ON THE STRUCTURE OF AN IMPORTANT CLASS OF EXHAUSTIVE PROBLEMS AND ON WAYS OF SEARCH REDUCTION FOR THEM

G.M. Adelson-Velskiy, V.L. Arlasarov, M.V. Donskoy
Institute of Control Science
Moscow, USSR

Abstract

The paper discusses necessity of structuring a search tree. A theorem is stated that the α-β procedure is the only search reduction procedure for non-structured minimay problems. For a class of problems structure in some way a non-trivial search reduction method is described.

Introduction

In most problems of artificial intellegence an exhaustive search is an important (we think, main) method of choosing a solution among certain alternatives. The central problem, which arrises here is the problem of search reduction without prejudice to the quality of solution. A search reduction is called absolute if the search graph is certainely lessened, and it is called heuristic if reduction of the search graph depends on the good luck. (We do not consider here search reduction techniques which may lead to the loss of solution, although they are sometimes also called heuristic). The subject of the theory of exhaustive search (considered as part of the artificial intelligence theory) should be, naturally, heuristic search reductions. In this connection the following problems arrise: the problem of formalisation, the problem of inventing a search reduction method, (heuristic by itself) the problem of analysis of a reduction method. A analysis consists apparently of the following parts: an applicability domain, uniqueness results (under such and such conditions no other method exists), results on optimal effect, examples of the absence of effect, results on "rentability" of a method whether the time spent to answer question: "To reduce or not" is saved by essential reduction of the size of searched set). In the present talk we ohose a simple (but important) case of exhaustive search - namely minimax problems - and α-β method as a heuristic search reduction method, to make a part of such an analysis. We show that if a corresponding method is used for a too wide class of problems, then it is the only search reduction method for this class (more precisely it majorates all other methods). A class is too wide if, roughly speaking, the structure of problems of the class is subject only to trivial restrictions. Example: there is no search reduction method applicable to all cooperative games (a degenerate case, where all vertices in the game tree are maximal).

As philosophical implication of this result we conclude that to construct a non-trivial search reduction methods one needs to use a structure of a problem, A method of search reduction discussed in the second part of the talk confirms our conclusion. It uses a sort of symmetries of some problems. This agrees with the P. Klein's general principle according to which Mathematics studies symmetries of the World. (However unlike the situation in geometry, in our case those symmetries do not form a group).

It is important that this method is compatible with the α-β algorithm and supplies essentially different, additional possibilities for search reduction. This method distinguishes our class of problems from earlier classes of problems with restrictions (traveling salesman etc).

In those problems restrictions were used to strengthen application of method and do lead to any innovation in it.

1. Notations, assumptions, definitions

We go over now to a formal description of the problem and of the results. All trees to be met in the talk are finite, directed and have the unique root. If T is a tree, $V_0 = V_0(T)$ denotes its root, V(T) (resp. E(T)) denotes the set of vertices (resp. edges) of T, End T denotes the set of end vertices. For $v \in V(T)$ $T(v)$ is the tree "having" at v, EN(v) is the set of edges exitting from v, $N(v)$ is the set of end vertices of $e \in$ EN(v). For $v_1, v_2 \in V(T)$, $[v_1, v_2]$ denotes the (directed) path from v_1 to v_2. Suppose we are given a map $i: V(T) \to \{\pm\}$. Set $V^+ = i^{-1}(+)$, $V^- = i^{-1}(-)$. Suppose further we are given a completely ordered set D such that for any $D' \subseteq D$, inf $D' \in D$ and sup $D' \in D$ are defined (e.g. $D = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$). For $d_1, d_2 \in D$ we set $-(-d_1) = d_1$, and we assume that $-d_1 \leqslant -d_2$ is equivalent to $d_1 \geqslant d_2$. The letter f denotes always a evaluation function f : End T $\to$ D and F denotes a transition function $F: E(T) \times D \to D$. Δ denotes the set of pairs (F, f) such that F is monotone non-decreasing with its second argument and $F(e \times D) = D$, $\forall e \in E(T)$. For $(F, f) \in \Delta$ the function $\varphi_{F,f}: V(T) \to D$ is defined inductively by $\varphi_{F,f}(v) = f(v)$, $v \in END\,T$;

$$\varphi_{F,f}(v) = i(v) \sup_{w \in N(v)} i(v)\, F([v,w], \varphi_{F,f}(w))$$

$v \in V(T) - END\,T$. A functions' family $\Sigma \subset \Delta$ is called too wide if the following holds. Suppose an algorithm A is given (with value-set D x D), which for any $v \in V(T)$ and for any subtree $T_1 \subset T$ and for any $(F, f) \in \Sigma$ computes sup and inf of $\varphi_{F_1, f_1}(v)$ over all $(F_1, f_1) \in \Sigma$ which coincide with (F, f) on T_1. Our condition has a form

(g1) $\forall (F, f) \in \Sigma \quad \forall T_1 \subset T \; \forall v_1, \ldots, v_m \in V(T)$ with $E(T_{v_i} \cap T_1) = \emptyset, v_i \notin [v_0, v_j]$ for $i \neq j$, for any choice of $b_i \in [A(T_1, F, f, v_i)^-, A(T_1, F, f, v_i)^+]$ there exists $(F_1, f_1) \in \Sigma$ such that $(F_1, f)|_{T_1} = (F, f)|_{T_1}$ and $\varphi_{F_1, f_1}(v_i) = b_i$.

If any interval $[d_1, d_2]$, $d_1 \neq d_2$ is infinite, (g1) may be replaced (as far as our aims are concerned) by a more weak assumption of absence in Σ of relations of the equality of inequality type: (g2) $\forall (F, f) \in \Sigma \quad \forall T_1 \subset T \quad \forall v_1, \ldots, v_m \in V(T)$ $E(T_{v_i} \cap T_1) = \emptyset, v_i \notin [v_0, v_j]$ for $i \neq j$ for any function $\rho: D^m \to D$ for any sign $\square$ taken from the set $\geqslant, =, \leqslant$ for any $d \in D$ there exists $(F_1, f_1) \in \Sigma$ such that $(F_1, f_1)|_{T_1} = (F, f)|_{T_1}$ and $\rho(\varphi_{F_1, f_1}(v_1), \ldots, \varphi_{F_1, f_1}(v_m)) \not\!\square\, d$

An exhaustive search working on $\Sigma \subset \Delta$ a rule of walking around T. Explicitly: it is a computable function $\pi: \mathbb{N} \times \Sigma \to V(T)$ such that $\forall (F, f) \in \Sigma$, $\forall n \in \mathbb{N}$ one has $\pi(n+1, F, f) \in \bigcup_{i \leqslant n} N(\pi(i, F, f))$.

A search reduction or cut-off of the search π is a computable function $\beta_\pi: \mathbb{N} \times \Sigma \to \{\pm 1\}$ such that $\forall n \in \mathbb{N}$, $\forall (F, f) \in \Sigma, \forall (F_1, f_1) \in \Sigma$ such that $(F_1, f_1)|\bigcup_{i \leqslant n} \pi(i, F, f) = (F, f)|\bigcup_{i \leqslant n} \pi(i, F, f)$, $\forall (F_2, f_2) \in \Sigma$ such that $(F_2, f_2)|_{T - T_{\pi(n, F, f)}} = (F_1, f_1)|_{T - T_{\pi(n, F, f)}}$ one has implication $\beta_\pi(n, F, f) = -1 \Rightarrow \varphi_{F_2, f_2}(v_0) = \varphi_{F_1, f_1}(v_0)$. Given cut-off β_π one constructs the new search π' (also exhaustive) which walks around less vertices: if $\beta_\pi(n, F, f) = -1$ then for all $i > n$ such that $\pi(i, F, f) \in T_{\pi(n, F, f)}$ we set $\pi'(i, F, f) = \pi(n, F, f)$. Repetitions may be afterwards excluded.

2. Results on α-β method

The α-β method is applied to the problem of computing $\varphi_{F,f}(v_0)$. This problem is solved by exhaustive search. Suppose we are given a tree T, a family $\Sigma \subset \Delta$ on which we are working, an exhaustive search (a way of walking around T) π and an algorithm A with properties described above. To construct a cut-off called α-β method, we construct at first for $(F, f) \in \Sigma$ the set $T(n) = \bigcup_{i \leqslant n} \pi(i, F, f)$ Then we construct a system of Bounds

$\ell \cdot \{A(T(n), F, f, v)^{\varepsilon}\}$ on functions of Σ. Then we construct the set $V_{AB}(T(n), \beta)$ equal to the union of trees T_v over those $v \in V(T)$ for which there exists $w \in [v_0, w]$ such that $F([v_0, v], \beta^{\iota(w)}(v)) \geqslant \beta^{-\iota(w)}(w)$. At last we put
$\rho_{AB}(n, F, f) = -1 \Leftrightarrow \pi(n, F, f) \in V_{AB}(T(n), \beta)$
To construct an A one uses restrictions on functions from Σ and explicit computations of $\varphi_{F, f}$ in certain vertices.

Theorem 1. If Σ is too wide then the only search reduction for an exhaustive search π working on Σ is $\rho_{AB, \pi}$ (in the sence that all other search reductions are wearser in the obvious sense).

This theorem in a slightly different form is contained already in [1].

Without going into technicalities let us note another result which can be either deduced from Theorem 1 or proved independently. Under assumption that the number of edges in min and max-vertices is aproximately equal, the number of end vertices in the optimal search tree is not less than $\sqrt{\sharp ENDT}$.

Simple examples of restrictions on Σ : a) the sum of values of $\varphi_{F, f}(v)$ $\forall (F, f) \in \Sigma$ over some fixed set of vertices (e.g. over vertices of the given level) is bounded by a given element; b) the problem of traveling salesman: $V^- = V(T)$ $\mathcal{D} = R \cup \{+\infty\} \cup \{-\infty\}$ a function $c . V(T) \to \mathcal{D}$ (the transport fares) is given, $c(V) \geqslant 0$, such that $F([v, w], d) = d + c(w)$. Then restriction follows from the evident relations

$$F([v_0, v], \varphi_{F, f}(v)) = \sum_{w \in [v_0, v]} c(w) + \varphi_{F, f}(v) \geqslant \sum_{w \in [v_0, v]} c(w)$$

3. Structures

Theorem 1 shows (as we have mentioned in the Introduction)that new methods of search reduction should be sought for families Σ distinguished by nontrivial relations.

As an model which (in a reasonable approach) includes all non-trivial restrictions known to us we propose the following schema: Let M be a set, possibly endowed with a structure of partially ordered set or some other suitable structure. Suppose that a map (structure map) σ : E(T) $\to$ M is given. Requiring of F and f a "good" behaviour with respect to σ and structure on M, we shall get a non-trivial restrictions on Σ. A travelling salesman problem is an example. We came now to a detailed description of an example of conditions on and of ways of its application to search reduction.

These conditions have naturally arrisen in analysis of exhaustion of variants in a chess programm (cf. [1]). For chess $\sigma(e)$, $e \in E(T)$, is a move from initial position of edge to its final position, considered on the empty board and containing a indication of which piece was captured; M is the set of all moves on an empty board with capture indications.

One has in this case $|V(T)| \geqslant 10^{100}$ $|M| \leqslant 10^4$. σ and M are analogously defined for checkers and card plays with cards being open.

Suppose that V^+ and V^- are interlaced in V(T), i.e. $\iota(w) = -\iota(v)$ for $w \in N(v)$. Suppose further that M = $M^+ \cup M^-$ and that for $w \in N(v)$ one has $\sigma([v, w]) \in M^+ \Leftrightarrow \iota(v) = +$. Let us write $\iota(m) = +$ for $m \in M^+$ and $\iota(m) = -$ otherwise. The set of non-empty paths $[v, w]$ $v, w \in V(T)$ is denoted Paths T. Seq M denotes the set of ordered sequences of elements from M (possibly with repetitions) whose signs alternate. Seg M is endowed with the natural structure of a tree and we shall consider Seq M with this structure. Then σ induces the map $\tilde{\sigma}$: Paths T $\to$ Seq M or, the same, the map $\tilde{\sigma}$: T $\to$ Seq M. Suppose that σ^{-1} is unambiguous on edges $e \in EN(v)$, $\forall v \in V(T)$ (i.e. e $e' \in EN(v)$ $\sigma(e) = \sigma(e') \Rightarrow e = e'$).

Then for $v \in V(T)$, $s \in$ Seq M one denotes by $v + s$ the unique vertex $w \in V(T_v)$ such that $\tilde{\sigma}([v, w]) = s$(if, of course, $s \in \tilde{\sigma}$ (Paths T_v)). Let us introduce on Seq M operation $+$ by: $m = m_1 + m_2$ if m is obtained by writing down m_2 after m_1 and if $|m_1|$ is even. We shall write also $m = m_1 \bullet m_2$ if $m_1 = m_{11} + \ldots + m_{1k}$, $m_2 = m_{21} + \ldots + m_{2n}$, $m = \sum_{i,j} m_{ij}$ if in the latter sum all lengths, except possibly last, are even.

4. Relation of influence

Suppose that on the tree Seq M a relation of influence is given and that a relation of influence of elements from Seq M on elements of M is given. Suppose that relation of influence satisfies the following conditions (where $m, m' \in M$, $s, s', s_i, \tilde{s} \in$ Seq M):

Axiom of symmetry of influence:

s influences $s' \Rightarrow s'$ influences s.

Axioms of extension of influence:

$m \in s \Rightarrow s$ influences m

$m \in s$ and s' influences $m \Rightarrow s'$ influences s.

Axioms of the transfer of influence:

s influences s' and $s = s_1 + s_2 \Rightarrow$ at least one s_i influences s'.

s influences m and $s = s_1 \bullet s_2 \Rightarrow$ either some s_i influences m or s_1 influences s_2.

s influences s' and $s = s_1 \bullet s_2 \Rightarrow$ either some s_i influences s' or s_1 influences s_2.

This relation of influence is extended with the help of σ to relation of influence in the Paths T and to influence of elements of Paths T on elements of M. For this extension some additional axioms are to hold (where $v \in V(T)$, $w_1, w_2, w_3 \in V(T_v)$

$$\tilde{\sigma}[v, w_3] = \tilde{\sigma}[v, w_1] \bullet \tilde{\sigma}[v, w_2]$$

Axioms connecting admissibility of moves and influence: a) $\exists m \in M^{\iota(v)}$ $m \in \sigma(EN(w_i)), i = 1, 2$ $m \notin \sigma(EN(w_3)) \Rightarrow$ $\Rightarrow [v, w_1]$ influences $[v, w_2]$ b) $\exists m \in M^{\iota(v)}$ $m \in \sigma(EN(v))$ $m \in \sigma(EN(w_i))$ $i = 1, 3$ $m \notin \sigma(EN(w_2)) \Rightarrow [v, w_1]$ influences either $[v, w_2]$ or m c) $\exists m \in M^{-\iota(v)}$ $m \in \sigma(EN(w_1))$, ~~influences~~ $m \notin \sigma(EN(w_3)) \Rightarrow [v, w_1] + m$ influences $[v, w_2]$

5. Method of geometrical relations

Let T_1 be a subtree in T_v, $v \in V(T_1)$. Suppose that we are given an algorithm B (for games B is the choice of the best move) which constructs for any $v \in V(T)$ and any subtree $T_1 \subset T$ a vertex $w \in N(v)$ such that $\varphi_{F,f,T_1}(v) = F([v, w], \varphi_{F,f,T_1}(w))$. (Subscript T_1 in φ denotes that computation of this φ involves edges and vertices only from T_1). Let us define for T_1 two subtrees $\delta^{\pm}(T_1)$, $\delta^{+}(T_1)$ is defined inductively beginning from the root by conditions:

$$v \in V(\delta^{+}(T_1))\ ;\ \forall u \in V^{\iota(v)}(\delta^{+}(T_1))\ [N(u, \delta^{+}(T_1)) := \{B(u, T_1)\}]\ ;\ \forall u \in V^{-\iota(v)}(\delta^{+}(T_1))\ [N(u, \delta^{+}(T_1)) := N(u, T_1)]$$

Analogically $\delta^{-}(T_1)$ is defined inductively by conditions: $v \in V(\delta^{-}(T_1))$,

$$\forall u \in V^{\iota(v)}(T_1)\ [N(u, \delta^{-}(T_1)) := N(u, T_1)]$$

$$\forall u \in V^{-\iota(v)}(T_1)\ [N(u, \delta^{-}(T_1)) := \{B(u, T_1)\}]$$

If $T_1 \subset T_v$, $T_2 \subset T_w$, $w \in V^{\iota(v)}(T)$ we set

$$\varepsilon(T_1, T_2) = \tilde{\sigma}(\delta^{-}(T_1)) \cap \tilde{\sigma}(\delta^{+}(T_2)) \in \text{Seq } M$$

If T_1, T_2 are subtrees in T, $P \in$ Paths T, $m \in M$, we say that T_1 influences m (resp. P) if there exists $P_1 \in$ Paths $\delta^{-}(T_1)$ such that P_1 influences m (resp. P). We say that T_1 influences T_2 if there exist $P_i \in$ Paths $\delta^{-}(T_i)$ $i = 1, 2$ such that P_1 influences P_2.

Let us now distinguish some special subtrees. Set $O(w, v) = \sigma(EN(w)) \setminus \sigma(EN(v))$,

$$L(w, v) = \{m \in \sigma(EN(w)) : [v, w] \text{ influences } m\},\ w \in T_v.$$

Subtree $T_1 \subset T_v$ is called testing if $\sigma(EN(w, T_1)) \supseteq O(w, v) \cup L(w, v)$ for $w \in V^{\iota(v)}(T_1)$ and $EN(w, T_1) = EN(w)$ for $w \in V^{-\iota(v)}(T_1)$

Further, subtree $T_2 \subset T_w$, $w \in V^{\iota(v)}(T_v)$ is called parallel to $T_1 \subset T_v$ if $N(u, T_2) = N(u)$ for $u \in V^{-\iota(w)}(T_2)$ and $N(u, T_2) = \sigma^{-1}(O(u, w) \cup N(v + \tilde{\sigma}([w, u]), T_1))$ for $u \in V^{\iota(w)}(T_2)$. Suppose now that we are given a function $\varkappa : M \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

Suppose that $F(e, d) = d$, $\forall e \in E(T)$ and put $S(v_0) = c$, $S(v) = \sum_{e \in [v_0, v]} \varkappa(\sigma(e))$

Suppose that a condition of majoration holds: $S(v) > S(w) \Rightarrow \varphi_f(v) > \varphi_f(w)$

E.G. for chess $S(v)$ may be taken to be material balance).

Theorem 2. Let $T_1 \subset T_v$, T_1 be testing subtree, $w \in V^{\iota(v)}(T_v)$. If T_1 does not influence $[v,w]$, $S(w) \leq S(v)$ and $v \in V_{\alpha\beta}(T_1)$ then $w \in V_{\alpha\beta}(T_2)$ for subtree $T_2 \subset T_w$ which is parallel to T_1.

This theorem menas, in terms of search, that if at the completion of search over T_1 we have established that $T_1 \subset V_{\alpha\beta}(T)$ then, in conditions of Theorem 2 the search over T_2 is superfluous and T_2 can be cut-off.

Let us describe one more result using notions introduced above. Let π be a fixed search computing $\varphi_{F,f}(v_0)$. Let $\pi(n,F,f)=u$, $\beta = A(T(n,F,f),u)^-$, $e \in EN(u)$. Let us assume, to fix setting, that $u \in V^+$. Put $K_0 = \emptyset$, $T_0 = T_n$, $G_0 = EN(u)$. Define for each $e \in G_{i-1}$ inductivily (for I = 1, 2, . . .) subtree $T_{i,e} \subset T_u$ and a set $K_{i,e}(u,e)$ in the following manner: $u \in V(T_{i,e})$, $EN(u,T_{i,e}) = e$, $\forall v \in V^-(T_{i,e})$ $[EN(v,T_{i,e}) = EN(v)]$; $\forall v \in V^+(T_{i,e}) - u$ $[e' \in EN(v,T_{i,e}) \Leftrightarrow$

$\Leftrightarrow (\sigma(e') \in \sigma(K_{i-1}(u,e)) \cap (\sigma(e') \notin \sigma(EN(u)) \cap$

$\cap (\tilde{\sigma}([u,v])$ influences $\sigma(e'))]$;

$G_i = \{e' \in G_{i-1} : \varphi_{F,f,T_{i,e'}}(u) \leq \beta\}$

$K_i(u,e) = \{e' \in G_i : T_{i,e'}$ influences $T_{i,e}\}$. Set at last $T_e = \lim T_{i,e}$, $G = \lim G_i$

Theorem 3. Let $G' \subset G$ and π' be a search over subtree $T_v \subset T_u$ such that $T_{\pi'}(N,F,f) \cap G' = \emptyset$. Suppose $\tilde{\sigma}([u,v])$ and $T_{\pi'}$ do not influence T_e, $\forall e \in G'$. If $\varphi_{\pi'}(v) \leq \beta$ then $\varphi_{\pi''}(v) \leq \beta$ where π'' is a search such that

$E_{\pi''}(T_{\pi''} \cap T_{\pi'}) \setminus E_{\pi'}(T_{\pi''} \cap T_{\pi'}) \subset G'$

The following method of search reduction is based on Theorem 3. Determine for a vertex $u \in V(T)$ subset G. Edges $e \in G$ will not be taken into search while the current path from u does not influence T_e.

Besides in the moment of return in a vertex one should check whether $T_{\pi'}$ influences T_e or not. If it influences one should include e into search. In the contrary Theorem 3 permits one to consider the search from the vertex under condideration as finished. It should be noted that different edges are rejected independently.

<u>References</u>

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