Axiomatic Foundations for Qualitative/Ordinal Decisions with Partial Preferences

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Abstract

The representational issues of preferences in the framework of a possibilistic (qualitative/ordinal) decision model under uncertainty, were originally introduced few years ago by Dubois and Prade, and more recently linked to case-based decision problem by Dubois et al.. In this approach, the uncertainty is assumed to be of possibilistic nature. Uncertainty (or similarity) and preferences on consequences are both measured on commensurate ordinal scales. However, in case-based decision problems, similarity or preferences on consequences may sometimes take values that are incomparable. In order to cope with some of these situations, we propose an extension of the model where both preferences and uncertainty arc graded on distributive lattices, providing axiomatic settings for characterising a pessimistic and an optimistic qualitative utilities. Finally, we extend our proposal to also include belief states that may be partially inconsistent, supplying elements for a qualitative case-based decision methodology.

1 Introduction

Assuming that uncertainty about the actual state may be represented by possibility distributions, Dubois and Prade [1995] proposed a qualitative counterpart to Von Neumann and Morgenstern's Expected Utility Theory [Von Neumann and Morgenstern, 1944]. Both uncertainty and preferences arc valued on linear ordinal scales of plausibility and preferences on decision consequences.

Gilboa and Schmeidler [1995] have proposed a Case-Based Decision Theory (CBDT) based on the choice of decisions according to their performance in previously experienced decision problems. This theory assumes a set M of decision problem instances storing the performance of decisions taken in different past situations as triples *(situation, decision, consequence),* and some measure *Sim* of similarity between situations. The Decision Maker (DM), in face of a new situation so, is proposed to choose a decision d which maximises a counterpart of the expected utility, namely the expression

$$U_{s_0,M}(d) = \sum_{(s,d,x)\in M} Sim(s_0,s)\cdot u(x)$$

where *Sim* is a non-negative function which estimates the similarity of situations and u provides a numerical utility for each consequence x. Gilboa and Schmeidler axiomatically characterise the preference relation induced by this U-maximisation.

In a recent paper [Dubois *et al,* 1998], an adaptation of the mentioned possibilistic decision model to the framework of case-based decision problems was suggested. But it was pointed out that some problems may appear in doing that. In order to cope with such problems, it has been proposed [Zapico and Godo, 1998] an extension of the possibilistic decision model to deal with non-normalised possibility distributions, i.e. distributions accounting for partial information that can be inconsistent to some extent.

In these proposals it is assumed that both uncertainty (similarity) and preferences are measured on linearly ordered scales, however, these hypotheses may not hold in many problems. There are real problems where we are not able to measure similarity and preferences in such linearly ordered sets but only in partially ordered ones. This situation may occur in case-based decision when the degrees of similarity on problems are only partially ordered. For example, consider that each situation is described as $s = (s^1,\ldots,s^b)$. Suppose we arc provided with a similarity function on situations *Sim:* $S \times S \rightarrow V$, defined in function of the b-features similarity functions. That is, let $S=S^1 \times\ldots\times S^b$, given $\mathcal{S}^k$: $S^k \times S^k \rightarrow E$, that measures the degree of similarity between two k-features, where E is a finite linear scale, the similarity on situations is defined by Sim(s,s') = $(\mathcal{S}^1(s^1,s'^1),\ldots, \mathcal{S}^b(s^b,s'^b))$, being $V= E \times\ldots\times E$, with $\leq_V$ the product ordering on V. In this case, the set of values for similarity, $(V, \leq_V)$, is a lattice. If we arc not provided with an aggregation criterion for similarity vectors that summarises the criteria on an ordinal linear scale, we are not able to apply the previously mentioned models. In a similar way, we may have that DM's preferences on consequences are only partially ordered, maybe as a consequence of a previous aggregation of various criteria. Indeed, a preference relation among consequences is usually modelled by a utility function u: $X \rightarrow U$, where U is a preference scale, frequently a (numerical or a qualitative) linear scale.

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However, in many cases, this preference function may be a vectorial one on a lattice. Hence, we are now interested in a qualitative decision model that let us make decisions in cases where the DM's preferences on consequences are only partially ordered or when the uncertainty on the consequences is valued on a lattice.

In this work, we propose axiomatic settings for qualitative decision making under uncertainty, requiring only finite distributive lattices for valuing uncertainty and preferences. Two qualitative criteria are axiomatized: a pessimistic and an optimistic one, respectively obeying an uncertainty aversion axiom and an uncertainty-attraction axiom. In order to be able to apply the model to case-based decision, we extend our initial proposal to include belief states that may be partially inconsistent.

In the following section we provide a background on lattices. In section 3 we propose two axiomatic settings for characterising both pessimistic and optimistic qualitative utilities, requiring only finite distributive and commensurate lattices for assessing uncertainty and preferences. An extension that lets us make decisions in contexts in which possibly partially inconsistent belief states are involved, is summarised in section 4. In section 5, it is shown how this extended model may be applied to case-based decision problems.

2 Background on Lattices

Lets us recall many definitions related with lattices (for more details [Davey and Priestley, 1990]), that we will use in the following.

2.1 Some Previous Definitions and Results

$(L, \wedge, \vee)$ is a *lattice* if $\wedge, \vee$ are associative, commutative, satisfy idempotency and the absorption laws. The induced order in a lattice is: $x \le y$ iff $x \wedge y = x$.

$(L, \wedge, \vee, n_L, 0, 1)$ will denote a *bounded lattice with involution,* i.e. L satisfies that $0, 1 \in L$ and $0 \le x \le 1\ \forall x \in L$, being $n_L: L \to L$ a decreasing function s.t. $nL(nL(x)) = x$.

Observations. Given $(L, \wedge, \vee)$ a lattice, then

- $\vee$ an $\wedge$ are non-decreasing.
- If $(L, \wedge, \vee, n_L, 0, 1)$ is a lattice with involution, nL satisfies that $n_L(0) = 1$ and $n_L(1) = 0$, $n_L(x \wedge y) = n_L(x) \vee n_L(y)$ and $n_L(x \vee y) = n_L(x) \wedge n_L(y)$.

Definition. Given a partially pre-ordered set $(L, \le)$, i.e. $\le$ is reflexive and transitive, the associated *indifference relation* $\sim$ is defined by $a \sim b$ iff $a \le b$ and $b \le a$.

Now we introduce some results that will be used in our proposal.

Proposition 1. Let $(L, \le)$ be a partially pre-ordercd set, then $\sim$ is an equivalence relation.

Definition. Given $(L, \le)$ a partially pre-ordered set, we denote by $L/\sim$ the quotient set w . $\sim$. $(L, \le)$ s a *preAattice* iff $(L/\sim, \sqsubseteq)$ is a lattice, defining $\sqsubseteq$ as: $[a] \sqsubseteq [b]$ iff $a \le b$.

Theorem 1. (A, $\le$) is a pre-lattice iff (A, $\le$) is a partially pre-ordered set, such that satisfies

I) For all a, b $\in$ A there exists an unique not empty subset SUP(a,b) $\subseteq$ A s.t.

- the SUP(a,b) elements are indifferent one to each other, i.e. $c \sim d$, $\forall$ c, d $\in$ SUP(a,b).
- if c $\in$ SUP(a,b), $c \sim d$, then d $\in$ SUP(a,b).
- $\forall$ c $\in$ SUP(a,b), $a \le c$ and $b \le c$.
- if $a \le e$ and $b \le e$, then ($e \sim c\ \forall c \in$ SUP(a,b)) or ($e >^1 c\ \forall c \in$ SUP(a,b)).

II) For all a, b $\in$ A there exists an unique not empty subset INF(a,b) $\subseteq$ A s.t.

- the INF(a,b) elements are indifferent one to each other, i.e. $c \sim d$ for all c, d $\in$ INF(a,b).
- if c $\in$ INF(a,b), $c \sim d$ then d $\in$ INF(a,b).
- if $e \le a$ and $e \le b$ then, ($e \sim c\ \forall c \in$ INF(a,b)) or ($c > e$ for all c $\in$ INF(a,b)).
- $\forall$ c $\in$ INF(a,b), $c \le a$ and $c \le b$.

2.2 Possibility Distributions and Lattices

Now, let us introduce the context of our work. Let $X = \{x_1, \ldots, x_p\}$ be a finite *set of consequences.* We will denote by $(V, \vee, \wedge, 0, 1, n_V)$ *a finite distributive lattice of uncertainty values* with minimum 0, maximum 1 and an involution n_V, and $\le_V$ the order induced by $\wedge$ in V. $(U, \le_U, 0, 1, n_U)$ will be a *finite distributive lattice of preference values* with involution nU. The *indifference* and *incomparability relations* are:

- $\lambda \sim \lambda'$ iff $\lambda \le_V \lambda'$ and $\lambda' \le_V \lambda$.
- $\lambda <> \lambda'$ iff $\lambda \not\le_V \lambda'$ and $\lambda' \not\le_V \lambda$.

Now, we consider the set of *consistent possibility distributions* on X over V, i.e.

$$Pi(X) = \{\pi: X \to V \mid \bigvee_{x \in X} \pi(x) = 1\}.$$

As usual, $\pi \le \pi'$ iff $\forall x \in X\ \pi(x) \le_V \pi'(x)$, with $\le_V$ the order induced by $\wedge$ in V.

At first, we will be interested in a subset of Pi(X), the set of *normalised possibility distributions,* i.e.

$$Pi^*(X) = \{\pi \in Pi(X) / \exists x \text{ s.t. } \pi(x) = 1\}.$$

For the sake of simplicity, we shall use x for denoting both an element belonging to X and the normalised possibility distribution on X such that $\pi(x) = 1$ and $\pi(z) = 0$ for $z \ne x$. In general, we shall also denote by A both a subset $A \subseteq X$ and the normalised possibility distribution on X such that $\pi(x) = 1$ if $x \in A$ and $\pi(x) = 0$ otherwise. Hence, we can consider X as included in $Pi^*(X)$.

Given $x, y \in X$, $x \ne y$, $\lambda, \mu \in V$ s.t. $\lambda \vee \mu = 1$, the *qualitative lottery* $(\lambda/x, \mu/y)$ is the consistent possibility distribution on X, defined as $(\lambda/x, \mu/y)(z) = \lambda$ if $z = x$, equal to μ if z = y and 0 otherwise.

The so-called *Possibilistic Mixture* is an operation defined on Pi(X) that combines two consistent possibility distributions π_1 and π_2 into a new one, denoted $(\lambda/\pi_1, \mu/\pi_2)$, with $\lambda, \mu \in V$ and $\lambda \vee \mu = 1$, and defined as

$$(\lambda/\pi_1, \mu/\pi_2)(x) = (\lambda \wedge \pi_1(x)) \vee (\mu \wedge \pi_2(x)).$$

[1] $e > c$ iff $c \le e$ and $c \not\le e$.

In order to have a closed operation on $Pi^*(X)$, the mixture operation is restricted to $Pi^*(X)$ requiring the scalars to satisfy an additional condition, i.e., if $\pi, \pi' \in Pi^*(X)$, we consider $(\lambda/\pi, \mu/\pi')$ with $\lambda, \mu \in$ V and $\lambda = 1$ or $\mu = 1$.
It is not difficult to verify that *reduction of lotteries* always holds, i.e. $\forall \lambda_1, \lambda_2 \in V$ s.t. $\lambda_1 \vee \lambda_2 = 1$, $\forall \pi \in Pi(X)$

$$(\lambda_1/(1/\pi, \mu_1/X), \lambda_2/(1/\pi, \mu_2/X)) = = (1/\pi, (\lambda_1 \wedge \mu_1) \vee (\lambda_2 \wedge \mu_2)/X).$$

Note. In order to simplify notation, we use $\wedge, \vee$ for denoting both operations on V and U, although they may be different, hoping they may be understood by the context.

3 Our Proposal

Consider u: $X \rightarrow U$ a preference function that assigns to each consequence of X a preference level of U, requiring V and U to be commensurate, i.e. there exists $h: V \rightarrow U$ a $\{0,1\}$-homomorphism relating both lattices V and U. Let n be the reversing homomorphism n: $V \rightarrow U$ defined as $n(\lambda) = n_U(h(\lambda))$. It also verifies n(0) = 1, n(l) = 0. For any $\pi \in Pi^*(X)$, consider the qualitative utility functions

$$QU^-(\pi) = \bigwedge_{x \in X} (n(\pi(x)) \vee u(x)),$$
$$QU^+(\pi) = \bigvee_{x \in X} (h(\pi(x)) \wedge u(x)).$$

Now, we will introduce the axioms that characterise the preferences relations induced by these functions and some results that we need for the representation theorems proofs.

Note. As U is a distributive lattice with involution, QU" and QU^+ preserve the possibilistic mixture in the sense that the following expressions hold,

$$QU^-(\lambda/\pi_1, \mu/\pi_2) = (n(\lambda) \vee QU^-(\pi_1)) \wedge (n(\mu) \vee QU^-(\pi_2)),$$
$$QU^+(\lambda/\pi_1, \mu/\pi_2) = (h(\lambda) \wedge QU^+(\pi_1)) \vee (h(\mu) \wedge QU^+(\pi_2)).$$

Proposition 2. Let $(Pi^*(X), \sqsubseteq)$, satisfying

- **A1**: $(Pi^*(X), \sqsubseteq)$ is a pre-lattice.
- A2 (uncertainty aversion): if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.

Then

a) The maximal elements of $(Pi^*(X), \sqsubseteq)$ are equivalent.
b) The maximal elements of $(X, \sqsubseteq)$ are equivalent, and they are equivalent to the maximal ones of $(Pi^*(X), \sqsubseteq)$.

Axiomatic setting. Let **AXP** be the following set of axioms on $(Pi^*(X), \sqsubseteq)$:

- Al: $(Pi^*(X), \sqsubseteq)$ is a pre-lattice.
- A2 (uncertainty aversion): if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.
- A3 (independence):

  $\pi_1 \sim \pi_2 \Rightarrow (\lambda/\pi_1, \mu/\pi) \sim (\lambda/\pi_2, \mu/\pi)$.
- **A4:** if $\pi_\lambda \sqsubseteq \pi_{\lambda'} \Rightarrow \pi_{n_V(\lambda)} \sqsupseteq \pi_{n_V(\lambda')}$,

  with $\pi_\lambda = (1/\bar{\pi}, \lambda/X)$ and $\bar{\pi}$ a maximal element of $(Pi^*(X), \sqsubseteq)$.
- **A5:** if $\lambda <> \lambda' \Rightarrow \pi_\lambda \sqsubset\sqsupset \pi_{\lambda'}$.
- **A6**: $\forall \pi \in Pi^*(X), \exists \lambda \in V$, such that $\pi \sim (1/\bar{\pi}, \lambda/X)$.

Observation. If A4 holds then, $\pi_\lambda \sim \pi_{\lambda'} \Rightarrow \pi_{n_V(\lambda)} \sim \pi_{n_V(\lambda')}$.

[2] π is a maximal element iff $\forall \pi' \in Pi^*(X)$, $\pi \sqsubseteq \pi' \Rightarrow \pi \sim \pi'$.

Lemma 1. Let $(U, \leq_U, 0, 1, n_U)$ and $(V, \leq_V, 0, 1, n_V)$ be two distributive lattices with involution, n: $V \rightarrow U$ a reversing epimorphism, and u: $X \rightarrow U$. Consider $QU^-(\pi) = \bigwedge_{x \in X}(n(\pi(x)) \vee u(x))$, if $(QU^-)^{-1}(1) \neq \emptyset$ and $(QU^-)^{-1}(0) \neq \emptyset$, then

a) there exists $x \in X$ s.t. $u(x) = 1$ and $\bigwedge_{x \in X} u(x) = 0$.
b) QU" is onto.

Lemma 2. Let n: $V \rightarrow U$ be an onto decreasing function also satisfying that if $\lambda <> \lambda'$ then $n(\lambda) <>_U n(\lambda')$. Then, n is a reversing epimorphism.

Finally, let $\preccurlyeq_{QU^-}$ be the preference ordering on $Pi^*(X)$ induced by QU", i.e. $\pi \preccurlyeq_{QU^-} \pi'$ iff $QU^-(\pi) \leq_U QU^-(\pi')$. In the following, we state that the set of axioms AXP characterise these preference ordcrings

Representation Theorem 2. *(Pessimistic Utility)* A preference relation $(Pi^*(X), \sqsubseteq)$ satisfies axioms AXP iff there exist

(i) a finite distributive utility lattice $(U, \wedge, \vee, n_U, 0, 1)$.
(ii) a preference function u: $X \rightarrow U$, s.t. $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x \in X} u(x) = 0$,
(iii) an onto order reversing function n: $V \rightarrow U$ s.t. $n(0) = 1$ and n(l) = 0, also satisfying

$$\text{if } \lambda <> \lambda' \text{ then } n(\lambda) <>_U n(\lambda'), \quad (1)$$
$$\text{and } n_U \circ n \circ n_V = n, \quad (2)$$

in such a way that it holds:

$$\pi' \sqsubseteq \pi \quad \text{iff} \quad \pi' \preccurlyeq_{QU^-} \pi.$$

<u>Proof</u>: $\leftarrow$) Now, we verify that the preference ordering on Pi*(X) induced by QU" satisfies the above set of axioms.

As $\leq_U$ is a partial order, $\preccurlyeq_{QU^-}$ is reflexive and transitive. QU" is onto, so we may define $SUP(\pi, \pi') = (QU^-)^{-1}(QU^-(\pi) \vee QU^-(\pi'))$, and $INF(\pi, \pi') = (QU)^{-1} (QU^-(\pi) \wedge QU^-(\pi'))$. Then, by theorem 1, $(Pi^*(X), \preccurlyeq_{QU^-})$ is a pre-lattice.

A2 results from the fact that $\vee$ and $\wedge$ are non-decreasing in U and n is a reversing function. While, A3 is a consequence of the fact that QU" preserves mixtures.

A4: if $\pi_\lambda \preccurlyeq_{QU^-} \pi_{\lambda'} \Rightarrow \pi_{n_V(\lambda)} \succcurlyeq_{QU^-} \pi_{n_V(\lambda')}$.

Let $\bar{\pi}$ be a maximal element of $Pi^*(X)$, so $QU^-(\bar{\pi}) = 1$. As QU^- preserves mixtures and $QU^-(X) = 0$, we have that $QU^-(\pi_\lambda) = n(\lambda)$. As $n_U \circ n \circ n_V = n$, and n_V and n_U are involutive, if $n(\lambda) \leq n(\lambda')$, then $n(n_V(\lambda)) = n_U(n(\lambda)) \geq n_U(n(\lambda')) = n(n_V(\lambda'))$.

A5 is a consequence of $\prec\succ_{QU^-}$ definition and that n satisfies (1). Now, we check A6. Let $\bar{\pi}$ be maximal element of $Pi^*(X)$ w.r.t. $\preccurlyeq_{QU^-}$. As $QU^-(1/\bar{\pi}, \lambda/X) = n(\lambda)$, then $QU^-(\pi) = n(\lambda) = QU^-(1/\bar{\pi}, \lambda/X)$ $\forall \lambda \in n^{-1}(QU^-(\pi))$.

$\rightarrow$) The proof is analogous with the one given in [Dubois *el al,* 1998] for the linear case. We structure the proof in the following three steps.

I) We define the distributive utility lattice U, with involution nU, and a reversing mapping n from V to U, satisfying if $\lambda <> \lambda'$ then $n(\lambda) <>_U n(\lambda')$, and $n_U \circ n \circ n_V = n$. By lemma 2, n results a reversing epimorphism.

II) A function QU": $Pi^*(X) \rightarrow U$ representing $\sqsubseteq$, i.e. such that $QU^-(\pi) \leq QU^-(\pi')$ iff $\pi \sqsubseteq \pi'$, is defined.

III) Finally, we prove that $QU^-(\pi) = \bigwedge_{x\in X}(n(\pi(x))\vee u(x))$, where u: $X \to U$ is the restriction of QU^- on X. u also satisfies that $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x\in X} u(x) = 0$.

Now, we develop these steps.

I) We consider on Pi*(X) the equivalence relation ~, defined as $\pi \sim \pi'$ iff $\pi \sqsubseteq \pi'$ and $\pi' \sqsubseteq \pi$, by A1 Pi*(X)/~ is a lattice. As in the linear case, we take as utility lattice U = Pi*(X)/~. As Theorem 1 guarantees the existence of SUP and INF, we consider in U the operations $\wedge$ and $\vee$ induced by SUP and INF, i.e.

$$[\pi] \vee [\pi'] = SUP(\pi, \pi') \text{ and } [\pi] \wedge [\pi'] = INF(\pi, \pi').$$

The $\leq_U$ induced from $\vee$ coincides with $\sqsubseteq$. It is not difficult to verify that [X] is minimum of U, and if $\bar{\pi}$ is a maximal element of Pi*(X), as all maximal elements are equivalent, $[\bar{\pi}]$ is maximum on U.

Let $\pi_\lambda = (1/\bar{\pi}, \lambda/X)$, and let **n: V → U** be defined as $n(\lambda) = [\pi_\lambda]$. It is not difficult to see that n is onto, n(1) = 0, n(0) = 1, and that n actually reverses the order. Now, we define n_U from n and n_V. Given $w\in U$, there exists $\lambda \in V$ s.t. $n(\lambda) = w$. We define $n_U(w) = n(n_V(\lambda))$.

By A5, n satisfies if $\lambda <> \lambda'$ then $n(\lambda) <>_U n(\lambda')$, and by definition of n_U, $n_U \circ n \circ n_V = n$. Hence, as n is a reversing epimorphism, and V is distributive, so is U.

II) As usual, QU^- may be defined on Pi*(X) in two steps. First, we define it on lotteries of type π_λ, as $QU^-(1/\bar{\pi}, \lambda/X) = n(\lambda)$. A6 lets us to extend this definition. Since $\forall\pi \exists \lambda$ s.t. $\pi \sim (1/\bar{\pi}, \lambda/X)$, we define $QU^-(\pi) = n(\lambda)$.

It is not difficult to verify that QU^- represents $\sqsubseteq$.

III) Consider u: $X \to U$ defined as $\mathbf{u(x) = QU^-(x)}$.

It remains to prove that $QU^-(\pi) = \bigwedge_{x\in X} (n(\pi(x))\vee u(x))$. To verify this, we will prove the following equality:

- $QU^-(\lambda_1/\pi_1,\lambda_2/\pi_2) = (n(\lambda_1)\vee QU^-(\pi_1))\wedge (n(\lambda_2)\vee QU^-(\pi_2))$ with $\lambda_1 = 1$ or $\lambda_2 = 1$.

By A6, $\exists\ \mu,\gamma$ s.t. $\pi_1 \sim (1/\bar{\pi}, \mu/X)$ and $\pi_2 \sim (1/\bar{\pi}, \gamma/X)$ by A3, $(\lambda_1/\pi_1, \lambda_2/\pi_2) \sim (\lambda_1/(1/\bar{\pi}, \mu/X), \lambda_2/(1/\bar{\pi}, \gamma/X))$, and reducing lotteries we obtain

$$(\lambda_1/\pi_1, \lambda_2/\pi_2) \sim (1/\bar{\pi}, ((\lambda_1\wedge\mu)\vee(\lambda_2\wedge\gamma))/X)).$$

Therefore, as n is distributive,

$$\begin{aligned} QU^-(\lambda_1/\pi_1, \lambda_2/\pi_2) &= n((\lambda_1 \wedge \mu) \vee (\lambda_2 \wedge \gamma)) = \\ &= (n(\lambda_1)\vee n(\mu)) \wedge (n(\lambda_2)\vee n(\gamma)) = \\ &= (n(\lambda_1)\vee QU^-(\pi_1)) \wedge (n(\lambda_2)\vee QU^-(\pi_2)) \end{aligned}$$

Hence, $QU^-(\pi_1\vee \pi_2)) = QU^-(\pi_1) \wedge QU^-(\pi_2))$

More generally, $QU^-(\bigvee_{i=1,p}\pi_i)= \bigwedge_{i=1,p}QU^-(\pi_i)$.

- $\mathbf{QU^-(\pi) = \bigwedge_{i=1,p} (n(\pi(x_i))\vee u(x_i))}$.

As $\pi \in Pi^*(X)$, then $\exists\ x_j \in X$ s.t. $\pi(x_j) = 1$, without loss of generality assume j = 1. Let $\pi_i = (1/x_1, \pi(x_i)/x_i)$. As $\pi = \bigvee_{i=1,p} \pi_i$, we have that

$$\begin{aligned} QU^-(\pi) &= QU^-(\bigvee_{i=1,p} \pi_i) = \\ &= \bigwedge_{i=1,p}\{(u(x_1) \wedge (n(\pi(x_i)) \vee u(x_i))\} = ^3 \\ &= \bigwedge_{i=1,p} (n(\pi(x_i))\vee u(x_i)). \end{aligned}$$

Finally, as $\bar{\pi}$ is normalised, exists $x_0 \in X$ s.t. $\bar{\pi}(x_0) = 1$, so $x_0 \leq \bar{\pi}$. Then by A2, $x_0 \sqsupseteq \bar{\pi}$. As QU^- represents $\sqsubseteq$, $QU^-(x_0)\geq QU^-(\bar{\pi}) = n(0)= 1$, hence $u(x_0)=1$, so $u^{-1}(1)\neq \emptyset$.

As $QU^-(X) = n(1) = 0$, and $QU^-(X) = \bigwedge_{x\in X} u(x)$, then $\bigwedge_{x\in X} u(x) = 0$. This ends the proof. □

[3] Note that $\pi(x_1) = 1$, so $u(x_1) = u(x_1)\vee n(\pi(x_1))$.

In order to represent an optimistic preference criterion, we consider now the distribution π_λ defined as $\pi_\lambda = (\lambda/X, 1/\underline{\pi})$, where $\underline{\pi}$ is minimal of $(Pi^*(X), \sqsubseteq)$, and we have to change the uncertainty aversion axiom A2 by an uncertainty-prone postulate

- **A2⁺**: if $\pi \leq \pi'$ then $\pi \sqsubseteq \pi'$,

and to modify the continuity axiom A6 into

- **A6⁺**: $\forall \pi \in Pi^*(X)\ \exists \lambda \in V$ such that $\pi \sim (\lambda/X, 1/\underline{\pi})$, where $\underline{\pi}$ is minimal of $(Pi^*(X), \sqsubseteq)$.

For an optimistic behaviour, we consider $\preccurlyeq_{QU^+}$ the preference ordering on Pi*(X) induced by QU^+, i.e.

$$\pi \preccurlyeq_{QU^+} \pi' \textbf{ iff } QU^+(\pi) \leq_U QU^+(\pi').$$

Representation Theorem 3. *(Optimistic Utility)* A preference relation $\sqsubseteq$ on Pi*(X) satisfies axioms set AXP+ = {A1, A2⁺, A3, A4, A5, A6⁺} iff there exist

(i) a finite distributive utility lattice $(U, \vee, \wedge, 0, 1, n_U)$,

(ii) a preference function u: $X \to U$, s.t. $u^{-1}(0) \neq \emptyset$ and $\bigvee_{x\in X} u(x) = 1$,

(iii) an onto order preserving function h: $V \to U$, s.t. h(0)=0, h(1) = 1, $n_U \circ h \circ n_V = h$, and also satisfying $\lambda <> \lambda'$ then $h(\lambda) <>_U h(\lambda')$,

in such a way that it holds:

$$\pi' \sqsubseteq \pi \quad \text{iff} \quad \pi' \preccurlyeq_{QU^+} \pi.$$

The proof is very analogous to the one for pessimistic utility, hence it is omitted.

Observation. As n is onto and decreasing, if V is linear, so is U (i.e. U non linear, then V is non linear). Moreover, as a consequence of the condition "if $\lambda <> \lambda'$ then $n(\lambda) <>_U n(\lambda')$", if V is non linear so is U. Hence, V and U are both linear lattices or both non linear lattices.

4 Extension for Partially Inconsistent Belief States

Sometimes, the Decision Maker may only have partial information about the possible consequences of decisions, for example, by having the performance of decisions taken in different past situations stored as a set M of triples *(situation, decision, consequence).* As previously mentioned, in such a framework, Gilboa and Schmeidler [1995] proposed a case-based decision model where the Decision Maker, faced with a new situation S_0, is supposed to choose a decision d which maximises a counterpart of classical expected utility. Another approach to Case-Based Decision, which proposes looking for decisions that always gave good results in similar experienced situations, was suggested by Dubois and Prade [1997].

In [Dubois *et al*, 1998] a link is established between Case-based Decision Theory and Qualitative Decision Theory by estimating how much plausible x is a consequence of a decision d in the current situation so in terms of what extent SQ is similar to situations in which x was experienced after taking the decision d. Being U and V finite linearly ordered scales that are commensurate, they consider a similarity function on situations *Sim:* $S \times S \to V$, and a preference function u: $X \to U$ which represents DM's preferences on consequences. So, a decision or action d can be identified with a possibility distribution on consequences. They define the *distribution associated to d and* S_0(and obviously depending on *Sim* and **M**) $\pi_{d,s_0}: X \to V$ on the set of consequences, as

$$\pi_{d,s_0}(x) = \max\{Sim(s_0, s) \mid (s, d, x) \in M\},$$

where, by convention, max $\emptyset = 0$. $\pi_{d,s_0}(x)$ represents the plausibility of x of being the consequence of s_0 by d. Hence, the proposal was to evaluate d, in terms of π_{d,s_0}.

If π_{d,s_0} is normalised, then it is always the case that optimistic criterion scores a decision higher than the pessimistic one, but if the distribution is not normalised it may not. This problem is solved in [Zapico and Godo, 1998] extending the model to include non-normalised distributions that represent belief states that may be partially inconsistent. A similar analysis is valid for our work, hence, in order to apply the model to case-based decision that may involve non-normalised distributions, we provide now the corresponding extension of our proposal. First, let us introduce the concepts of normalisation and height of a distribution. Define h, the *height of a distribution,* as $h(\pi) = \bigvee_{x \in X} \pi(x)$, and for each distribution consider the subset of consequences with maximal plausibility

$$X_{h(\pi)} = \{ x \in X / \forall y \in X\ \pi(y) \not> \pi(x)\}.$$

We define $N(\pi)$, *the normalisation of* as the normalised distribution

$$N(\pi)(x) = \begin{cases} 1 & \text{if } x \in X_{h(\pi)} \\ \pi(x) & \text{otherwise.} \end{cases}$$

We extend the set of possibilistic lotteries to the set $Pi^{ex}(X)$ of non necessarily normalised distributions on V. Hence, we need to extend the concept of possibilistic mixture PME on $Pi^{ex}(X)$ to combine π_1 and π_2 with $(\lambda, \mu) \in \Phi_V$, with $\Phi_V = \{(\alpha, \beta) \in V \times V \mid \alpha \vee \beta = 1\}$, i.e.

PME: $Pi^{ex}(X) \times Pi^{ex}(X) \times \Phi_V \to Pi^{ex}(X)$,

$$PME(\pi_1, \pi_2, \lambda, \mu) = (\lambda/\pi_1, \mu/\pi_2) = (\lambda \wedge \pi_1(x)) \vee (\mu \wedge \pi_2(x))$$

Given a function F: V→ V, such that F(1) = 0, now we may consider the qualitative (or ordinal) utility functions on $Pi^{ex}(X)$, corresponding to those considered previously

$$\underline{QU}_F^-(\pi) = QU^-(N(\pi)) \wedge n(F(h(\pi)))$$

$$\underline{QU}_F^+(\pi) = QU^+(N(\pi)) \vee h(F(h(\pi))).$$

Let $\sqsubseteq_F$ be a preference relation in $Pi^{ex}(X)$. We will denote by *Q* its restriction to $Pi^*(X)$, $\sim_F$ and $\sim$ the corresponding indifference relations. In order to characterise the preference orderings induced by the utilities $\underline{QU}_F^-$and $\underline{QU}_F^+$, we extend the axiom sets AXP and AXP^+, defined on $(Pi^*(X), \sqsubseteq)$ in Section 3, with

- **AZEE:** for all $\pi \in Pi^{ex}(X)$ $\pi \sim_F (1/N(\pi), F(h(\pi))/X)$.

The intuitive idea behind this axiom is that, according to the above utility functions, a non-normalised possibilistic lottery π is indifferent to the corresponding normalised lottery $N(\pi)$ provided that this is modified in terms of an uncertainty level related with the normality degree of the lottery expressed by its height $h(\pi)$. For, example if we consider $F = n_V$, $N(\pi)$ is weighted by the "negation" of the height of the original distribution.

We say that a preference relation $\sqsubseteq_F$ on $Pi^{ex}(X)$ satisfies axiom set $\underline{AXP} = AXP \cup \{\underline{A7PF}\}$ ($\underline{AXP}^+ = AXP^+ \cup \{\underline{A7PF}\}$ resp.) iff its restriction to $Pi^*(X)$, denoted by $\sqsubseteq$, satisfies AXP (AXP^+ resp.) and $\sqsubseteq_F$ also satisfies A7PF.

Representation Theorem 4 . A preference relation $\sqsubseteq_F$ on $Pi^{ex}(X)$ satisfies axiom set AXP ($\underline{AXP}^+$ **resp.**) iff there exist

(i) a finite distributive utility lattice $(U, \vee, \wedge, 0, 1)$ with involution n_U,

(ii) a preference function u: $X \to U$, s.t. $u^{-1}(1) \neq \emptyset$ and $\bigwedge_{x \in X} u(x) = 0$,

(iii) an onto order preserving function h: $V \to U$ s.t. **$h(0)=0$, $h(1)=1$, and $n_U \circ h \circ n_V = h$, also satisfying**

$$\text{if } \lambda <> \lambda' \text{ then } h(\lambda) <>_U h(\lambda'),$$

in such a way that it holds:

$$\pi' \sqsubseteq_F \pi \text{ iff } \underline{QU}_F^-(\pi') \leq_U \underline{QU}_F^-(\pi)$$

$(\pi' \sqsubseteq_F \pi$ iff $\underline{QU}_F^+(\pi') \leq_U \underline{QU}_F^+(\pi)$ resp.), with $n = n_U \circ h$.

5 Case-based Decision: an Application of the Model

Now, we may apply this model to case-based decision, for example, in the context4 of the *COMR1S Project* [Plaza, *et a*/., 1998]. Suppose we have different agents, called Personal Representative Agents (PRA for short), each of one pursuing a different interest for a same user, and a PA (Personal Assistant) agent coordinating the proposals presented by PRAs. Each PRA presents its most relevant proposal among one of the following :

- an appointment with a person (app)
- an alert about the proximity of a person or event of interest for the user (pro)
- a proposal of receiving propaganda related with events like demonstrations, future conferences, etc. (rp)
- a reminder of an event that will happen soon (rem),

together with a degree of the estimated proposal relevance: great importance (gi), moderate importance (mi), doubtful importance (di), null. (For more details see [Plaza, *et a*/., 1998]).

The PA has to choose one of the PRAs' proposals to send it to the user, with its own evaluation of relevance: gi, mi, di, null.

Suppose we have a memory of cases storing the performance of proposals made in the past by the PA with the respective user opinions about PA's behaviour. A case is represented as a triple c = (vs, **winner, x)**, with:

- vs = $((d^1, rel_1), \ldots, (d^n, rel_n))$, where (di rel_i) describes the proposal made by the PRAi and the importance that it assigned to its proposal,
- **winner** = (PA's proposal, PA's evaluation of the importance of its proposal).
- Finally, x is a pair reflecting the user opinion. Its first component is user's evaluation on PA's proposal, while the second one is his evaluation of the relevance PA has assigned to it. User opinion is measured on $U = E \times E$, being $E = \{0, \lambda, \mu, 1\}$, w $0 < \lambda < \mu < 1$, d n_E its reversing involution.

[4]Actually, we will consider a simplified perspective of the problems involved in this project.

The similarity function on proposals, S_prop, defined over E, is described in Table 1,

Table 1: similarity on proposals

S_prop	app	pro	rem	rp
app	1	μ	λ	0
pro	μ	1	λ	0
rem	λ	λ	1	0
rp	0	0	0	1

while the similarity on labels of relevance, S_rel, is defined in Table 2.

Table 2: similarity on relevance labels.

S rel	IP	mi	di	null
gi	1	U	*X*	0
mi	U	1	*X*	0
di	*X*	*X*	1	0
null	0	*0*	0	1

Now, we define the similarity on states as:

$$SIM(vs,vs') = \bigwedge_{i=1,n}(S\_prop(d^i, d'^i), S\_rel(rel_i, rel'_i)).$$

Suppose there are 3 PRAs, being available the memory of cases M described in Table 3.

Table 3: Memory of cases

cases	vs	winner	user_ opinion
c1	((app1,gi),(app2,gi), (app3,gi))	(app2,gi)	(1, 1)
c2	((app1,gi),(rem2,gi), (app3,gi))	(rem2,mi)	(1, μ)
c3	((rp1,di),(rem2,di), (pro3,di))	(pro3,mi)	(μ, λ)
c4	((app1,gi),(rem2,di), (rp3,di)	(rem2,mi)	(λ, λ)
c5	((pro1,mi), (pro2, di), (app3,mi))	(pro1,di)	(1, λ)
c6	((pro1,di),(rem2,di), (app3, mi))	(pro1,di)	(λ, μ)
c7	((app1,mi),(rem2,null), (app3,mi)	(rem2,di)	(0, 0)
c8	((app1,gi)(rem2,di) (rem3, null))	(app1,gi)	(1, μ)

Let vs_0 = ((app1, mi),(rem2, mi), (rem3, di)). Now, we evaluate some of PA's available options. As we may see, many of these distributions are non-normalised, so we apply $\underline{QU}_F^+$ and $\underline{QU}_F^-$. Consider $F = n_V$, with $n_V = (n_E, n_E)$. So, $U_F^-{}_{s_0}(d) = \underline{QU}_F^-(\pi_{d,vs_0}) = \mathcal{L}(\pi_{d,s_0}))\wedge QU^-(\mathcal{N}(\pi_{d,s_0}))$ and $U_F^+{}_{s_0}(d) = QU^+(\mathcal{N}(\underline{\pi}))\vee n_V(\mathcal{L}(\underline{\pi}))$.

The distributions associated to PA's proposals not made before, like (rp3, mi), (app2, di), are null. Hence, their utilities are 0 and 1 w.r.t. pessimistic and optimistic criteria respectively. While for (app1,gi), $\pi_{d,vs_0}(1, \mu) = (1, 0)$ and (0, 0) otherwise, so $\underline{QU}_F^-(\pi_{d,vs_0}) = (1, 0) \wedge (1, \mu) = (1, 0)$, while $\underline{QU}_F^+(\pi_{d,vs_0}) = (1, 0) \vee n_V(1, \mu) = (1, \lambda)$.

6 Conclusions

In this paper, we propose a framework that allows us to make decisions in contexts in which we only have partially ordered information, in the sense that DM's preferences are valued on a distributive lattice, and that the valuation set of uncertainty (or similarity) is partially ordered too.

We axiomatically characterise these criteria.

As the problem of partially ordered information may have been originated in a case-based decision problem with similarity degrees valued on a lattice involving belief states partially inconsistent, we extended our initial proposal to non-normalised distributions, obtaining two criteria for case-based decisions.

Up to now, we have considered $\wedge$ and $\vee$ as the available operations, now we are considering other operations defined in the lattices, letting us defining other utility functions which we are characterising.

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