Computational Aspects of the Implementation of Disk Inversions*

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Abstract

More than three decades the implementation of iterative methods for the simultaneous inclusion of polynomial zeros in circular complex interval arithmetic is carried out using the exact inversion of disks. Based on theoretical analysis and numerical examples, we show that the centered inversion gives smaller inclusion disks. This surprising result is the consequence of better convergence of the midpoints of produced disks when the centered inversion is employed. Some examples of inclusion methods with the centered and exact inversion, together with numerical results, are given.

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1 Introduction

A great importance of numerical methods for determining polynomial zeros in the theory and practice (for example, in solving many problems of applied and finance mathematics, control theory, signal processing, nonlinear circuits, bioscience, and other disciplines) has led to the development of a great number of zero-finding methods in this field, see, e.g., the books [7], [12], [14]. These numerical methods have become practically applicable together with the rapid growth of digital computers some fifty years ago. However, the computed solution of an algebraic equation is only an approximation to the exact solution due to the errors originating from discretization, truncation and from rounding. This naturally leads to the question *"what is the error in the result?"*

Solving polynomial equations, apart from the work engaged in the procedure applied to improve the approximate result, a considerable amount of work is involved in determining error bounds of the improved approximations to the polynomial zeros. An

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efficient approach that overcomes the aforementioned problem and gives satisfactory results is based on the use of interval arithmetic. Particularly, it turns out that iterative interval methods for the simultaneous inclusion of polynomial zeros, realized in circular complex interval arithmetic, are efficient in the case of complex zeros. These methods produce disks that contain the wanted zeros in each iteration. For this reason, such methods can be regarded as a self-validated numerical tool that provides automatic computation of rigorous error bounds (given by radii of resulting inclusion disks) to the approximate solutions. This very useful (inclusion) property is the main advantage of interval methods.

The aim of this note is to point to the efficient use of a "proper" inversion of a disk in the implementation of a class of simultaneous inclusion root-finding methods based on fixed point relations. Although interval methods started being developed since the 1970's, they were realized using the inversion based on Möbius's transformation of a disk Z by the function $z \mapsto 1/z$, the so-called *exact inversion*, denoted by Z^{-1}. In circular interval arithmetic (arithmetic which deals with disks) this operation is the exact operation since the image Z^{-1} completely coincides with the exact range $\{1/z : z \in Z\}$. Note that only a couple of authors tried to deal with some other type of inversions in order to obtain smaller inclusion disks. The reason probably lies in the fact that the exact inversion gives the smallest disks compared with other inversions so that it seemed that its application is quite reasonable. In this paper we show that the size of inclusion disks depend heavily on some other (extra-arithmetical) features, not only of the employed arithmetic.

2 Circular complex interval arithmetic

We start with a short review of the basic operations in circular interval arithmetics. For more details see the books [2] and [12]. A circular closed region (disk) $Z := \{z : |z-c| \le r\}$ with center $c := \operatorname{mid} Z$ and radius $r := \operatorname{rad} Z$ we will denote in this paper by parametric notation $Z := \{c; r\}$.

Using the Möbius transformation we introduce the exact inversion

$$\{c;r\}^{-1} = \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\} = \{1/z : z \in \{c;r\} \quad (0 \notin \{c;r\}). \tag{1}$$

As we will see in this paper, in some applications it is more convenient to take an inverse disk $\{c;r\}^{I_c} := \{1/c; \rho\}$ whose center is just $1/c$, where c is the center of the original disk $Z = \{c;r\}$. Denote the circumference of such inverse disk with D_c, and let

$$D_e = \left\{ z \ : \ \left| z - \frac{\bar{c}}{|c|^2 - r^2} \right| = \frac{r}{|c|^2 - r^2} \right\}$$

be the circumference of the exact inverse disk $\{c;r\}^{-1}$ given by (1). Since $\{c;r\}^{-1}$ is the exact range, it has to be

$$\{c;r\}^{I_c} := D_c \cup \operatorname{int} D_c \supseteq \{c;r\}^{-1}.$$

According to this and Fig. 1, the radius $\rho = \operatorname{rad} \{c;r\}^{I_c}$ is equal to

$$\begin{aligned} \rho &= \max_{w \in \{c;r\}^{-1}} \left| \frac{1}{c} - w \right| = \max_{\theta \in [0,2\pi)} \left| \frac{1}{c} - \frac{\bar{c} + r\exp(i\theta)}{|c|^2 - r^2} \right| = \frac{r}{|c|} \max_{\theta \in [0,2\pi)} \left| \frac{r + c\exp(i\theta)}{|c|^2 - r^2} \right| \\ &= \frac{r}{|c|} \max_{\alpha \in [0,2\pi)} \left| \frac{r + |c|\exp(i\alpha)}{|c|^2 - r^2} \right| = \frac{r}{|c|(|c| - r)}. \end{aligned}$$

This formula, often used by Rokne, Wu, Ratschek, Rump and others, can be also derived using a general approach to circular centered forms of elementary complex functions, see [11] and [13].

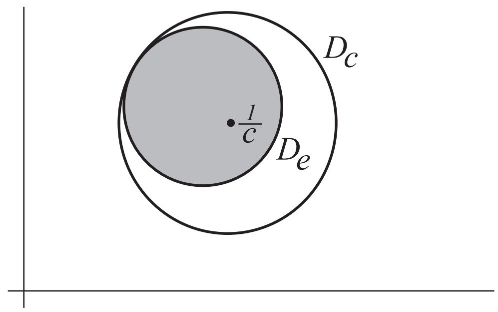

Fig. 1 The exact and centered inversion

It is necessary to check that the disk $\{c; r\}^{I_c} = \{1/c; \rho\}$ completely contains the exact range $\{1/z : z \in \mathbb{C}\} = \{c; r\}^{-1}$, in other words, we have to prove the inequality

$$\Big|\text{mid}\, D_c - \text{mid}\, D_e\Big| \le \text{rad}\, D_c - \text{rad}\, D_e,$$

that is,

$$\left|\frac{1}{c} - \frac{\bar{c}}{|c|^2 - r^2}\right| \le \frac{r}{|c|(|c| - r)} - \frac{r}{|c|^2 - r^2},$$

which reduces to the equality. This means that the circle D_e touches (inside) D_c (see Fig. 1). Therefore, the so-called *centered inversion* is given by

$$\{c; r\}^{I_c} = \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \supset \{c; r\}^{-1} \quad (0 \notin \{c; r\}). \tag{2}$$

Actually, the inversion defined in this way coincides with the Taylor form of inversion derived in [11]. One observes that the centered inversion always produces larger disks than the exact inversion (1).

If $Z_k = \{c_k; r_k\}$ $(k = 1, 2)$, then

$$\begin{aligned}
&Z_1 \pm Z_2 = \{c_1 \pm c_2; r_1 + r_2\},\\
&w \cdot Z = \{w\, \text{mid}\, Z; |w|\text{rad}\, Z\} \quad (w \in \mathbb{C}),\\
&Z_1 \cdot Z_2 = \{c_1c_2; |c_1|r_2 + |c_2|r_1 + r_1r_2\},\\
&Z_1 : Z_2 = Z_1 \cdot \text{INV}\, Z_2 \quad (0 \notin Z_2,\ \text{INV} \in \{()^{-1}, ()^{I_c}\}).
\end{aligned}$$

The addition, subtraction and inversion Z^{-1} are exact operations, that is, $Z_1 * Z_2 = \{z_1 * z_2 \,:\, z_1 \in Z_1, z_2 \in Z_2\}$, $* \in \{+, -, ()^{-1}\}$. We will use the abbreviation INV to denote the inversion of a disk.

3 Simultaneous inclusion of polynomial complex zeros

Let P be a monic polynomial of degree n with simple real or complex zeros $\zeta_1, \ldots, \zeta_n$ and assume that we have found an array of n disks $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ such that

$\zeta_i \in Z_i$ ($i \in \boldsymbol{I}_n := \{1,\dots,n\}$). Denote by $\boldsymbol{\zeta} = (\zeta_1,\dots,\zeta_n)$ and $\boldsymbol{z} = (z_1,\dots,z_n)$ the vectors of the exact zeros of P and the centers of disks, $z_i = \operatorname{mid} Z_i$, and let us represent a fixed point relation in a general form

$$\zeta_i = F_i(\boldsymbol{z},\boldsymbol{\zeta}) \quad (i \in \boldsymbol{I}_n). \tag{3}$$

Let $N(z) = P(z)/P'(z)$ denote Newton's correction. Our study will be carried out in particular cases of the following two examples of the fixed point relations

$$\zeta_i = z_i - \frac{P(z_i)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z_i-\zeta_j)} \quad (i \in \boldsymbol{I}_n), \tag{4}$$

$$\zeta_i = z_i - \frac{1}{\dfrac{1}{N(z_i)} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n}\dfrac{1}{z_i-\zeta_j}} \quad (i \in \boldsymbol{I}_n), \tag{5}$$

which can be easily obtained from the factorization

$$P(z) = \prod_{j=1}^{n}(z-\zeta_j)$$

applying the logarithmic derivative in the case of (5).

Substituting the zeros on the right side of (3) by their inclusion disks and using the inclusion property, we obtain the inclusion

$$\zeta_i \in \hat{Z}_i := F_i(\boldsymbol{z},\boldsymbol{Z}) \quad (i \in \boldsymbol{I}_n). \tag{6}$$

Under suitable initial conditions (taking into account the size of initial disks and their distribution), the set $\hat{Z}_i$ is a new contracted disk containing the zero ζ_i. In general, we will use the symbol $\widehat{\ }$ to denote quantities in the subsequent iteration.

Setting $(Z_1,\dots,Z_n) =: (Z_1^{(0)},\dots,Z_n^{(0)})$, from (6) we can construct the following iterative methods for the simultaneous inclusion of all simple zeros of the polynomial P :

$$Z_i^{(m+1)} = F_i(\boldsymbol{z}^{(m)},\boldsymbol{Z}^{(m)}) \quad (m = 0,1,\dots;\ i \in \boldsymbol{I}_n). \tag{7}$$

Let $Z_i^{(m)} := \{c_i^{(m)}; r_i^{(m)}\}$ be inclusion disks produced by the iterative method (7) such that $r_i^{(m)} \to 0$ ($i \in \boldsymbol{I}_n$) when $m \to \infty$, and let $r^{(m)} = \max\limits_{1\le i\le n} r_i^{(m)}$. If there exists a real number k and a nonzero constant γ such that

$$\frac{r^{(m+1)}}{\left(r^{(m)}\right)^k} \to \gamma,$$

then k is called the *order of convergence* of the iterative interval method (7). In practice, for small enough $r^{(m)}$ it is sufficient to show that $r^{(m+1)} = O\Big(\big(r^{(m)}\big)^k\Big)$, where O is the Landau symbol. This definition of the order of convergence is satisfactory for the class of interval methods considered in this paper. A more general definition of the convergence speed, expressed by the so-called *R-order*, can be found in [2].

Having in mind (6) and (7), we start from the fixed point relations (4) and (5) and construct the following particular methods for the simultaneous inclusion of all simple zeros of the polynomial P :

Weierstrass-like method [12], [15], the convergence order 2:

$$\hat{Z}_i = z_i - P(z_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{INV}(z_i - Z_j) \quad (i \in \boldsymbol{I}_n). \tag{8}$$

Gargantini-Henrici's method [5], the convergence order 3:

$$\hat{Z}_i = z_i - \mathrm{INV}_2\left(1/N(z_i) - \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{INV}_1\Big(z_i - Z_j)\Big)\right) \quad (i \in \boldsymbol{I}_n). \tag{9}$$

Here we assume that $\mathrm{INV}, \mathrm{INV}_1, \mathrm{INV}_2 \in \{()^{-1}, ()^{I_c}\}$. The subscript indices "1" and "2" in (9) point to the order of application of inversions. The interval method (9) (with the exact inversions) was proposed in [5] so that it is often refereed to as Gargantini-Henrici's method. Let us note that original methods (8) and (9) presented in the papers cited above, as many other similar methods based on fixed point relations, used only the exact inversion, that is, $\mathrm{INV} = \mathrm{INV}_1 = \mathrm{INV}_2 = ()^{-1}$.

We could consider some other fixed point relations and corresponding interval methods, but the conclusions are entirely the same as in the case of interval methods (8) and (9).

Remark 1 The main advantage of interval methods (8) and (9) is the inclusion property; namely, in each iteration these interval methods produce the array of disks $Z_1^{(m)}, \ldots, Z_n^{(m)}$ such that $\zeta_i \in Z_i^{(m)}$ $(m = 0, 1, 2, \ldots;\ i \in \boldsymbol{I}_n)$. In this way the automatic control of error is provided since $|\mathrm{mid}\, Z_i^{(m)} - \zeta_i| \leq \mathrm{rad}\, Z_i^{(m)}$, taking the midpoints of disks to be approximations to the zeros.

From the convergence analysis of interval methods (8) and (9) we can find that

$$\hat{r}_i = \mathrm{rad}\, \hat{Z}_i = O(|P(z_i)|r), \quad r = \max_{1 \leq i \leq n} r_i \tag{10}$$

for the Weierstrass-like method (8) and

$$\hat{r}_i = \mathrm{rad}\, \hat{Z}_i = O(|P(z_i)|^2 r) \tag{11}$$

for the Gargantini-Henrici method (9), see the book [12] for details. Obviously, since

$$|P(z_i)| = |z_i - \zeta_i| \prod_{j \neq i} |z_i - \zeta_j| = O(r),$$

from (10) and (11) we conclude that the convergence order of the methods (8) and (9) is two and three, respectively.

Besides the study of simultaneous inclusion methods, let us consider iterative methods for the simultaneous determination of complex zeros realized in ordinary complex arithmetic. Without loss of generality, we will restrict out attention to the methods corresponding to the inclusion methods (8) and (9). If we start from the fixed point relations (4) and (5) and substitute the exact zeros $\zeta_1, \ldots, \zeta_n$ by their ("point") approximations $z_1, \ldots, z_n$, then we obtain the following two methods for the simultaneous approximation of polynomial zeros:

Weierstrass-Durand-Kerner method [6], the convergence order 2:

$$\hat{z}_i = z_i - \frac{P(z_i)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} (z_i - z_j)} \quad (i \in \boldsymbol{I}_n); \tag{12}$$

Ehrlich-Aberth's method [1], [4], the convergence order 3:

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{N(z_i)} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \dfrac{1}{z_i - z_j}} \quad (i \in \boldsymbol{I}_n). \tag{13}$$

For more details on these methods see the recent book [7].

From (10) and (11) we infer that the convergence of radii strongly depends on the centers of inclusion disks; when the centers are closer to the zeros, the convergence of radii is faster. Let us examine now the convergence behavior of the centers of disks $\hat{Z}_i$ produced by the inclusion methods (8) and (9) distinguishing two cases: (i) the exact inversion (1) is applied; (ii) the centered inversion (2) is applied.

This behavior can be simply examined considering the resulting disks obtained by the inversions (1) and (2). Starting from (1) we find (assuming that r is sufficiently small and $|c| > r$).

$$\begin{aligned} \{c; r\}^{-1} &= \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\} \\ &= \left\{ \frac{1}{c} + \underbrace{\frac{(r/|c|)^2}{c} \left(1 + (r/|c|)^2 - (r/|c|)^4 + \cdots \right)}; \frac{r}{\left| |c|^2 - r^2 \right|} \right\}. \end{aligned} \tag{14}$$

Bearing in mind (2) and the mapping function $z \mapsto 1/z$, we note that the centered inversion preserves the property of centering, while the exact inversion does not. This means that the centers of disks produced by the inclusion methods (8) and (9) coincide with the iterative methods (12) and (13), respectively, when the centered inversion is applied. On the other hand, applying the exact inversion, we observe by comparing (1) and (14) that the centers of disks obtained by the methods (8) and (9) are "removed" (for underlined part) and will not coincide with (12) and (13). Therefore, the convergence of centers will be spoiled when the exact inversion is employed. Consequently, taking into account the estimation (10) and (11), the inclusion methods (8) and (9) show faster convergence when the centered inversion is applied. At first sight this is a paradox since the centered inversion always produces larger disks than the centered inversion (see (2)). However, we have shown that the convergence speed of interval methods depends not only of the disk size but also of the convergence behavior of centers of disks.

According to the presented analysis it follows that the better convergence of centers of resulting inclusion disks provides the faster convergence. Hence, the following natural question could be posed: can the improvement of convergence of centers accelerate the convergence speed of interval methods? The answer is *yes*, which was demonstrated for the first time in [3] where the following inclusion method of Gargantini-Henrici's type with Newton's corrections $N(z_j) = P(z_j)/P'(z_j)$ was stated:

$$\hat{Z}_i = z_i - \mathrm{INV}_2\left(1/N(z_i) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{INV}_1\left(z_i - Z_j + N(z_j) \right) \right) \quad (i \in \boldsymbol{I}_n). \tag{15}$$

We note that the centers $\text{mid}\,(Z_j - N(z_j)) = z_j - N(z_j)$ behave as the approximations obtained by Newton's method, that eventually provides the acceleration of convergence of the sequences of radii $\left\{\text{rad}\, Z_i^{(m)}\right\}$. The following statement was proved in [3]:

Theorem 1 *If initial inclusion disks $Z_1^{(0)}, \ldots, Z_n^{(0)}$ are reasonable small, then the R-order of convergence $O_R(15)$ of the interval method* (15) *is given by*

$$O_R(15) \geq \begin{cases} (3+\sqrt{17})/2 \cong 3.562 & \textit{if } \text{INV}_1 = ()^{-1}, \\ 4 & \textit{if } \text{INV}_1 = ()^{I_c}. \end{cases}$$

In essence, the increase of the convergence rate is the result of the accelerated convergence of the centers of the disks $\hat{Z}_i$ calculated by (15). In particular, when the centered inversion is applied in (15), then the sequences $\left\{\text{mid}\, \hat{Z}_i^{(m)}\right\}$ behave as the sequences of approximations defined by the fourth-order Nourein's method [9]

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{N(z_i)} - \displaystyle\sum_{\substack{j \neq i \\ j=1}}^{n} \frac{1}{z_i - z_j + N(z_j)}} \quad (i \in \boldsymbol{I}_n).$$

In general, it is desirable to accelerate the convergence of centers of disks appearing in iterative interval formulas. The application of the centered inversion moves the center of the improved disk $\hat{Z}_i$ very close to the zero ζ_i.

Further improvement of the convergence rate of the interval methods (9) and (15) can be achieved by applying more rapid method instead of Newton's method. The following iterative method for solving algebraic equation $P(z) = 0$, proposed by Ostrowski [10], is convenient in the acceleration of convergence of interval methods:

$$\hat{z} = z - N(z)\frac{P(z - N(z)) - P(z)}{2P(z - N(z)) - P(z)} = z - g(z), \quad N(z) = \frac{P(z)}{P'(z)}. \tag{16}$$

The order of convergence of the Ostrowski method (16) is four. The term $g(z)$ in (16) is called Ostrowski's correction. Let us note that the iterative method (16) can be also applied to arbitrary (real or complex) function.

In a similar way as in the construction of the interval method (15), we can derive the Gargantini-Henrici method with Ostrowski's corrections $g(z_j)$ in circular complex arithmetic:

$$\widehat{Z}_i = z_i - \text{INV}_2\left(1/u(z_i) - \sum_{\substack{j=1 \\ j \neq i}}^{n} \text{INV}_1\Big(z_i - Z_j + g(z_j)\Big)\right) \quad (i \in \boldsymbol{I}_n). \tag{17}$$

Applying circular arithmetic operations, it can be proved that the choice $\text{INV}_1 = \text{INV}_2 = ()^{I_c}$ in (17) produces the disks $\hat{Z}_i$ whose centers behave as the approximations obtained by the simultaneous method

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{N(z_i)} - \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{z_i - z_j + g(z_j)}} \quad (i \in \boldsymbol{I}_n). \tag{18}$$

The convergence order of the method (18) is six so that we can expect very fast convergence of the interval method (17) since Ostrowski's approximation $z_j - g(z_j)$ is very close to the exact zero ζ_j. More precisely, we can state the following result:

Theorem 2 *Let* $(Z_1^{(0)}, \ldots, Z_n^{(0)}) := (Z_1, \ldots, Z_n)$ *be an array of disjoint initial disks containing the zeros* $\zeta_1, \ldots, \zeta_n$ *of* P. *If the midpoints of initial disks are close enough to the zeros of* P, *then the R-order of convergence of the iterative method* (17) *is given by*

$$O_R(17) \geq \begin{cases} (3+\sqrt{17})/2 \cong 3.562 & \text{if } \ \mathrm{INV}_1 = ()^{-1}, \\ 6 & \text{if } \ \mathrm{INV}_1 = ()^{I_c}. \end{cases}$$

We note that the R-order of the interval method (13) is not increased when $\mathrm{INV}_1 = ()^{-1}$ although the corrections of the method of higher order is applied. Numerical examples confirms this fact, see, e.g., Table 2. Detailed theoretical explanation of this phenomenon is given in [8] and [12].

4 Numerical examples

The presented analysis will be illustrated by the following examples.

Example 1 We have applied Weierstrass-like method (8) with $\mathrm{INV} = ()^{-1}$ and $\mathrm{INV} = ()^{I_c}$ for the inclusion of zeros of the polynomial

$$P(z) = z^7 + z^5 - 10z^4 - z^3 - z + 10.$$

We have started with the initial disks $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$ that contain the exact zeros $2, \pm 1, \pm i, -1 \pm 2i$. The maximal radii of the obtained disks are given in Table 1, where $A(-q)$ means $A \times 10^{-q}$.

Methods	$\max r_i^{(1)}$	$\max r_i^{(2)}$	$\max r_i^{(3)}$	$\max r_i^{(4)}$	$\max r_i^{(5)}$	$\max r_i^{(6)}$
(8) with $()^{-1}$	0.39	0.22	1.77(−2)	3.21(−5)	6.18(−11)	1.17(−22)
(8) with $()^{I_c}$	0.44	0.28	7.84(−3)	1.17(−6)	8.84(−15)	3.77(−31)

Table 1: Weierstrass-like method (8) with exact and centered inversion

Example 2 We have applied two versions of the Gargantini-Henrici method (9) for the inclusion of zeros of the following polynomial of the 25th degree with initial disks with the same radii $r_i^{(0)} = 0.3$,

$$\begin{aligned} P(z) \;=\; & (z-4)(z^4-1)(z^4-81)(z^2-8z+17)(z^2-6z+13)(z^2-4z+5)(z^2-2z+5) \\ & \times (z^2-4z+13)(z^2+2z+5)(z^2+4z+5)(z^2+4z+13). \end{aligned}$$

The maximal radii are displayed in Table 2.

Methods	$\max r_i^{(1)}$	$\max r_i^{(2)}$	$\max r_i^{(3)}$	$\max r_i^{(4)}$
(9) with $()^{-1}$	9.27(−2)	6.96(−4)	1.99(−12)	1.42(−39)
(9) with $()^{I_c}$	1.70(−1)	1.08(−4)	1.18(−15)	8.99(−50)

Table 2: Gargantini-Henrici method (9) with exact and centered inversion

Example 3 Apart from the Gargantini-Henrici method (9), we have also applied the accelerated methods (15) and (17) with $\mathrm{INV}_1 = \mathrm{INV}_2 = ()^{-1}$ and $\mathrm{INV}_1 = \mathrm{INV}_2 = ()^{I_c}$ for inclusion of the zeros of the polynomial

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3x^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300.$$

We have started with the initial disks $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$ that contain the zeros -3, ± 1, $\pm 2i$, $\pm 2 \pm i$. The maximal radii of disks are given in Table 3.

Methods	$\max r_i^{(1)}$	$\max r_i^{(2)}$	$\max r_i^{(3)}$	$\max r_i^{(4)}$
(9) with $()^{-1}$	6.20(−2)	8.13(−5)	4.45(−15)	1.47(−46)
(9) with $()^{I_c}$	1.10(−1)	5.73(−5)	6.12(−16)	1.52(−50)
(15) with $()^{-1}$	6.20(−2)	5.65(−5)	1.21(−17)	5.05(−62)
(15) with $()^{I_c}$	1.10(−1)	4.57(−5)	2.16(−19)	3.01(−76)
(17) with $()^{-1}$	6.10(−2)	1.78(−5)	2.01(−18)	3.90(−64)
(17) with $()^{I_c}$	1.10(−1)	6.40(−6)	1.70(−31)	6.19(−189)

Table 3: The methods (9), (15) and (17) with exact and centered inversion

¿From Tables 1, 2 and 3 we observe that the centered inversion (2) gives smaller disks in the case of all tested methods (8), (9), (15) and (17). The improvement is especially stressed when the methods (15) and (17) with correction is applied. Slightly larger disks in the first iteration, when the centered inversion is applied, is the results of relatively slow convergence of the centers at the beginning of iterative process.

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