# The Extrapolated Taylor Model

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### Abstract

The Taylor model [8] is one of the inclusion functions available to compute the range enclosures. It has the property of $(m + 1)^{th}$ convergence order, where, $m$ is the order of the Taylor model used. It computes a high order polynomial approximation to a multivariate Taylor expansion, with a remainder term that rigorously bound the approximation error. The sharper bounds on the enclosures computed using the Taylor model can be obtained either by successively partitioning the domain $x$ using suitable subdivision factors, or by increasing the convergence rate of the Taylor model using higher order Taylor models. However, higher order Taylor forms require higher degrees of the polynomial part, which in turn require more computational effort and more memory. This is the major drawback of increasing the order $m$ of Taylor models for obtaining range enclosures with higher order convergence rates.

In this paper, we attempt to overcome these drawbacks by using a lower order Taylor model, and then using *extrapolation* to accelerate the convergence process of the sequences generated with the *lower order* Taylor model. The effectiveness of all the proposed algorithms is tested on various multivariate examples and compared with the conventional methods. The test results show that the proposed extrapolation-based methods offer considerable speed improvements over the conventional methods.

## 1 Introduction

Taylor forms are higher degree generalizations of the centered forms [10]. The Taylor model [8] is a kind of Taylor form available to compute the range enclosures. It computes a high order polynomial approximation to a multivariate Taylor expansion, with a remainder term that rigorously bounds the approximation error. It has the property of $(m + 1)^{th}$ convergence order, where, $m$ is the order of the polynomial part.

The enclosures computed using the Taylor model can be tightened to the desired accuracy in two ways: first, by successively partitioning the domain $x$ using suitable

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subdivision factors, and second, by increasing the convergence rate of the Taylor model using higher order Taylor models. However, higher order Taylor forms require higher degrees of the polynomial part, which in turn require more computational effort and more memory. This is the major drawback of increasing the order $m$ of Taylor models for obtaining range enclosures with higher order convergence rates.

In this paper, we attempt to overcome these drawbacks by using a lower order Taylor model, and then using *extrapolation* to accelerate the convergence process of the sequences generated with the *lower order* Taylor model. The main features of the proposed method are:

- It uses the lowest order Taylor model - the *first order* Taylor model - to compute the initial range enclosures.

- For the polynomial part of the first order Taylor model, as every variable occurs only once, the *exact range* of the polynomial can be simply obtained using the natural interval evaluation [12]. In contrast, in typical higher order Taylor models, the conventional Taylor model uses the Linear Dominated Bounder (LDB) [7] and the Quadratic Dominated Bounder (QDB) [9] to bound the range of the polynomial part.

- It generates extrapolated sequences that, in theory, converge with one order higher than the corresponding Taylor model. For instance, while the existing first order Taylor model has a quadratic convergence order, the extrapolated first-order Taylor model has a convergence order of three (cubic convergence order). In practice, even higher orders may be obtained with the extrapolated *first order* Taylor model, as seen in the examples considered later in this paper.

- An identical method to the one proposed here can be used for second and higher order Taylor models, to obtain accelerated convergence speeds by atleast one order. This would however require the polynomial range bounding to be done using, for instance, the LDB or QDB methods mentioned above. We therefore choose to describe and demonstrate the proposed method for first order Taylor models, where the polynomial range bounding is simply done using the NIE.

The rest of the paper is organized as follows. In Section 2, we present the notations and preliminaries of interval analysis. In Section 3, we present the definitions and the properties of Taylor forms. In Section 4, we give a brief introduction to the Taylor model. In Section 5, we show the existence of an asymptotic expansion for the sequences obtained using the Taylor model. In Section 6, we present the extrapolation process. In Section 7, we discuss the Brezinski's error control criterion. In Section 8, we present the proposed MAIN algorithm. In Section 9, we compare the performance of the proposed method versus that of the existing Taylor model on several multidimensional examples. Lastly, in Section 10, we draw the conclusions. The numerical results are listed in Appendix A and the brief background of asymptotic expansion is given in Appendix B.

# 2    Interval Analysis

## 2.1    Notation and Definitions

$\mathbb{R}$ denotes the field of real numbers, and $\mathbb{R}^n$ the vector space of column vectors of length $n$ with real entries. A *real interval* $\boldsymbol{x}$ is a closed and bounded set of real

numbers, $\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}$ where $\underline{x}$ and $\overline{x}$ are called the lower and upper endpoints of the interval $\boldsymbol{x}$ The set of all real intervals is denoted by $\mathbb{IR}$. An interval vector $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)^T$ with components $\boldsymbol{x}_k = [\underline{x}_k, \overline{x}_k]$ is also called a *box*. The set of all boxes of dimension $n$ is denoted by $\mathbb{IR}^n$.

The *lower bound* of a box $\boldsymbol{x}$ is $\inf \boldsymbol{x} := \underline{x}$, its *upper bound* is $\sup \boldsymbol{x} := \overline{x}$, and its midpoint is mid $\boldsymbol{x} := \frac{1}{2}(\underline{x} + \overline{x})$. The *width* of an interval is wid $\boldsymbol{x} = \overline{x} - \underline{x} \geq 0$. A set inclusion $\boldsymbol{x} \subseteq \boldsymbol{y}$ is true only when $\underline{y} \leq \underline{x}$ and $\overline{x} \leq \overline{y}$.

Two intervals are equal if their corresponding endpoints are equal. The intersection of two intervals $\boldsymbol{x}$ and $\boldsymbol{y}$ is empty, $\boldsymbol{x} \cap \boldsymbol{y} = \emptyset$, if either $\underline{x} > \overline{y}$ or $\underline{y} > \overline{x}$. Else, the intersection of $\boldsymbol{x}$ and $\boldsymbol{y}$ is again an interval $\boldsymbol{x} \cap \boldsymbol{y} = \left[\max\left(\underline{x}, \underline{y}\right), \min\left(\overline{x}, \overline{y}\right)\right]$.

**Definition 2.1** *Let $f : \boldsymbol{x}\mathbb{R}$ be a function defined for $x \in \boldsymbol{x}$. Then the* range *of $f$ over $\boldsymbol{x}$ is denoted as*

$$range(f, \boldsymbol{x}) = \{f(x) | x \in \boldsymbol{x}\}$$

**Definition 2.2** *(Inclusion function) A function $f(\boldsymbol{x})$ is an* inclusion function *for $f : \boldsymbol{x} \to \mathbb{R}$ if $range(f, \boldsymbol{y}) \subseteq f(\boldsymbol{y})$ for all $\boldsymbol{y} \subseteq \boldsymbol{x}$.*

**Definition 2.3** *(Inclusion monotonicity) An inclusion function $f(\boldsymbol{x})$ is* inclusion monotonic *if $\boldsymbol{x} \subseteq \boldsymbol{y} \Rightarrow f(\boldsymbol{x}) \subseteq f(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}^l$.*

If $f(\boldsymbol{x})$ is an inclusion function of $f$ for $\boldsymbol{x} \in \mathbb{IR}^l$, then we can always write

$$f(\boldsymbol{x}) = range(f, \boldsymbol{x}) + e(\boldsymbol{x})$$

for some interval $e(\boldsymbol{x})$ with $0 \in e(\boldsymbol{x})$. We call wid $e(\boldsymbol{x})$ the *excess width* of $f(\boldsymbol{x})$.

# 3   Taylor Forms

Let $f : \boldsymbol{x} \to \mathbb{R}$ be a function that is $m + 1$ times differentiable on $\boldsymbol{x} \in \mathbb{IR}^l$. Let $x = (x_1, x_2, \ldots, x_l) \in \boldsymbol{x}$. Let

$$|\lambda| = \lambda_1 + \ldots + \lambda_l, \ \lambda! = \lambda_1, \ldots, \lambda_l, \ D^\lambda f(x) = \frac{\partial^{\lambda_1 + \ldots + \lambda_l} f(x)}{\partial x_1^{\lambda_1} \ldots \partial x_l^{\lambda_l}}$$

Then, the Taylor expansion of $f$ of order $m$ is given as

$$f(x) = p(x) + r(x) \tag{1}$$

where $p(x)$ and $r(x)$ are defined as

$$p(x) = f(c) + \sum_{|\lambda|=1}^{m} \frac{D^\lambda f(x)}{\lambda!} (x - c)^\lambda,$$

$$r(x) = \sum_{|\lambda|=m+1} \frac{f^{(\lambda)}(\xi)}{\lambda!} (x - c)^{m+1}$$

with $c = $ mid $\boldsymbol{x}$ and $\xi \in \boldsymbol{x}$. We call $p(x)$ the $m^{th}$ *order polynomial part of $f$* and $r(x)$ the *remainder part* of the Taylor expansion.

Assume an inclusion function of the $(m + 1)$-th derivative of $f$ exists and is bounded over $\boldsymbol{x}$. Then, the corresponding Taylor form of order $m$, denoted by $f_{Taylor}(\boldsymbol{x})$, can be expressed as [6]:

$$f_{Taylor}(\boldsymbol{x}) = \text{range}(p, \boldsymbol{x}) + r(\boldsymbol{x}) \tag{2}$$

where $\text{range}(p, \boldsymbol{x}) = \{p(x) \mid x \in \boldsymbol{x}\}$ is the exact range of the polynomial part $p(\boldsymbol{x})$ on $\boldsymbol{x}$, and $r(\boldsymbol{x})$ is any inclusion function for the range of the remainder part on $\boldsymbol{x}$.

The Taylor form has convergence order $(m + 1)$, as shown in [6]:

**Theorem 3.1** *[6] Assume that the Taylor form of order $m$ is as defined above. Then,*

$$range(f, \boldsymbol{x}) \subseteq f_{Taylor}(\boldsymbol{x})$$

$$wid\ f_{Taylor}(\boldsymbol{x}) - wid\ range(f, \boldsymbol{x}) \leq L\,(wid\ \boldsymbol{x})^{m+1} \tag{3}$$

*where $L$ is some constant independent of $\boldsymbol{x}$.*

# 4   The Taylor Model

The Taylor model (TM) is defined as follows.

**Definition 4.1** *(Taylor model) [8] Let $f : \boldsymbol{x} \to \mathbb{R}$ be a function that is $(m + 1)$ times continuously partially differentiable on an open set containing the domain $\boldsymbol{x}$. Let $c$ be the point in $\boldsymbol{x}$ and $P$ the $m$-th order Taylor polynomial of $f$ around $c$. Let $I$ be an interval such that*

$$f(x) \in P(\boldsymbol{x} - c) + I \text{ for all } x \in \boldsymbol{x}$$

*Then, the pair $(P, I)$ is called an $m$-th order Taylor model of $f$ around $c$. Apparently, $P + I$ encloses $f$ between two hypersurfaces on $\boldsymbol{x}$.*

Similar to the Taylor form, the Taylor model can be expressed in terms of the polynomial part and the remainder part, as in (1). Bounding of the polynomial part $p(\boldsymbol{x})$ is done in the COSY INFINITY package with the Linear Dominated Bounder (LDB) [7] and Quadratic Dominated Bounder (QDB) [9], whereas the remainder part $r(\boldsymbol{x})$ is computed using interval arithmetic [2].

One of the fundamental properties of the Taylor model as described in [8] is that the sharpness of the enclosure of the range of the function computed using the Taylor model scales with the $(m + 1)$-th order in the width of the domain. That is, Theorem 3.1 also applies to the Taylor model, provided sharp bounds on the polynomial part are obtained.

# 5   Asymptotic Expansion for the Taylor Model

Let $f_{TM}(\boldsymbol{x})$ denote the Taylor model of a function $f$ on box $\boldsymbol{x}$. Denote the exact range of the polynomial part as $\text{range}(p, \boldsymbol{x})$, and an enclosure for the remainder term as $R_{TM}(\boldsymbol{x})$. Then, similar to (1) and (2), we can express $f_{TM}(\boldsymbol{x})$ as

$$f_{TM}(\boldsymbol{x}) = \left[\underline{f_{TM}(\boldsymbol{x})},\ \overline{f_{TM}(\boldsymbol{x})}\right] = \text{range}(p, \boldsymbol{x}) + R_{TM}(\boldsymbol{x}) \tag{4}$$

Suppose we uniformly subdivide the box using the subdivision factor $N$, as follows [11]:

$$\boldsymbol{x}_{i,j} \;=\; \left[ \underline{\boldsymbol{x}}_i + (j-1) \; \text{wid} \; \frac{\boldsymbol{x}_i}{N}, \; \underline{\boldsymbol{x}}_i + j \; \text{wid} \; \frac{\boldsymbol{x}_i}{N} \right], \quad j = 1, 2, \ldots, N$$

$$\boldsymbol{x}_i \;=\; \bigcup_{j=1}^{N} \boldsymbol{x}_{i,j},$$

$$\boldsymbol{x} \;=\; \bigcup_{j_i=1}^{N} \left( \boldsymbol{x}_{1,j_1}, \; \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \tag{5}$$

The range enclosure computed with the Taylor model for this partition, denoted by $f_{TM(N)}(\boldsymbol{x})$, is

$$
\begin{aligned}
f_{TM(N)}(\boldsymbol{x}) \;&=\; \bigcup_{j_i=1}^{N} f_{TM} \left( \boldsymbol{x}_{1,j_1}, \; \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \\
&=\; \bigcup_{j_i=1}^{N} \big\{ \text{range} \left( p, \boldsymbol{x}_{1,j_1}, \; \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \\
&\qquad\qquad + R_{TM} \left( \boldsymbol{x}_{1,j_1}, \; \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \big\} \\
&=\; \text{range} \left( f, \boldsymbol{x} \right) + e_{TM(N)}(\boldsymbol{x}) \tag{6}
\end{aligned}
$$

where $e_{TM(N)}(\boldsymbol{x})$ is the excess width associated with the above uniform subdivision of the interval vector $\boldsymbol{x}$, i.e.,

$$e_{TM(N)}(\boldsymbol{x}) := \bigcup_{j_i=1}^{N} e_{TM} \left( \boldsymbol{x}_{1,j_1}, \; \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \tag{7}$$

From Theorem 3.1,

$$\text{wid} \; e_{TM(N)}(\boldsymbol{x}) = \sigma \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{(m+1)} + O \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{(m+2)} \tag{8}$$

where $\sigma$ is some constant independent of $\boldsymbol{x}$. From (6) and (8),

$$\text{wid} \; f_{TM(N)}(\boldsymbol{x}) - \text{wid} \; \text{range} \left( f, \boldsymbol{x} \right) = \sigma \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{(m+1)} + O \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{(m+2)} \tag{9}$$

so,

$$\underline{f_{TM(N)}(\boldsymbol{x})} = \underline{\text{range} \left( f, \boldsymbol{x} \right)} + \sigma_1 \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{m+1} + O \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{m+2} \tag{10}$$

and

$$\overline{f_{TM(N)}(\boldsymbol{x})} = \overline{\text{range} \left( f, \boldsymbol{x} \right)} + \sigma_2 \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{m+1} + O \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{m+2} \tag{11}$$

Comparing the formula (15) from the Appendix with (10) for $s = 1$, we get

$$A(y) \leftarrow \underline{f_{TM(N)}(\boldsymbol{x})}, \; A \leftarrow \underline{\text{range} \left( f, \boldsymbol{x} \right)}, \; \alpha_k \leftarrow \sigma_1, \tag{12}$$

$$y^{\sigma_k} \leftarrow \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{m+1}, \; O \left( y^{\sigma_{s+1}} \right) \leftarrow O \left( \frac{\text{wid} \; \boldsymbol{x}}{N} \right)^{m+2}.$$

Similarly, from (11) and (15), we get

$$A(y) \leftarrow \overline{f_{TM(N)}(\boldsymbol{x})}, \; A \leftarrow \overline{\text{range} \left( f, \boldsymbol{x} \right)}, \; \alpha_k \leftarrow \sigma_2, \tag{13}$$

$$y^{\sigma_k} \leftarrow \left(\frac{\operatorname{wid} \boldsymbol{x}}{N}\right)^{m+1} , \ O\left(y^{\sigma_{s+1}}\right) \leftarrow O\left(\frac{\operatorname{wid} \boldsymbol{x}}{N}\right)^{m+2} .$$

**Remark 5.1** *It can be seen from the expressions for the infimum (12) and for the supremum (13) that there exists an asymptotic expansion for the method based on Taylor model and uniform subdivision. Further, we can accelerate the convergence process by* atleast *one order higher than the original sequence i.e. from $O\left(\frac{1}{N^{m+1}}\right)$ to $O\left(\frac{1}{N^{m+2}}\right)$, where $N$ is the subdivision factor and $m$ is the order of the Taylor model. Thus, for instance, while the first order Taylor model ($m = 1$) is quadratically convergent, the extrapolated first order Taylor model is cubically convergent.*

**Remark 5.2** *From (10) and (11), we see that by using a Taylor model of order $m$ for increasing subdivision factors $N$, two separate sequences can be constructed which converge to two different limits. One is the sequence of lower bounds on the range enclosure converging to the range infimum, and the other is the sequence of upper bounds on the range enclosure converging to the range supremum. In our work, we shall construct these two separate sequences of lower and upper bounds of the range enclosure using Taylor model and extrapolate them to their respective limits (we do not directly apply extrapolation to the sequence of intervals enclosing the range).*

# 6    Extrapolation Process — Sequence Transformation

Extrapolation methods (equivalently, *convergence acceleration methods* or *sequence transformations*) are popularly used for accelerating the convergence process of sequences [4, 15, 17, 16]. Extrapolation methods basically transform the original sequence into another one which converges to the limit more quickly (when the limit exists).

## 6.1    Richardson Extrapolation Process for the Romberg Sequence

Amongst the various extrapolation methods [4, 15], perhaps the most popular and widely used method is the Richardson extrapolation process (REP). Let $K \in \mathbb{N}$, $\rho \geq 2$, and $\{A_j\}$, $j = 0, 1, \ldots, K$ be the sequence to be accelerated. For $\rho = 2$, this becomes the *Romberg sequence* defined as choice A. The REP can be given as:

ALGORITHM REP-GEOMETRIC:

1. Set $A_0^{(j)} = A_j$, $j = 0, 1, \ldots, K$

2. Compute $A_k^{(j)}$ by recursion

$$A_k^{(j)} = A_{k-1}^{(j)} + \frac{\left(A_{k-1}^{(j)} - A_{k-1}^{(j-1)}\right)}{\rho^k - 1}, \quad \left\{ \begin{array}{l} k = 1, 2, \ldots, K, \\ \ \ j = k, \ldots, K. \end{array} \right.$$

which is similar to the Aitken's $\Delta^2$ process for the first extrapolated column $k = 1$.

Let $y = 1/\rho$ in (15). Then, the order of convergence for the extrapolated sequences is given as:

$$A_k^{(j)} - A = O\left(y^{(k+1)}\right)$$

The sequences $\left\{ A_k^{(j)} \right\}$ computed using Step 2 of above Algorithm can be arranged in a two-dimensional array called the Romberg Table, denoted $[A]^k$, cf. Table 1. The arrows in the table show the flow of computations. The $k^{th}$ column of the Romberg Table is referred to as the $(k-1)^{th}$ *extrapolated* column.

Table 1: The Romberg Table, $[A]^K$ with $K = 5$ (i.e., with 5 extrapolated columns)

$$
\begin{array}{llllll}
A_0^{(0)} & & & & & \\
& \searrow & & & & \\
A_0^{(1)} & \to & A_1^{(1)} & & & \\
& \searrow & & \searrow & & \\
A_0^{(2)} & \to & A_1^{(2)} & \to & A_2^{(2)} & \\
& \searrow & & \searrow & & \searrow \\
A_0^{(3)} & \to & A_1^{(3)} & \to & A_2^{(3)} & \to & A_3^{(3)} \\
& \searrow & & \searrow & & \searrow & & \searrow \\
A_0^{(4)} & \to & A_1^{(4)} & \to & A_2^{(4)} & \to & A_3^{(4)} & \to & A_4^{(4)} \\
& \searrow & & \searrow & & \searrow & & \searrow & & \searrow \\
A_0^{(5)} & \to & A_1^{(5)} & \to & A_2^{(5)} & \to & A_3^{(5)} & \to & A_4^{(5)} & \to & A_5^{(5)}
\end{array}
$$

# 7 Error Control in Convergence Acceleration Process

From the user's point of view, it is not sufficient to know that for a given sequence $A_n$, the sequence $\widehat{A}_n$ will converge faster. It would be better to have an estimate of the error $(\widehat{A}_n - S)$ or, still better, to know a sequence of intervals containing the unknown limit $S$ of the sequence $A_n$. Another peculiar characteristic observed in extrapolation is that the extrapolated sequences usually converge to the true limit from either side of $S$. Hence, it is necessary to have an error estimate for the extrapolated sequences.

Amongst the many researchers, Brezinski [3, 4] offered an error control criterion which was found more useful and practical in almost all cases. He proposed to construct the sequence of intervals containing the *true limit* of the sequence under consideration. The same is discussed in detail in the following sub-section.

## 7.1 Brezinski's Error Control Criterion

Let $\{S_n\}$ be the sequence under consideration. Let $S$ be the limit of the sequence $\{S_n\}$. Let $\{T_n\}$ and $\{V_n\}$ be two other sequences obtained by applying REP to $\{S_n\}$. Suppose the sequence $\{T_n\}$ converges faster than $\{S_n\}$, and $\{V_n\}$ converges faster than $\{T_n\}$, both to the same limit $S$. Thus, $\{S_n\}$, $\{T_n\}$, and $\{V_n\}$ can be successive columns of the Romberg Table 1.

Let $b \in \mathbb{R}$ ($b$ is called the *Brezinski's factor*). Define

$$V_n(b) = V_n - b(V_n - T_n), \qquad n \in \mathbb{N}$$

and construct the interval

$$J_n(b) = [\min(V_n(b), V_n(-b)), \max(V_n(b), V_n(-b))] \qquad (14)$$

**Theorem 7.1** *[3] If $T_n - S = o(S_n - S)$ and $V_n - S = o(T_n - S)$, then $\forall b \neq 0, \exists N :$ $\forall n \geq N, S \in J_n(b)$. Moreover $V_n(\pm b) - S = o(S_n - S)$.*

**Remark 7.1** *Brezinski has pointed out a fundamental practical point in [3]: "Under some assumptions, the theorem given above says that for all $n$ greater than $N$, $S$ belongs to some interval. However, such a $N$ is not known without adding supplementary assumptions. Such an $N$ has been attained if the interval at the step $n+1$ is contained in the interval obtained at the step $n$, whatever $n \geq N$ may be. This is a good test for having attained this $N$".*

**Remark 7.2** *As pointed out in Theorem 7.1, the Brezinski's sequence of intervals $V_n(\pm b)$ (so, also $J_n(b)$) can have a rate of convergence faster than $\{S_n\}$, at the most of $\{T_n\}$, but not faster than $\{T_n\}$. Hence, we lose the benefit of extrapolation by one column.*

**Remark 7.3** *The value of Brezinski's factor $b$ in turn affects two factors in constructing the Brezinski's sequence of intervals $J_n(b)$ in (14). One is the width of the sequence of intervals $J_n(b)$, and the other is the value of $N$ referred to in Theorem 7.1. Larger the value of $b$, wider is the interval $J_n(b)$, but smaller is $N$. Whereas, smaller the value of $b$, tighter is the interval $J_n(b)$, but larger is $N$. In general, the suggested value of $b$ is between $0$ and $1$, cf. [3].*

# 8    The Proposed REP Based Algorithm for the Taylor Model

The proposed Algorithm MAIN in Section 8.4 below accepts as inputs the initial box $\boldsymbol{x}$, the first order Taylor model $f_{TM}$, a tolerance parameter $\varepsilon$, and the number $K$ of extrapolated columns in the Romberg table. It generates as output the table $[TM\_Range\_approx]^K$ containing the range enclosing intervals. The sequences of the range enclosing intervals in this table converge (columnwise) increasingly faster than the sequence of range enclosures obtained with the existing Taylor model. All entries of this table are range enclosures of desired accuracy $\varepsilon$, with the bottom rightmost entry possessing the highest accuracy.

We shall now describe the working of Algorithm MAIN. At Step 1, using Algorithm *Sequence_infsup*, we first obtain the range enclosures with the *first order* Taylor model, where the subdivision factor $N$ is increased geometrically. We construct two separate sequences of lower and upper bounds from the obtained range enclosures. Next, at Step 2 we extrapolate these sequences to their respective limits (the limits are the range infimum and range supremum) using the REP. Algorithms *Romberg_inf* and *Romberg_sup* produce the Romberg Tables for the range infimum and supremum. At Step 3, we apply Brezinski's error control criterion to these Romberg Tables, and generate the Brezinski's tables of nested intervals (referred to as **BTNII** and **BTNIS**), using Algorithm *Range_Approx_infsup*. These tables comprise of nested intervals $[C_{nested}]^K$ and $[D_{nested}]^K$, respectively. At Step 4, we check the width of the Brezinski's nested intervals in the **BTNII** Table $[C_{nested}]^K$ generated at Step 3. For

each interval in the table having a width less than the specified tolerance $\varepsilon$, we note down its infimum and form another Romberg-like Table $[C_L]^K$ with these infimums as the corresponding entries. Similarly, at Step 5 we check the width of the Brezinski's nested intervals in the **BTNIS** Table $[D_{nested}]^K$, and do likewise to construct another Romberg-like Table $[D_U]^K$ with supremums as the corresponding entries. At Step 6, from the Tables $[C_L]^K$ and $[D_U]^K$, we construct a Table $[TM\_Range\_approx]^K$ of intervals enclosing the range. If Table $[TM\_Range\_approx]^K$ is empty, then it means that the sequences generated using the extrapolation have not yet reached the specified accuracy, and we therefore need to generate some more elements in the sequence for extrapolation. To obtain a new sequence element, at Step 7, we increase the subdivision factor to $N_{K+1} = 2^{K+1}$, go back to Algorithm *Sequence_infsup* and execute Steps 2b and 2c to generate range enclosure $f_{NIE(N_{K+1})}(\boldsymbol{x})$ for the new partition $N_{K+1}$. With this updated sequence having $K+1$ elements, we again execute Steps 2 through 7 of the current MAIN algorithm. The process continues till we obtain range enclosing interval(s) of desired accuracy $\varepsilon$.

The Table $[TM\_Range\_approx]^K$ may contain several interval entries, as the Brezinski's Tables $[C_{nested}]^K$ and $[D_{nested}]^K$ for the infimum and supremum, respectively, may generate more than one interval satisfying the tolerance parameter. These interval entries enclose the *true* range with increasing accuracy, as we go column-wise in the table. The interval at the bottom-right of this table therefore encloses the range with the highest accuracy (atleast of accuracy $\varepsilon$).

## 8.1 The Sequence Generation

The algorithm *Sequence_infsup* accepts as inputs the initial box $\boldsymbol{x}$, the *first order* Taylor model $f_{TM}$ of the given function, and number $K$ of extrapolated columns in the Romberg table. It returns the sequences of lower bounds $\left\{ A_0^{(j)} \right\}_{j=0}^{K}$ and the sequences of upper bounds $\left\{ B_0^{(j)} \right\}_{j=0}^{K}$, that are generated for a geometrically increasing subdivision factor $N_j = 2^j$, $j = 0, 1, \ldots, K$ which is given in Section 6.1.

**ALGORITHM** Sequence_infsup

$$\left[ \left\{ A_0^{(j)} \right\}_{j=0}^{K}, \ \left\{ B_0^{(j)} \right\}_{j=0}^{K} \right] = \text{Sequence\_infsup}(\boldsymbol{x}, \ f_{TM}, \ K)$$

**Inputs:** The initial box $\boldsymbol{x}$, the first order Taylor model $f_{TM}$, and the number $K$ of extrapolated columns in the Romberg table.

**Outputs:** The sequences of infimums $\left\{ A_0^{(j)} \right\}_{j=0}^{K}$ and supremums $\left\{ B_0^{(j)} \right\}_{j=0}^{K}$.

**BEGIN Algorithm**

1. Using the COSY INFINITY package [1], automatically construct the first order Taylor model $f_{TM}(\boldsymbol{x})$ as

$$\begin{aligned} f_{TM}(\boldsymbol{x}) &= \left[ \underline{f_{TM}(\boldsymbol{x})}, \ \overline{f_{TM}(\boldsymbol{x})} \right] \\ &= \text{range}(p, \boldsymbol{x}) + R_{TM}(\boldsymbol{x}) \end{aligned}$$

where the range$(p, \boldsymbol{x})$ is computed sharply using the NIE of the first order polynomial $p$. Next, set

$$A_0^{(0)} = \underline{f_{TM}(\boldsymbol{x})}, \qquad B_0^{(0)} = \overline{f_{TM}(\boldsymbol{x})}$$

2. FOR   $j = 1, 2, \ldots, K$

   (a) Compute the number of elements in the uniform subdivision partition as $N_j = 2^j$.

   (b) Using $N_j$, uniformly partition the initial box $\boldsymbol{x}$ as per (5)

   (c) For this partition of $\boldsymbol{x}$, obtain the range enclosure $f_{TM(N_j)}(\boldsymbol{x})$ as per (6) (using again COSY INFINITY as in Step 1)

$$
\begin{aligned}
f_{TM(N_j)}(\boldsymbol{x}) &= \left[ \underline{f_{TM(N_j)}(\boldsymbol{x})}, \quad \overline{f_{TM(N_j)}(\boldsymbol{x})} \right] \\
&= \bigcup_{j_i=1}^{N_j} f_{TM}\left( \boldsymbol{x}_{1,j_1}, \ \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \\
&= \bigcup_{j_i=1}^{N} \left\{ \begin{array}{l} \text{range}\left( p, \boldsymbol{x}_{1,j_1}, \ \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \\ + R_{TM}\left( \boldsymbol{x}_{1,j_1}, \ \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right) \end{array} \right\}
\end{aligned}
$$

   where, range$\left( p, \boldsymbol{x}_{1,j_1}, \ \boldsymbol{x}_{2,j_2}, \ldots, \boldsymbol{x}_{l,j_l} \right)$ is computed sharply using the NIE of the first order polynomial $p$.

   (d) Set

$$A_0^{(j)} \longleftarrow \underline{f_{TM(N_j)}(\boldsymbol{x})}, \ B_0^{(j)} \longleftarrow \overline{f_{TM(N_j)}(\boldsymbol{x})}$$

   END FOR

3. RETURN $\left\{ A_0^{(j)} \right\}_{j=0}^{K}$ and $\left\{ B_0^{(j)} \right\}_{j=0}^{K}$.

**END Algorithm**

In practice, we need to process in Step 2c only those boxes that remain after the cutoff test described next.

### 8.1.1   Cut-off Ttest

This is identical to the well-known cutoff test [14] used in interval global optimization algorithms to find the minimum of a function, except that we also apply it here to find the maximum of the function.

Let $f_{TM}$ denote the range enclosure obtained using Taylor model over a box $\boldsymbol{x}$, with $f_{TM} = \left[ \underline{f_{TM}}, \overline{f_{TM}} \right]$. Let $c_{max}$ denote the maximum of $\underline{f_{TM}}$ obtained over all the subboxes generated with a given $N$; similarly, let $c_{min}$ denote the minimum of $\overline{f_{TM}}$ obtained over all these subboxes. Now, if for any box $\boldsymbol{x}$ among these subboxes we have

$$c_{min} < \underline{f_{TM}} < \overline{f_{TM}} < c_{max}$$

then, clearly, the box $\boldsymbol{x}$ is irrelevant in our range finding problem, and so can be discarded.

## 8.2 Algorithm for Romberg Tables for Infimum and Supremum

After constructing the sequences of lower and upper bounds on the range enclosure, we obtain the respective Romberg tables by executing Algorithms *Romberg_inf* and *Romberg_sup* based on the REP:

**ALGORITHM** Romberg_inf

$$[A]^K = \text{Romberg\_inf}\left( \left\{ A_0^{(j)} \right\}_{j=0}^K, \ K \right)$$

**Inputs:** The sequence of lower bounds $\left\{ A_0^{(j)} \right\}_{j=0}^K$, and the number $K$ of columns in the Romberg Table.

**Output:** The Romberg Table $[A]^K$ containing the extrapolated sequences.

**BEGIN Algorithm**

1. Construct the Romberg Table for the range infimum, using the REP described in section 6.1:

$$A_k^{(j)} = A_{k-1}^{(j)} + \frac{A_{k-1}^{(j)} - A_{k-1}^{(j-1)}}{(2^k - 1)}, \ \left\{ \begin{array}{l} k = 1, 2, \dots, K, \\ j = k, \dots, K. \end{array} \right.$$

$$[A]^K = \left\{ A_k^{(j)}, \ k = 0, 1, \dots, K, \ j = k, \dots, K \right\}$$

2. RETURN the Romberg Table $[A]^K$

**END Algorithm**

We can have a similar algorithm *Romberg_sup* with inputs as $\left\{ B_0^{(j)} \right\}_{j=0}^K$ to generate the Romberg Table $[B]^K$ of extrapolated sequences for the range supremum (the description of this algorithm is omitted here).

## 8.3 Algorithm for Generating the Brezinski Tables of Intervals

Based on Theorem 7.1 and Remark 7.1, we can have an algorithm to construct the Brezinski's Table of nested intervals for the infimum (**BTNII**) and Brezinski's Table of nested intervals for the supremum (**BTNIS**).

As mentioned in Section 7.1, the number of nested elements in Brezinski's tables varies with the value of $b$. Our aim is to get as many nested elements as possible. Therefore, in the algorithm, the value of $b$ is varied over a range (chosen here as 0 to 5), and the Brezinski's tables **BTII** and **BTIS** are constructed for each $b$ value, as per section 7.1. From the **BTII** and **BTIS** tables, only the nested intervals are then picked to create respectively two tables $[C_{nested}]^K$ and $[D_{nested}]^K$. These tables contain Brezinski's nested intervals for the range infimum and supremum, and we therefore refer to them as Brezinski's Table of Nested Intervals for Infimum (**BTNII**) and (**BTNIS**) for Supremum. Lastly, from among all the pairs of **BTNII** and **BTNIS** constructed for the various values of $b$, we output that pair having the maximum number of nested elements.

**ALGORITHM** Range_Approx_infsup

$$\left[ \left[C_{nested}\right]^K, \; \left[D_{nested}\right]^K \right] = \text{Range\_Approx\_infsup} \left( \left[A\right]^K, \; \left[B\right]^K \right)$$

**Inputs:** The Romberg Tables $[A]^K$ and $[B]^K$.
**Outputs:** The **BTNII** and **BTNIS** Tables $[C_{nested}]^K$ and $[D_{nested}]^K$ containing Brezinski's nested intervals for the range infimum and supremum, respectively.

**BEGIN Algorithm**

1. FOR b $= 0, 0.1, \ldots, 5$ DO

    (a) From the Romberg Table $[A]^K$, construct Brezinski's table $[C]^K$ of intervals for the infimum (**BTII**) as follows (cf. equation 14):

    $$V_{k+2}^{(j)}(b) = A_{k+2}^{(j)} - b \left( A_{k+2}^{(j)} - A_{k+1}^{(j)} \right), \; \left\{ \begin{array}{l} k = 0, 1, \ldots, K-2, \\ j = k+2, \ldots, K. \end{array} \right.$$

    $$C_{k+2}^{(j)} = \left[ \begin{array}{c} \min \left( V_{k+2}^{(j)}(+b), \; V_{k+2}^{(j)}(-b) \right), \\ \max \left( V_{k+2}^{(j)}(+b), \; V_{k+2}^{(j)}(-b) \right) \end{array} \right], \; \left\{ \begin{array}{l} k = 0, 2, \ldots, K-2, \\ j = k+2, \ldots, K. \end{array} \right.$$

    (b) Similarly, from the Romberg Table $[B]^K$, construct Brezinski's Table $[D]^K$ of intervals for the supremum (**BTIS**).

    (c) Check for nestedness[1] of the intervals in Tables $[C]^K$. Form another Table $[C_{nested}]_b^K$ (referred to as **BTNII**) with these nested intervals as the corresponding entries.

    (d) Do likewise for the nested intervals in $[D]^K$, to form the Table $[D_{nested}]_b^K$ (referred to as **BTNIS**).

    END FOR

2. Find the value of $b$ for which the maximum number of nested intervals exist in the **BTNII** and **BTNIS** Tables $[C_{nested}]_b^K$ and $[D_{nested}]_b^K$. Set $b'$ equal to this $b$ value.

3. Set $[C_{nested}]^K \leftarrow [C_{nested}]_{b'}^K$, $[D_{nested}]^K \leftarrow [D_{nested}]_{b'}^K$

4. RETURN $[C_{nested}]^K$ and $[D_{nested}]^K$.

**END Algorithm**

## 8.4   Proposed MAIN Algorithm

We now present the Algorithm *MAIN* to obtain range enclosures of higher order convergence.

**ALGORITHM** MAIN

---

[1]Nestedness is checked columnwise, for consecutive intervals in each column.

$[TM\_Range\_approx]^K = \text{MAIN}\,(\boldsymbol{x},\ f_{TM},\ K,\ \varepsilon)$

**Inputs:** Initial box $\boldsymbol{x}$, the first order Taylor model $f_{TM}$, a tolerance parameter $\varepsilon$, the number $K$ of extrapolated columns in Romberg table (initially, one may start with, say, $K = 5$).

**Output:** The Table $[TM\_Range\_approx]^K$ containing the range approximating intervals. The bottom right most entry of the table gives the range enclosure of highest accuracy (atleast of accuracy $\varepsilon$).

   **BEGIN Algorithm**

1. Execute the algorithm *Sequence_infsup* given in section 8.1 to generate two separate sequences of infimums and supremums:

$$\left[ \left\{ A_0^{(j)} \right\}_{j=0}^{K},\ \left\{ B_0^{(j)} \right\}_{j=0}^{K} \right] = \text{Sequence\_infsup}\,(\boldsymbol{x}, f_{TM},\ K)$$

2. Execute the algorithm *Romberg_inf* and *Romberg_sup* described in section 8.2 to generate two separate Romberg tables for the lower bound and upper bound of the range enclosures:

$$[A]^K = \ \text{Romberg\_inf}\left( \left\{ A_0^{(j)} \right\}_{j=0}^{K},\ K \right)$$

and

$$[B]^K = \ \text{Romberg\_sup}\ \left( \left\{ B_0^{(j)} \right\}_{j=0}^{K},\ K \right)$$

3. Execute the algorithm *Range_Approx_infsup* described in section 8.3 to compute Brezinski's Tables of nested intervals for the infimum and supremum of the *true* range enclosure. (**BTNII** and **BTNIS**):

$$\left[ [C_{nested}]^K,\ [D_{nested}]^K \right] = \text{Range\_Approx\_infsup}\ \left( [A]^K,\ [B]^K \right)$$

4. Check the width of the Brezinski's nested intervals in the **BTNII** Table $[C_{nested}]^K$: For each interval in the table having a width less than $\varepsilon$, note down its infimum. From another Romberg-like Table $[C_L]^K$ with these infimums as the corresponding entries.

5. Likewise, check the width of the nested intervals in the **BTNIS** Table $[D_{nested}]^K$. For each interval having a width less than $\varepsilon$, note down its supremum. Form another Romberg-like Table $[D_U]^K$ with these supremums as the corresponding entries.

6. Construct intervals whose lower and upper endpoints are the corresponding entries of $[C_L]^K$ and $[D_U]^K$, respectively. Construct a Table of range approximations $[TM\_Range\_approx]^K$ based on these intervals:

7. IF Table $[TM\_Range\_approx]^K$ is non-empty THEN go to the next Step 8, ELSE do the following:

   (a) Set $j = K + 1$.

   (b) Go to algorithm *Sequence_infsup* (cf. section 8.1) and do Steps 2a to 2c to generate range enclosure $f_{TM(N_{K+1})}(\boldsymbol{x})$ for the new partition $N_{K+1} = 2^{K+1}$. Then, execute Step 2d to obtain

$$A_0^{(K+1)} \longleftarrow \underline{f_{TM\left(N_{K+1}\right)}}\,(\boldsymbol{x}),\ B_0^{(K+1)} \longleftarrow \overline{f_{TM\left(N_{K+1}\right)}\,(\boldsymbol{x})}$$

(c) With $\left\{A_0^{(j)}\right\}_{j=0}^{K+1}$ and $\left\{B_0^{(j)}\right\}_{j=0}^{K+1}$, and setting $K = K + 1$, execute Step 2 through Step 7 of the current MAIN algorithm (in practice, additional processing is required only for the newest entries, as the existing entries of the tables remain unchanged).

8. RETURN $[TM\_Range\_approx]^K$.

**END Algorithm**

In the above algorithm, instead of partitioning the box $\boldsymbol{x}$ uniformly with geometric progression i.e., for $N_j = 2^j$, $j = 1, 2, \ldots$, we can also partition it with $N_j$ varying in arithmetic progression [16], i.e., for $N_j = j + 1$, $j = 1, 2, \ldots$ and compute the range enclosures. The order of convergence for the sequences of lower and upper bounds (generated from these enclosures) also can be accelerated using the same MAIN algorithm discussed in section 8.4 with a little modification.

It is found from the numerical experiments that more computational cost and time is involved in the extrapolation of sequences generated using arithmetic progression.

# 9    Numerical Results

We demonstrate the effectiveness of the proposed technique on several multidimensional examples. All computations are carried out on a SUN FIRE 280 R machine with dual 750 MHz UltraSparc III processors and 2 GB RAM for the computations. In all our examples, we construct the approximate range with the tolerance of $\varepsilon = 10^{-10}$.

**Example 9.1** *The 1-dimensional example of Cornelius and Lohner [5]*

$$f(x) = \frac{x^2 - 5x + 9}{x - 5}$$

*Domain:* $[1, 3]$,
*True Range:* $[-1.5, -1.0]$,
*TM Order:* 1.

**Example 9.2** *The 1-dimensional example of Gritton [8]*

$$
\begin{aligned}
f_2(x) \;=\; & (-371.93625 - 791.2465656x + 4044.944143x^2 + 978.1375167x^3 \\
& -16547.8928x^4 + 22140.72827x^5 - 9326.549359x^6 - 3518.536872x^7 \\
& +4782.532296x^8 - 1281.47944x^9 - 283.4435875x^{10} + 202.6270915x^{11} \\
& -16.17913459x^{12} - 8.88303902x^{13} + 1.575580173x^{14} + 0.1245990848x^{15} \\
& -0.03589148622x^{16} - 0.0001951095576x^{17} + 0.0002274682229x^{18}
\end{aligned}
$$

*Domain:* $[0.4, 2.4]$,
*True Range:* $[-216.7375283770099247 \ldots, \; 6.2829979690900121 \ldots]$,
*TM Order:* 1

**Example 9.3** *The 3-dimensional example of Makino and Berz [8]*

$$
\begin{aligned}
f(x, y, z) \;=\; & \frac{4\tan(3y)}{3x + x\sqrt{\frac{6x}{-7(x-8)}}} - 120 - 2x - 7z(1 + 2y) - \sinh\left(0.5 + \frac{6y}{8y + 7}\right) \\
& + \frac{(3y + 13)^2}{3z} - 20z(2z - 5) + \frac{5x\tanh(0.9z)}{\sqrt{5y}} - 20y\sin(3z)
\end{aligned}
$$

$$f_4(x, y, z) = f(x, y, z) + \sum_{j=1}^{10} (f(x, y, z) - f(x, y, z))$$

*Domain:* $[1.75, 2.25] \times [0.75, 1.25]^2$,
*True Range:* $[-10.390145290239591\ldots,\ 17.296315827104715\ldots]$,
*TM Order:* 1

**Example 9.4** *The 6-dimensional example (trigonometric) of Makino and Berz [8]*

$$f_5(x) = \sum_{i=1}^{6} f_i(x)^2, f_i(x) = 6 - \sum_{j=1}^{6} \cos x_j + i(1 - \cos x_i) - \sin x_i$$

*Domain:* $[0.75, 2.75]^6$,
*True Range:* $[22.181076453959111\ldots,\ 1987.092100251149480\ldots]$,
*TM Order:* 1.

**Example 9.5** *The 7-dimensional example (trigonometric) of Makino and Berz [8]*

$$f_5(x) = \sum_{i=1}^{7} f_i(x)^2, f_i(x) = 7 - \sum_{j=1}^{7} \cos x_j + i(1 - \cos x_i) - \sin x_i$$

*Domain:* $[0.75, 2.75]^7$,
*True Range:* $[38.079169919521753\ldots, 3127.9894468042180\ldots]$,
*TM Order:* 1.

The results generated with the proposed algorithm are reported in Tables 2 to 7. Tables 8 and 9 report the total number of boxes processed and the computational time taken to reach the tolerance $\varepsilon = 10^{-10}$, with the proposed extrapolation based method and with the existing Taylor model and uniform subdivision method. The description of the tabulated results is given below:

- For each example, the results are separately listed as 'Table a', 'Table b', and 'Table c'.

- 'Table $a$' shows the Brezinski's table $[C]^K$ of intervals for the infimum (**BTII**), the overestimation of the intervals in $[C]^K$ (which is the width of the Brezinski's interval given in $[C]^K$) and the order of convergence of the same.

- Similarly, 'Table $b$' shows the Brezinski's table $[D]^K$ of intervals for the supremum (**BTIS**), the overestimation of the intervals in $[D]^K$ (which is again the width of the Brezinski's interval given in $[D]^K$) and the order of convergence of the same.

- A star entry in 'Table $a$' and 'Table $b$' signifies that the corresponding $(n+1)^{th}$ interval is not nested in the previous $n^{th}$ interval, and hence does not satisfy the Brezinski's nestedness condition mentioned in Remark 7.1. Similarly, a hash (#) entry in the tables signifies that the computed quotient value is erratic, because the numerical zero (i.e., zero within machine precision) is already reached for the corresponding range overestimations.

- In 'Table $c$' we show the range enclosing intervals constructed from the infimums and supremums of the nested intervals (non-nested intervals are marked with a 'star') given in Tables **BTII** and **BTIS**. The first column titled 'Original' shows

the range computed using the Taylor model and uniform subdivision, whereas columns $k = 1, \ldots, K$, show the range enclosures constructed from Tables $[C]^K$ and $[D]^K$. The rate of convergence for the respective columns are shown in the columns titled '*Ratio*'.

- All the intervals of 'Table c' enclose the *true function range*, but only the entries towards the bottom of the last few columns are found to satisfy the desired accuracy (this can be seen from the overestimation given in parenthesis). In any case, the remaining intervals are also listed to clearly show the convergence order for each extrapolated column. The final output of the algorithm is Table $[TM\_Range\_approx]^K$ consisting of range enclosures of the desired accuracy.

Due to lack of space we have reported the results of only two examples (Examples 9.1 and 9.5) in the appendix. The results of the remaining examples (Example 9.2 to 9.4) are available on http://www.sc.iitb.ac.in/ nataraj/.

## 9.1 Discussion

Based on the results in Tables 2 to 7, we make the following observations.

- As given by theory, the range enclosures computed using the existing *first order* Taylor model converge *quadratically*, as can be seen from the columns titled 'Ratio' in 'Table c' for each example.

- From the existence of an asymptotic expansion shown for the proposed extrapolation-based *first order* Taylor model, we see that using extrapolation it is possible to obtain atleast *cubically* converging sequences i.e., one order higher than the original one. We can see this happening practically from the results given in 'Table c' for each example. The column titled 'Ratio' corresponding to the second extrapolated column ($k = 2$) shows the sequence converging with the order $O\left(\frac{1}{N^3}\right)$.

- Even though the theory says that we can extrapolate the sequence by just one column, we find that extrapolation did work beyond one column, when the *original sequence* converges *smoothly* with the order $O\left(h^2 = \frac{1}{4}\right)$. We observe the same from the ratio entries in the 'a', 'b' and 'c' parts of Tables 2 to 7 that the sequences converge with $O\left(\frac{1}{N^{k+2}}\right)$.

- With the proposed extrapolation based Taylor model technique, the number of subdivisions required to achieve the desired accuracy is significantly less than for the conventional *first order* Taylor model method. From Tables 8 and 9, we see that to achieve the desired accuracy, the former gives the average percentage reduction in the total number of boxes processed as 51.4%, and the average reduction in case of the computational time as 51.3%.

- In all the examples, the intervals in Table $[TM\_Range\_approx]^K$ enclose the true range of the function.

- The expressions given for the REP involves only a few elementary arithmetic operations, such as addition, subtraction, and division. Therefore, little additional work is required to construct the Romberg tables. Similarly, little additional computational cost is needed to construct the Brezinski's Tables of approximating intervals for infimum (**BTII**) and supremum (**BTIS**), since it involves only a few additions and multiplications. Hence, with little additional computational work, we can achieve much better accuracy than with the original sequence.

# 10 Conclusions

In this paper, we proposed the extrapolated Taylor model based on the *first order* Taylor model and extrapolation. We showed that using extrapolation, we can accelerate the convergence rate from *quadratic* to *cubic*. Although theoretically, the proposed method accelerates convergence order by one, in practice we find that even higher convergence orders are usually obtainable.

We also numerically demonstrated the proposed method on univariate and multivariate examples. Summarizing the results of the numerical tests, we see that the proposed technique based on extrapolation works well, and generates range approximating intervals of high accuracy. We see that there is a significant reduction in the computational burden in two ways. Firstly, the number of subdivisions required by the proposed method is significantly smaller than for the existing Taylor model. Secondly, the polynomial part is bounded using the simplest tool of NIE; therefore there is no need to apply LDB, QDB or Bernstein method for polynomial range bounding as in [7, 9, 13]. Moreover, the proposed method can be similarly applied to higher order Taylor models, but then the LDB, QDB or Bernstein method may needed to be used for the polynomial range bounding.

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# A    Numerical Examples

Table 2: 'a' Example 9.1: The Brezinski's Table of Intervals for Infimum (BTII) with maximum error (width of the interval) and convergence order for various subdivision factors with $b = 3$

| $N$ | $k = 1$ | Ratio |
|---|---|---|
| 4 | $[-1.51315641534, -1.48844411376]$  (2.5e–2) | |
| 8 | $[-1.50188332584, -1.49824053907]$  (3.6e–3) | 6.78 |
| 16 | $[-1.50025361337, -1.49975507311]$  (5.0e–4) | 7.30 |
| 32 | $[-1.50003296762, -1.49996760882]$  (6.5e–5) | 7.63 |
| 64 | $[-1.50000420462, -1.49999583253]$  (8.4e–6) | 7.81 |
| 128 | $[-1.50000053096, -1.49999947140]$  (1.1e–6) | 7.90 |
| 256 | $[-1.50000006671, -1.49999993344]$  (1.3e–7) | 7.95 |
| | $O(h^3) = 8$ | |

| $N$ | $k = 2$ | Ratio |
|---|---|---|
| 8 | $[-1.50016037673, -1.49986504389]$  (3.0e–4) | |
| 16 | $[-1.50001202180, -1.49998898612]$  (2.3e–5) | 12.82 |
| 32 | $[-1.50000082889, -1.49999920688]$  (1.6e–6) | 14.20 |
| 64 | $[-1.50000005453, -1.49999994667]$  (1.1e–7) | 15.04 |
| 128 | $[-1.50000000349, -1.49999999654]$  (7.0e–9) | 15.50 |
| 256 | $[-1.50000000022, -1.49999999978]$  (4.4e–10) | 15.75 |
| | $O(h^4) = 16$ | |

| $N$ | $k = 3$ | Ratio |
|---|---|---|
| 16 | $[-1.50000129147, -1.49999892894]$  (2.4e–6) | |
| 32 | $[-1.50000004924, -1.49999995517]$  (9.4e–8) | 25.11 |
| 64 | $[-1.50000000171, -1.49999999837]$  (3.3e–9) | 28.12 |
| 128 | $[-1.50000000006, -1.49999999995]$  (1.1e–10) | 29.93 |
| 256 | $[-1.50000000000, -1.49999999999]$  (3.6e–12) | 30.89 |
| | $O(h^5) = 32$ | |

| $N$ | $k = 4$ | Ratio |
|---|---|---|
| 32 | $[-1.50000000563, -1.49999999535]$ (1.0e–8) | |
| 64 | $[-1.50000000011, -1.49999999990]$ (2.1e–10) | 49.88 |
| 128 | $[-1.50000000000, -1.49999999999]$ (3.7e–12) | 55.98 |
| 256 | $[-1.50000000000, -1.49999999999]$ (6.8e–14) | 54.56# |
| | $O(h^6) = 64$ | |

| $N$ | $k = 5$ | Ratio |
|---|---|---|
| 64 | $[-1.50000000001, -1.49999999999]$ (2.3e–11) | |
| 128 | $[-1.50000000000, -1.49999999999]$ (2.4e–13) | 96.78 |
| 256 | $[-1.50000000000, -1.49999999999]$ (9.1e–15) | 26.37# |
| | $O(h^7) = 128$ | |

| $N$ | $k = 6$ | Ratio |
|---|---|---|
| 128 | $[-1.50000000000, -1.49999999999]$ (2.4e–13) | |
| 256 | $[-1.50000000000, -1.49999999999]$ (9.3e–15) | 3.71# |
| | $O(h^8) = 256$ | |

| $N$ | $k = 7$ | Ratio |
|---|---|---|
| 256 | $[-1.50000000000, -1.49999999999]$ (1.1e–14) | – |
| | $O(h^9) = 512$ | |

**Comments:** In the above Table 2 'a', we observe the following: *i*) Sufficient nestedness in all the columns ($k = 1$ to $k = 6$). *ii*) In the $6^{th}$ extrapolated column ($k = 6$), for the uniform subdivision factor $N = 256$ the reduction in the overestimation for range infimum is 1.40e+7 times (from 1.3e–7 to 9.3e–15). *iii*) The rate of convergence of the range infimum sequences (obtained using first order TM) given in column ($k = 1$) is one order higher $O\left(\frac{1}{N^{m+2}}\right)$ than the theoretical rate of convergence $O\left(\frac{1}{N^{m+1}}\right)$ and is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N^3}\right)$ in column ($k = 1$) to $O\left(\frac{1}{N^9}\right)$ in column ($k = 7$).

Table 3: 'b' Example 9.1: The Brezinski's Table of Intervals for Supremum (BTIS) with maximum error (width of the interval) and convergence order for various subdivision factors with $b = 3$

| $N$ | $k = 1$ | Ratio |
|---|---|---|
| 4 | $[-1.091340220385, -0.910191460055]$ (1.8e–1) | |
| 8 | $[-1.022529166945, -0.977996492562]$ (4.5e–2) | 4.06 |
| 16 | $[-1.005431630769, -0.994639419273]$ (1.1e–2) | 4.12 |
| 32 | $[-1.001330263297, -0.998678743133]$ (2.7e–3) | 4.07 |
| 64 | $[-1.000329050679, -0.999672078543]$ (6.6e–4) | 4.04 |
| 128 | $[-1.000081821655, -0.999918319598]$ (1.6e–4) | 4.02 |
| 256 | $[-1.000020400239, -0.999979617420]$ (4.1e–5) | 4.01 |
| | $O(h^3) = 8$ | |

| $N$ | $k = 2$ | Ratio |
|---|---|---|
| 8 | $[-1.000406547029, -0.999975395201]$ (4.3e–4)* | |
| 16 | $[-1.000100469231, -0.999905636603]$ (1.9e–4)* | |
| 32 | $[-1.00013366588, -0.999986776468]$ (2.7e–5) | 7.32 |
| 64 | $[-1.000001689927, -0.999998313981]$ (3.4e–6) | 7.87 |
| 128 | $[-1.000000211765, -0.999999788349]$ (4.2e–7) | 7.97 |
| 256 | $[-1.000000026486, -0.999999973517]$ (5.3e–8) | 7.99 |
| | $O(h^3) = 8$ | |

| $N$ | $k = 3$ | Ratio |
|---|---|---|
| 16 | $[-1.000028108677, -0.999952941397]$ (7.5e–5) | |
| 32 | $[-1.000000469047, -0.999999276491]$ (1.2e–6) | 63.03 |
| 64 | $[-1.000000011230, -0.999999983400]$ (2.8e–8) | 42.85 |
| 128 | $[-1.000000000310, -0.999999999552]$ (7.6e–10) | 36.69 |
| 256 | $[-1.000000000009, -0.999999999986]$ (2.2e–11) | 34.19 |
| | $O(h^4) = 16$ | |

| $N$ | $k = 4$ | Ratio |
|---|---|---|
| 32 | $[-1.000001078928, -0.999999269689]$ (1.8e–6) | |
| 64 | $[-1.000000013386, -0.999999989280]$ (2.4e–8) | 75.05 |
| 128 | $[-1.000000000268, -0.999999999762]$ (5.0e–10) | 47.62 |
| 256 | $[-1.000000000007, -0.999999999993]$ (1.3e–11) | 38.87 |
| | $O(h^5) = 32$ | |

| $N$ | $k = 5$ | Ratio |
|---|---|---|
| 64 | $[-1.000000006824, -0.999999990350]$ (1.6e–8) | |
| 128 | $[-1.000000000057, -0.999999999932]$ (1.3e–10) | 131.3 |
| 256 | $[-1.000000000007, -0.999999999999]$ (1.4e–12) | 87.82 |
| | $O(h^6) = 64$ | |

| $N$ | $k = 6$ | Ratio |
|---|---|---|
| 128 | $[-1.000000000039, -0.999999999972]$ (6.6e–11) | |
| 256 | $[-1.000000000002, -0.999999999999]$ (2.8e–13) | 235.7 |
| | $O(h^7) = 128$ | |

| $N$ | $k = 7$ | Ratio |
|---|---|---|
| 256 | $[-1.000000000006, -0.999999999999]$ (1.3e–13) | – |
| | $O(h^8) = 256$ | |

**Comments:** In the above Table 3 'b', we observe the following: *i*) Sufficient nestedness in all the columns ($k = 1$ to $k = 7$) except for the first two entries in the column ($k = 2$). *ii*) In the $7^{th}$ extrapolated column ($k = 7$) for the uniform subdivision factor $N = 256$, the reduction in the overestimation for range infimum is 3.15e+8 times (from 4.1e–5 to 1.3e–13). *iii*) The range supremum sequences (obtained using first order TM) given in column ($k = 1$) converge with the same rate as that of the theoretical rate of convergence $O\left(\frac{1}{N^{m+1}}\right)$ and is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N^2}\right)$ in column ($k = 1$) to $O\left(\frac{1}{N^8}\right)$ in column ($k = 7$).

Table 4: 'c' Example 9.1: Range enclosures (and overestimation) with convergence order for $b = 3$

| $N$ | Original | Ratio |
|---|---|---|
| 2 | $[-1.533750000001, -0.879999999999]$ (1.6e–1) | |
| 4 | $[-1.504918981483, -0.970574380165]$ (3.4e–2) | 4.48 |
| 8 | $[-1.500669063581, -0.992840717357]$ (7.8e–3) | 4.39 |
| 16 | $[-1.500087433282, -0.998236823105]$ (1.9e–3) | 4.23 |
| 32 | $[-1.500011181352, -0.999562583188]$ (4.5e–4) | 4.13 |
| 64 | $[-1.500001413921, -0.999891069255]$ (1.1e–4) | 4.07 |
| 128 | $[-1.500000177772, -0.999972820284]$ (2.7e–5) | 4.03 |
| 256 | $[-1.500000022286, -0.999993211693]$ (6.8e–6) | 4.02 |
| | $O(h^2) = 4$ | |

| $N$ | $k = 1$ | Ratio |
|---|---|---|
| 8 | $[-1.501883325839, -0.977996492562]$ (2.4e–2) | |
| 16 | $[-1.500253613367, -0.994639419273]$ (5.6e–3) | 4.25 |
| 32 | $[-1.500032967617, -0.998678743133]$ (1.4e–3) | 4.15 |
| 64 | $[-1.500004204615, -0.999672078543]$ (3.3e–4) | 4.08 |
| 128 | $[-1.500000530957, -0.999918319598]$ (8.2e–5) | 4.04 |
| 256 | $[-1.500000022286, -0.999993211693]$ (6.8e–6) | 4.02 |
| | $O(h^2) = 4$ | |

| $N$ | $k = 2$ | Ratio |
|---|---|---|
| 32 | $[-1.500000828888, -0.999986776468]$ (1.4e–5) | |
| 64 | $[-1.500000054526, -0.999998313981]$ (1.7e–6) | 8.07 |
| 128 | $[-1.500000003498, -0.999999788349]$ (2.2e–7) | 8.09 |
| 256 | $[-1.500000000222, -0.999999973518]$ (2.7e–8) | 8.06 |
| | $O(h^3) = 8$ | |

| $N$ | $k = 3$ | Ratio |
|---|---|---|
| 32 | $[-1.500000049244, -0.999999276491]$ (7.7e–7) | |
| 64 | $[-1.500000001712, -0.999999983400]$ (1.8e–8) | 42.2 |
| 128 | $[-1.5000000000566, -0.999999999552]$ (5.1e–10) | 36.3 |
| 256 | $[-1.500000000002, -0.999999999987]$ (1.5e–11) | 33.9 |
| | $O(h^4) = 16$ | |

| $N$ | $k = 4$ | Ratio |
|---|---|---|
| 64 | $[-1.500000000108, -0.999999989280]$ (1.1e–8) | |
| 128 | $[-1.500000000002, -0.999999999762]$ (2.4e–10) | 45.7 |
| 256 | $[-1.500000000004, -0.999999999994]$ (6.4e–12) | 37.7 |
| | $O(h^5) = 32$ | |

| $N$ | $k = 5$ | Ratio |
|---|---|---|
| 128 | $[-1.500000000001, -0.999999999932]$ (6.9e–11) | |
| 256 | $[-1.500000000001, -0.999999999993]$ (7.7e–13) | 89.6 |
| | $O(h^6) = 64$ | |

| $N$ | $k = 6$ | Ratio |
|-----|---------|-------|
| 256 | $[-1.500000000001, -0.999999999999]$ (1.5e–13) | – |
| | $O(h^7) = 128$ | |

**Comments:** In the above Table 4 'c', for the uniform subdivision factor $N = 256$, the range overestimation (for the sequences obtained using first order Taylor model) in the first column ($Original$) is 6.81e–6. The second extrapolated column ($k = 2$) gives a reduction in the overestimation by 764 times (from 2.04e–5 to 2.67e–8), whereas in the $6^{th}$ extrapolated column ($k = 6$) the reduction is 1.34e+8 times (from 2.04e–5 to 1.52e–13). The rate of convergence of excess width is given in the column ($Ratio$). Here, we see that the excess width obtained with the first order Taylor model (given in column $Ratio$) goes down with $O\left(\frac{1}{N^2}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N^2}\right)$ in column ($k = 1$) to $O\left(\frac{1}{N^7}\right)$ in column ($k = 6$).

Table 5: 'a' Example 9.5: The Brezinski's Table of Intervals for Inimum (BTII) with maximum error (width of the interval) and convergence order for various subdivision factors with $b = 5$

| $N$ | $k = 1$ | Ratio |
|-----|---------|-------|
| 4 | $[-278.26849379651, 408.1274547577]$ (686.2) | |
| 8 | $[-14.645048779726, 94.70429526917]$ (109.1) | 6.28 |
| 16 | $[26.785409506447, 49.87820284031]$ (23.1) | 4.74 |
| 32 | $[35.500422521271, 40.72146570850]$ (5.2) | 4.42 |
| 64 | $[37.465264060335, 38.70102141865]$ (1.2) | 4.23 |
| 128 | $[37.929542194080, 38.22979028988]$ (3.0e–1) | 4.12 |
| 256 | $[38.042243807610, 38.11622005554]$ (7.4e–2) | 4.06 |
| 512 | $[38.069998466246, 38.08835687159]$ (1.8e–2) | 4.03 |
| | $O(h^2) = 4$ | |

| $N$ | $k = 2$ | Ratio |
|-----|---------|-------|
| 8 | $[18.686888471132, 54.258113093786]$ (35.6) | |
| 16 | $[36.876534397946, 39.301987357007]$ (2.4) | 14.67 |
| 32 | $[37.921633779039, 38.237151005457]$ (3.2e–1) | 7.69 |
| 64 | $[38.059312989152, 38.099029239720]$ (3.9e–2) | 7.94 |
| 128 | $[38.076686386975, 38.081652811989]$ (4.9e–3) | 7.997 |
| 256 | $[38.078859665509, 38.079480108951]$ (6.2e–4) | 8.005 |
| 512 | $[38.079131158077, 38.079208676152]$ (7.7e–5) | 8.004 |
| | $O(h^3) = 8$ | |

| $N$ | $k = 3$ | Ratio |
|---|---|---|
| 16 | [37.658124852138, 38.735964915484] (1.1) | |
| 32 | [38.075444998157, 38.082023988309] (6.6e–3) | 163.8 |
| 64 | [38.079082603311, 38.079230121852] (1.5e–4) | 44.60 |
| 128 | [38.079168993501, 38.079170003470] (1.0e–6) | 146.1 |
| 256 | [38.079169810497, 38.079170002329] (1.9e–7) | 5.26 |
| 512 | [38.079169909145, 38.079169929068] (1.9e–8) | 9.63 |
| | $O(h^4) = 16$ | |

| $N$ | $k = 4$ | Ratio |
|---|---|---|
| 32 | [38.055835707960, 38.094000350082] (3.8e–2) | |
| 64 | [38.079101927827, 38.079238014713] (1.4e–4) | 280.4 |
| 128 | [38.079167803529, 38.079172040918] (4.2e–6) | 32.12 |
| 256 | [38.079169853777, 38.079169985367] (1.3e–7) | 32.20 |
| 512 | [38.079169917468, 38.079169921563] (4.1e–9) | 32.13 |
| | $O(h^5) = 32$ | |

| $N$ | $k = 5$ | Ratio |
|---|---|---|
| 64 | [38.078900006680, 38.079574918151] (6.7e–4) | |
| 128 | [38.079169917552, 38.079169925338] (7.7e–9) | 8.6e+4 |
| 256 | [38.079169919319, 38.079169919741] (4.2e–10) | 18.48# |
| 512 | [38.079169919510, 38.079169919519] (9.4e–12) | 45.06# |
| | $O(h^6) = 64$ | |

| $N$ | $k = 6$ | Ratio |
|---|---|---|
| 128 | [38.079166730533, 38.079172048720] (5.3e–6) | |
| 256 | [38.079169919439, 38.079169919590] (1.5e–10) | 3.4e+4 |
| 512 | [38.079169919514, 38.079169919516] (1.7e–12) | 89.86# |
| | $O(h^7) = 128$ | |

| $N$ | $k = 7$ | Ratio |
|---|---|---|
| 256 | [38.079169911203, 38.079169931983] (2.1e–8) | |
| 512 | [38.079169919515, 38.079169919516] (6.3e–13) | 3.2e+4# |
| | $O(h^8) = 256$ | |

| $N$ | $k = 8$ | Ratio |
|---|---|---|
| 512 | [38.079169919490, 38.079169919531] (4.1e–11) | – |
| | $O(h^9) = 512$ | |

**Comments:** In the above Table 5 'a', we observe the following: *i*) Sufficient nestedness in all the columns ($k = 1$ to $k = 8$). *ii*) In the $7^{th}$ extrapolated column ($k = 7$) for the uniform subdivision factor $N = 512$, the reduction in the overestimation for range infimum is 2.9e+10 times (from 1.84e–2 to 6.32e–13). *iii*) The rate of convergence of the range infimum sequences (obtained using first order TM) given in column ($k = 1$) is same as that of the theoretical rate of convergence $O\left(\frac{1}{N^{m+1}}\right)$ and is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N^2}\right)$ in column ($k = 1$) to $O\left(\frac{1}{N^9}\right)$ in column ($k = 8$).

Table 6: 'b' Example 9.5: The Brezinski's Table of Intervals for Supremum (BTIS) with maximum error (width of the interval) and convergence order for various subdivision factors with $b = 5$

| $N$ | $k = 1$ | Ratio |
|---|---|---|
| 4 | [2723.6116728775, 3549.5489453687]  (826.4) | |
| 8 | [3015.7875147606, 3241.7725387735]  (225.9) | 3.65 |
| 16 | [3098.9485661174, 3157.2040806185]  (58.3) | 3.88 |
| 32 | [3120.6220639144, 3135.3774093629]  (14.7) | 3.95 |
| 64 | [3126.1349921112, 3129.8464133026]  (3.7) | 3.975 |
| 128 | [3127.5242985951, 3128.4549055184]  (9.3e–1) | 3.988 |
| 256 | [3127.8729703049, 3128.1059619096]  (2.3e–1) | 3.994 |
| 512 | [3127.9603041396, 3128.0185942819]  (5.8e–2) | 3.997 |
| | $O(h^2) = 4$ | |

| $N$ | $k = 2$ | Ratio |
|---|---|---|
| 8 | [3122.0940704619, 3133.2373309705]  (11.1) | |
| 16 | [3127.4731490258, 3128.4784395960]  (1.0) | 11.08 |
| 32 | [3127.9340908707, 3128.0435004839]  (1.1e–1) | 9.19 |
| 64 | [3127.9829593369, 3127.9958649536]  (1.3e–2) | 8.48 |
| 128 | [3127.5242985951, 3128.4549055184]  (9.3e–1) | 3.988 |
| 256 | [3127.9893495791, 3127.9895437927]  (1.9e–4) | 8.10 |
| 512 | [3127.9894347281, 3127.9894588659]  (2.4e–5) | 8.05 |
| | $O(h^3) = 8$ | |

| $N$ | $k = 3$ | Ratio |
|---|---|---|
| 16 | [3127.8931026856, 3128.0998317488]  (2.1e–1) | |
| 32 | [3127.9853286462, 3127.9939962239]  (8.7e–3) | 23.85 |
| 64 | [3127.9892477538, 3127.9896587325]  (4.1e–4) | 21.09 |
| 128 | [3127.9894361075, 3127.9894578913]  (2.2e–5) | 18.87 |
| 256 | [3127.9894461886, 3127.9894474318]  (1.2e–6) | 17.52 |
| 512 | [3127.9894467673, 3127.9894468415]  (7.4e–8) | 16.78 |
| | $O(h^4) = 16$ | |

| $N$ | $k = 4$ | Ratio |
|---|---|---|
| 32 | [3127.9883453805, 3127.9905404715]  (2.2e–3) | |
| 64 | [3127.9894127543, 3127.9894802356]  (6.8e–5) | 32.53 |
| 128 | [3127.9894457909, 3127.9894478051]  (2.0e–6) | 33.50 |
| 256 | [3127.9894467736, 3127.9894468346]  (6.1e–8) | 32.99 |
| 512 | [3127.9894468032, 3127.9894468051]  (1.9e–9) | 32.16 |
| | $O(h^5) = 32$ | |

| $N$ | $k = 5$ | Ratio |
|---|---|---|
| 64 | [3127.9894462684, 3127.9894468349]  (5.7e–7) | |
| 128 | [3127.9894467788, 3127.9894468269]  (4.8e–8) | 11.77 |
| 256 | [3127.9894468037, 3127.9894468047]  (9.9e–10) | 48.41 |
| 512 | [3127.9894468042, 3127.9894468043]  (4.4e–11) | 22.77 |
| | $O(h^6) = 64$ | |

| $N$ | $k = 6$ | Ratio |
|---|---|---|
| 128 | [3127.9894467949, 3127.9894468147] (1.9e–8) | |
| 256 | [3127.9894468042, 3127.9894468043] (1.4e–10) | 137.40 |
| 512 | [3127.9894468042, 3127.9894468043] (3.5e–11) | 4.11# |
| | $O(h^7) = 128$ | |

| $N$ | $k = 7$ | Ratio |
|---|---|---|
| 256 | [3127.9894468042, 3127.9894468043] (5.8e–11) | |
| 512 | [3127.9894468042, 3127.9894468043] (4.0e–11) | 1.45# |
| | $O(h^8) = 256$ | |

| $N$ | $k = 8$ | Ratio |
|---|---|---|
| 512 | [3127.9894468042, 3127.9894468043] (4.6e–11) | – |
| | $O(h^9) = 512$ | |

**Comments:** In the above Table 6 'b', we observe the following: $i$) Sufficient nestedness in all the columns ($k = 1$ to $k = 8$). $ii$) In the $7^{th}$ extrapolated column ($k = 7$) for the uniform subdivision factor $N = 512$, the reduction in the overestimation for range supremum is 1.46e+9 times (from 5.83e–2 to 4.00e–11). $iii$) The rate of convergence of the range supremum sequences (obtained using first order TM) given in column ($k = 1$) is same as that of the theoretical rate of convergence $O\left(\frac{1}{N^{m+1}}\right)$ and is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N^2}\right)$ in column ($k = 1$) to $O\left(\frac{1}{N^9}\right)$ in column ($k = 8$).

Table 7: 'c' Example 9.5: Range enclosures (and overestimation) with convergence order for $b = 5$

| $N$ | Original | Ratio |
|---|---|---|
| 2 | [−209.62889894109, 3466.9552181196] (586.7) | |
| 4 | [−3.7101143748365, 3219.1740363722] (132.9) | 4.41 |
| 8 | [29.094688839834, 3151.3785291683] (32.4) | 4.11 |
| 16 | [36.022526839995, 3133.9018748180] (7.9) | 4.06 |
| 32 | [37.5888397961666, 3129.475271183] (1.9) | 4.03 |
| 64 | [37.959567003661, 3128.3618448261] (4.9e–1) | 4.02 |
| 128 | [38.049641432402, 3128.0826627491] (1.2e–1) | 4.008 |
| 256 | [38.071834306781, 3128.0127652677] (3.1e–2) | 4.004 |
| 512 | [38.077341828386, 3127.9952782250] (7.7e–3) | 4.002 |
| | $O(h^2) = 4$ | |

| $N$ | $k = 1$ | Ratio |
|---|---|---|
| 8 | $[-14.645048779726, 324.1772538773]$ (166.5) | |
| 16 | $[26.785409506447, 315.7204080618]$ (40.5) | 4.11 |
| 32 | $[35.500422521271, 313.5377409363]$ (9.9) | 4.06 |
| 64 | $[37.465264060335, 312.9846413302]$ (2.5) | 4.03 |
| 128 | $[37.929542194080, 312.8454905518]$ (6.2e–1) | 4.02 |
| 256 | $[38.042243807610, 312.8105961909]$ (1.5e–1) | 4.009 |
| 512 | $[38.069998466246, 312.8018594282]$ (3.8e–2) | 4.004 |
| | $O(h^2) = 4$ | |

| $N$ | $k = 2$ | Ratio |
|---|---|---|
| 16 | $[36.876534397946, 3128.4784395960]$ (1.7) | |
| 32 | $[37.921633779039, 3128.0435004839]$ (2.1e–1) | 7.99 |
| 64 | $[38.059312989151, 3127.9958649536]$ (2.6e–2) | 8.05 |
| 128 | $[38.076686386975, 3127.9902309997]$ (3.3e–3) | 8.04 |
| 256 | $[38.078859665509, 3127.9895437927]$ (4.1e–4) | 8.02 |
| 512 | $[38.079131158076, 3127.9894588659]$ (5.1e–5) | 8.01 |
| | $O(h^3) = 8$ | |

| $N$ | $k = 3$ | Ratio |
|---|---|---|
| 32 | $[38.075444998157, 3127.9939962239]$ (8.3e–3) | |
| 64 | $[38.079082603310, 3127.9896587325]$ (2.9e–4) | 27.65 |
| 128 | $[38.079168993500, 3127.9894578913]$ (1.2e–5) | 24.91 |
| 256 | $[38.079169810497, 3127.9894474318]$ (7.4e–7) | 16.31 |
| 512 | $[38.079169909145, 3127.9894468415]$ (4.8e–8) | 15.47 |
| | $O(h^4) = 16$ | |

| $N$ | $k = 4$ | Ratio |
|---|---|---|
| 64 | $[38.079101927826, 3127.9894802356]$ (1.0e–4) | |
| 128 | $[38.079167803529, 3127.9894478051]$ (3.1e–6) | 32.54 |
| 256 | $[38.079169853777, 3127.9894468346]$ (9.6e–8) | 32.40 |
| 512 | $[38.079169917468, 3127.9894468051]$ (3.0e–9) | 31.77 |
| | $O(h^5) = 32$ | |

| $N$ | $k = 5$ | Ratio |
|---|---|---|
| 128 | $[38.079169917553, 3127.9894468269]$ (2.5e–8) | |
| 256 | $[38.079169919319, 3127.9894468047]$ (7.2e–10) | 34.32 |
| 512 | $[38.079169919510, 3127.9894468043]$ (6.0e–11) | 11.97# |
| | $O(h^6) = 64$ | |

| $N$ | $k = 6$ | Ratio |
|---|---|---|
| 256 | $[38.079169919439, 3127.9894468043]$ (1.9e–10) | |
| 512 | $[38.079169919514, 3127.9894468043]$ (5.2e–11) | 3.56# |
| | $O(h^7) = 128$ | |

| $N$ | $k = 7$ | Ratio |
|---|---|---|
| 512 | $[38.079169919515, 3127.9894468043]$ (5.4e–11) | – |
| | $O(h^8) = 256$ | |

**Comments:** In the above Table 7 'c', for the uniform subdivision factor $N = 512$, the range overestimation (for the sequences obtained using first order Taylor model) in the first column (*Original*) is 7.66e–3. The second extrapolated column ($k = 2$) gives a reduction in the overestimation by 754 times (from 3.83e–2 to 5.08e–5), whereas in the $6^{th}$ extrapolated column ($k = 6$) the reduction is 1.4e+8 times (from 3.83e–2 to 5.18e–11). The rate of convergence of excess width is given in the column (*Ratio*). Here, we see that the excess width obtained with the first order Taylor model (given in column *Ratio*) goes down with $O\left(\frac{1}{N^2}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N^2}\right)$ in column ($k = 1$) to $O\left(\frac{1}{N^8}\right)$ in column ($k = 7$).

Table 8: Comparison of the number of boxes processed to achieve range accuracy of $\varepsilon = 10^{-10}$ with the Taylor model method of Berz [8] and proposed extrapolated Taylor model method

| Example Number | *dim* $l$ | No. of boxes processed | | |
|---|---|---|---|---|
| | | Taylor model method [8] | Proposed extrapolated Taylor model method | % Reduction with the proposed method |
| 9.1 | 1 | 107 | 51 | 52.34 |
| 9.2 | 1 | 595 | 397 | 33.23 |
| 9.3 | 3 | 14059 | 4874 | 65.33 |
| 9.4 | 6 | 103 | 44 | 57.28 |
| 9.5 | 7 | 108 | 44 | 59.26 |

Table 9: Computational time taken (in secs.) to achieve range accuracy of $\varepsilon = 10^{-10}$ with the Taylor model method of Berz [8] and proposed extrapolated Taylor model method

| Example Number | *dim* $l$ | Computational time required (in secs.) | | |
|---|---|---|---|---|
| | | Taylor model method [8] | Proposed extrapolated Taylor model method | % Reduction with the proposed method |
| 9.1 | 1 | 0.39 | 0.18 | 54.57 |
| 9.2 | 1 | 2.74 | 1.85 | 32.50 |
| 9.3 | 3 | 390.38 | 124.05 | 68.22 |
| 9.4 | 6 | 0.69 | 0.32 | 53.09 |
| 9.5 | 7 | 0.75 | 0.40 | 48.11 |

# B   Asymptotic Expansion

A given infinite sequence $\{A_n\}$ can be related to a function $A(y)$ that is known, and hence is computable, for $0 < y \leq b$ with some $b > 0$, the variable $y$ being continuous or discrete. This relation takes the form $A_n = A(y_n)$, $n = 0, 1, \ldots$, for some monotonically decreasing sequence $\{y_n\} \subset (0, b]$ that satisfies $\lim_{n \to \infty} y_n = 0$. Thus, in case $\lim_{y \to 0+} A(y) = A$, $\lim_{n \to \infty} A(n) = A$ as well. Consequently, computing $\lim_{n \to \infty} A(n)$ amounts to computing $\lim_{y \to \infty} 0 + A(y)$ in such a case, and this is precisely what we want to do.

Again, in many cases of interest, the function $A(y)$ may have a well-defined expansion for $y \to 0+$ whose *form* is known. For example $A(y)$ may satisfy for some positive integer $s$

$$A(y) = A + \sum_{k=1}^{s} \alpha_k y^{\sigma_k} + O\left(y^{\sigma_{s+1}}\right) \text{ as } y \to 0+, \tag{15}$$

where $\sigma_k \neq 0$, $k = 1, 2, \ldots, s + 1$, with $\operatorname{Re} \sigma_1 < \operatorname{Re} \sigma_2 < \ldots < \operatorname{Re} \sigma_{s+1}$, and $\alpha_k$ are constants *independent of $y$*. This expansion is also known as *asymptotic expansion*. Obviously, $\operatorname{Re} \sigma_1 > 0$ guarantees that $\lim_{y \to 0+} A(y) = A$. The $\sigma_k$ are assumed to be known, but we need not know the coefficients $\alpha_k$. The existence of an asymptotic expansion for some given sequence is a *necessary theoretical condition* for the applicability of an extrapolation method.

The idea of *extrapolation* is to somehow eliminate the term $y^{\sigma_1}$ from the expansion in (15) and obtain a new approximation $A_1(y)$ to $A$ whose error is $A_1(y) - A = O\left(y^{\sigma_2}\right)$ as $y \to 0+$. This is the procedure followed in the well-known *Richardson extrapolation process*.