# A NEW LOGIC OF BELIEFS: MONOTONIC AND NON-MONOTONIC BELIEFS - Part I

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## Abstract

We present here a new formalization of beliefs, which has a simpler semantics than the previous formalizations, and develop an inference method for it by generalizing the resolution method. The usual prepositional formulas are embedded in our logic as a special type of belief formulas. One can obtain a non-monotonic logic of beliefs by applying, say, circumscription to the basic belief-logic developed here which is monotonic in nature. One can also apply the technique repeatedly to construct a hierarchy of belief-logics $BL_k$, $k \geq 1$, where $BL_k \supset BL_{k-1}$ and $BL_k$ can handle formulas involving up to level k nested applications of the belief operator B.

## 1. Introduction

The fundamental assumption in this paper is that the distinction between the notion of truth of the belief of a proposition P and that of P lies in the underlying contexts of worlds. In the case of beliefs, the context is a set of worlds whereas in the case of propositions the context is a single world. We say that P is *believed* by an agent in the context W, which is a set of worlds, if P is true in every world $W_j \in$ W. Put another way, the agent believes in any thing unless there is evidence to the contrary. Why does the context being a set of worlds make a significant difference? First, the meaning of negation as in $B\neg P$ is now different from the meaning of negation as in ~BP.[1] By $B\neg P$ = "believe in negation of P", we mean that P is false in each $w_j \in W$, which is quite different from ~BP, meaning that P is false for at least one $w_j \in W$. Thus, $B\neg P \neq$ ~BP. Second, it is possible that an agent believes neither P nor $\neg P$. It is also possible that he believes both P and $\neg P$ simultaneously, i.e., assert $BP \wedge B\neg P$, which can happen only in the extreme case $W = \emptyset$. Note that every thing is believed when $W = \emptyset$, including the propositional false-formula, denoted by □.

[1] We use "~" for negations applied to beliefs, to distinguish it from the negation "¬" applied to propositions] formulas.

In defining a logic for beliefs, one of the first questions that arises is what is its relationship to the standard logic. For example, if $\phi_1$ and $\phi_2$ are two equivalent prepositional formulas, then is $B\phi_1$ considered to be equivalent to $B\phi_2$? More generally, if $\phi_1$ is believed and $\phi_1$ logically implies $\phi_2$, then should $\phi_2$ be also believed? Note that if the answer to the second question is 'yes', then the same is true for the first question. For the belief-logic defined here, both the answers are 'yes*. A different belief-logic is defined in [Levesque, 1984] specifically to allow $B\phi_1$ not to be equivalent to $B\phi_2$ even though $\phi_1$ and $\phi_2$ may be equivalent as propositional formulas. This is achieved by considering a general notion of a world $W_j$ in which the truth value of a proposition P may be true (T), false (F), undefined, or simultaneously true and false (i.e., P does not have a unique truth value). These general worlds are used for modeling the "explicit" beliefs whereas the "implicit" beliefs are modeled (for the most part) by the standard worlds, with each proposition having a unique truth value.

For the logic of belief described here, only the standard worlds are considered. We do not distinguish thus the explicit beliefs from the implicit beliefs. More importantly, we consider each propositional formula $\phi$ as a special kind of belief-formula $\phi'$ whose truth value is evaluated in the same way as that of the general belief-formulas. This is not the case in [Levesque, 1984]. Another interesting property of the belief-logic given here is that one can apply the construction repeatedly to obtain a hierarchy of belief-logics $BL_k$, $k \geq 1$, where $BL_k$ can handle formulas involving up to level k nested applications of the belief-operator B. Thus, the formula $B(\phi' \vee B\psi)$ - note the use $\phi'$ instead of $\phi$ - can be handled in $BL_2$. The belief-logic in [Levesque, 1984] can consider such nested formulas directly. However, we feel that our formulation of belief-logic is actually more natural in that it explicitly recognizes the inherent higher complexity of the formulas with higher nested levels of B and handles them in a higher level (larger value of k) logic $BL_k$.

We give here a simple inference technique for the new belief-logic ($= BL_1$) by generalizing the resolution method in propositional logic. A similar generalization of the resolution rules for $\mathbf{BL}_{k-1}$ immediately gives the resolution rules for $\mathbf{BL}_k$, $k \geq 2$. In Part-I, we present only the basic monotonic form of the belief-logic $\mathbf{BL}_1$. The construction of a non-monotonic form of $\mathbf{BL}_1$ (and other $\mathbf{BL}_k$, $k > 1$) by using the circumscription is presented elsewhere [Kundu, 1991b]. Note that the separation of the non-monotonic aspects from the monotonic aspects of beliefs makes it easier to model a world and control the inferencing in it.

## 2. Basic Notions

We formally define a formula of the form $B\phi$, where $\phi$ is a propositional formula, to be an *atomic* belief-formula. A general *belief-formula* is defined to be an atomic belief-formula or a formula which is obtained by logical combination of other belief-formulas via $\wedge$, $\vee$, and $\sim$. We use "$\rightarrow$" as an abbreviation for the usual logical combination of $\vee$ and $\neg$ (or $\sim$). Thus, $\sim B\neg P \vee B(P \rightarrow Q)$ is a belief-formula, but $BB\neg P$ is not. Note that the operator B is considered to have higher precedence than $\wedge$, $\vee$, and $\rightarrow$.

Let $\{P_1, P_2, ..., P_n\}$ be the basic propositions in our universe of discourse. We denote by $\Omega_n$ the set of $2^n$ individual worlds defined by the combination of T/F values of the $P_j$'s. For $W \subseteq \Omega$, we say $B\phi$ has the truth value T at W if $\phi$ is true at each $w_j \in W$. In that case, the set W is said to be a *model* of $B\phi$. The truth value of a complex belief-formula is defined in the usual way by combining the truth values of its subformulas via the connectives $\{\wedge, \vee, \neg$, and $\rightarrow\}$. In particular, $\sim B\phi$ is true at W if and only if $\phi$ is false at at least one world $w_j \in W$. Clearly, the only model of $B\Box$ is the empty subset $\varnothing \subseteq \Omega$ and the models of $\sim B\Box$ are all non-empty subsets of $\Omega$. On the other hand, the models of $B\phi \wedge \sim B\Box$ are the set of all non-empty models of $B\phi$, i.e, the non-empty subsets of the set of models of $\phi$; the only model of $B\phi \wedge B\neg\phi$ is $W = \varnothing$. We write $\Omega(\beta)$ for the set of models of a belief-formula $\beta$ and write $\Omega(\phi)$ also for the set of models of a propositional formula $\phi$. By abuse of notation, we let $\Box$ denote also the false belief-formula, which should not cause any confusion; similarly, we write $T = \sim\Box$. The belief-formula $\beta$ is said to be a *tautology* or valid if $\Omega(\beta) = 2^{\Omega}$. It is easy to see that $B\phi$ is valid if and only if $\phi$ is a valid propositional formula. The notion of satisfiability of a set of belief-formulas and its models are defined in the usual way.

We say a belief-formula $\beta_1$ is *equivalent* to another belief-formula $\beta_2$ if they have the same models, in which case we write $\beta_1 = \beta_2$. Table 1 shows some simple equivalences of belief-formulas. Note that $B\Box \neq \Box$ and, in general, $B\phi_1 \vee B\phi_2 \neq B(\phi_1 \vee \phi_2)$.

We say $\beta_1$ *implies* $\beta_2$ or $\beta_2$ can be *inferred* from $\beta_1$, if $\beta_1 \rightarrow \beta_2$ is valid. In symbols, we write $\beta_1 \Rightarrow \beta_2$. It is clear that $B\phi_1 \vee B\phi_2$ implies $B(\phi_1 \vee \phi_2)$, but as we noted above the converse may not be true. One can derive $[B\phi_1 \vee B\phi_2] \Rightarrow B[\phi_1 \vee \phi_2]$ by repeated application of Theorem 1 below. First, since $\phi_1 \Rightarrow \phi_1 \vee \phi_2$, we get $B\phi_1 \Rightarrow B[\phi_1 \vee \phi_2]$ and, similarly, $B\phi_2 \Rightarrow B[\phi_1 \vee \phi_2]$. In propositional logic, we

TABLE 1. Some simple equivalences of belief-formulas.

| | |
|---|---|
| $B\phi_1 \wedge B\phi_2$ | $= B(\phi_1 \wedge \phi_2)$ |
| $B\phi_1 \wedge (B\phi_2 \vee B\phi_3)$ | $= [B\phi_1 \wedge B\phi_2] \vee [B\phi_1 \wedge B\phi_3]$ |
| $B\phi_1 \vee (B\phi_2 \wedge B\phi_3)$ | $= [B\phi_1 \vee B\phi_2] \wedge [B\phi_1 \vee B\phi_3]$ |
| $\sim\sim B\phi_1$ | $= B\phi_1$ |
| $\sim[B\phi_1 \wedge B\phi_2]$ | $= \sim B\phi_1 \vee \sim B\phi_2$ |
| $\sim[B\phi_1 \vee B\phi_2]$ | $= \sim B\phi_1 \wedge \sim B\phi_2$ |
| $B\phi_1 \wedge B\phi_2$ | $= B\phi_2 \wedge B\phi_1$ |
| $B\phi_1 \vee B\phi_2$ | $= B\phi_2 \vee B\phi_1$ |
| $B\Box \vee B\phi$ | $= B\phi$ |
| $\sim B\neg\Box$ | $= \Box$ (i.e., $BT = T$) |

also have $\phi_1 \Rightarrow \psi$ and $\phi_2 \Rightarrow \psi$ implies $[\phi_1 \vee \phi_2] \Rightarrow \psi$. The same is true if $\phi_1$, $\phi_2$, and $\psi$ are replaced by arbitrary belief-formulas. By putting $B\phi_1$ in place of $\phi_1$, $B\phi_2$ in place of $\phi_2$, and $B[\phi_1 \vee \phi_2]$ in place of $\psi$, we get that $B\phi_1 \vee B\phi_2 \Rightarrow B[\phi_1 \vee \phi_2]$. The following Modus-ponens for belief-formulas is also immediate from the definitions.

> If $\beta_1$ and $\beta_1 \rightarrow \beta_2$ are valid belief-formulas,
> then $\beta_2$ is a valid belief-formula.

**Theorem 1.** For any two propositional formulas $\phi_1$ and $\phi_2$, we have

(1) $\phi_1 \Rightarrow \phi_2$ if and only if $B\phi_1 \Rightarrow B\phi_2$ if and only if $\sim B\phi_2 \Rightarrow \sim B\phi_1$.
(2) $\sim B(\phi_1 \vee \phi_2) \Rightarrow \sim B(\phi_1) \wedge \sim B(\phi_2)$.
(3) $B(\neg\phi_1) \wedge \sim B\Box \Rightarrow \sim B\phi_1$.

Proof. Immediate from the definitions. ■

The following theorem shows that belief formulas are sufficiently powerful to describe any family K of subsets of $\Omega$, i.e., $K \subseteq 2^{\Omega}$. In particular, if $N = |\Omega| = 2^n$, then the number of distinct (non-equivalent) belief-formulas over the propositions $\{P_1, P_2, ..., P_n\}$ is $2^{2^N}$ whereas the number of distinct propositional formulas is only $2^N$.

**Theorem 2.** If K is an arbitrary family of subsets of $\Omega$, then there is a belief-formula $\beta_K$ whose set of models equals K, i.e., $\Omega(\beta_K) = K$. Moreover, $\beta$ is unique up to equivalence.

Proof. If $K = \varnothing$, then we take $\beta_K = \Box$. Now let $K = \{W_1, W_2, ..., W_m\}$. We first show that there is a belief-formula $\beta_W$ which has a unique model W for any $W \subseteq \Omega$. If $W = \varnothing$, then $\beta_W = B\square$. Now assume that $W \neq \varnothing$. For each world $w_j \in W = \{w_1, w_2, ..., w_q\}$, let $C_j$ be the conjunction of the form $Z_1 \wedge Z_2 \wedge ... \wedge Z_n$, where each $Z_i = P_i$ or $\neg P_i$, such that the propositional formula $C_j$ has exactly one model $w_j$. The belief-formula $\beta_W = B(C_1 \vee C_2 \vee ... \vee C_q) \wedge {\sim}B\neg C_1 \wedge {\sim}B\neg C_2 \wedge ... \wedge {\sim}B\neg C_q$ then has the unique model W. If we write $\beta_j$ for the formula whose unique model is $W_j \in K$, then $\beta_K = \beta_1 \vee \beta_2 \vee ... \vee \beta_m$ satisfies the theorem. The uniqueness of $\beta_K$ is immediate. ■

The belief-formula $\beta_K$ obtained in the proof of Theorem 2 may be called the *disjunctive normal form* of a belief-formula whose models are $K = \{W_1, W_2, ..., W_m\}$. Example 1 shows some families of subsets of $\Omega_2$ and the associated disjunctive normal form belief-formulas. The proof of Theorem 2 suggests the notation $N\phi = {\sim}B\neg C_1 \wedge {\sim}B\neg C_2 \wedge ... \wedge {\sim}B\neg C_q$, where $\phi = C_1 \vee C_2 \vee ... \vee C_q$ is a disjunctive normal form propositional formula. The formula $N\phi$, which may be read as "necessarily contains (or implied by) $\phi$", has the models $\{W: \Omega(\phi) \subseteq W\}$. In particular, the models of $N\square$ is $2^\Omega$ and thus $N\square = \mathbf{T}$.

**Example 1.** Consider the universe of discourse consisting of two proposition $\{P_1, P_2\}$. Thus $\Omega = \{w_0, w_1, w_2, w_3\}$, where $w_0$ corresponds to $P_1 = F = P_2$, $w_1$ corresponds to $P_1 = F$ and $P_2 = T$, etc. Table 2 shows some of the families $K \subseteq 2^\Omega$ and their associated disjunctive normal form belief-formulas $\beta_K$. Here, we write $P_1 \wedge P_2$ in short as $P_1P_2$, $\neg P_1 \wedge P_2$ as $\neg P_1P_2$, etc. ■

TABLE 2. Examples of disjunctive normal form belief-formulas in the propositions $P_1$ and $P_2$.

| $K \subseteq 2^\Omega$ | The disjunctive normal form $\beta_K$ |
|---|---|
| $\varnothing$ | $\square$ |
| $\{\varnothing\}$ | $B\square$ |
| $\{\{w_0\}\}$ | $B(\neg P_1\neg P_2) \wedge {\sim}B(P_1 \vee P_2)$ |
| $\{\{w_0\}, \{w_1\}\}$ | $[B(\neg P_1\neg P_2) \wedge {\sim}B(P_1 \vee P_2)] \vee$ $[B(\neg P_1P_2) \wedge {\sim}B(P_1 \vee \neg P_2)]$ |
| $\{\{w_0, w_1\}\}$ | $B(\neg P_1\neg P_2 \vee \neg P_1P_2) \wedge$ ${\sim}B(P_1 \vee P_2) \wedge {\sim}B(P_1 \vee \neg P_2)$ |
| $\{\{w_0\}, \{w_0, w_1\}\}$ | $[B(\neg P_1\neg P_2) \wedge {\sim}B(P_1 \vee P_2)] \vee$ $[B(\neg P_1\neg P_2 \vee \neg P_1P_2) \wedge {\sim}B(P_1 \vee P_2)$ $\wedge {\sim}B(P_1 \vee \neg P_2)]$ |

In view of Theorem 2, we can associate a belief-formula $\phi'$ with each propositional formula $\phi$ in an one-to-one fashion such that the set of models of $\phi'$ is given by $\Omega(\phi') = \{\{w_j\}: w_j \in \Omega(\phi)\}$. In particular, $\square' = \square$ though $\mathbf{T}' \neq \mathbf{T}$. The mapping from $\phi$ to $\phi'$ is consistent with the logical operations $\wedge$ and $\vee$ in the sense that

$$(\phi_1 \wedge \phi_2)' = \phi_1' \wedge \phi_2' \quad \text{and} \quad (\phi_1 \vee \phi_2)' = \phi_1' \vee \phi_2'.$$

In general, $(\neg\phi)' \neq {\sim}(\phi')$, which should not be surprising since $B(\neg\phi) \neq {\sim}B\phi$. Nevertheless, if $\phi_1$ and $\phi_2$ are two propositional formulas and $\phi_1 \Rightarrow \phi_2$, i.e., $\neg\phi_1 \vee \phi_2$ is a tautology in propositional logic, then we have $\phi_1' \Rightarrow \phi_2'$ for the associated belief formulas, i.e., ${\sim}(\phi_1') \vee \phi_2'$ is a tautology in belief-logic. The following theorem, where $\phi$ may also be $\square$ or a tautology, is immediate and is illustrated in Table 3.

**Theorem 3.** For any propositional formulas $\phi_1$ and $\phi_2$, we have $\phi_1' \Rightarrow B\phi_1$ and $B\neg\phi_1 \Rightarrow {\sim}(\phi_1')$. Also, $\phi_1 \Rightarrow \phi_2$ if and only if $\phi_1' \Rightarrow \phi_2'$. ■

TABLE 3. Illustration of Theorem 3 for the universe of propositions $\{P_1, P_2\}$; $\Omega = \{w_0, w_1, w_2, w_3\}$.

| Formula | Models |
|---|---|
| $P_1$ | $\{w_2, w_3\}$ |
| $\neg P_1$ | $\{w_0, w_1\}$ |
| $P_1'$ | $\{\{w_2\}, \{w_3\}\}$ |
| ${\sim}(P_1')$ | $2^\Omega - \{\{w_2\}, \{w_3\}\}$ |
| $BP_1$ | $\{\varnothing, \{w_2\}, \{w_3\}, \{w_2, w_3\}\}$ |
| $B\neg P_1$ | $\{\varnothing, \{w_0\}, \{w_1\}, \{w_0, w_1\}\}$ |

## 3. Resolution Method for Beliefs

The inference problem in a belief-logic consists of determining whether a given goal belief-formula $\beta_0$ is implied by a finite set of belief-formulas $S = \{\beta_1, \beta_2, ...\beta_k\}$, called facts. We present a resolution method for inferencing in the belief-logic, which is a generalization of the resolution method for the propositional logic. The derivation of $\square$ in the belief-logic often involves as a substep the derivation of $B\square$, meaning that the facts which are believed are contradictory to each other (as in $B\phi$ and $B\neg\phi$). The final derivation of $\square$ shows that the beliefs themselves are in contradiction with each other (as in $B\phi$ and ${\sim}B\phi$).

We begin by defining the notion of literals, clauses, and the resolvent. If C is a propositional-clause, including the case $C = \square$, then BC and ~BC are said to be *belief-literals*; BC is said to be a *positive* literal and ~BC a *negative* literal. A *belief-clause* is a disjunction of zero or more belief-literals. We sometimes use the term "literal" as a short form of "belief-literal" and also of "propositional-literal", when no confusion is likely; similarly, the term "clause" is used for both "belief-clause" and "propositional-clause". It is clear from Table 1 that each belief-formula is equivalent to a conjunction of belief-clauses.

We define the notion of subsumption between two belief-clauses $\beta_1$ and $\beta_2$ in such a way that the following holds: If $\beta_1$ subsumes $\beta_2$, then each model of $\beta_1$ is a model of $\beta_2$, i.e., $\beta_1 \rightarrow \beta_2$ is valid. This means, in particular, that we can remove a clause $\beta_j$ from a set S if it is subsumed by some other clause in S without affecting the satisfiability or unsatisfiability of S. Formally, we define the subsumption as follows:

(1) If $C_1$ and $C_2$ are two propositional clauses such that $C_1$ subsumes $C_2$, then we say $BC_1$ *subsumes* $BC_2$ and $\sim BC_2$ *subsumes* $\sim BC_1$. (It is not possible for a positive belief-literal $BC_1$ to subsume a negative belief-literal $\sim BC_2$ because $\square \in \Omega(BC_1) - \Omega(\sim BC_2)$. The only case where a negative belief-literal subsumes a positive belief-literal is $\sim BT = \square$ subsuming $BC_2$, $C_2$ being arbitrary, and $\sim BC_1$ subsuming $BT = T$.)

(2) More generally, a belief-clause $\beta_1$ is said to *subsume* another belief-clause $\beta_2$ if each literal in $\beta_1$ equals or subsumes some literal in $\beta_2$. The empty belief-clause $\square$, which has no literals in it, subsumes all other belief-clauses.

Thus, $\sim BP_1 \vee BP_2$ is subsumed by $\sim BP_1$; also, $\sim BP_1$ is subsumed by $\sim B(P_1 \vee P_2)$ or, more generally, by a clause of the form

$$\sim B(P_1 \vee C_1) \vee \sim B(P_1 \vee C_2) \vee \ldots \vee \sim B(P_1 \vee C_k),\ k \geq 1.$$

We first define two types of resolvents for belief-literals, called Type-I and Type-II. These are then generalized to the case of arbitrary belief-clauses. Finally, we define a third type of resolvents, called Type-III. One can actually generalize the other two resolvents to include the Type-III resolvent, but we choose to formulate it separately for the sake of clarity. In each case, we make sure that the resolvent is logically implied by the parent clauses from which it is derived.

(1°) *Type-I* resolvent.
If $BC_1$ and $BC_2$ are two belief-literals such that $C_1$ and $C_2$ are propositionally resolvable, then we say that $BC_1$ and $BC_2$ are *resolvable* and we define their *resolvent* to be the literal given by (R.1).

(R.1) $\quad \mathrm{res}(BC_1, BC_2) = B(\mathrm{res}(C_1, C_2)).$

Thus, we have $\mathrm{res}(B(P \vee \neg Q), B(Q \vee R)) = B(P \vee R)$. It is clear that $BC_1 \wedge BC_2 \Rightarrow B(\mathrm{res}(C_1, C_2))$. Note that this type of resolvent can be at best $B\square$, but not $\square$.

(2°) *Type-II* resolvent.
Let $C_1 = L_1 \vee L_2 \vee \ldots \vee L_m$ and $C_2 = L_{k+1} \vee L_{k+2} \vee \ldots \vee L_m \vee C'_2$ be two propositional clauses, where $0 \leq k < m$ and each $L_j$ is a propositional literal; $C'_2$ may be the empty clause. Then, we say that $BC_1$ and $\sim BC_2$ are *resolvable* and we define their *resolvent* to be the clause given by (R.2).

(R.2) $\quad \mathrm{res}(BC_1, \sim BC_2) = \sim B(\neg L_1 \wedge \neg L_2 \wedge \ldots \wedge \neg L_k)$
$\quad = \sim B(\neg L_1) \vee \ldots \vee \sim B(\neg L_k)$
$\quad = \square$, if $k = 0$.

Two special cases of (R.2) that are worth noting are: $\mathrm{res}(B\square, \sim BC_2) = \square = \mathrm{res}(BC_2, \sim BC_2)$, and $\mathrm{res}(BC_1, \sim B\square) = \sim B(\neg L_1) \vee \sim B(\neg L_2) \vee \ldots \vee \sim B(\neg L_m)$. Once again, we have $BC_1 \wedge \sim BC_2 \Rightarrow \mathrm{res}(BC_1, \sim BC_2)$.

(3°) Generalization of (R.1) and (R.2).
Two belief-clauses $\beta_1 = X_1 \vee X_2 \vee \ldots \vee X_m$ and $\beta_2 = Y_1 \vee Y_2 \vee \ldots \vee Y_n$ are said to be *resolvable* if there is a literal in $\beta_1$ (say, $X_m$) which can be resolved with a literal in $\beta_2$ (say, $Y_1$) using (R.1) or (R.2). We define the resolvent $\mathrm{res}(\beta_1, \beta_2)$ to be the clause given by (R.3) below. We say (R.3) is of Type-I or Type-II according as $\mathrm{res}(X_m, Y_1)$ is of Type-I or Type-II. Note that $\beta_1$ and $\beta_2$ may contain several pairs of literals with respect to which we can form $\mathrm{res}(\beta_1, \beta_2)$ and we may obtain different resolvents in this way. (This is unlike the case of propositional logic, where one obtains a unique resolvent no matter how it is resolved.)

(R.3) $\quad \mathrm{res}(\beta_1, \beta_2) = X_1 \vee X_2 \vee \ldots \vee X_{m-1} \vee \mathrm{res}(X_m, Y_1) \vee Y_2 \vee \ldots \vee Y_n.$

If the right hand side of (R.3) contains duplicate literals, i.e., $X_i = Y_j$ for some $i \leq m-1$ and $j \geq 2$, or $\mathrm{res}(X_m, Y_1)$ equals some of the other literals, then we simplify it by removing the duplicate literals. That $\beta_1 \wedge \beta_2 \Rightarrow \mathrm{res}(\beta_1, \beta_2)$ follows from the corresponding properties of (R.1) and (R.2).

(4°) *Type-III* resolvent.
Let $\beta = X_1 \vee X_2 \vee \ldots \vee X_m$, $m \geq 2$, where each $X_j = \sim B(C \vee C_j)$ is a negative literal and C is the largest common subclause of the propositional clauses $\{C \vee C_j : 1 \leq j \leq m\}$, i.e., there is no literal common to all $C_j$. Then $\beta$ is subsumed by $\sim BC$ and we define the resolvent $\mathrm{res}(\beta)$ to be the literal given by (R.4).

(R.4) $\quad \mathrm{res}(\beta) = \sim BC.$

A *derivation* of a belief-clause $\beta$ from a set S of belief-clauses is defined to be a sequence $D = (D_1, D_2, \ldots, D_m)$, where each $D_k \in S$ or is a resolvent of some previous clause(s) in the sequence D and $D_m = \beta$. A *refutation proof* of a goal $\beta_0$ from a set of facts $\{\beta_1, \beta_2, \ldots, \beta_k\}$ is a derivation of $\Box$ from the union of the sets of clauses obtained from $\beta_j$, $1 \leq j \leq k$, and the clauses obtained from $\neg\beta_0$. There is no restriction on how the clauses of $\beta_j$ and of $\sim\beta_0$ are formed.

**Example 2.** Shown below are the minimal unsatisfiable sets of clauses S in a single proposition P, which do not contain $\Box$.[2]

(1) $B\Box$ and one of $\{\sim BP, \sim B\neg P, \sim B\Box\}$
(2) $\{\sim BP, BP\}$
(3) $\{\sim B\neg P, B\neg P\}$
(4) $\{BP, \sim B\Box, B\neg P\}$
(5) $\{\sim BP \vee B\Box, BP \vee B\neg P, \sim B\neg P \vee B\Box, \sim B\Box\}$

It is easy to see that $\Box$ can be derived in each case. There are three ways of deriving $\Box$ in case (4). One of the many possible ways of deriving $\Box$ in case (5) is shown in Fig. 1. If we replace one or more of the input clauses by their alternate representations (e.g., replace $\sim BP \vee B\Box$ by $\sim BP \vee B\neg P$), then one can still derive $\Box$. ■

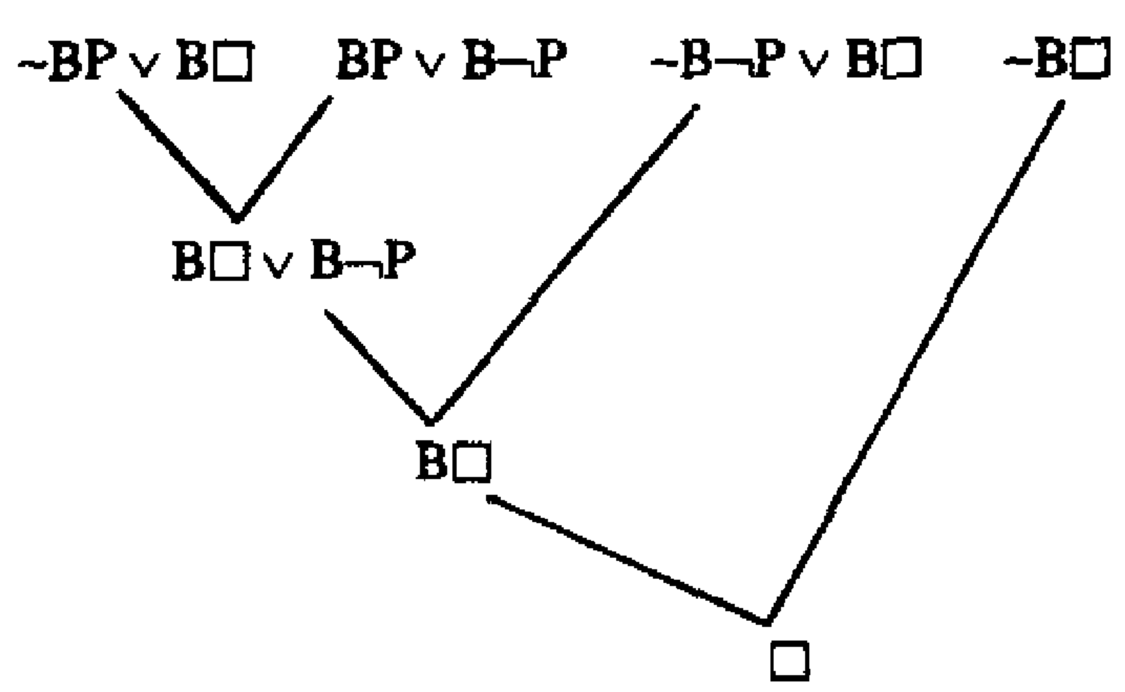



Figure 1. A derivation of $\Box$ from the belief-clauses $\{\sim BP \vee B\Box, BP \vee B\neg P, \sim B\neg P \vee B\Box, \sim B\Box\}$.

**Example 3.** Suppose our universe of discourse consists of the basic propositions P = "Tweety has wings" and Q = "Tweety flies". Let B be the belief operator "Kundu believes ...". We consider the following facts (1)-(2) and the goal (3):

(1) "Kundu believes that if Tweety has wings, then Tweety flies". This is represented by $B(P \rightarrow Q) = B(\neg P \vee Q)$.
(2) "Kundu does not believe that Tweety does not have wings". This is represented by $\sim B\neg P$.
(3) "Kundu does not believe that Tweety cannot fly". This is represented by $\sim B\neg Q$.

[2] The minimality of S means that S does not contain a proper subset which is also unsatisfiable. In particular, no clause in S is subsumed by another clause in S.

One expects that perhaps (3) can be proved from (1)-(2). Fig. 2 shows a refutation proof of (3). If we replace (2) by the stronger fact $BP \wedge \sim B\Box$, then we can indeed still prove the goal (3). We first resolve BP with $B(\neg P \vee Q)$, giving the resolvent BQ. Now, obtain $B\Box = res(BQ, B\neg Q)$, which can be then resolved with $\sim B\Box$ to obtain the desired empty-clause $\Box$. Note the use of $\sim B\Box$ here, without which we cannot obtain $\Box$ in this case. This is not surprising because the empty set of worlds $\varnothing$ is a model for $\{BP, B(P \rightarrow Q), B\neg Q\}$. We can actually derive a stronger goal $BQ \wedge \sim B\Box$ in this case.

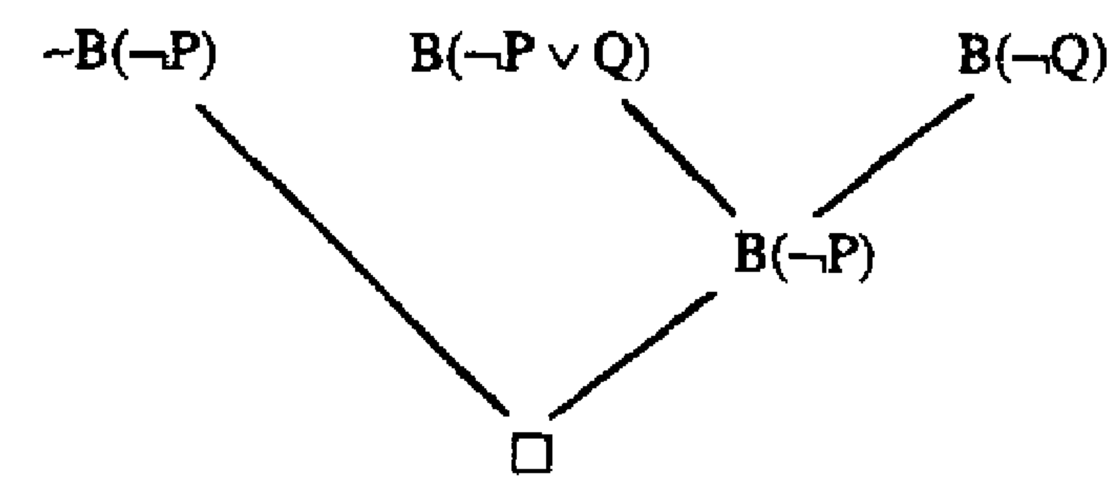



Figure 2. An illustration of resolution proof for beliefs.

If we consider the goal (3′) = BQ = "Kundu believes that Tweety flies", which is stronger than (3), then it may not be so obvious if (3′) can be proved from (1)-(2). Fig. 3 shows all possible resolvents that can be obtained from $\{(1), (2), \sim(3')\}$ and, in particular, that $\Box$ cannot be derived. The set of worlds W = {TT, FF}, where TT means P = T = Q and similarly for FF, forms a model of $\{(1), (2), \sim(3')\}$. ■

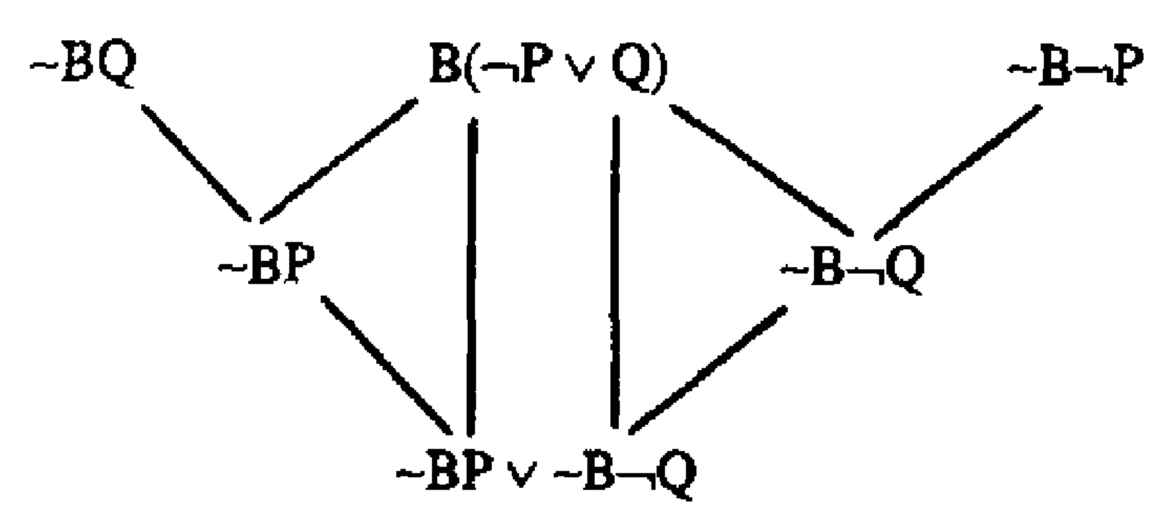



Figure 3. An example where $\Box$ cannot be derived.

The main result of this paper is the following theorem. We omit the proof for want of space, but it can be found in [Kundu, 1991a], We point out that the resolution rules given here can be seen to be closely related to that for predicate logic (without function symbols). One could almost say that the completeness of the belief-logic inference method follows from that of predicate logic without function symbols, except for the slight difference of the empty set of worlds being a possible model for a given belief-formulas.

Theorem 4 (Completeness and soundness of the resolution method). Let S be an arbitrary finite set of belief-clauses. The set S is unsatisfiable if and only if there is a derivation of □ from S. ■

## 4. Conclusion

We have presented here a new belief-logic for propositional facts (i.e., without variables and quantifiers), including a resolution proof method for this logic, which can be thought of as a generalization of the resolution method for propositional logic. Our formulation of the belief-logic differs from the other belief-logics in two fundamental ways: (1) The ordinary propositional logic is imbedded within our belief-logic. Each propositional formula $\phi$ is mapped to a belief-formula $\phi'$ in an one-to-one to fashion in such a way that if $\phi$ implies $\psi$ as propositional formulas, then $\phi'$ implies $\psi'$ as belief-formulas, and conversely. Also, the truth value of $\phi'$ is evaluated in the same way as for general belief-formulas. (2) The belief-logic developed here is mono tonic in nature and its construction can be applied repeatedly to obtain a hierarchy of belief-logics $BL_k$, $k \geq 1$, such that $BL_k \supset BL_{k-1}$ and $BL_k$ can handle all belief-formulas involving up to level k nested applications of the belief-operator B. We obtain a non-monotonic belief-logic from $BL_k$ by applying circumscription to it, for instance. This is the approach taken in [Kundu, 1991b].

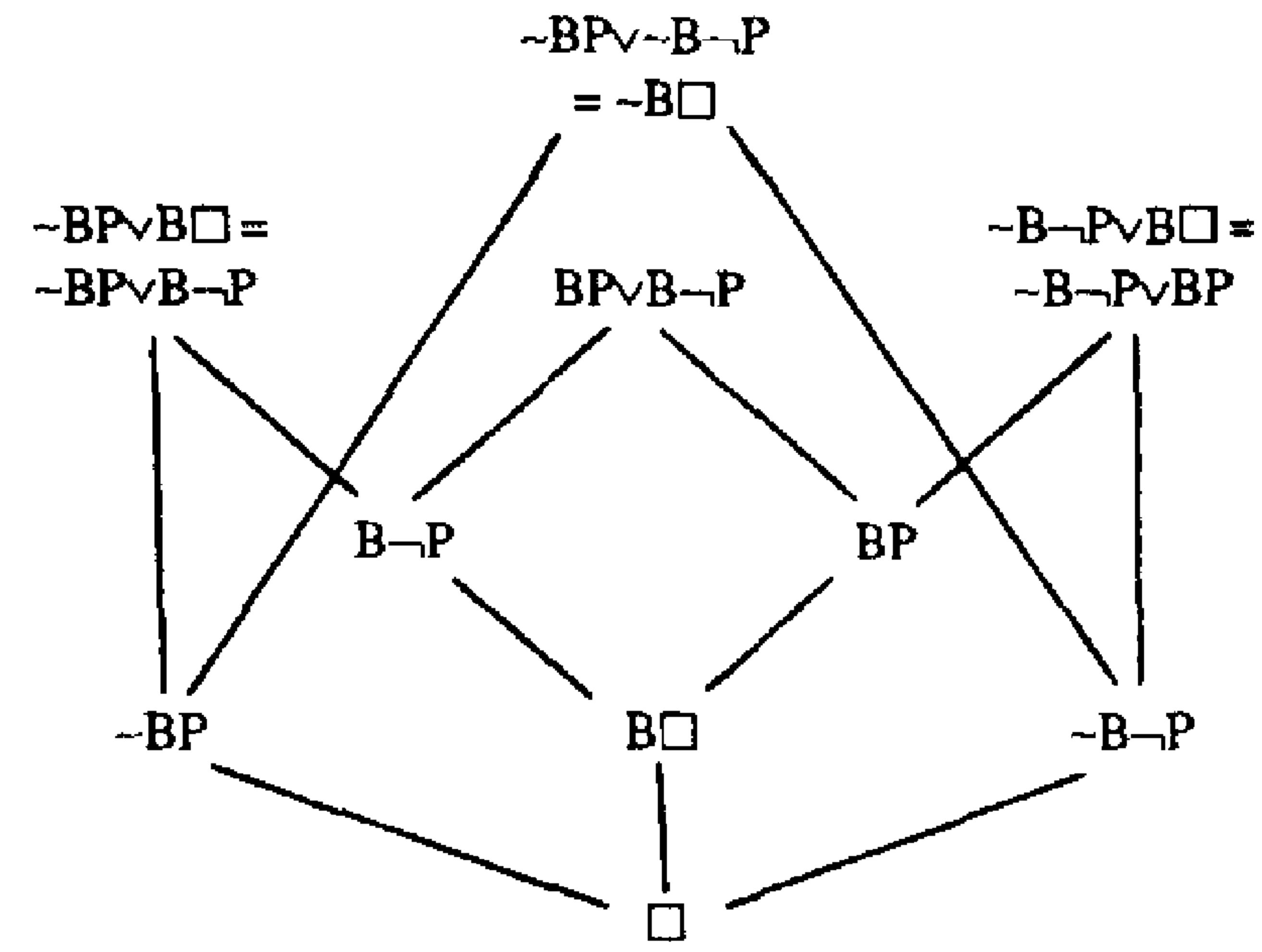
Acknowledgements: The author wishes to express sincere thanks to an anonymous referee for indicating the close connection between the resolution rules given here and those of predicate logic.

**Appendix:** Consider the universe $\Omega = \{w_0, w_1\}$ of a single proposition P. Shown below are all the belief-literals and their models. There are only three belief-clauses which are not equivalent to a belief-literal or **T**, namely, ~BP ∨ B□ = ~BP ∨ B¬P, ~B¬P ∨ B□ = ~B¬P ∨ BP, and BP ∨ B¬P. The set of models for these clauses are given by $\{\emptyset, \{w_0\}, \{w_0, w_1\}\}$, $\{\emptyset, \{w_1\}, \{w_0, w_1\}\}$, and $\{\emptyset, \{w_0\}, \{w_1\}\}$, respectively.

The belief-literals in proposition P.

| B□ | ~B□ | BP | ~BP | B¬P | ~B¬P |
|---|---|---|---|---|---|
| $\emptyset$ | $\{w_0\}$ | $\emptyset$ | $\{w_0\}$ | $\emptyset$ | $\{w_1\}$ |
| | | $\{w_1\}$ | $\{w_1\}$ | $\{w_0, w_1\}$ | $\{w_0\}$ |
| | | $\{w_0, w_1\}$ | | | |

The following diagram shows the subsumption relationships among the belief-clauses other than **T**, where a clause $\beta_1$ is shown below another clause $\beta_2$ if $\beta_1$ subsumes $\beta_2$.




The subsumption relationships among the belief-clauses in a single proposition P.

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