On an Expression for the Midpoint and the Radius of the Product of Two Intervals*

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Abstract

The paper deals with expressions for the midpoint and the radius of the product of two intervals. Based on these expressions two applications are considered: One leads to an explicit representation of the fixed point of the function $[A][x]+[b]$ if zero is contained in the interval vector $[b]$ but not in the interior of the entries of the interval matrix $[A]$. The second application proves the semi–convergence of an interval matrix $[A]$ under certain assumptions which are not studied up to now.

Keywords: midpoint of an interval, radius of an interval, total step method, powers of an interval matrix, semi–convergence of matrices, interval analysis
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1 Introduction

Compact real intervals $[a] = [\underline{a}, \overline{a}]$ are basic objects for verifying and enclosing solutions of various mathematical problems such as linear or nonlinear systems of equations, initial value problems, integral equations. With enclosures in mind one defines an interval arithmetic by

$$[a] \circ [b] = \{\, a \circ b \mid a \in [a],\, b \in [b] \,\},\ \circ \in \{\, +, -, \cdot, / \,\},\ 0 \notin [b] \text{ in case of division.}$$

The result $[a] \circ [b]$ can be expressed by means of the bound $\underline{a}, \overline{a}, \underline{b}, \overline{b}$, of the intervals $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$; cf. [1] or [16]. Unfortunately, the set $\mathbb{IR}$ of such intervals together with the addition and the multiplication do not form a field although the set $\mathbb{R}$ of real numbers is imbedded isomorphically in $\mathbb{IR}$ via $a \equiv [a, a]$. Therefore, numerous auxiliary functions are used when dealing with intervals. Among them are the midpoint $\check{a} = \mathrm{mid}([a]) = (\underline{a}+\overline{a})/2$ of an interval, its radius $r_a = \mathrm{rad}([a]) = (\overline{a}-\underline{a})/2$, i.e., half of its length, and its absolute value $|[a]| = \max\{|\underline{a}|, |\overline{a}|\} = |\check{a}| + r_a$. In view of the radius of a matrix–vector product, essential formulas are those concerning the representation of the midpoint and the radius of the product $[c] = [a] \cdot [b]$ of two intervals. While the operations $\mathrm{mid}(\cdot)$ and $\mathrm{rad}(\cdot)$ are additive, unfortunately they are far from being

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multiplicative. Thus $\mathrm{mid}([a]+[b]) = \check{a}+\check{b}$, $\mathrm{rad}([a]+[b]) = r_a + r_b$ but generally $\check{c} \neq \check{a}\cdot\check{b}$, $r_c \neq r_a \cdot r_b$. In [16] the formulas

$$\begin{aligned} \check{c} &= \check{a}\check{b} + \mathrm{sign}(\check{a}\check{b}) \cdot \min\{\, r_a|\check{b}|,\, |\check{a}|r_b,\, r_a r_b \,\}, && (1) \\ r_c &= \max\{\, r_a|[b]|,\, |[a]|r_b,\, r_a|\check{b}| + |\check{a}|r_b \,\} && (2) \end{aligned}$$

can be found. Since they need products before max and min are applied they are often unwieldy for applications such as the computation of $[x]$ from the fixed point equation

$$[x] = [A][x] + [b], \tag{3}$$

where $[A]$ is an $n \times n$ interval matrix and $[b]$ is an interval vector with n components. Therefore, we will look for a representation of $\check{c}$, r_c – at least for selected cases – which does not maximize/minimize over three operands. This is done in Section 2 where also a global upper bound is given for r_c which can be reformulated to a less known expression found by Ris in [18]; cf. also [19]. In Section 3 we apply the results of Section 2 to the problem (3) representing $[x]$ explicitly if 0 is contained in $[b]$ but not in the interior of the entries of $[A]$. We also derive a system of equations for the midpoint and the radius of $[x]$ if the assumption on $[b]$ (but not on $[A]$) is dropped. We outline an algorithm to solve this system iteratively and report on numerical experiments which yield to the solution of (3) within at most 4 iterations. Moreover, we apply the results of Section 2 to the powers of an interval matrix $[A]$ in reducible normal form proving a criterion for the semi–convergence of $[A]$ if this matrix fulfills some additional assumptions.

2 Midpoint–Radius Representation of the Product of Intervals

We first present some notation and basic facts. The set $\mathbb{IR}$ consists of all real compact intervals $[a] = [\underline{a}, \overline{a}]$, $\underline{a} \leq \overline{a}$. Analogously, the setS of all real interval vectors $[x] = [\underline{x}, \overline{x}] = ([x]_i) = ([\underline{x}_i, \overline{x}_i])$ and of all $m \times n$ interval matrices $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}])$ are denoted by $\mathbb{IR}^n$ and $\mathbb{IR}^{m\times n}$, respectively. Midpoint, radius and absolute value are written as in (1), (2). For vectors and matrices, these quantities are defined entrywise and are denoted in the same way. In addition to (1), (2) standard relations are

$$\begin{array}{ccccccc} |[a]|r_b & \leq & \mathrm{rad}([a][b]) & \leq & |[a]|r_b + r_a|\check{b}| & \leq & |[a]|r_b + r_a|[b]|, \\ r_a|[b]| & \leq & \mathrm{rad}([a][b]) & \leq & r_a|[b]| + |\check{a}|r_b & \leq & |[a]|r_b + r_a|[b]|, \end{array} \tag{4}$$

and

$$\mathrm{rad}([a][b]) \quad \leq \quad 2r_a r_b \quad \text{if } 0 \in [a]\cap[b], \tag{5}$$

where $[a]$, $[b]$ are intervals; cf. [1] or [16]. By virtue of the additivity of the midpoint and the radius operations, these inequalities transfer immediately to the product of two matrices $[A]$, $[B]$, where "$\leq, <, \geq, >$" between real vectors or real matrices are to be understood entrywise.

For the proof of our first theorem, we will need the following two tables with properties of $[c] = [a]\cdot[b]$, which can be derived by simple computations.

Table 1: Properties of $[c] = [a] \cdot [b]$

	$\overline{b} \le 0$	$\underline{b} < 0 < \overline{b}$	$0 \le \underline{b}$
$\overline{a} \le 0$	$\check{a} \le 0,\ \check{b} \le 0$ $[c] = [\overline{a}\overline{b}, \underline{a}\underline{b}]$ $\check{c} = \check{a}\check{b} + r_a r_b$ $r_c = r_a\lvert\check{b}\rvert + \lvert\check{a}\rvert r_b$	$\check{a} \le 0$ $[c] = [\underline{a}\overline{b}, \underline{a}\underline{b}]$ $\check{c} = \check{a}\check{b} - r_a\check{b}$ $r_c = \lvert[a]\rvert r_b$	$\check{a} \le 0,\ \check{b} \ge 0$ $[c] = [\underline{a}\overline{b}, \overline{a}\underline{b}]$ $\check{c} = \check{a}\check{b} - r_a r_b$ $r_c = r_a\lvert\check{b}\rvert + \lvert\check{a}\rvert r_b$
$\underline{a} < 0 < \overline{a}$	$\check{b} \le 0$ $[c] = [\overline{a}\underline{b}, \underline{a}\underline{b}]$ $\check{c} = \check{a}\check{b} - \check{a} r_b$ $r_c = r_a\lvert[b]\rvert$	$[c] = [\min\{\overline{a}\underline{b}, \underline{a}\overline{b}\},$ $\max\{\underline{a}\underline{b}, \overline{a}\overline{b}\}]$	$\check{b} \ge 0$ $[c] = [\underline{a}\overline{b}, \overline{a}\overline{b}]$ $\check{c} = \check{a}\check{b} + \check{a} r_b$ $r_c = r_a\lvert[b]\rvert$
$0 \le \underline{a}$	$\check{a} \ge 0,\ \check{b} \le 0$ $[c] = [\overline{a}\underline{b}, \underline{a}\overline{b}]$ $\check{c} = \check{a}\check{b} - r_a r_b$ $r_c = r_a\lvert\check{b}\rvert + \lvert\check{a}\rvert r_b$	$\check{a} \ge 0$ $[c] = [\overline{a}\underline{b}, \overline{a}\overline{b}]$ $\check{c} = \check{a}\check{b} + r_a\check{b}$ $r_c = \lvert[a]\rvert r_b$	$\check{a} \ge 0,\ \check{b} \ge 0$ $[c] = [\underline{a}\underline{b}, \overline{a}\overline{b}]$ $\check{c} = \check{a}\check{b} + r_a r_b$ $r_c = r_a\lvert\check{b}\rvert + \lvert\check{a}\rvert r_b$

The case

$$\underline{a} < 0 < \overline{a}, \quad \underline{b} < 0 < \overline{b} \tag{6}$$

in the middle of Table 1 can be further resolved according to the four possible combination of the bounds of $[c]$, leading to Table 2. Since here $r_a > 0,\ r_b > 0$, the inequalities in (6) are equivalent to

$$-1 < \frac{\check{a}}{r_a} < 1, \quad -1 < \frac{\check{b}}{r_b} < 1.$$

If $\underline{c} = \min\{\underline{a}\overline{b}, \overline{a}\underline{b}\} = \underline{a}\overline{b}$ then trivially $\underline{a}\overline{b} \le \overline{a}\underline{b}$, whence $(\check{a}-r_a)(\check{b}+r_b) \le (\check{a}+r_a)(\check{b}-r_b)$. This is equivalent to $\check{a}/r_a \le \check{b}/r_b$, which occurs in the left column of Table 2.

In view of our subsequent results we will write $\rho(A)$ for the spectral radius of a real square matrix A and $\mathrm{int}(S)$ for the interior of a set $S \subseteq \mathbb{R}$ (with respect to the standard topology). In addition, we will use the definitions

$$r_a^- = \min\{|\check{a}|, r_a\}, \quad r_a^+ = \max\{|\check{a}|, r_a\}, \quad s_{\check{a}} = \mathrm{sign}(\check{a}) = \begin{cases} 1 & \text{if } \check{a} > 0, \\ 0 & \text{if } \check{a} = 0, \\ -1 & \text{if } \check{a} < 0 \end{cases}$$

Table 2: Properties of $[c] = [a] \cdot [b]$ in the case $\underline{a} < 0 < \overline{a}$, $\underline{b} < 0 < \overline{b}$

	$-1 < \frac{\check{a}}{r_a} \le -\frac{\check{b}}{r_b} < 1$	$-1 < -\frac{\check{b}}{r_b} < \frac{\check{a}}{r_a} < 1$
$-1 < \frac{\check{a}}{r_a} \le \frac{\check{b}}{r_b} < 1$	$\check{a} \le 0$ $\lvert\check{a}\rvert r_b \ge r_a \lvert\check{b}\rvert$ $[c] = [\underline{a}\overline{b}, \underline{a}\underline{b}]$ $\check{c} = \check{a}\check{b} - r_a\check{b}$ $r_c = \lvert[a]\rvert r_b$	$\check{b} > 0$ $\lvert\check{a}\rvert r_b < r_a \lvert\check{b}\rvert$ $[c] = [\underline{a}\overline{b}, \overline{a}\overline{b}]$ $\check{c} = \check{a}\check{b} + \check{a} r_b$ $r_c = r_a \lvert[b]\rvert$
$-1 < \frac{\check{b}}{r_b} < \frac{\check{a}}{r_a} < 1$	$\check{b} < 0$ $\lvert\check{a}\rvert r_b < r_a \lvert\check{b}\rvert$ $[c] = [\overline{a}\underline{b}, \underline{a}\underline{b}]$ $\check{c} = \check{a}\check{b} - \check{a} r_b$ $r_c = r_a \lvert[b]\rvert$	$\check{a} > 0$ $\lvert\check{a}\rvert r_b > r_a \lvert\check{b}\rvert$ $[c] = [\overline{a}\underline{b}, \overline{a}\overline{b}]$ $\check{c} = \check{a}\check{b} + r_a\check{b}$ $r_c = \lvert[a]\rvert r_b$

for intervals $[a]$ and an analogous entrywise definition for vectors and matrices. Note that the relations

$$r_a^- = |\,|\overline{a}| - |\underline{a}|\,|/2 \quad \text{and} \quad r_a^+ = (|\underline{a}| + |\overline{a}|)/2 \tag{7}$$

hold. In the form (7) the quantity r_a^+ is defined as size of $[a]$ in [17], [19] and denoted by $s([a])$. We will not use this notation in the rest of the paper.

Based on this notation, we can formulate our first theorem which lists expressions for the midpoint $\check{c}$ and the radius r_c of a product $[c] = [a] \cdot [b]$ depending on whether zero is contained in the interior of $[a] \cap [b]$ or not. As a direct consequence, a general inequality for r_c is added.

Theorem 2.1

Let $[a]$, $[b] \in \mathbb{IR}$. Then the following relations hold for $[c] = [a] \cdot [b]$.

a) If $0 \notin \operatorname{int}([a] \cap [b])$ then

$$\begin{aligned} \check{c} &= \check{a}\check{b} + (s_{\check{a}} r_a^-)(s_{\check{b}} r_b^-), \\ r_c &= r_a^+ r_b + r_a r_b^+. \end{aligned}$$

b) If $0 \in \operatorname{int}([a] \cap [b])$ then

$$\begin{aligned} \check{c} &= \check{a}\check{b} + s_{\check{a}} s_{\check{b}} \min\{|\check{a}|r_b, r_a|\check{b}|\}, \\ r_c &= r_a r_b + \max\{|\check{a}|r_b, r_a|\check{b}|\} = \max\{|[a]|r_b, r_a|[b]|\} \\ &< r_a^+ r_b + r_a r_b^+ = 2 r_a r_b\,. \end{aligned}$$

c) The inequality

$$r_c \le r_a^+ r_b + r_a r_b^+$$

holds without any restrictions.

Theorem 2.1 is easily proved using the Tables 1 and 2. Notice that $0 \notin \mathrm{int}([a])$ implies $|\check{a}| \ge r_a$ while $0 \in \mathrm{int}([a])$ implies $|\check{a}| < r_a$, hence $r_a^- = r_a,\ r_a^+ = |\check{a}|$ in the first case and $r_a^- = |\check{a}|,\ r_a^+ = r_a$ in the second. Similarly, the assumption $0 \in \mathrm{int}([a] \cap [b])$ of Table 2 implies $|\check{a}| < r_a,\ |\check{b}| < r_b$, whence $r_a > 0,\ r_b > 0,\ r_a^- = |\check{a}|,\ r_a^+ = r_a,\ |[a]| < 2r_a,\ |[b]| < 2r_b$. Moreover, adding the product $r_a r_b$ to both sides of an inequality like $|\check{a}|r_b \ge r_a|\check{b}|$ in Table 2 implies $|[a]|r_b \ge r_a|[b]|$, i.e., one gets the corresponding inequality with absolute values instead of midpoints.

By virtue of the second formula in (7) the representation of r_c in Theorem 2.1 a) and the inequalities of r_c in Theorem 2.1 b) and c) can be reformulated in terms of the size of $[a]$ and $[b]$. One can then recognize that they coincide with results in [18] and [19][1].

Notice that the inequality in Theorem 2.1 c) is certainly not worse than the corresponding ones in (4) if $0 \notin \mathrm{int}([a] \cap [b])$. If $0 \in \mathrm{int}([a] \cap [b])$ it is sometimes better, sometimes worse than those in (4) as the examples $[a] = [b] = [-2, 8]$ and $[a] = [b] = [-1, 1]$ show. By no means it is worse than the last inequality in (4). If $0 \in \mathrm{int}([a] \cap [b])$ it coincides with (5).

Additional expressions for $\mathrm{rad}([c])$ involving Ratschek's χ–function are presented in [17].

Theorem 2.1 transfers directly to the diameter of $[c]$ defined as $2r_c$, and parts of it to interval matrices, where here and in the sequel the symbol '$\circ$' denotes the Hadamard product of two real matrices.

Theorem 2.2

Let $[A] \in \mathbb{IR}^{m\times\ell}$, $[B] \in \mathbb{IR}^{\ell\times n}$. Then the following relations hold for $[C] = [A] \cdot [B] \in \mathbb{IR}^{m\times n}$, $s_{\check{A}} = (s_{\check{a}_{ij}}) \in \mathbb{R}^{m\times\ell}$, $s_{\check{B}} = (s_{\check{b}_{ij}}) \in \mathbb{R}^{\ell\times n}$.

a) If $0 \notin \mathrm{int}([a]_{ik} \cap [b]_{kj})$ for all indices i, j, k then

$$\begin{aligned} \check{C} &= \check{A}\check{B} + (s_{\check{A}} \circ r_A^-)(s_{\check{B}} \circ r_B^-), \\ r_C &= r_A^+ r_B + r_A r_B^+. \end{aligned}$$

b) The matrix inequality

$$r_C \le r_A^+ r_B + r_A r_B^+$$

holds without any restrictions, with strict inequality for those entries $(r_C)_{ij}$ for which $0 \in \mathrm{int}([a]_{ik} \cap [b]_{kj})$ occurs for at least one index k.

3 Fixed Point Equation

In this section we present a representation of a solution $[x]^* \in \mathbb{IR}^n$ of the fixed point equation

$$[x] = [A][x] + [b], \tag{8}$$

where $[A] \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$ still have to be specified. This fixed point equation results from the standard iteration

$$[x]^{k+1} = [A][x]^k + [b], \quad k = 0, 1, \ldots \tag{9}$$

[1] The author had access to these two papers when he had already finished his own one.

which is the prototype of iterative processes in order to enclose all solutions of linear systems

$$\tilde{A}x = \tilde{b}, \quad \tilde{A} \in [\hat{A}] \in \mathbb{IR}^{n\times n},\ \tilde{b} \in [\hat{b}] \in \mathbb{IR}^n.$$

Use for instance $[A] = I - [\hat{A}]$, $[b] = [\hat{b}]$ (I identity matrix) in the case of the total step method considered in [1], pp. 143 ff (where $[a]_{ii} \neq 0$ is allowed) or $[A] = I - R[\hat{A}]$, $R \approx (\mathrm{mid}([\hat{A}]))^{-1}$, $[b] = R[\hat{b}]$ in the case of Krawczyk's method (without intersection) studied in [16], pp. 125 ff.

In [15] it was shown that (9) is globally convergent to a unique fixed point $[x]^*$ of (8) if and only if the spectral radius of $|[A]|$ satisfies $\rho(|[A]|) < 1$. In [12] and [13] representations of $[x]^*$ are collected for particular matrices $[A] \in \mathbb{IR}^{n\times n}$ with $\rho(|[A]|) < 1$ and vectors $[b] \in \mathbb{IR}^n$. See also [1], [5], [8], [16]. In [4] and [14] existence and uniqueness of fixed points $[x]^*$ are studied for $\rho(|[A]|) \geq 1$. In this case for interval matrices $[A]$ with irreducible absolute value $|[A]|$ it turned out that no fixed point exists if $r_b \neq 0$. This is certainly not the case for reducible matrices $|[A]|$ as the example

$$[x] = \begin{pmatrix} 2 & 0 \\ 0 & [0, 1/2] \end{pmatrix} [x] + \begin{pmatrix} 0 \\ [0, 1/2] \end{pmatrix}$$

shows with the unique solution $[x]^* = (0, [0, 1])^T$. Since the situation for general reducible matrices $|[A]|$ with $\rho(|[A]|) \geq 1$ is a little bit complicated (cf. Theorem 3.3 in [4]) and since in our next theorem we must restrict to $0 \in [b]$ which together with $r_b = 0$ yields $b = 0$ we only consider the case $\rho(|[A]|) < 1$. Based on the results of Section 2 we prove the following properties on $[x]^*$ in the case $\underline{a}_{ij} \geq 0$ or $\overline{a}_{ij} \leq 0$ (depending on the indices i, j).

Theorem 3.1

Let $[A] \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$, $0 \notin \mathrm{int}([a]_{ij})$ for $i, j = 1, \ldots, n$, $\rho(|[A]|) < 1$. Then the unique solution $[x]^* = \check{x}^* + [-r_{x^*}, r_{x^*}]$ of (8) has the following properties, where $s_{\check{A}} = (s_{\check{a}_{ij}}) \in \mathbb{R}^{n\times n}$ and where $\circ$ denotes the Hadamard product of two matrices.

a) The vector $[x]^*$ is determined by the unique solution of the $2n \times 2n$ system

$$\check{x} = \left(\check{A} + s_{\check{A}} \circ r_A D_x^{(1)}\right) \check{x} + \left(s_{\check{A}} \circ r_A D_x^{(2)} D_{\check{x}}\right) r_x + \check{b}, \tag{10}$$

$$r_x = \left(r_A D_x^{(2)} D_{\check{x}}\right) \check{x} + \left(|\check{A}| + r_A D_x^{(1)}\right) r_x + r_b. \tag{11}$$

Here, $D_x^{(1)}$, $D_x^{(2)}$, $D_{\check{x}} \in \mathbb{R}^{n\times n}$ are diagonal matrices with $(D_x^{(1)})_{ii} = 1$ if $0 \in \mathrm{int}([x]_i)$ and 0 otherwise while $D_x^{(2)} = I - D_x^{(1)}$,
$D_{\check{x}} = \mathrm{diag}(\mathrm{sign}(\check{x}_1), \ldots, \mathrm{sign}(\check{x}_n))$. The system (10), (11) is equivalent to the system (8).

b) If $[y]^*$ is determined by the unique solution of the linear system

$$\check{x} = (\check{A} + s_{\check{A}} \circ r_A)\check{x} + \check{b}, \tag{12}$$

$$r_x = |[A]| r_x + r_b, \tag{13}$$

(which coincides with (10), (11) for $D_x^{(1)} = I$, $D_x^{(2)} = O$, $D_{\check{x}}$ arbitrary) then $[y]^*$ is the solution of the interval equation

$$[y] = \tilde{A}[y] + [b], \quad \text{with } \tilde{A} = \check{A} + s_{\check{A}} \circ r_A \tag{14}$$

and satisfies

$$[y]^* = (I - \tilde{A})^{-1}\check{b} + (I - |[A]|)^{-1} r_b[-1,1] \subseteq [x]^*.$$

In particular, $0 \in [y]^*_i$ *implies* $0 \in [x]^*_i$.

c) *If* $0 \in [b]$ *then* $0 \in [x]^*$, *whence* $[x]^*$ *is determined by (12), (13) and has the representation*

$$[x]^* = (I - (\check{A} + s_{\check{A}} \circ r_A))^{-1}\check{b} + [-1,1](I - |[A]|)^{-1} r_b.$$

Proof:

a) Since $0 \notin \mathrm{int}([a]_{ij})$ for all entries of $[A]$ we obtain $|\check{A}| \ge r_A,\ r_A^- = r_A,\ r_A^+ = |\check{A}|$, whence, by virtue of Theorem 2.2 we get

$$\begin{aligned} \mathrm{mid}([A][x]) &= \check{A}\check{x} + (s_{\check{A}} \circ r_A^-)(s_{\check{x}} \circ r_x^-) \\ &= \check{A}\check{x} + \left(s_{\check{A}} \circ r_A)(s_{\check{x}} \circ (D_x^{(1)}|\check{x}| + D_x^{(2)} r_x)\right), \\ \mathrm{rad}([A][x]) &= r_A r_x^+ + r_A^+ r_x = r_A(D_x^{(1)} r_x + D_x^{(2)}|\check{x}|) + |\check{A}| r_x. \end{aligned}$$

Now the system (10), (11) follows immediately by applying the midpoint/radius operations to (8), whence its equivalence with (8) is obvious. Since (8) is uniquely solvable, the same holds for (10), (11).

b) The first part of the assertion follows analogously to a) with (8), (10), (11) being replaced by (14), (12), (13). Note that $|\tilde{A}| = |\check{A}| + r_A = |[A]|$, where the critical case $(s_{\check{A}})_{ij} = 0$ implies $(r_A)_{ij} = 0$ by virtue of the assumption on $[A]$, whence certainly $|\tilde{a}_{ij}| = |\check{a}_{ij}| = |[a]_{ij}|$.

   In order to prove the subset property we start the iteration (9) with $[x]^0 = [y]^*$. This results in $[x]^0 = [y]^* = \tilde{A}[y]^* + [b] \subseteq [A][y]^* + [b] = [x]^1$ which implies $[x]^k \subseteq [x]^{k+1},\ k = 0, 1, \ldots$, and, finally, $[y]^* \subseteq [x]^*$. The remaining parts of the assertion are obvious.

c) First we recall that $[x]^*$ contains every solution of linear systems $x = Ax + b$ with $A \in [A]$ and $b \in [b]$. Therefore, since $b = 0 \in [b]$ we obtain $0 \in [x]^*$. Hence $|\check{x}^*| \le r_{x^*},\ r_{x^*}^- = |\check{x}^*|,\ r_{x^*}^+ = r_{x^*}$, and (10), (11) reduce to (12), (13) from which the representation of $[x]^*$ follows immediately. □

We illustrate Theorem 3.1 c) by two examples.

Example 3.2

Consider (8) with $[A] = \frac{1}{4}\begin{pmatrix} [0,1] & 1 \\ -1 & [-1,0] \end{pmatrix}$ and $[b] = ([0,2],[-2,8])^T$. Here $\rho(|[A]|) = 1/2 < 1$, hence Theorem 3.1 c) applies. It yields $\check{x}^* = (2,2)^T$ and $r_{x^*} = (4,8)^T$, whence $[x]^* = ([-2,6],[-6,10])^T$. □

Example 3.3

Apply Theorem 3.1 c) formally to $[A] = \begin{pmatrix} [2,4] & [0,2] \\ [-2,0] & [-3,-1] \end{pmatrix}$ and $[b] = ([0,2],[0,4])^T$. This time we get $r_{x^*} = (I - |[A]|)^{-1} r_b = (1,-2)^T$. Since r_{x^*} cannot have negative components, the assumption $\rho(|[A]|) < 1$ must be hurt. In fact we have $\rho(|[A]|) > 1$, and Theorem 5 in [14] shows that the equation (8) can only have a solution $[x]^*$ for the given matrix if $r_b = 0$ which is not the case in our example. □

Usually, $D_x^{(1)}$ and $D_{\check{x}}$ are not known unless more information on the solution $[x]^*$ of (8) is given. Therefore, in order to compute $[x]^*$ via (10), (11) one can only start an iterative process with a trial along the following algorithm, where the updates at the end of the while–loop result from the *computed* vector $[x]$, i.e., if $0 \in \mathrm{int}([x]_j)$ then $(D_x^{(1)})_{ii} = 1$ else $(D_x^{(1)})_{ii} = 0$ for $i = 1, \ldots, n$, etc. . We first list the algorithm assuming a computation without rounding errors ('theoretical algorithm'). Later on we will comment on its practical realization in our tests.

Algorithm

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if ( ρ(|[A]|) ≥ 1 )  or  ( 0 ∈ int([a]_{i0 j0}) for some index pair (i0, j0) )
    then  stop                              % safety condition
initialize k = 0,  x̌ = 0,  D_x^(1) = I,  D_x^(2) = O,  stop = false
while (k < k_max) and (stop = false)
    k = k + 1
    compute D_x̌                             % eventually still to be changed
    solve (10),(11)                         % new candidates for x̌, r_x
    compute [x] = x̌ + [-r_x, r_x]           % used in the next line
    if [x] = [A][x] + [b]  then stop = true  else  update D_x^(1),  D_x^(2)
end while
```

The choice $\check{x} = 0$ for the initialization in the algorithm results from the fact that $D_{\check{x}}$ does not influence the first cycle of the while–loop since $D_x^{(2)}$ is initialized by $D_x^{(2)} = O$. Thus $\check{x}$ can be chosen arbitrarily. The vector $[x]$ resulting from this first cycle is the solution $[y]^*$ in Theorem 3.1 b).

If $[z]^*$ denotes the vector $[x]$ resulting from the second cycle and if

$$0 \notin \mathrm{int}([y]_i^*) \text{ implies both } 0 \notin \mathrm{int}([z]_i^*) \text{ and } \mathrm{sign}(\check{y}_i) = \mathrm{sign}(\check{z}_i) \tag{15}$$

then

$$r_{y^*} \le r_{z^*}, \tag{16}$$

i.e., the radius of the vector $[x]$ in the algorithm increases at the beginning. This follows from

$$r_{y^*} = (|\check{A}| + r_A) r_{y^*} + r_b = |[A]| r_{y^*} + r_b \tag{17}$$

and

$$\begin{aligned} r_{z^*} &= r_A D_{y^*}^{(2)} D_{\check{y}^*} \check{z}^* + (|\check{A}| + r_A D_{y^*}^{(1)}) r_{z^*} + r_b \\ &= r_A D_{y^*}^{(2)} |\check{z}^*| + (|\check{A}| + r_A D_{y^*}^{(1)}) r_{z^*} + r_b \\ &\ge r_A D_{y^*}^{(2)} r_{z^*} + (|\check{A}| + r_A D_{y^*}^{(1)}) r_{z^*} + r_b = |[A]| r_{z^*} + r_b, \end{aligned} \tag{18}$$

where we used the assumption (15) and the fact that only those diagonal entries $(D_{y^*}^{(2)})_{ii}$ differ from zero for which $0 \notin \mathrm{int}([y]_i^*)$, i.e., for which $r_{y_i^*} \le |\check{y}_i^*|$ holds. Subtracting (17) from (18) yields

$$r_{z^*} - r_{y^*} \ge |[A]| (r_{z^*} - r_{y^*}).$$

Represent $(I - |[A]|)^{-1}$ by its Neumann series in order to see that it is nonnegative, whence (16) follows.

In our practical realization of the theoretical algorithm we do not totally account for rounding errors. In fact, we mainly compute with interval arithmetic in INTLAB but partly use ordinary floating point arithmetic in MATLAB, for instance in order to solve the linear system (12), (13). In addition, we replace the stopping criterion $[x] = [A][x] + [b]$ by

$$\max \{ q([x]_i, ([A] \cdot [x] + [b])_i) \mid i = 1, \dots, n \} \leq \varepsilon,$$

where ε is a small positive number and $q([a], [b])$ denotes the Hausdorff distance between two intervals $[a]$, $[b]$ (cf. [1] or [16]). Thus we end up with an interval vector which normally is not the exact fixed point $[x]^*$ but is expected to approximate $[x]^*$ well.

Up to now we do not know whether our theoretical algorithm always leads to $[x]^*$ if we drop the safety bound $k_{\max}$ in the while–loop and let k tend to infinity. In any case there are at most $2^n \cdot 3^n$ linear systems (10), (11) to be solved according to the possible choices of $D_x^{(1)}$ and $D_{\tilde{x}}$. Therefore, the vector $[x]$ in the algorithm must repeat cyclically after some initial phase. If one can show that $[x]$ in the while–loop increases monotonically with respect to '$\subseteq$' with increasing k then $[x] = [x]^*$ is guaranteed after at most $n \cdot 3^n$ cycles, since the final vector $[x]$ of the algorithm satisfies (10), (11) and therefore (8), and since $0 \in \mathrm{int}([x]_i)$ remains true for all cycles to follow such that – aside from the initialization in the first cycle $k = 1$ – the entry $(D_x^{(1)})_{ii} = 1$ does no more change if it appears for the first time in some cycle $k \geq 2$. In each of our more than thousand numerical examples the subset property of $[x]$ was always fulfilled, hence the radius r_x increased throughout the iteration. The practical realization of the algorithm always terminated for $k \leq 4 < k_{\max} = 10$, where the worst case $k = 4$ appeared in less than 5 % of our MATLAB–tests. In these tests we generated the dimension $n \leq 200$ and the entries of $[A]$ and $[b]$ (pseudo–) randomly with diameters up to 100 000. We computed $\rho(|[A]|)$ using MATLAB's function `eig` and rescaled $[A]$ in the case $\rho(|[A]|) \geq 1$. As bound for the stopping criterion, we chose $\varepsilon = \max\{10c, c^2\} \cdot 10^{-15}$, where c is the maximal admissible diameter of the entries of $[A]$, $[b]$ (here, $c = 10^5$). The square c^2 in the definition of ε is due to the product $[A] \cdot [x]$.

4 Powers of Interval Matrices

Let $[A] \in \mathbb{IR}^{n \times n}$ and define the powers of $[A]$ by

$$[A]^0 = I, \quad [A]^{k+1} = [A]^k [A], \; k = 0, 1, \dots .$$

In [9] a necessary and sufficient criterion was derived for $[A]$ to be *convergent*, i.e., $\lim_{k\to\infty} [A]^k = O$. In [2], [3], [10] the so–called *semi–convergence* of $[A]$ was studied, that is the convergence of the sequence $([A]^k)$ to a limit which is not necessarily the zero matrix (cf. [6]). For irreducible interval matrices $[A]$, i.e., those with irreducible absolute value, the subject was handled exhaustively in [10]. For reducible interval matrices, i.e., those for which $|[A]|$ is reducible, sufficient criteria for the semi–convergence as well as for the divergence could be derived in [2], [3], but there remained left some situations for which sufficient criteria could only be studied in specific cases. In order to understand their description we recall the following definitions from [3], [6] and [20].

The (directed) graph $G([A]) = (X, E)$ of an interval matrix $[A] \in \mathbb{IR}^{n\times n}$ is the same as that of $|[A]|$, i.e., $X = \{1, \dots, n\}$ is the set of nodes and $E = \{(i, j) \mid [a]_{ij} \neq 0\}$

is the set of edges. We write $i \to j$ if $(i,j) \in E$. A sequence

$$(i, i_1), (i_1, i_2), \ldots, (i_{l-2}, i_{l-1}), (i_{l-1}, j) \tag{19}$$

of edges is called a path of length l which connects the node i with the node j. We abbreviate (19) by

$$i \to i_1 \to i_2 \to \ldots \to i_{l-2} \to i_{l-1} \to j. \tag{20}$$

A graph $G_B([A]) = (X_B, E_B)$ is called a block graph of a block matrix $[A] = ([A]_{ij})_{i,j=1,\ldots,s} \in \mathbb{IR}^{n\times n}$ with blocks $[A]_{ij}$ if $X_B = \{1, 2, \ldots, s\}$ and $E_B = \{\,(i,j) \mid [A]_{ij} \neq O\,\} \subseteq X_B \times X_B$. We use the notation $i \underset{B}{\to} j$ and $i \underset{B}{\to} i_1 \underset{B}{\to} i_2 \underset{B}{\to} \ldots \underset{B}{\to} i_{m-1} \underset{B}{\to} j$ analogously to (20). In the latter case we say that i is connected with j in $G_B([A])$.

By means of an appropriate permutation matrix P reducible interval matrices $[A]$ can be transformed into the so–called reducible normal form

$$R([A]) = P[A]P^T = \begin{pmatrix} [A]_{11} & [A]_{12} & \ldots & [A]_{1s} \\ O & [A]_{22} & \ldots & [A]_{2s} \\ \vdots & \ddots & \ddots & \vdots \\ O & \ldots & O & [A]_{ss} \end{pmatrix} \tag{21}$$

which is a block form with square diagonal block matrices $[A]_{ii}$ which are either irreducible or 1×1 zero matrices.

We say that the j-th column of $[A] \in \mathbb{IR}^{n\times n}$ has the $*$-property if there exists a power $[A]^k$ containing in the same j-th column at least one non-degenerate interval. It can be seen that this is the case if and only if there is a path $i = i_0 \to i_1 \to i_2 \to \ldots \to i_{m-1} \to i_m = j$ in the graph $G([A])$ which contains two neighboring nodes i_l, i_{l+1} such that $[a]_{i_l,i_{l+1}}$ is non-degenerate. Moreover, it is known for the reducible normal form $R([A])$ that if some column j has the $*$-property then all columns of the same block column of $[A]$ have this property. Therefore, we say that a block column i of $R([A])$ has the $*$–property if some (and therefore all) of its columns has this property. For details see [9].

As usual we call the matrix $A \in \mathbb{R}^{m\times n}$ non-negative if $A \geq O$. By $A > O$ we denote non-negative matrices whose entries all are positive. We call them positive. For vectors we apply these definitions analogously.

According to the Theorem of Perron and Frobenius for irreducible non-negative matrices A the spectral radius $\rho(A)$ is a positive simple eigenvalue of A, and there are two positive eigenvectors x, y such that

$$Ax = \rho(A)x, \quad y^TA = \rho(A)y^T, \quad y^Tx = 1 \tag{22}$$

holds (see, e.g., [20], p. 30, and [7], p. 500).

In matrix theory one often divides non-negative irreducible matrices into two classes: the primitive matrices, which have, by definition, only $\rho(A)$ as (simple) eigenvalue λ with $|\lambda| = \rho(A)$, and the cyclic matrices of index $h > 1$ with the (simple) eigenvalues $\lambda_j = \rho(A)\, e^{\frac{j}{h}\cdot 2\pi \mathrm{i}}$, $j = 0, 1, \ldots, h-1$ as the only ones with absolute value equal to $\rho(A)$. The theory guarantees that other cases cannot occur for such matrices. By means of some appropriate permutation matrix P cyclic matrices A of index h can be brought into the so-called cyclic normal form

$$PAP^T = \begin{pmatrix} O & A_{12} & O & O & \dots & O \\ O & O & A_{23} & O & \dots & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & \dots & O & A_{h-2,h-1} & O \\ O & O & \dots & O & O & A_{h-1,h} \\ A_{h,1} & O & \dots & O & O & O \end{pmatrix} \tag{23}$$

with square diagonal blocks $A_{ii} = O,\ i = 1, \dots, h$, and $A_{i,i+1} \neq O,\ i = 1, \dots, h-1$, $A_{h1} \neq O$.

Note that cyclic matrices $A \in \mathbb{R}^{n\times n}$ with index h and $\rho(A) = 1$ cannot be semi–convergent. But their powers are bounded and the limit

$$A^{h\cdot\infty} := \lim_{k\to\infty} A^{h\cdot k} \tag{24}$$

exists. If $A = (A_{ij})_{i,j=1,\dots,h}$ is in cyclic normal form (23), this limit satisfies

$$A^{h\cdot\infty} = h \cdot \sum_{j=1}^{h} x^{(j)}(y^{(j)})^T \tag{25}$$

as was shown in Lemma 3.8 of [3]. Here $x^{(j)},\ y^{(j)} \in \mathbb{R}^n,\ j = 1, \dots, h$, are those non-negative vectors which have the positive j-th block (corresponding to the block partition (23) of $A = (A_{ij})_{i,j=1,\dots,h}$) in common with x and y from (22), respectively, and which are zero otherwise. The position of the zero entries coincides in $x^{(j)}$ and $y^{(j)}$ for the same index j.

Now we are ready to formulate the class of matrices whose powers could not yet be handled in a satisfactory way.

Assumptions A. Let $[A] = ([A]_{ij})_{i,j=1,\dots,s} \in \mathbb{IR}^{n\times n}$ be in reducible normal form (21) with semi–convergent diagonal blocks $[A]_{ii},\ i = 1, \dots, s$. Assume that there is at least one block column i which has the $*$-property and which fulfills the following conditions:

$$[A]_{ii} \equiv A_{ii}, \quad \rho(|A_{ii}|) = 1, \quad |A_{ii}| \text{ is cyclic of some index } h.$$

Let $|A_{ii}|$ be in cyclic normal form (23). □

These assumptions were named Assumptions B. in [3]. The remark below (23) and the Assumptions A. show that $\rho(A_{ii}) < 1$ must hold for the specific block mentioned there. By means of the results in Section 2 we will prove now the following theorem.

Theorem 4.1
Let

$$[A] = \begin{pmatrix} [Q] & [R] \\ O & A_{ss} \end{pmatrix} = ([A]_{ij})_{i,j=1,\dots,s}$$

satisfy the Assumptions A. with $i = s$. Assume that $[Q]$ is convergent, and let one of the following three properties be fulfilled:

(i) $0 \notin \mathrm{int}([r]_{ij})$ *for all entries of* $[R] = ([r]_{ij})$,

(ii) $[R] = -[R]$,

(iii) $[Q] = -[Q]$.

Define $x,\ x^{(j)},\ j=1,\dots,h$, as in (22) and (25) for $A=|A_{ss}|$. If the equalities

$$|\check{R}|\,x^{(j)}=c,\quad r_R\,x^{(j)}=\tilde{c},\quad j=1,\dots,h, \tag{26}$$

hold for some constant vectors c, $\tilde{c}$ then $[A]^\infty=\lim_{k\to\infty}[A]^k$ exists.

Proof:

We proceed similarly as in the proof of Theorem 4.16 in [3]. To this end let

$$[A]^k=\begin{pmatrix}[Q]^k & [R]^{(k)}\\ O & A_{ss}^k\end{pmatrix}.$$

From [3] it is known that $\lim_{k\to\infty}\operatorname{mid}([A]^k)=O$ even if (i), (ii), or (iii) do not hold. Therefore, by virtue of $\lim_{k\to\infty}[Q]^k=O,\ \lim_{k\to\infty}A_{ss}^k=O$ it only remains to show that $\lim_{k\to\infty}\operatorname{rad}([R]^{(k)})$ exists. Even without (i), (ii), or (iii) we have

$$\begin{aligned}\operatorname{rad}([R]^{(k+1)}) &= \operatorname{rad}([Q]^k|[R])+\operatorname{rad}([R]^{(k)})\,|A_{ss}|=\dots \qquad (27)\\ &= \sum_{j=0}^{k}\operatorname{rad}([Q]^j[R])\,|A_{ss}|^{k-j}\\ &= \sum_{j=0}^{k_0}\operatorname{rad}([Q]^j[R])\,|A_{ss}|^{k-j}+\sum_{j=k_0+1}^{k}\operatorname{rad}([Q]^j[R])\,|A_{ss}|^{k-j}, \qquad (28)\end{aligned}$$

where $k_0<k$ is some positive integer. The last sum in (28) can be made arbitrarily small if one chooses k_0 sufficiently large since the inequality

$$\sum_{j=k_0+1}^{k}\operatorname{rad}([Q]^j[R])\,|A_{ss}|^{k-j}\le\left(\sum_{j=k_0+1}^{\infty}|[Q]^j|\right)|[R]|\,K$$

holds. Here K is some constant bound for the powers of $|A_{ss}|$, $\operatorname{rad}([Q]^j[R])\le|[Q]^j[R]|\le|[Q]^j|\cdot|[R]|$, and $\sum_{j=0}^{\infty}|[Q]^j|$ is convergent according to Lemma 3.9 in [3], or [11]. Let $k=k_0+lh+m$ with $m\in\{0,1,\dots,h-1\}$. Fix k_0,m and let k tend to infinity. With the notation of (25) and the definition

$$F=\lim_{k\to\infty}\left(\sum_{j=0}^{k_0}\operatorname{rad}([Q]^j[R])\,|A_{ss}|^{k-j}\right)$$

we obtain

$$\begin{aligned}F &= \lim_{l\to\infty}\left(\sum_{j=0}^{k_0}\operatorname{rad}([Q]^j[R])\,|A_{ss}|^{lh}|A_{ss}|^{k_0+m-j}\right)\\ &= \sum_{j=0}^{k_0}\operatorname{rad}([Q]^j[R])\,|A_{ss}|^{h\cdot\infty}|A_{ss}|^{k_0+m-j}\\ &= \sum_{j=0}^{k_0}\operatorname{rad}([Q]^j[R])\,h\sum_{i=1}^{h}x^{(i)}(y^{(i)})^T|A_{ss}|^{k_0+m-j}. \qquad (29)\end{aligned}$$

This proves that F is well–defined with the restrictions above.

Now we assume (i) and apply Theorem 2.2 in order to get

$$\mathrm{rad}([Q]^j[R]) = \mathrm{rad}^+([Q]^j)r_R + \mathrm{rad}([Q]^j)r_R^+ = \mathrm{rad}^+([Q]^j)r_R + \mathrm{rad}([Q]^j)|\check{R}|.$$

Hence (26) and (29) imply

$$\begin{aligned} F &= h\sum_{j=0}^{k_0}\{(\mathrm{rad}^+([Q]^j)\tilde{c} + \mathrm{rad}([Q]^j)c)\,(\sum_{i=1}^{h}(y^{(i)})^T)|A_{ss}|^{k_0+m-j}\} \\ &= h\sum_{j=0}^{k_0}\{(\mathrm{rad}^+([Q]^j)\tilde{c} + \mathrm{rad}([Q]^j)c)\,y^T|A_{ss}|^{k_0+m-j}\} \\ &= h\left(\sum_{j=0}^{k_0}\{\mathrm{rad}^+([Q]^j)\tilde{c} + \mathrm{rad}([Q]^j)c\}\right)y^T\ . \end{aligned}$$

Since the last expression is independent of m and since k_0 was large but arbitrary we finally get

$$\lim_{k\to\infty}\mathrm{rad}([R]^{(k)}) = h\left(\sum_{j=0}^{\infty}\{\mathrm{rad}^+([Q]^j)\tilde{c} + \mathrm{rad}([Q]^j)c\}\right)y^T, \tag{30}$$

in particular, $[A]^\infty$ exists. Note that the infinite sums are convergent since $\sum_{j=0}^{\infty}|[Q]^j|$ is a convergent majorant.

Next we assume (ii). This time we have

$$\mathrm{rad}([Q]^j[R]) = |[Q]^j|r_R,$$

and (26), (29) imply

$$\begin{aligned} F &= h\sum_{j=0}^{k_0}\{|[Q]^j|\tilde{c}\,(\sum_{i=1}^{h}(y^{(i)})^T)|A_{ss}|^{k_0+m-j}\} \\ &= h\sum_{j=0}^{k_0}\{|[Q]^j|\tilde{c}\,y^T|A_{ss}|^{k_0+m-j}\} \\ &= h\left(\sum_{j=0}^{k_0}|[Q]^j|\tilde{c}\right)y^T\ . \end{aligned}$$

which again proves convergence.

Finally we assume (iii). From

$$\mathrm{rad}([Q]^j[R]) = \mathrm{rad}([Q]^j)|[R]| = \mathrm{rad}([Q]^j)(|\check{R}| + r_R),$$

we get

$$\begin{aligned} F &= h\sum_{j=0}^{k_0}\{\mathrm{rad}([Q]^j)(c+\tilde{c})\,(\sum_{i=1}^{h}(y^{(i)})^T)|A_{ss}|^{k_0+m-j}\} \\ &= h\sum_{j=0}^{k_0}\{\mathrm{rad}([Q]^j)(c+\tilde{c})\,y^T|A_{ss}|^{k_0+m-j}\} \\ &= h\left(\sum_{j=0}^{k_0}\mathrm{rad}([Q]^j)(c+\tilde{c})\right)y^T \end{aligned}$$

which finishes the proof. □

Example 4.2
Let

$$[A] = \begin{pmatrix} [0,1/2] & [1,5] & [-8,-2] & [2,10] & [a] \\ 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 & 0 \end{pmatrix}.$$

Here $[Q] = [0,1/2]$ and $A_{ss} = A_{22}$ are convergent while $|A_{22}|$ is 2–cyclic and has spectral radius one. Hence 1 is an eigenvalue of $|A_{22}|$, and $x = (1,1,1,1)^T$, $y = (1,1,1,1)^T/4$ are corresponding positive right and left eigenvectors as mentioned in (22). Moreover, $x^{(1)} = (1,1,0,0)^T$, $x^{(2)} = (0,0,1,1)^T$.

If $[a] = [1,3]$ then the assumptions of Theorem 4.1 are fulfilled including (i) with $[R] = ([1,5],[-8,-2],[2,10],[a])$. We get $c = 8$ and $\tilde{c} = 5$. Thus $[A]$ is semi–convergent with limit

$$[A]^\infty = \begin{pmatrix} 0 & [b] & [b] & [b] & [b] \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $[b] = [-23,23]/4$ according to (30). Obviously $[Q]^j = [0,1/2^j]$ implies $|\text{mid}([Q]^j)| = \text{rad}([Q]^j) = \text{rad}^+([Q]^j) = 1/2^{j+1}$ for $j \geq 1$. For $j = 0$ take into account $[Q]^0 = 1$ whence $\text{rad}^+([Q]^0) = 1$ while $\text{rad}([Q]^0) = 0$.

If one replaces $[a] = [1,3]$ by $[a] = [0,2]$ the first equality in (26) is no longer true while the second still holds. Here the powers of $[A]$ show a cyclic behavior with

$$\lim_{k\to\infty} [R]^{(2k)} = ([c],[c],[d],[d]), \qquad \lim_{k\to\infty} [R]^{(2k+1)} = ([d],[d],[c],[c]),$$

where $[c] = [-17/3, 17/3]$, $[d] = [-67/12, 67/12]$.

Replacing $[a] = [1,3]$ by $[a] = [0,4]$ the second equality in (26) is no longer true while the first still holds. Again the powers of $[A]$ show a cyclic behavior with

$$\lim_{k\to\infty} [R]^{(2k)} = ([c],[c],[d],[d]), \qquad \lim_{k\to\infty} [R]^{(2k+1)} = ([d],[d],[c],[c]),$$

where $[c] = [-19/3, 19/3]$, $[d] = [-71/12, 71/12]$. □

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