GENERAL APPROACH TO NONMONOTONIC LOGICS

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Abstract

Nonmonotonic logics are logics in which theorems can be invalidated by new axioms. In this paper we propose a general framework in which a large class of non-monotonic logics can be specified.

0. Introduction

Traditional logics are <u>monotonic</u> - i.e., theorems of any theory belong to the set of theorems of any extension of the theory. In human reasoning the situation is different: one's knowledge often makes impossible to infer on the pure logical basis, yet some conclusion has to be drawn. Such a conclusion, called <u>belief</u> in AI terminology, can be invalidated when new logical information is added. It means that common sense reasoning violates the principle of monotonicity.

A possible approach to the problem of formalization of beliefs is that of McDermott and Doyle [3J. They consider the concept of consistency as modality. Unfortunately, they have no logical axioms and inference rules specific for that notion. In consequence, their nonmonotonic logic is too weak.

McDermott [2] improves the above logic by supplying some axioms and inference rules. This extension results in nonmonotonic versions of the traditional modal systems T, *S4,* and S5. Unfortunately, the systems S4 and T are too weak to capture all properties of the notion of consistency, and his nonmonotonic version of S5 turns out to be equivalent to the monotonic modal S5 logic.

In the following paper we propose a general framework in which, we think, a formalization of beliefs based on the notion of consistency, can properly be handled. Our approach differs from that of McDermott in the way in which the notion of nonmonotonic theorem is specified. While both approaches make use of an algebraic operator for that purpose, our operator is more general. By means of it a large class of nonmonotonic logics can be defined. For each logic a model-theoretic semantics is given and the completeness theorem is stated.

Proposing such a general approach we do not suggest that all of those logics are of some importance. In fact, only very few of them are interesting. The problem of choosing such an ideal logic is very complex. In this paper we propose a solution to the problem for a specific class of theories.

For the sake of simplicity we work within propositional language. It should be, however, remarked that all results can easily be generalized to the first-order language,

1. Language

The language under consideration contains: the set of proposition letters denoted by p,q,p1,q1,..; the connectives ^ (not) , > (if ... then); the modal operator M (it is consistent that); the brackets (,). A <u>formula</u> is either a proposition letter, an expression (*«α)*, where α is a formula, or an expression (ACB), where α and B are formulas. Other connectives are defined as usual. The modal operator L (it is provable that) is an abbreviation for M α. The brackets can be omitted if convenient.

2. The monotonic component

Each nonmonotonic theory is identified with a set of formulas, the <u>proper axioms</u> of the theory. A deductive structure of the theory is given in two steps, first specifying a set of <u>logical axioms</u> and <u>monotonic inference rules</u>, common to all theories, and then defining a theorem of the theory by means of special algebraic operator. We start with the first part of the above specification.

Logical axiom

A1. $\alpha \supset (\beta \supset \alpha)$
A2. $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
A3. $(\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)$
A4. $L(\alpha \supset \beta) \supset (L\alpha \supset L\beta)$
A5. $L\alpha \supset \alpha$
A6. $M\alpha \supset LM\alpha$

<u>Monotonic inference rules</u>

R1. From $\alpha \supset \beta$ and α to infer
R2. From α to infer $L\alpha$

The axioms A1-A3 and the rule R1 form a possible axiomatization of classical propositional logic. The axioms A4-A6 and the rule R2 are minimal conditions which should be valid if modal operators are to meet their interpretation as consistency and provability. It is difficult to discuss such conditions without a reference to an inference system. For a time being, however, we can assume that we have some abstract operator Y mapping theories into sets of formulas regarded as theorems of the theories. Assuming that for each theory A the set Y(A) contains all propositional tautologies and is closed under the rule R1, there are two further conditions which should be fulfilled:

(L) $\alpha \in Y(A)$ iff $L\alpha \in Y(A)$
(M) $\sim\alpha \notin Y(A)$ iff $M\alpha \in Y(A)$, assuming that $\beta \notin Y(A)$ for some formula β.

Now the rule R2 is the only-if half of (L). A4 is the modus ponens rule R1 stated in the object language. A5 states the obvious fact that whatever is provable is true. A6 follows from (L) and (M) (note that it is equivalent to $L\sim\alpha \vee LM\alpha$, take any theory A and consider two cases: (i) $\sim\alpha \in Y(A)$; (ii) $\sim\alpha \notin Y(A)$). Observe that R2, and A5 together with R1 guarantee (L). The if half of (M) is also satisfied (assume that $M\alpha \in Y(A)$ and $\beta \notin Y(A)$ for some β; thus $\sim M\alpha \notin Y(A)$; but $\sim M\alpha \equiv L\sim\alpha$, and hence $\sim\alpha \notin Y(A)$ by R2). It should be remarked that the only-if half of (M) is not assured. We shall discuss this problem later.

We write $S \vdash \alpha$ iff α is provable in the usual sense from the set of formulas S and instances of A1-A6 by repeated application of R1-R2. We define $Th(S) = \{\alpha : S \vdash \alpha\}$.

Formally, A1-A6 and R1-R2 form an axiomatization of traditional propositional S5 modal logic (see [1]).

3. Kripke models

We review Kripke semantics for propositional S5 modal logic. It will play an important role in further development.

A model is a system $K = (W, m)$, where W is a non-empty set of possible worlds, and m is a mapping from proposition letters into 2^W.

The value of formula α in $K = (W, m)$ with respect to $w \in W$, denoted by $V(K, w, \alpha)$, is defined as follows: (i) $V(K,w,p) = 1$ iff $w \in m(p)$, where p is any propositional letter; (ii) $V(K,w,\sim\alpha) = 1 - V(K,w,\alpha)$; (iii) $V(K,w,\alpha \supset \beta) = 1$ iff $V(K,w,\alpha) = 0$ or $V(K,w,\beta) = 1$; (iv) $V(K,w,M\alpha) = 1$ iff $V(K,u,\alpha) = 1$ for some $u \in W$.

A formula α is true in K ($K \models \alpha$), iff $V(K,w,\alpha) = 1$ for each $w \in W$. A set S of formulas is true in K, and we say that K is a model for S, iff $K \models \alpha$ for each $\alpha \in S$.

4. Generalized nonmonotonic operator

We noted that the specified monotonic component guarantees (L) and the if half of (M). There still remains the problem of the only-if half of (M), which we denote by (M⊃). McDermott tries to resolve the problem by means of a special operator NM_A defined for each theory A as $NM_A(S) = Th(A \cup As_A(S))$, where $As_A(S)$, the set of assumptions from S is given by $As_A(S) = \{M\alpha : \sim\alpha \notin S\} - Th(A)$.

Let S be a fixed point of NM_A. It can be regarded as a set of formulas obtainable from A by means of monotonic rules augmented by (M⊃), and thus satisfies all conditions required from the set of theorems nonmonotonically derivable from A. Unfortunately, for a given theory A there are, in general, many fixed points under that operator. Moreover, each of them is minimal in the following sense: if S1, S2 are fixed points of NM_A, then $S1 \subseteq S2$ implies $S1 = S2$ (see [3]). Hence, the only plausible solution is to identify the set of theorems nonmonotonically derivable from A with the intersection of all fixed points of NM_A. Unfortunately, the resulting logic is the monotonic S5 modal logic (see [2]).

To resolve the problem we propose to change the conceptual emphasis in our approach. Instead of formalizing the notion of consistency, and treating the nonmonotonicity as some kind of side effect of a resulting system, we shall look at nonmonotonic logic as at a formalization of beliefs based on the concept of consistency. To fix some ideas, consider a theory A. From the strictly logical point of view, the whole of valid information about A is contained in Th(A). Now if this information is insufficient and, moreover, no new axioms can be added, there is only one possibility: to extend the theory on a heuristic basis. Of course, if S is such a heuristic extension of A, then formulas from the set Th(S) - Th(A) should be regarded as beliefs.

There still is the problem how such an extension is to be specified. We think that the syntactic form of axioms can give us some hints. Consider, for instance, the theory $A = \{Mp \supset q\}$. Although A has infinitely many extensions, one feels that extensions containing Mp, and hence q, should be preferred to all others. It is due to the fact that we are tempted to regard Mp as the logical truth. Of course, Mp is not valid. The problem is that to decide whether something is consistent one has to answer the crucial question: consistent with what? The simplest answer: with the theory itself, is unacceptable. Note that p and ~q are both consistent with A, yet regarding Mp and M~q as simultaneuosly valid leads to inconsistency. The correct interpretation of consistency is to view it with respect to the theory itself together with all beliefs. Having this in mind, it is clear that the truth value of Mp depends on beliefs we hold about the theory. Nevertheless, Mp can consistently be added to A and, moreover, it seems to be the only candidate for such an addition.

The above discussion suggest that the set of theorems nonmonotonically derivable from a theory should be identified with the set of theorems monotonically derivable from some extension of the theory. We propose to specify such an extension by choosing a set of <u>preferable assumptions</u>, i.e., a set of assumptions one would like to add to the theory. To make this idea workable we generalize the operator NM_A as follows.

Let A be a theory, S and T sets of formulas. We define the <u>generalized nonmonotonic operator</u> $NM_{A,T}$ by

$NM_{A,T}(S) = Th(A \cup As_{A,T}(S))$, where $As_{A,T}(S)$, the <u>set of assumptions from</u> S <u>with respect to</u> T is given by

$As_{A,T}(S) = \{M\alpha:\ \alpha \in T \text{ and } \sim\alpha \notin S\} - Th(A)$.

Intuitively, the set $\{M\alpha: \alpha \in T\}$ forms the set of preferable assumptions for a theory A.

We propose the following general definition of nonmonotonic logic. By <u>nonmonotonic logic</u> we understand any function f from sets of formulas into sets of formulas. Intuitively, f should be regarded as a mapping which assigns to each theory A a set f(A) such that the set $\{M\alpha: \alpha \in f(A)\}$ is the set of preferable assumptions of A.

If f is a nonmonotonic logic and A is a theory, then the <u>set of theorems nonmonotonically derivable from</u> A, denoted by TH(A), is the intersection of all fixed points of $NM_{A,f(A)}$.

There still remains the problem what nonmonotonic logic f should be chosen to meet intuitive expectations. We shall propose a solution to this problem for a specific class of theories. But first we give a semantics of nonmonotonic logic.

5. Semantics of nonmonotonic logic

The semantics of nonmonotonic logic is based on the notion of model of S5 modal logic.

Let K be a model, T a set of formulas. By M(K,T) we denote the set $\{M\alpha: \alpha \in T \text{ and } K \models M\alpha\}$.

A model K for a theory A is said to be <u>M-maximal with respect to</u> T iff for each model K1 for A, $M(K,T) \subseteq M(K1,T)$ implies $M(K,T) = M(K1,T)$.

The following completeness theorem holds:

<u>Theorem 5.1.</u> Let f be a nonmonotonic logic. For each theory A and formula α, α∈TH(A) iff α is true in each M-maximal model for A with respect to f(A).

We have a detailed proof of the above theorem but we are not able to give it in this brief paper.

6. Default theories

The problem of choosing the appropriate nonmonotonic logic is, in general, very complex. In this paper we limit ourselves to a specific class of theories. Following Reiter's terminology (see [4]) we call them <u>default theories</u>.

We denote by PC the classical propositional calculus. A theory A is said to be a default theory iff each axiom of A belongs to PC or is of the form

$\delta \& M\beta_1 \& \ldots \& M\beta_n \supset \gamma$ or

$L\delta \& M\beta_1 \& \ldots \& M\beta_n \supset \gamma$ or

$M\beta_1 \& \ldots \& M\beta_n \supset \gamma$

where $\delta, \beta_1, \ldots, \beta_n, \gamma \in PC$.

Let α be an axiom of a default theory. By P(α) we denote the set of formulas given by

$P(\alpha)$ = if $\alpha \in PC$ then the empty set else $\{\beta_1, \ldots, \beta_n\}$.

If A is a default theory, then by P(A) we denote the set $\bigcup_{\alpha \in A} P(\alpha)$.

For default theories the problem of choosing the appropriate nonmonotonic logic is simple. The only reasonable candidate seems to be the function which assigns to a theory A the set P(A).

From the Theorem 5.1 we thus obtain the following completeness theorem for default theories:

<u>Theorem 6.1.</u> Let A be a default theory. For each formula α, α∈TH(A) iff α is true in every M-maximal model for A with respect to P(A).

References

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