A Set-Theoretic Approach to Automated Deduction in Graded Modal Logics

A. Montanari and **A. Policriti**
Dipartimento di Matematica e Informatica, Universita di Udine
Via delle Scienze 206, 33100 Udine, Italy

Abstract

In the paper, we consider the problem of supporting automated reasoning in a large class of knowledge representation formalisms, including terminological and epistemic logics, whose distinctive feature is the ability of representing and reasoning about finite quantities. Each member of this class can be represented using graded modalities, and thus the considered problem can be reduced to the problem of executing graded modal logics. We solve this problem using a set-theoretic approach that first transforms graded modal logics into poly modal logics with infinitely many modalities, and then reduces derivability in such polymodal logics to derivability in a suitable first-order set theory.

1 Introduction

The general theme of this paper is the description of a novel approach to the problem of supporting the automation of reasoning in a family of knowledge representation formalisms. Such a family is characterized by the fact that its members need to represent and reason about finite quantities, and it includes terminological logics, epistemic logics, universal modalities, van der Hoek and de Rijke have shown that all these languages can be represented using graded modalities [Fattorosi-Barnaba and De Caro, 1985] (cf. [Hoek and de Rijke, 1995] for a complete description of this kind of reductions). In this paper, we propose an approach to automated deduction in graded modal logics which is based on a set-theoretic translation method introduced by D'Agostino et al. in [D'Agostino *et a/.*, 1995] to support derivability in propositional modal' logic.

Most inference systems for modal logic are defined in the style of sequent or tableaux calculi, e.g. [Fitting, 1983; Wansing, 1994]. As an alternative, a number of *translation* methods for modal logic into classical first-order logic have been proposed in the literature (for a comprehensive survey, cf. [Ohlbach, 1993]). Such methods allow the use of Predicate Calculus mechanical theorem provers to implement modal theorem provers. Compared with the direct approach of finding a proof algorithm for a specific class of modal logics, the translation methods have the advantage of being *independent* of the particular modal logic under consideration: a single theorem prover may be used for any translatable modal logic.

In the standard approach, the first-order language *C* into which the translation is carried out contains a constant r denoting the initial world in the frame, a binary relation *R(x,y)* denoting the accessibility relation, and a denumerable number of unary predicates $P_i(x)$. The translation function is defined by induction on the structural complexity of the modal formula as follows:

- $\pi(P_j, x) \equiv P_j(x)$;
- $\pi(-, x)$ commutes with the boolean connectives;
- $\pi(\Box\psi, x) \equiv \forall y(xRy \rightarrow \pi(\psi, y))$.

Efficiency concerns have motivated further investigations on the above (relational) translation method. Such studies (e.g. [Ohlbach, 1991]) suggested a "functional" semantics for modal logic and resulted in a family of more efficient and general translation methods. From the computational point of view, the functional translation may still cause some problem when using a first-order theorem prover, due to the presence of equalities in the translation of the axioms. A method for limiting the complexity induced by the introduction of equality using a mixed relational/functional translation is proposed in [Nonnengart, 1993].

A common feature of all the methods mentioned above is that, in order to be applied directly, the underlying modal logic must have a first-order semantics. All attempts to apply them to logics not having a first-order semantics have required *ad-hoc* techniques. Moreover, if the logic has a first-order semantics, but it is only specified by Hilbert axioms, a preliminary step is necessary to find the corresponding first-order axioms. The question of automatically solving this last problem has been extensively studied and algorithms have been proposed, e.g. [Benthem, 1985; Gabbay and Ohlbach, 1992].

The above analysis can be easily tailored to the case of graded modalities. The semantics of graded modalities is very natural and intuitive, but it has a disadvantage: the inference systems based on it deal with O_n and operators by generating a number of terms that, in general, can be very large. This problem can be overcome by using a Hilbert-style axiomatic system, which allows one to perform arithmetic symbolic reasoning; in such a case, however, the search space for proving even very simple theorems can grow very much and it is usually rather unstructured. In view of the previous points, a translational approach to automated reasoning with graded modalities has been considered by Ohlbach et al. (cf. [Ohlbach *et al,* 1995]). Such an approach provides the possibility of using a standard deductive system — thereby guaranteeing symbolic reasoning — for which optimizations and good implementations are available.

In this paper, we exploit an alternative translation method whose basic idea is to map modal formulae into set-theoretic terms. Such a method works for all normal complete finitely axiomatizable modal logics, regardless of the first-order axiomatizability of their semantics. It also works if the modal logic under consideration is only specified by Hilbert axioms. Furthermore, it can be easily generalized to polymodal logics with finitely many modalities [D'Agostino *et al.*, 1995].

Even though graded modal logics can be seen as polymodal logics ([Ohlbach *et al,* 1995]), the set-theoretic translation method cannot be applied directly, because the number of modalities involved in their translation is infinite. In the following, we show how to adapt the set-theoretic translation for polymodal logics with finitely many modalities to encompass an infinite number of accessibility relations (each one corresponding to a different "grade"). As a matter of fact, graded modal logics are treated as a special case of a more general technique able to deal with polymodal logics with infinitely many modalities.

2 Graded modal logics

Graded modal logics have been introduced in the 60's by Goble [Goble, 1970], who proposed a logic with a fixed number of modalities, each one associated with a natural number and representing a different degree of necessity. As an example, the formula $N_n\varphi \wedge N_m\psi$, with $m > n$, espresses the fact that both φ and ψ are necessary, but ψ is *more* necessary than φ. This approach has been later generalized by Fine (cf. [Fine, 1972]) who, inspired by Tarskian numerical quantifiers, introduced modal operators associated with natural numbers: the so-called *graded* modal operators $\Box_n$ and $\Diamond_n$, with $n \in \mathbb{N}$. Finally, in the 80's, Fattorosi-Barnaba, De Caro, and Cerrato provided sound and complete axiomatizations of graded modal logics, together with some interesting decidability results [Fattorosi-Barnaba and De Caro, 1985; De Caro, 1988; Cerrato, 1990; 1992].

Graded modal logics allow one to express conditions on the number of objects satisfying a given property such as: "at least n elements (satisfying relation R) have property y". Formally, the basic system of graded modal logic $\overline{K}$ is an extension of K obtained by adding graded modalities. The language of $\overline{K}$ is obtained from the standard language of pure modal logic by substituting $\Box_n$ and $\Diamond_n$ ($n \in \mathbb{N}$) for $\Box$ and $\Diamond$. Formulae of $\overline{K}$ are:

$$Form(\overline{K}, \Phi) = p, q, \ldots, \neg\varphi, \varphi \vee \psi, \Diamond_n\varphi,$$

where $p, q, \ldots$ stand for propositional letters, and modal formulae are defined inductively as usual.

It is convenient to define the following abbreviation:

$$\Diamond!_n\varphi = \begin{cases} \neg\Diamond_0\varphi & \text{if } n = 0; \\ \Diamond_{n-1}\varphi \wedge \neg\Diamond_n\varphi & \textit{otherwise}, \end{cases}$$

whose intuitive meaning is that φ *holds at exactly n R-accessible worlds.*

$\overline{K}$ is a normal modal logic with respect to $\Box_0$, and it is characterized by the following axioms (cf. [Fattorosi-Barnaba and De Caro, 1985]):

A_1 $\vdash_{\overline{K}} \varphi$ for any propositional tautology φ;

A_2 $\Box_n\varphi \rightarrow \Box_{n+1}\varphi$;

A_3 $\Box_0(\varphi \rightarrow \psi) \rightarrow (\Diamond_n\varphi \rightarrow \Diamond_n\psi)$;

A_4 $\Box_0\neg(\varphi \wedge \psi) \rightarrow ((\Diamond!_n\varphi \wedge \Diamond!_m\psi) \rightarrow \Diamond!_{n+m}(\varphi \vee \psi))$,

and by the rules of modus ponens, substitution and necessitation:

MP $\vdash_{\overline{K}} \varphi \rightarrow \psi, \vdash_{\overline{K}} \varphi \Rightarrow \vdash_{\overline{K}} \psi$;

SUB $\vdash_{\overline{K}} \alpha \leftrightarrow \beta \Rightarrow \vdash_{\overline{K}} \varphi \leftrightarrow [\alpha/\beta]\varphi$;

N $\vdash_{\overline{K}} \varphi \Rightarrow \vdash_{\overline{K}} \Box_0\varphi$.

The semantics of $\overline{K}$ is given in terms of Kripke frames. In particular, the satisfiability relation is defined as usual over atomic formulae and boolean connectives, while the clauses for graded modalities are the following ones:

$w \models_{\overline{K}} \Diamond_n\varphi \Leftrightarrow |\{v \in W : wRv \wedge w \models_{\overline{K}} \varphi\}| > n$;

$w \models_{\overline{K}} \Box_n\varphi \Leftrightarrow |\{v \in W : wRv \wedge w \models_{\overline{K}} \neg\varphi\}| \leq n$.

It is worth noting that the standard modal operators $\Box$ and $\Diamond$ correspond to $\Box_0$ and $\Diamond_0$, respectively, thereby showing that $\overline{K}$ is an extension of K.

It can be showed that $\overline{K}$ is sound and complete with respect to the class of all Kripke frames.

THEOREM 2.1 *For any formula $\varphi \in Form(\overline{K}, \Phi)$, it holds that*

$$\vdash_{\overline{K}} \varphi \Leftrightarrow \models \varphi.$$

Soundness is proved by induction on the structural complexity of φ, while completeness is proved by using an argument *à la* Henkin.

The soundness and completeness proof given for $\overline{K}$ can be immediately generalized to deal with all those graded modal logics whose accessibility relation either excludes pairwise *distant* worlds (e.g. $\overline{K}$, $\overline{D}$, $\overline{T}$) or is transitive (e.g. all systems of graded modal logic over $\overline{S5}$). In order to prove the completeness of the remaining graded modal logics, it is necessary to work with a weakened notion of canonical model (cf. [Cerrato, 1990]).

3 A set-theoretic translation method

In [D'Agostino *et al.*, 1995] D'Agostino et al. proposed a set-theoretic translation method ($\Box$-as-*Pow* translation, from now on) to execute modal logics. The main idea underlying the $\Box$-as-*Pow* translation is to formalize the notion of validity in Kripke frames by a set-theoretic formula that is provable in the underlying set theory if and only if the original formula is modally derivable. According to the $\Box$-as-*Pow* translation, any Kripke frame *is* the set of its worlds and any world in a frame *is* the set of those worlds accessible from it. The theory driving the translation is a very weak, finitely (first-order) axiomatizable set theory, called Ω [D'Agostino *et al.*, 1995], whose axioms, in the language with relational symbols $\in$ and $\subseteq$, and functional symbols $\cup, \setminus$, and *Pow*, are:

$x \in y \cup z \iff x \in y \vee x \in z;$

$x \in y \setminus z \iff x \in y \wedge x \notin z;$

$x \subseteq y \iff \forall z(z \in x \rightarrow z \in y);$

$x \in Pow(y) \iff x \subseteq y.$

A peculiarity of the technique is the weakness of the theory Ω: it consists of only four axioms describing the most rudimentary and basic among the operators of naive set theory. In particular, notice that neither the extensionality axiom nor the axiom of foundation are in Ω. Given a modal formula $\phi(P_1, ..., P_n)$, its translation is defined as the set-theoretic *term* $\phi^*(x, x_1, ..., x_n)$, with variables $x, x_1, ..., x_n$, built using $\cup, \setminus$, and *Pow*. Intuitively, the term $\phi^*(x, x_1, ..., x_n)$ represents the set of those worlds (in the frame x) in which the formula ϕ holds. The inductive definition of $\phi^*(x, x_1, ..., x_n)$ is the following:

- $P_i^* = x_i$;
- $(\phi \vee \psi)^* = \phi^* \cup \psi^*$;
- $(\neg\phi)^* = x \setminus \phi^*$;
- $(\Box\phi)^* = Pow(\phi^*)$.

For all modal formulae ϕ, ψ, the following results hold, showing, respectively, the completeness and the soundness of the translation [D'Agostino *et al.*, 1995]:

$$\phi \vdash_{K_*} \psi \ \Rightarrow$$

$$\Omega \vdash \forall x(Trans(x) \wedge \forall \vec{z}(x \subseteq \phi^*(x, \vec{z})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z}))),$$

and

$$\Omega \vdash \forall x(Trans(x) \wedge \forall \vec{z}(x \subseteq \phi^*(x, \vec{z})) \rightarrow \forall \vec{z}(x \subseteq \psi^*(x, \vec{z})))$$

$$\Rightarrow \ \phi \models_f \psi,$$

where $Trans(x)$ stands for $\forall y\ (y \in x \rightarrow y \subseteq x)$. It is immediate to see that, for frame-complete theories, the above translation captures exactly the notion of K_*-derivability.

3.1 Polymodal logics with finitely many modalities

In [D'Agostino *et al.*, 1995], D'Agostino et al. also show how to generalize the $\Box$-as-*Pow* translation to polymodal logics. The basic idea is to mimic a polymodal frame, provided with finitely many accessibility relations, with a set, provided with the membership relation only. To this end, D'Agostino et al. defined an alternative semantics for polymodal logic, called p-semantics, that replaces the plurality of accessibility relations $\lhd_1, \ldots, \lhd_k$ by a single accessibility relation R and k copies $U_1, \ldots, U_k$ of the universe U. The p-semantics is formally defined as follows.

DEFINITION 3.1 *A p-frame $\mathcal{F}$ is a $(k+2)$-tuple $(U, U_1, \ldots, U_k, R)$, where $U, U_1, \ldots, U_k$ are sets and R is a binary relation on $U \cup U_1 \cup \ldots \cup U_k$, such that, for all u, v, t in $U \cup U_1 \cup \ldots \cup U_k$, if $u \in U$, uRv and vRt, then $t \in U$ (we will denote this property by $Trans^2(U)$).*

The intuition behind the notion of p-frame is the following one. Consider two worlds $w, w' \in U$ such that $w \lhd_i w'$ in the original polymodal frame. Since only one accessibility relation is available in p-frames, we cannot directly access w' from w (via R) anymore. However, we can follow a two-step path: first we move to (the unique) $w_i \in U_i$ R-accessible from w; then we move from w_i to w' via R.

A *p-valuation* $\models_p$ is a subset of $U \times \Phi$, where Φ is the set of propositional variables. In the case of boolean combinations, the p-valuation $\models_p$ may be lifted to the set of all polymodal formulae in the canonical fashion. In the case of $\Box_i$, with $i = 1, \ldots, k$, for all $u \in U$ we put

$$u \models_p \Box_i\phi \Leftrightarrow \forall v(uRv \wedge v \in U_i \rightarrow \forall t(vRt \rightarrow t \models_p \phi)).$$

A polymodal formula ϕ is p-valid in a p-frame $(U, U_1, \ldots, U_k, R)$ if and only if for all p-valuations $\models_p$ and all worlds $u \in U$, $u \models_p \phi$ holds.

The link between polymodal frames and p-frames is formally expressed by the following theorem [D'Agostino *et al.*, 1995]:

THEOREM 3.2 *If ψ, ϕ are polymodal formulae, then $\psi \models \phi$ if and only if ϕ is p-valid in all p-frames in which ψ is p-valid.*

A set-theoretic counterpart of p-frames can be easily given in the standard way. As far as the translation of the modal operators is concerned, on the ground of the definition of $\models_p$, the set-theoretic semantics of $\Box_i\phi$ becomes:

$$(\Box_i\phi)^* \equiv Pow((\bar{x} \setminus y_i) \cup Pow(\phi^*)),$$

where $\bar{x} = x \cup y_1 \cup \ldots \cup y_k$.

THEOREM 3.3 *Let H be a k-dimensional polymodal logic extending $K \otimes \ldots \otimes K$ with the axiom schema $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$. For any polymodal formula ϕ,*

(soundness)

$$\Omega \vdash \forall x \forall y_1 \ldots \forall y_k(Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \rightarrow \forall \vec{z}(x \subseteq \phi^*(x, y_1, \ldots, y_k, \vec{z}))) \ \Rightarrow \ \psi \models \phi$$

(completeness)

$$\vdash_H \phi \Rightarrow \Omega \vdash \forall x \forall y_1 \ldots \forall y_k(Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \rightarrow \forall \vec{z}(x \subseteq \phi^*(x, y_1, \ldots, y_k, \vec{z}))),$$

where
$Axiom_H(x, y_1, \ldots, y_k)$ *is* $\forall \vec{y}(x \subseteq \psi^*(x, y_1, \ldots, y_k, \vec{y})))$, *and* $Trans^2(x)$ *stands for* $\forall y \forall z\ (y \in z \wedge z \in x \rightarrow y \subseteq x)$, *that is,* $x \subseteq Pow(Pow(x))$.

4 Translating graded modalities

The general scheme followed for applying the set theoretic translation is the one suggested by [Ohlbach *et al.*, 1995]: a two-step translation that first transforms graded modal logic into a polymodal logic with infinitely many modalities, and then reduces derivability in such a polymodal logic to derivability in a suitable first-order set theory.

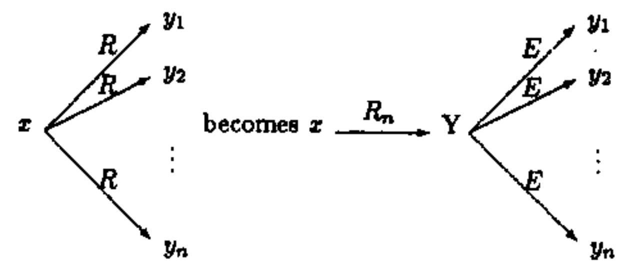
A graded modal logic expresses properties of different (infinitely many) modalities which are all referring to the same accessibility relation. In other words, infinitely many Kripke semantics are provided over the same accessibility relation scheme. The task of the first step of the translation is that of rewriting the semantics of a graded modal logic in such a way to introduce a different accessibility relation for each different modality. Once this step has been performed, the next task is to generalize the existing translation for polymodal logics with finitely many modalities, to the case of infinitely many ones.

The advantange of using the $\Box$-as-*Pow* translation is its ability of dealing with non-first-order axiomatizable polymodal logics. This fact has two important consequences: on the one hand, it allows one to naturally translate the polymodal counterpart of $\overline{K}$ ($\overline{K}_E$), which is indeed non-first-order axiomatizable; on the other hand, it can be applied to non-first-order axiomatizable logics over $\overline{K}$. A further advantage is the fact that the technique introduced is very general and can in fact be employed to translate polymodal logics with infinitely many modalities.

4.1 Polymodal logics with infinitely many modalities

According to Kripke semantics, a $\overline{K}$-formula of the form $\Diamond_n \varphi$ is true at a given world x of a frame $\mathcal{F} = (W, R)$ if and only if there exists $Y \subseteq R(x)$ of cardinality greater than n and such that φ is true at any world $y \in Y$.

An alternative interpretation for $\overline{K}$ can be obtained introducing a new class of worlds, denoted by W_Y, representing *sets* of accessible worlds ($W_Y \subseteq Pow(W)$). The single accessibility relation R can now be replaced by the denumerable set of relations $\{R_n : n \in \mathbb{N}\}$, where R_n associates a given world x with those elements of W_Y having cardinality greater than n. A further accessibility relation E will be used to associate a given element of W_Y with its elements. The situation is described by the following picture:



The alternative semantics described above suggests the introduction of the following modal logic $\overline{K}_E$. The language of $\overline{K}_E$ is obtained from that of $\overline{K}$ by substituting $\langle n \rangle$, $[n]$, $\Diamond$, and $\Box$ for $\Diamond_n$ and $\Box_n$. Formulae are defined as usual, and denoted by:

$$Form(\overline{K}_E, \Phi) = p, q, \ldots, \neg\varphi, \varphi \vee \psi, \langle n \rangle \varphi, \Diamond\varphi.$$

The intuitive meaning of the newly introduced symbols is the following:

$\langle n \rangle \varphi$ is true at a given world if and only if there exists an R_n-accessible world where φ holds;

$\Diamond\varphi$ is true at a given world if and only if there exists an E-accessible world where φ holds.

It is immediate to see that $\Diamond_n p$ and $\Box_n p$ correspond to $\langle n \rangle \Box p$ and $[n] \Diamond p$, respectively. Clearly, on the ground of this definition, $\overline{K}_E$ can turn out much more expressive than $\overline{K}$, since one can combine the modal operators of $\overline{K}_E$ arbitrarily.

DEFINITION 4.1 *Axioms and rules of* $\overline{K}_E$ *are the following:*

N1 propositional logic axioms together with modus ponens;

N2 the axioms of K for $[n]$ *and* $\Box$*:*
$$[n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi),$$
$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi);$$

N3 the rule of necessitation for $[n]$ *and* $\Box$*:*
$$\text{if} \vdash_{\overline{K}_E} \varphi \text{ then} \vdash_{\overline{K}_E} [n]\varphi,$$
$$\text{if} \vdash_{\overline{K}_E} \varphi \text{ then} \vdash_{\overline{K}_E} \Box\varphi;$$

N4 $[0]\Diamond\varphi \rightarrow [n]\Diamond\varphi$;

N5 $\langle n \rangle \Box\varphi \rightarrow \langle n \rangle \Diamond\varphi$;

N6 $[n]\varphi \rightarrow [n+1]\varphi$;

N7 $\langle n+m \rangle \Box(\varphi \vee \psi) \rightarrow (\langle n \rangle \Box\varphi \vee \langle m \rangle \Box\psi)$;

N8 $(\langle n \rangle \Box(\varphi \wedge \psi) \wedge \langle m \rangle \Box(\varphi \wedge \neg\psi)) \rightarrow \langle n+m+1 \rangle \Box\varphi$.

The presence of axioms N1-N3 ensures the possibility to give a semantics to $\overline{K}_E$ simply considering it as a particular polymodal logic.

It can be shown (cf. [Ohlbach *et al.*, 1995]) that $\overline{K}_E$ is not first-order axiomatizable.

The next question to answer is relative to the soundness and completeness of $\overline{K}_E$ with respect to the chosen semantics. The soundness of $\overline{K}_E$ can be easily established. As for completeness, a partial result can be obtained making use of the following translation function:

DEFINITION 4.2 Let $\Pi : \mathcal{L}(\overline{K}, \Phi) \to \mathcal{L}(\overline{K}_E, \Phi)$ be the function that maps formulae of $\overline{K}$ in formulae of $\overline{K}_E$ according to the following rules:

$\Pi(p) = p$ for all $p \in \Phi$;

$\Pi(\neg\varphi) = \neg\Pi(\varphi)$;

$\Pi(\varphi \circ \psi) = \Pi(\varphi) \circ \Pi(\psi)$ per $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$;

$\Pi(\Diamond_n \varphi) = \langle n \rangle \Box \Pi(\varphi)$;

$\Pi(\Box_n \varphi) = [n] \Diamond \Pi(\varphi)$.

The above translation function is sound and complete (cf. [Ohlbach *et al.*, 1995]):

THEOREM 4.3 *For any formula* $\varphi \in \mathcal{L}(\overline{K}, \Phi)$

$$\vdash_{\overline{K}} \varphi \textit{ if and only if } \vdash_{\overline{K}_E} \Pi(\varphi).$$

From the above result, we derive the completeness of $\overline{K}_E$ for the fragment of the language consisting of the translation of $\overline{K}$-formulae (cf. [Ohlbach *et al.*, 1995]):

THEOREM 4.4 *For any formula* $\varphi \in \mathcal{L}(\overline{K}, \Phi)$

$$\textit{if } \models_{\overline{K}_E} \Pi(\varphi) \textit{ then } \vdash_{\overline{K}_E} \Pi(\varphi).$$

The above theorem guarantees the possibility of using the system $\overline{K}_E$ as an intermediate system for the translation described in the next section.

4.2 The set-theoretic translation of graded modal logics

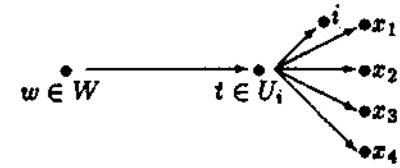
The logic $\overline{K}_E$ can be seen as the extension of a normal modal logic $K_{\overline{\mathcal{R}}}$, where $\overline{\mathcal{R}}$ is the set of accessibility relations $\{R_i : i \in \mathbb{N}\} \cup \{E\}$, with the axiom $\Psi = $ N4 $\wedge \ldots \wedge$ N8. From a general point of view, the problem we want to solve is to design a (set-theoretic) translation method that can be applied to a logic $\tilde{H}$ extending $K_{\overline{\mathcal{R}}}$ with a (possibly non-first-order) axiom $\psi(p_{j_1}, \ldots, p_{j_m})$. $\overline{K}_E$ will thus be considered as a particular case in which $\psi(p_{j_1}, \ldots, p_{j_m})$ is Ψ.

A frame for $\tilde{H}$ is a structure $\mathcal{F} = (W, \{R_i\}_{i \in \mathbb{N}^E})$, where $\mathbb{N}^E$ stands for $\mathbb{N} \cup \{E\}$. Since $\tilde{H}$ is a polymodal logic with infinitely many modalities, in order to apply the set-theoretic translation method is necessary to define a semantics that allows us to consider only one accessibility relation, to be interpreted by the membership relation $\in$. The fact that we must deal with infinitely many modalities implies that we cannot use the technique introduced in [D'Agostino *et al.*, 1995]. The key definition for the proposed semantics is the following notion of *p-frame*.

DEFINITION 4.5
Given a frame $\mathcal{F} = (W, \{R_n\}_{n \in \mathbb{N}^E})$ *for* $\tilde{H}$, *a p-frame is a pair* (S, R), *where:*

- $S = W \cup \bigcup_{i \in \mathbb{N}^E} U_i \cup \mathbb{N}^E$, *where the* U_i*'s are pairwise distinct copies of* W;
- R *is a binary relation on* S *such that*
  - *i.* $\forall w \in W, t \in S \setminus (W \cup \mathbb{N}^E), x \in S\ (wRt \wedge tRx \rightarrow x \in W \cup \mathbb{N}^E)$;
  - *ii.* $\forall w \in W, i \in \mathbb{N}^E$ *it is not the case that* wRi *and* $\exists t \in S \setminus (W \cup \mathbb{N}^E)\ (wRt \wedge tRi)$;
  - *iii.* $\forall w \in W, t, u, \in S \setminus (W \cup \mathbb{N}^E), i \in \mathbb{N}^E\ (wRt \wedge wRu \wedge tRi \wedge uRi \rightarrow t = u)$;
  - *iv.* $\forall t \in S \setminus (W \cup \mathbb{N}^E), i, j \in \mathbb{N}^E\ (tRi \wedge tRj \rightarrow i = j)$.

For example, the elements $x_1, \ldots, x_4$ below are those in relation R_i with w. The element t of U_i is introduced to simulate such a relation, and the element i is used to determine the index of the relation.



As for the semantics described in the case of finitely many modalities, a p-valuation $\models_p$ is a subset $W \times \Phi$ assigning truth values to propositional variables at W-worlds only. The extension of $\models_p$ to all $\tilde{H}$-formulae is inductively defined as follows:

$w \models_p \neg\varphi$ if and only if $w \not\models_p \varphi$;

$w \models_p \varphi \vee \psi$ if and only if $w \models_p \varphi$ or $w \models_p \psi$;

$w \models_p [i]\varphi$ if and only if $\forall t(wRt \wedge tRi \rightarrow \forall v(tRv \wedge v \neq i \rightarrow v \models_p \varphi))$.

The following results guarantee that the proposed semantics can be safely used in place of the usual one:

LEMMA 4.6 *Given a p-frame* (S, R), *there exists a frame* $(W, \{R_i\}_{i \in \mathbb{N}^E})$ *in which are valid all the formulae p-valid in* (S, R).

LEMMA 4.7 *For any frame* $(W, \{R_i\}_{i \in \mathbb{N}^E})$, *there exists a p-frame* (S, R) *in which are p-valid all the formulae valid in* $(W, \{R_i\}_{i \in \mathbb{N}^E})$.

Any p-frame for $\tilde{H}$ can be embedded in a model of a suitable set theory in such a way that W is a set, and any $w \in W$ is mapped in a set of elements of w_i's for $i \in \mathbb{N}^E$. Any w_i is of the form $w_i = \{v : wR_iv\} \cup \{i\}$.

The translation for propositional letters and boolean connectives is defined as usual:

$P_i^* = x_i$;

$(\neg\varphi)^* = x \setminus \varphi^*$;

$(\varphi \vee \psi)^* = \varphi^* \cup \psi^*$.

The translation of the different modalities must be given according to the p-semantics and replacing the accessibility relation by the membership relation:

$$w \in ([i]\varphi)^* \Leftrightarrow \forall t(t \in w \wedge i \in t \rightarrow \forall v(v \in t \setminus \{i\} \rightarrow v \in \varphi^*)),$$

from which it can be easily checked that the translation of $[i]\varphi$, for $i \in \mathbb{N}^E$, must be defined as follows:

$$([i]\varphi)^* = Pow((S \setminus Rng(\{i\} \times_\in S)) \cup Pow(\varphi^* \cup \{i\})).$$

Let $\overline{\Omega}$ be the sub-theory of the theory Ω_c introduced in [Benthem *et al.*, 1995] and defined as follows:

$$\overline{\Omega} = \Omega + \{\} + \times_{\in} + Rng.$$

The following two theorems state the completeness and the soundness of the proposed translation with respect to $\overline{\Omega}$:

THEOREM 4.8 (Completeness) *For any modal formula* $\varphi \in \mathcal{L}(K_{\overline{\mathbb{N}}})$ *we have that*

$$\vdash_{\tilde{H}} \varphi \quad \Rightarrow \quad \overline{\Omega} \vdash \forall y \forall x(Trans^2(x,y) \wedge Axiom_{\tilde{H}}(x,y) \rightarrow \forall \vec{z}(x \subseteq \varphi^*(x,y,\vec{z}))),$$

where $Trans^2(x,y)$ *stands for the conjunction of the clauses on R introduced in the definition of p-frame, and* $Axiom_{\tilde{H}}(x,y)$ *is the translation of the formula consisting of the conjunction of the axioms of* $\tilde{H}$.

THEOREM 4.9 (Soundness) *For any modal formula* $\varphi \in \mathcal{L}(K_{\overline{\mathbb{N}}})$

$$\overline{\Omega} \quad \vdash \quad \forall y \forall x(Trans^2(x,y) \wedge Axiom_{\tilde{H}}(x,y) \rightarrow \forall \vec{z}(x \subseteq \varphi^*(x,y,\vec{z}))) \Rightarrow \models_{\tilde{H}} \varphi.$$

Once we have defined the set-theoretic translation in the context of a generic extension $\tilde{H}$ of $K_{\overline{\mathbb{N}}}$, the result for $\overline{K}_E$ follows as a special case:

$$\overline{\Omega} \quad \vdash \quad \forall y \forall x(Trans^2(x,y) \wedge Axiom_{\overline{K}_E}(x,y) \rightarrow \forall \vec{z}(x \subseteq \varphi^*(x,y,\vec{z}))) \Rightarrow \models_{\overline{K}_E} \varphi,$$

$$\vdash_{\overline{K}_E} \varphi \quad \Rightarrow \quad \overline{\Omega} \vdash \forall y \forall x(Trans^2(x,y) \wedge Axiom_{\overline{K}_E}(x,y) \rightarrow \forall \vec{z}(x \subseteq \varphi^*(x,y,\vec{z}))),$$

where $Axiom_{\overline{K}_E}(x,y)$ is the translation of Ψ, namely $\forall \vec{z}(x \subseteq \Psi^*(x,y,\vec{z}))$.

Notice that the soundness of the translation is stated with respect to the validity in $\overline{K}_E$, since we do not have the completeness of $\overline{K}_E$ with respect to derivability. However, using the translation function Π we can prove the following theorem:

THEOREM 4.10 *For any formula* $\varphi \in \mathcal{L}(\overline{K})$ *we have that:*

$$\vdash_{\overline{K}} \varphi \quad \Leftrightarrow \quad \overline{\Omega} \vdash \forall x \forall y(Trans^2(x,y) \wedge Axiom_{\overline{K}_E}(x,y) \rightarrow \forall \vec{z}(x \subseteq (\Pi(\varphi))^*(x,y,\vec{z}))).$$

5 Conclusions and further directions

In this paper, we generalized the D-as-Pow translation, proposed by D'Agostino et al. in [D'Agostino *et al*, 1995], to apply it to graded modal logic. The resulting method allows us to support automated reasoning in a large class of knowledge representation formalisms that can be reduced to graded modal logic. It can actually be applied to polymodal logics with infinitely many modalities. Indeed, there are no axioms in the underlying set theory constraining the behavior of the different modalities; such a behavior is governed by (the translation of) the axioms of the considered polymodal logic. As an example, it can be exploited to execute two-sorted metric temporal logics [Montanari and de Rijke, 1995], provided that they are reinterpreted as (a special kind of) propositional dynamic logics.

References

[Benthem *et al*, 1995] J. van Benthem, G. D'Agostincv A. Montanari, and A. Policriti. Modal deduction in second-order logic and set theory. Technical Report ML-95-02, Institute for Logic Language and Information, 1995.

[Benthem, 1985] J. van Benthem. *Modal Logic and Classical Logic.* Bibliopolis, Napoli, Italy, 1985.

[Cerrato, 1990] C. Cerrato. General canonical models for graded normal logics (graded modalities IV). *Studia Logtea,* 49:241-252, 1990.

[Cerrato, 1992] C. Cerrato. Decidability by filtration for graded modal logics (graded modalities V), *Studia Logtea,* 53:61-74, 1992.

[D'Agostino *et al*, 1995] G. D'Agostino, A. Montanari, and A. Policriti. A set-theoretic translation method for polymodal logics. *Journal of Automated Reasoning,* 15:317-337, 1995.

[De Caro, 1988] F. De Caro. Graded modalities II (canonical models). *Studia Logica,* 47:1-10, 1988.

[Fattorosi-Barnaba and De Caro, 1985] M. Fattorosi-Barnaba and F. De Caro. Graded modalities I. *Studia Logica,* 44:197-221, 1985.

[Fine, 1972] K. Fine. In so many possible worlds. *Notre Dame Journal of Formal Logic,* 13:516-520, 1972.

[Fitting, 1983] M. Fitting. *Proofs Methods for Modal and Intuitionistic Logics.* D. Reidel Pub. Comp., Dordrecht, Boston, and Lancaster, 1983.

[Gabbay and Ohlbach, 1992] D. M. Gabbay and H. J. Ohlbach. Quantifier elimination in second-order predicate logic. In Morgan Kaufmann, editor, *Proc. of the* 4^{th} *International Conference on Principles of Knowldedge Representation and Reasoning, KR'92,* pages 425-436, 1992.

[Goble, 1970] L. F. Goble. Grades of modality. *Logique et Analyse,* 13:323-334, 1970.

[Hoek and de Rijke, 1995] W. van der Hoek and M. de Rijke. Counting objects. *Journal of Logic and Computation,* 5(3):325-345, 1995.

[Montanari and de Rijke, 1995] A. Montanari and M. de Rijke. Completeness results for two-sorted metric temporal logics. In Springer Verlag, editor, *Proc. of the* 4^{th} *International Conference on Algebraic Methodology and Software Technology, LNCS 936,* pages 385-399, 1995.

[Nonnengart, 1993] A. Nonnengart. First-order modal logic theorem proving and functional simulation. In *Proc. of* 13^{th} *International Joint Conference on Artificial Intelligence, IJCAI-93,* pages 80-85, Chambery, France, 1993.

[Ohlbach *et al*, 1995] H. J. Ohlbach, A. Schmidt, and U. Hustadt. Translating graded modalities into predicate logic. Technical Report MPI-I-95-2-008, Max-Planck-Institut fur Informatik, Saarbruken, Germany, 1995.

[Ohlbach, 199l] H. J. Ohlbach. Semantic-based translation methods for modal logics. *Journal of Logic and Computation,* 1(5), 1991.

[Ohlbach, 1993] H. J. Ohlbach. Translation methods for non-classical logics: An overview. *Bull, of the IGLP,* l(l):69-89, 1993.

[Wansing, 1994] H. Wansing. Sequent calculi for normal modal propositional logics. *Journal of Logic and Computation,* 4(2):125-142, 1994.