ACTA ARITHMETICA
XXI (1972)

Wacław Sierpiński's papers on the theory of numbers

by

A. Schinzel (Warszawa)

Sierpiński's papers on the theory of numbers date from the first (1906–1910) and the last (1948–1968) period of his creative activity. Exceptional in this respect are papers [19] and [20] connected by the subject matter with the first and the second period respectively, but written in the years 1917–1920. In the first period Sierpiński worked in the analytic theory of numbers and the theory of diophantine approximation, in the second period in elementary number theory. We shall discuss successively papers belonging to these three fields.

Papers on the analytic theory of numbers [1], [2], [3], [4], [5], [7], [8] have been written under the influence of Voronoï who was Sierpiński's teacher. [1] concerns the circle problem. Let $A(x)$ be the number of lattice points in a circle $u^2+v^2 \leqslant x$. Using the geometrical method of Voronoï [17] Sierpiński proved the estimation

$$R(x) = Ax - \pi x = O(x^{1/3}) \tag{1}$$

thus obtaining the first improvement of the classical result of Gauss $R(x) = O(x^{1/2})$. Nine years later E. Landau [8] gave another proof of this theorem. Informing about the circle problem in the introduction to his paper Landau wrote; "In dieser Richtung hat Herr Sierpiński 1906 die Grenzen des Wissens sehr erweitert: er bewies nämlich nicht nur

$$\lim_{x=\infty} \frac{A(x)-\pi x}{\sqrt{x}} = 0,$$

sondern sogar die Relation

$$A(x) = \pi x + O(\sqrt[3]{x}).$$

Ich stelle mir nun in meinem vorliegenden Arbeit die Aufgabe für den Sierpińskischen Satz, der gewiss nicht vielen Mathematikern dem Wortlaut nach bekannt ist, und dessen polnisch geschriebenen 40 Seiten longen Beweis gewiss nur sehr wenige Mathematiker gelesen haben einen neuen, ganz anderen Beweis zu geben. Nur eine Eigenschaft des Kreises habe ich in dieser Abhandlung zu beweisen: sie liegt aber so tief, dass zumal ich wie Herr Sierpiński im Bereich der reellen Analysis verbleibe — der Leser keinen kurzen Beweis erwarten darf".

Now, (1) can be proved simply by the use of Bessel functions (see [9]), but the lower bound ϑ of the exponents v such that $R(x) = O(x^v)$ is still unknown. It was J. G. van der Corput who first proved in 1923 that $\vartheta < \frac{1}{3}$. Successive improvements have been obtained by Walfisz, Nieland, Titchmarsh, Hua and Chen-Jing-run [3] who has proved $\vartheta \leqslant 12/37$.

Connected with [1] is the paper [8] which contains a proof of the formula

$$\sum_{l^2+m^2+n^2\leqslant x} \frac{1}{l^2+m^2+n^2} = 4\pi\sqrt{x} + \text{const} + o(1).$$

[2], Sierpiński's doctorate thesis is an enlarged version of his unpublished dissertation for the degree of candidate. For this dissertation, on the proposal of Voronoï Sierpiński was awarded a gold medal by the University of Warsaw. The paper is concerned with the sum $\sum \tau(n) f(n)$, where $\tau(n)$ is the number of decompositions of n into sum of two squares.

The following asymptotic formulae are obtained

$$\sum_{n\leqslant x} \frac{\tau(n)}{n} = \pi \log x + K + O\left(\frac{1}{\sqrt{x}}\right),$$

$$\sum_{n\leqslant x} \tau(n^2) = \frac{4}{\pi} x \log x + Bx + O(x^{2/3}),$$

$$\sum_{n\leqslant x} \tau(n)^2 = 4x\log x + \left(\frac{8K}{\pi} - \frac{48}{\pi^2}F + \frac{4}{3}\log 2 - 4\right)x + O(x^{3/4}\log x),$$

where

$$K = 2.5849817596\ldots, \qquad F = \sum_{n=1}^{\infty} \frac{\log n}{n^2},$$

$$B = \frac{4}{\pi}\left(C + \frac{K}{\pi} - \frac{12}{\pi^2}F + \frac{\log 2}{3} - \frac{1}{\pi}\right)$$

and C is Euler's constant.

The proof is based on an ingenious transformation of the sum $\sum \tau(n) f(n)$ namely

$$\sum_{n>a}^{k\leqslant b} \tau(n) f(n) = 4 \sum_{v=0}^{(b-1)/2} (-1)^v \sum_{\mu=0}^{\mu\leqslant \frac{b}{2v+1}} f\big((2v+1)\mu\big).$$

Motivating the proposed award to Sierpiński, Voronoï wrote as follows ([18], translation from Russian): "Using the theorem of Jacobi the author has succeeded in obtaining a remarkable transformation of the sum

$$\sum\sum f(m^2+n^2) = \sum_{k>a}^{k\leqslant b} \tau(k) f(k)$$

which comprises as a special case the identity of Liouville ([1])

$$\mathrm{E}\sqrt{x} + \mathrm{E}\sqrt{x-1^2} + \mathrm{E}\sqrt{x-2^2} + \ldots = \mathrm{E}\frac{x}{1} - \mathrm{E}\frac{x}{3} + \mathrm{E}\frac{x}{5} - \ldots$$

For the approximate evaluation of the sum occuring on the right-hand side of the former equality, the author uses a special device, which is nothing else but very successful generalization of the summatory formula of Euler–Maclaurin applicable to functions which have in the range of summation a finite number of points of discontinuity. The quoted results of the author's own investigations evidence his outstanding mathematical abilities and have unquestionable value for the science".

[4], [5] and [7] are close to [2] by their subject. In [5] Sierpiński considers the sum $\sum_{k=1}^{n} \varrho(k)$, where $\varrho(k)$ is the number of representations of k as difference of two squares and obtains the formula

$$\lim_{n=\infty} \frac{1}{n} \sum_{k=1}^{n} \big(\varrho(k) - d(k)\big) = 0,$$

where $d(k)$ is the number of divisors of k.

In [7] there are derived asymptotic formulae for $\sum_{n\leqslant x} f_i(n)$ $(i = 1, 2, 3)$, where $f_1(n)$ is the number of the positive integers m such that $m^2 | n$, $f_2(n)$ is the sum of these numbers and $f_3(n)$ the greatest among them.

The papers on diophantine approximation [10] — [17] date from the years 1909–1910.

[11] brings a proof that for every real x and positive integer n there are at most two fractions p/q such that

$$|x - p/q| < 1/nq, \qquad 1 \leqslant q \leqslant n.$$

The papers [10] and [12] announce and [13], 14] and [17] contain the proof of the formula

$$\lim_{n=\infty} \frac{1}{n} \sum_{k=1}^{n} (kx - \mathrm{E}kx) \begin{cases} < \frac{1}{2} \text{ for rational } x, \\ = \frac{1}{2} \text{ for irrational } x. \end{cases}$$

([1]) Here and in the sequel Ex denotes the integral part of x.

In [15] and [16] this formula for irrational x is generalized as follows

$$\lim_{n=\infty}\frac{1}{n}\sum_{k=1}^{n}\big(kx+y-\mathrm{E}(kx+y)\big)=\frac{1}{2}\quad (y\ \text{arbitrary}).$$

The above equality proves the equipartition of the sequence $kx \bmod 1$. Other proofs of this important fact have been given about the same time by P. Bohl and H. Weyl. Then followed the classical paper of Weyl of 1916 and the long series of papers by many authors (see Koksma [*7*]).

Related to the theory of diophantine approximation are the papers [19] (decimal expansions) and [80] (continued fractions). In the first Sierpiński deals with absolutely normal numbers (i.e. numbers normal in any integral scale of notation). This notion has been introduced by Borel, who proved that almost all real numbers are absolutely normal. In 1909 Borel [*1*] wrote: "Dans l'état actuel de la Science, la détermination effective d'un nombre absolument normal paraît un problème des plus difficiles". Sierpiński defined an absolutely normal number as the lower bound of a certain well determined set of real numbers. Interesting comments to this paper are to be found in a note of Lebesgue [*10*].

The papers on algorithms and expansions listed separately on pp. 21–22 have a certain arithmetical aspect but belong primarily to analysis. In one of them published in 1911 Sierpiński introduced expansions now called generally Engel series, although the relevant paper of Engel dates from 1913.

The remaining number theoretical papers of Sierpiński belong to elementary number theory and deal with divisibility and congruences, diophantine equations, arithmetical functions, prime numbers, additive theory and lattice points.

The first group includes papers [9] (properties of Legendre symbol), [26], [27] (sequences periodical mod m), [21], [74], [101], [107] (pseudoprimes and their generalizations), [98] (sequences of coprime integers), [23], [33] (algebraic congruences), [51], [65] (exponential congruences). [21] contains a very simple proof of the existence of infinitely many pseudoprimes (i.e. integers n dividing 2^n-2): if n is a pseudoprime 2^n-1 is also one. The same proof has been a little later given by R. Steuerwald [*14*]. The investigations of pseudoprimes have been continued by other writers, e.g. by P. Erdös and A. Rotkiewicz.

Diophantine equations are treated in the papers [6], [42], [68], [75], [78] (Pythagorean triangles), [47], [73], [85], [91], [102], [108], [109] (inhomogenous quadratic equations), [35], [37], [69], [84], [86], [95], [97], [118], [119] (systems of quadratic equations and equations of higher degree), [41], [48] (exponential equations). Investigating diophantine equations Sierpiński used only elementary methods. Of particular interest is the proof given in [86] and [95] that each of the equations

$$\binom{x}{3}+\binom{y}{3}=\binom{z}{3}\quad\text{and}\quad\binom{x}{3}+\binom{y}{3}=2\binom{z}{3}$$

has infinitely many nontrivial solutions in positive integers x, y, z.

The papers on arithmetical functions ([32], [44], [52], [56], [57], [89], [105]) all but [56] deal with special functions like φ, σ etc. and not with the general theory. [44] brings a proof that the equation $\varphi(x+k) = \varphi(x)$ has for every k at least one solution. The problems of this type have been further investigated by P. Erdös, A. Mąkowski, A. Schinzel. The paper [32] has a different, more analytic character. It is proved there that for suitable sequences m_k, n_k and for any choice of sign $\pm$

$$\lim_{k=\infty}\frac{\varphi(m_k\pm 1)}{\varphi(m_k)}=\infty,\quad \lim_{k=\infty}\frac{\varphi(n_k\pm 1)}{\varphi(n_k)}=0.$$

This subject has been taken further by several writers: P. Erdös, A. Schinzel, Shao-Pin Tsung and Y. Wang (see [*4*], [*12*], [*13*]).

Sierpiński devoted many papers to the investigation of primes. In [29] he gave an often quoted formula for the nth prime:

$$p_n=\mathrm{E}10^{2^n}a-10^{2^{n-1}}\mathrm{E}10^{2^{n-1}}a,\quad \text{where}\quad a=\sum_{k=1}^{\infty}p_k10^{-2^k}.$$

In [28], [62], [70] he studied the form of decimal expansions of primes and he proved ([62]) that one can choose arbitrarily the first k and the last l digits of a prime provided the last digit is 1, 3, 7 or 9. In [70] he considered the sum s_n of digits of the nth prime and asked whether for infinitely many n $s_n > s_{n+1}$, which has been settled in the affirmative by P. Erdös [*5*].

He investigated also the differences between consecutive primes $d_n = p_{n+1}-p_n$ ([22], [25], [38]) and proved in [22] that

$$\overline{\lim_{n=\infty}}\min(d_n, d_{n+1})=\infty.$$

This result was generalized by P. Erdös [*3*] to the equality

$$\overline{\lim_{n=\infty}}\min(d_n, d_{n+1},\ldots, d_{n+r})=\infty\quad (r\ \text{arbitrary})$$

and improved in another way by A. Walfisz [*16*] and K. Prachar [*11*].

Sierpiński had an abiding interest in Dirichlet's theorem on arithmetic progression ([20], [90], [94]) and in its hypothetical generalization to polynomials of higher degree, still unproven ([54]), from which he derived many consequences ([83], [87], [96], [103], [111], [112], [114],

[115]). In [100] he proved that for every k there exists an n such that x^2+n takes more than k prime values. He also studied the distribution of primes in sequences of exponential growth ([50], [66], [72], [81], [92], [106]) and in [66] he proved the existence of integers k such that all the numbers $2^n k+1$ are composite.

The papers [55], [99] are concerned with the distribution of squarefree numbers, [117] with that of powers of primes.

Five papers on sums of primes ([30], [58], [60], [67], [93]) are already on the border of additive number theory. The latter is represented by papers on the sums of squares, cubes and triangular numbers; [24], [45], [53], [59], [61], [71], [82], [110], on decompositions of rationals into unit fractions [46], [104] and on related problems [36], [113]. In [46] Sierpiński put forward the conjecture that for every $n>1$ the equation $\frac{5}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is solvable in positive integers x, y, z. This conjecture has been verified by G. Gentile [*6*] for all $n<10^6$. [36] brings a characterization (found also by B. M. Stewart [*15*]) of all positive integers n such that each $m\leqslant n$ is the sum of distinct divisors of n. [31], [34], [76] deal with additive properties of arbitrary sequences of integers.

The paper [49] ([64] being close to it) is concerned with the lattice points, more precisely with the number of these points not inside a domain (which is usual in the geometry of numbers) but on the boundary.

As it is clear from the above short survey, Sierpiński's interests were concentrated not on the structure of the theory but on problems and conjectures. Lists of unsolved problems are to be found in [63], [77] and some metamathematical comments in [79]. Besides, Sierpiński published several expository or popular articles [18], [39], [40], [43], [88], [116].

Last but not least he wrote 13 books on number theory: a monograph in Polish consisting of two parts [I] and [VIII], a monograph in English [XIII] based on the former but completely rewritten, a textbook of foundations of arithmetic [IV] containing an exposition of elements of number theory and nine popular booklets. Among these there are little monographs on Pythagorean triangles [III], diophantine equations [V], sums of unit fractions [VI], prime numbers [X], triangular numbers [XI], a collection of unsolved problems [IX] and a collection of exercises [XII].

References to other writers

[*1*] E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo, 27 (1909), pp. 247–271.

[*2*] Chen Jing-Run, *The lattice points in a circle*, Sc. Sinica 12 (1962), pp. 633–649.

[*3*] P. Erdös, *On some applications of Brun's method*, Acta Sc. Math. Szeged 13 (1949), pp. 57–63.

[*4*] P. Erdös and A. Schinzel, *Distributions of the values of a class of arithmetical functions*, Acta Arith. 6 (1961), pp. 393–413.

[*5*] — *On a problem of Sierpiński*, Acta Arith. 11 (1965), pp. 189–192.

[*6*] G. Gentile, *Sull indeterminate* $5/n=1/x+1/y+1/z$, Archimede 1959, pp. 222–223.

[*7*] J. F. Koksma, *Diophantische Approximationen*, Berlin 1935.

[*8*] E. Landau, *Über die Zerlegungen der Zahlen in zwei Quadrate*, Ann. Mat. Pura Appl. (3) 20 (1913), pp. 1–28.

[*9*] — *Vorlesungen über Zahlentheorie*, Bd. II, Leipzig 1927.

[*10*] H. Lebesgue, *Sur certaines démonstrations d'existence*, Bull. Soc. Math. France 45 (1917), pp. 132–144.

[*11*] K. Prachar, *Über ein Resultat von A. Walfisz*, Mh. Math. 58 (1954), pp. 114–116.

[*12*] A. Schinzel and Y. Wang, *A note on some properties of the functions* $\varphi(n), \sigma(n)$ *and* $\theta(n)$, Ann. Polon. Math. 4 (1958), pp. 201–213.

[*13*] Shao Pin-Tsung, *On the distribution of the values of a class of arithmetical functions*, Bull. Acad. Polon. Sci., Cl. III, 4 (1956), pp. 569–572.

[*14*] R. Steuerwald, *Über die Kongruenz* $2^{n-1}\equiv 1 \pmod{n}$, Sitz.-Ber. math. naturw. Kl. Bayer. Akad. Wiss. München 1947, p. 177.

[*15*] B. M. Stewart, *Sums of distinct divisors*, Amer. J. Math. 76 (1954), pp. 779–785.

[*16*] А. З. Вальфиш, *Изолированные простые числа*, Докл. АНСССР 90(1953), pp. 711–713.

[*17*] G. Voronoï, *Sur un problème de calcul des fonctions asymptotiques*, J. Reine Angew. Math. 126 (1903), pp. 241–282.

[*18*] — *Отзывъ о конкурсномъ сочинении подъ девизомъ "Summa"*, Годичный Актъ Импер. Варшавского Унив. 1904.