately expanded to $[1, n+1]$ gives a complete, unduplicated count of the convex topologies on $[1, n+1]$.

Step 1: *Re-defining minimal neighborhoods of* $j \in [1, n]$. We loop through all convex topologies τ on $[1, n]$. The simplest way to extend τ to $[1, n+1]$ so that the restriction of the extension is still τ would be to keep the minimum neighborhoods of each $j \in [1, n]$ unchanged. However, we may also expand some of the minimal neighborhoods of points $j \in [1, n]$ to include $n+1$. To maintain convexity and to guarantee a topology on $[1, n+1]$ whose restriction to $[1, n]$ agrees with τ, the minimal neighborhood $MN(j)$ of j can be expanded to include $n+1$ if and only if $MN(j)$ already includes n. If $n \in MN(j) \subseteq MN(k)$ and $MN(j)$ is expanded to include $n+1$, then $MN(k)$ must also be expanded to include $n+1$, for otherwise, $MN(k)$ would be a neighborhood of j not including $n+1$, contrary to the hypothesis that the minimal neighborhood of j was to include $n+1$.

As an immediate consequence, if $n \in MN(j) = MN(k)$, then $MN(j)$ is expanded to include $n+1$ if and only if $MN(k)$ is. That is, a single basis element which happens to be the minimal neighborhood of distinct points j and k is still treated as a single entity in the expansion process.

Thus, if $\mathcal{B} = \{MN(1), MN(2), \ldots, MN(n)\}$ has m distinct sets containing n, we expand the outermost k of these to include $n+1$, looping as k goes from 1 to m.

For example, consider the convex topology τ on $[1, 8]$ having a base of minimal neighborhoods $\mathcal{B} = \{\{1\}, [2, 8], [3, 4], \{5\}, [5, 8], \{8\}\}$, as shown in Figure 1.

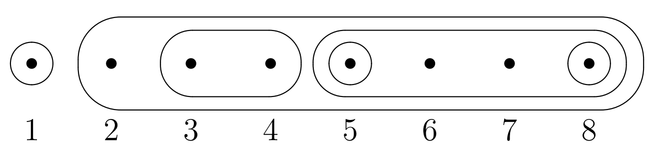

Figure 1: Example. A sample topology on $[1, 8]$

We may add 9 to this topology without changing any of the minimal neighborhoods of j for $j \in [1, 8]$, or since $MN(2), MN(6) = MN(7)$, and $MN(8)$ include the right endpoint 8, they may be extended to include the added point 9. Since $8 \in MN(8) \subset MN(7) = MN(6) \subset MN(2)$, we note that $MN(6)$ is expanded if and only if $MN(7)$ is expanded, so we do not need to treat $MN(6)$ and $MN(7)$ as distinct basis elements and we may effectively ignore the duplicate $MN(7)$. Also, if $MN(6)$ is expanded, then $MN(6) \subset MN(2)$ implies that $MN(2)$ would also have to be expanded. Repeating this idea, we may expand nothing, only the outermost (i.e., longest) minimal neighborhood containing 8, namely $MN(2)$, the outermost two minimal neighborhoods containing 8, namely $MN(2)$ and $MN(6)$, or the outermost three, $MN(2), MN(6)$, and $MN(8)$. See Figure 2.

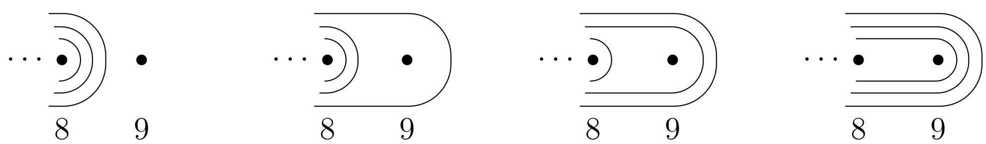

Figure 2: Example. Possible expansions of minimal neighborhoods containing previous right endpoint: None, outermost one, outermost two, outermost three.

Step 2: *Defining the minimal neighborhood of the added point.* Having determined the expansion of minimal neighborhoods of $j \in [1, n]$, it remains to define the minimal neighborhood $MN(n + 1)$ of $n + 1$. Clearly we must have $n+1 \in MN(n+1)$. The convexity condition and our need to retain the original topology τ on $[1, n]$ as a subspace imply that $MN(n + 1)$ must be of form $\{n + 1\} \cup I$ where I is increasing and open in τ. The final condition is the minimality of the neighborhood $MN(n + 1)$. In Step 1, we may have expanded some neighborhoods of n to contain $n + 1$ and, if so, the minimal neighborhood of $n + 1$ must be contained in each of these previously defined neighborhoods of $n+1$. Thus, $MN(n+1)$ must be of the form $\{n+1\}\cup I$ where I is increasing and τ-open, and I is contained in the innermost (shortest) neighborhood $MN(j)$ which was expanded in Step 1.

Continuing the example presented above, we may expand none of the original minimal neighborhoods of $j \in [1, 8]$ to include 9, and then the minimal neighborhood $MN(9)$ of 9 may be defined as $\{9\} \cup I$ where I is an increasing τ-open set in any of the six ways suggested in Figure 3.

Figure 4 shows the three possible choices for the minimal neighborhood $MN(9)$ if the outermost two minimal neighborhoods containing 8, namely $MN(2)$ and $MN(6)$ have been expanded to include 9.

A computer implementation of this algorithm yields the values for $T_{cvx}(n)$ shown in Table 2 below. With the $T_{cvx}(2) = 4$ convex topologies on $[1, 2]$ as

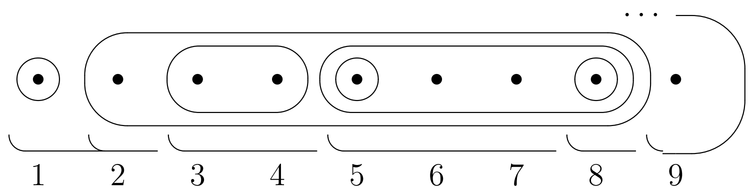

Figure 3: Example. Possible choices for $MN(9)$ if no minimal neighborhoods $MN(j)$ are expanded for $j \in [1, 8]$.

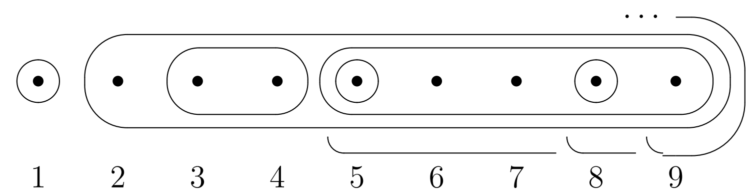

Figure 4: Example. Possible choices for $MN(9)$ if $MN(2)$ and $MN(6)$ are expanded to include 9.

input, the computer implementation loops through all the topologies τ on $[1, n]$, adds $n+1$, determines the number m of distinct minimal neighborhoods of $j \in [1, n]$ containing n, expands the outermost k of these to contain $n + 1$ (as k goes from 0 to m), determines the increasing τ-open sets, defines the minimal neighborhood $MN(n+1)$ of $n+1$ as $\{n+1\} \cup I$ where I is one of the increasing τ-open sets contained in the smallest $MN(j)$ previously expanded to include $n + 1$, and, at each selection of an option above, increments the $T_{cvx}(n + 1)$ counter and records the data for this new topology on $[1, n + 1]$ required for the next iteration.

The efficiency of this algorithm can be improved by eliminating duplication of computations. For example, if p is the largest integer with $MN(p) = X$ for two topologies s and t which agree to the right of p, then the computation for s duplicates that for t, as noted by a helpful referee.

The numbers $T_{cvx}(n)$ in Table 2 were also verified for $n \leq 8$ without the algorithm using an exhaustive generation scheme. For comparison, we also include the number $T_{Nest}(n)$ of nested convex topologies and the number $T(n)$ of topologies on n points in the table.

n	$T_{Nest}(n)$	$T_{cvx}(n)$	$T(n)$
1	1	1	1
2	3	4	4
3	10	21	29
4	34	129	355
5	116	876	6,942
6	396	6,376	209,527
7	1,352	48,829	9,535,241
8	4,616	388,771	642,779,354
9	15,760	3,191,849	63,260,289,423
10	53,808	26,864,936	8,977,053,873,043

Table 2: The numbers $T_{Nest}(n)$ and $T_{cvx}(n)$ of nested convex topologies and convex topologies on an n-point totally ordered set, and the number $T(n)$ of topologies on an n-point set.

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