CHANGES IN REPRESENTATION WHICH PRESERVE STRATEGIES IN GAMES

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ABSTRACT

One reason for changing the representation of a game is to make it similar to a previously solved one. As a definition of similarity, people have previously often proposed homomorphism-like structures. One such structure, the "s -homomorphism", is defined and studied in this paper.

It is indicated that a useful winning strategy exists for any game in a general class called, "positional games". A set of sufficient conditions is derived which a game has to fulfill to have an s_0-homomorphism with a positional game. The conditions are exemplified by applying it to a class of games shown by Newell to be representable as tic-tac-toe.

KEY WORDS

Game-playing, problem-solving, strategies, homomorphisms, representation, artificial intelligence.

1. Introduction

It- has been known for some time that the ease with which a problem can be solved is heavily dependent on the manner in which the problem is stated. To enable a serious study of this phenomenon, (which we shall call representation dependence), it is necessary to have a well-defined model of what a problem is. Three such models are available in the literature, Green[6], Amare] [1] and Ernst and Newell [5], The latter two have been used to study representation dependence. In the third one any problem is considered to be that of proving a statement in a first order predicate calculus. It is our belief that the study of representation dependence is possible in this model also in terms of what is known as the extension of theories by definition [10].

Any of these existing models can thus be used for the study of representation dependence. Indeed, some day we may be able to consider the relationship between the three models in terms of such a study. Meanwhile, studies on representation dependence will have to continue separately along these three different avenues. This paper will use a formal variant- of the "state space" model of GPS [5] and study certain issues of representation dependence.

The reason for changing the representation of a problem is not always to make the problem "simpler" (although in the examples of Amarel [1] the problem space is shrunk). In many cases changing the representation of a problem increases its search space, and yet makes the problem easier by allowing previously learned problem solving methods to be used in the problem. Polya [9] argues as follows: When a person is given a problem if he notices sufficient similarities between the given problem and a problem which he knows how to solve, then he can solve the given problem in the same way that he solved the known problem. In general, the two problems will have different, abstract- structures (in our case different state spaces), but they are sufficiently similar that the same strategies, heuristics, etc., can be used on both.

As a definition of this similarity we propose different kinds of homomorphism-like relationships between two problems which preserves certain strategies, heuristics, etc., for solving the problems. Por instance, Amarel [1] has suggested that what is a single transformation in one problem may correspond to a sequence of transformations in another problem. Other kinds of homomorphisms have also been described in the literature. For example, there is a well known similarity between the strategies for playing the games of staircase nim and nim. This similarity is a rather unusual homomorphism between the two gomes. This led some to develop general methods to determine if a given game is homomorphic to nim. The resulting methods led to the discovery of a homomorphism between the eight pawns game and nim (and several other homomorphisms [2,3]).

Encouraged by these results, we decided to study the conditions under which a game could have a tic-tac-toe-like game as a homomorphic image. Previous to this investigation, a generalized class of games, called positional games, had been studied by Koffman [71. This class included such games as Go-Moku, Hex, Bridgit and two and three dimensional tic-tac-toe. Koffman developed a game-independent learning technique which was effective for any positional game. The development of methods for recognizing homomorphisms with positional games would enable wider application of Koffman's method for playing these games.

Another reason for studying homomorphisms with positional games is to try to answer some of the questions raised by Simon [11] and by Newell [8], They claimed that the four games in Fig. 1 (one of which is tic-tac-toe) are "isomorphic" to one another in some *very* non-obvious ways. We found that a few of these games were not even obviously positional, although most- of them were. Therefore, we felt- that a deeper analysis of game structures was needed before homomorphisms with positional games could be studied.

During our investigation, it also became evident, that the homomorphisms one should be looking for are certainly not one-one maps or not even maps. In fact, they turned out to be general relations between the game spaces under consideration. Moreover, we found that whether two games were homomorphic or not depended heavily on the starting position of the game. In the interest of axiomatic simplicity, the state space of a formalized game, often has many more states than one needs for the purpose at hand. These illegal states often do not get mapped properly by homomorphisms. By restricting one's discussions only to states reached from a legal initial state, one can avoid this difficulty.

A formal definition of a game is given and discussed in the next section. Then we introduce the homomorphism that will form the central theorem of the paper. Next, the idea of a winning strategy is introduced, and it is shown that if a homomorphism exists between two games, then one can construct a winning strategy in one from a winning strategy in the other. In section 3 positional games are discussed. A set of sufficient conditions are developed for a game to have a positional game as a homomorphic image.

In all of these sections, examples are given for all formal definitions for motivation. The last section contains some concluding remarks.

2. A Homomorphism Between Games

In this section we define a homomorphism between games which, in some sense, preserves winning strategies. However, first we must define games; this is the purpose of the first subsection. Next, the homomorphism is described and in the last subsection we prove that it preserves winning strategies.

2.1 Definition of a Game

A game G is a 5-tuple, (S, R, P, W, L). S is the set of game situations. R is the legal move relation. That is, sRt if and only if t is the result of making a legal move in the game situation s. P is the set of game situations in which the first player is on move. (We are assuming that the game is a two person game.) W is the set of winning positions and L is the set of losses.

We assume that any game satisfies the following set of postulates:

G1: $Dom(R) \cap (W \cup L) = \emptyset$

G2: $L \subseteq P$

G3: $W \cap P = \emptyset$

G4: $sRt \rightarrow (s \in P \equiv t \notin P)$

G1 indicates that there are no legal moves from a winning or losing position. G2 requires the first player to be on move when he loses; G3 requires the second player to be on move when the first player wins. Note that S-P is the set of situations in which the second player is on move. G4 indicates that there is strict alternation between players. Although these postulates rule out certain games, they are satisfied by the games described in this paper and many other games, also.

Race (see Fig. 1) fits this definition of a game quite nicely. Each game situation consists of three parts: the board; who is on move; and which tickets are unbought. P is the set of situations in which the first player is on move. R is the legal move relation. Each situation in W has one of the first player's horses on the third rank; a situation in L has one of the second player's horses on the third rank.

The reader will note that it is not possible to have a horse of both players on the third rank. Although such positions are elements of S (see discussions in Sec. 3), they can never be obtained due to G1. That is, after the first player moves a horse to the third rank, he leaves the second player with a situation in which there are no legal moves; thus, the second player never has the opportunity to subsequently move one of his horses to the third rank also.

2.2 Definition of s_0-morphism

Given two games G and G', a s_0-morphism, h, is a homomorphism form game, G = (S, R, P, W, L), to game G' = (S', R', P', W', L'). Unlike most homomorphisms, it is not a map but a relation between S and S'. In other words, a situation in S may have several images in S'; h is only concerned with those situations in S that can be reached by applying a sequence of legal moves to some starting situation, s_0. We define T to be the set of situations that are reachable from s_0; i.e., $T = \hat{R}(s_0)$, where $\hat{R}$ is the transitive closure of $\hat{R}$ $(R = 1 \cup R \cup R^2 \cup R^3 \ldots)$.

An s_0-morphism, h, is a subset of SxS' that satisfies the following postulates:

H1: $s \in T \,\&\, s' \in h(s) \,\&\, s'R't' \rightarrow \exists t(t' \in h(t) \,\&\, sRt)$

H2: $s \in T \,\&\, sRt \,\&\, s' \in h(s) \rightarrow \exists t'(t' \in h(t) \,\&\, s'R't')$

H3: $s \in T \,\&\, s' \in h(s) \rightarrow (s \in W \equiv s' \in W') \,\&\, (s \in L \equiv s' \in L')$

H4: $s \in T \,\&\, s' \in h(s) \rightarrow (s \in P \equiv s' \in P')$

According to H2, whenever there is an arc from s to t in G, there must also be an arc from every image of s to some image of t in G'. H1 is similar to H2 except the roles of G and G' are reversed. H3 requires every image of a win (loss) to be a win (loss); conversely every reachable situation that has an image which is a win (loss), must be a win (loss). H4 requires the same player to be on move in a situation and its image.

An example will help clarify the properties of a s_0-morphism. Race is s_0-morphic to tic-tac-toe. (See Fig. 1.) Each tic-tac-toe square, i, corresponds to the ticket Mi where the tic-tac-toe squares are assigned the following numbers:

2	9	4
7	5	3
6	1	8

A tic-tac-toe situation is an image of a race situation when

i. each unbought ticket is an unmarked square and conversely;

ii. a tic-tac-toe square marked by a player corresponds to the ticket bought by a player owning a horse on the ticket.

Fig. 2 shows a race position and several of its tic-tac-toe images. The s_0-morphism requires squares 7 and 3 to be unmarked; square 5 to be marked with O; and squares 2 and 6 to be marked X. For this race position, it is easy to see that all of the properties of an s_0-morphism are satisfied. In either of the tic-tac-toe positions in Fig. 2, the second player may mark square 7 or 3 with an X, while the corresponding move of buying ticket M7 and M3 is legal in the race situation in Fig. 2. Thus, H1 is satisfied. H2 is satisfied because the second player buying a ticket corresponds to marking one of the empty squares in the tic-tac-toe positions.

To see that H3 is satisfied, note that each horse is affected by precisely three tickets whose corresponding squares constitute a winning line in tic-tac-toe. Thus, in order for a player to win at race, he must have bought one of these sets of three tickets. For example, if the first player succeeds in moving horse d to the goal rank, it follows that he has bought tickets, M2, M7 and M6. Any image of such a win must have X's on squares 2, 7 and 6 which is a win in tic-tac-toe. A similar argument shows that the losses of race have images that are losses in tic-tac-toe.

Obviously, H4 is satisfied because play is strictly alternating in both games.

2.3 Definition of a Strategy

Given a game, G, a strategy, Q, is a partial map on P with S as its range with the proviso that $Q(s) = t$ implies sRt. That is, given a player's move, the strategy determines the next situation for the player to go to.

Given a strategy, Q, and a situation, $s_1 \in P$, a Q-sequence for s_1 is a sequence, $s_1, s_2, \ldots$, of situations such that for all i, $s_i R s_{i+1}$: Moreover, for all odd i, $s_{i+1} = Q(s_i)$, and if s_n is the last element of the sequence, (i.e., the sequence is finite), then $s_n \notin dom(R)$. Intuitively, each of the first player's moves in the sequence is dictated by the strategy, Q, while the second player's moves are arbitrary legal moves.

A Q-sequence for s_1 is called winning if for some finite n, $s_n \in W$. It is easy to see that in such cases n is even.

A strategy is a winning strategy for s if every Q sequence for s is a winning sequence. It can be shown without any difficulty that, if Q is a winning strategy for a situation s, then it is a winning strategy for any situation $s_i \in P$ which occurs in a Q-sequence for s. Using the above definitions we can now prove:

Theorem 1. If there is an s_0-morphism, h, from game G, to game G', and there is a winning strategy, Q, for $s \in T$, then there is a strategy, Q', which is winning for each $s' \in h(s)$. Conversely, if Q' is a winning strategy for s', then there is a strategy which is winning for every s such that $s' \in h(s)$ and $s \in T$.

The proof of this theorem appears in Banerji & Ernst [4] and in Banerji [3]. For the immediate purposes of this paper, we shall now show how an s_0-morphism exists between certain classes of games.

3. Two Related Classes of Games

Koffman [7] developed a powerful method for learning how to play any game in a class of games which he called positional games. If we had some way of recognizing when a game is essentially positional, we could use Koffman's method for playing the game. In the first subsection we will define positional games and in the next subsection we will derive sufficient conditions for a game to be s_0-morphic to a positional game. Theorem 1 shows that Koffman's method is useful on any game that is s_0-morphic to a positional game.

3.1 Definition of a Positional Game

A positional game is defined by a 3-tuple, (N, A, B), where each element of A and B are subsets of N. Intuitively, one can view a positional game as being played on a board in which the set of squares is N. Each player on his move occupies an unoccupied square. A and B are the sets of winning paths for the first and second players, respectively. If the first player occupies all of the squares in a path in A, he wins. The second player wins upon occupying a path in B.

We will represent each situation by ((X, Y, #), m) where X is the set of squares occupied by the first player; Y is the set of squares occupied by the second player; # is the set of empty squares; and m is a 1 if the first player is on move, and 0 otherwise. Given any (N, A, B), we can construct a game (S, R, P, W, L) which satisfies the following postulates:

P0: $S = \{((X, Y, \#), m) \mid m \in \{0, 1\} \; \& \; \{X, Y, \#\} \text{ is a partition on } N\}$

P1: $((X, Y, \#), m) \in W$ iff $m=0 \; \& \; \exists a (a \in A \; \& \; a \subseteq X)$

P2: $((X, Y, \#), m) \in L$ iff $m=1 \; \& \; \exists b (b \in B \; \& \; b \subseteq Y)$

P3: $((X, Y, \#), m) \in P$ iff $m=1$

P4: $((X_1, Y_1, \#_1), m_1) R ((X_2, Y_2, \#_2), m_2)$ iff $((X_1, Y_1, \#_1), m_1) \notin W \cup L$ and either

i. $\exists n (n \in \#_1 \; \& \; X_2 = X_1 \cup \{n\} \; \& \; \#_2 = \#_1 - \{n\} \; \& \; m_1 = 1 \neq m_2$

or

ii. $\exists n (n \in \#_1 \; \& \; Y_2 = Y_1 \cup \{n\} \; \& \; \#_2 = \#_1 - \{n\} \; \& \; m_1 = 0 \neq m_2$

It is easy to show that any (S, R, P, W, L) so constructed satisfies G1-G4.

PO gives the set of game situations. P1 and P2 relate the wins and losses to A and B. P3 indicates that the first player is on move when m-1. According to P4, the first player's move consists of moving an element of # to X, while the second player's move consists of moving an element of # to Y. F4 also forbids moves from a win or a loss.

Tic-tac-toe can be represented as a positional game. N is the set of nine squares on a tic-tac-toe board. A is the set of horizontal, vertical and diagonal lines, and B - A; i.e., the set of winning paths are the same for both players.

3.2 Definition of Reducible Games

(Often a game can be described, conveniently, in terms of a set of properties. A chess position, for example, can be described by considering each square to be a property whose value is the piece on the square (or empty). A property then is a function which maps game situations into some set, e.g., chess pieces. Reducible games are described in terms of such properties.

A reducible game (S, R, P, W, L) is described in terms of n such properties, $f_{.}$, f_{2},, ..., f , and two classes, A and B, of subsets of these properties. For any position in S each of the n properties has a value of 0, 1, ? or 3. The postulates of a reducible game involve the set T of positions that are reachable from some starting position $s_{0e}S$. Space does not allow us to give a complete formal definition of the postulates of a reducible game. Instead, we give a brief informal description of a reducible game and a detail description of race, a typical reducible game.

The A and B of a reducible game are analogous to the A and [3 of a positional game. Each element acA is a winning set of properties in the sense that a reachable position s is a win if f(s)-1 i'or each f ga. Similarly, a reachable position, s is a loss if *f(s)-?* for each febeB.

Each legal move from a reachable position, s, to a new position t(i.e., sRt) changes the value of some properly f from 0 to non-zero. That is, there is an f such that. f(s)-0 and f(t)^0. In addition, once a property acquires a non-zero value if wil 1 remain non-zero for the remainder of the game. Due to these rules (and other rules described below) about the way that moves change the values of properties, a property can become inessential to the play of the game. An inessential property is one that can never contribute to a win (or a loss-) because every winning (losing) set containing it, contains at-least, one property whose value cannot be changed to a 1 *(2),* Moves affect the values of inessential properties in a different way than essential properties.

If a reachable position s is not a win nor a loss, and if f(s)-0 for some property, f, then there is a legal move from s to a new position t (i.e., sRt). In t all of the essential properties have the same values as they had in s except for f whose, value changes, i.e., f(t)=0. If f is an inessential property of t, f(t)=3. If f is an essential property of t, f(t)=1 if the player made the move (i.e., sgP). If the opponent, made the move and f is essential for t, then f(t)-2. In addition, all of the inessential properties of t acquire a value of 3. Thus, we see that a single move may change the values of several properties.

An example, race, will help to clarify the above description of reducible games. One way to describe a race position is to tell the players who is on move, what tickets are still available for purchase, and how the horses stand, i.e., to whom, if any, each horse belongs and where it, stands on the track, which horses are disqualified and which horses are unowned. Each situation, then, has three components, and we shall designate a situation by a triple (t, b, m) where m is the usual move indicator, 0 or 1. The first component,, t, is the set of unbought tickets. The second component,, b, is the board, i.e., an assignment to each horse of P1 , P.?, P3, E1 , E2, E.3, U0 or DQ. These values stand for "owned by the first player and in position 1, *71* or 3", "owned by the second player and in position 1, ?, or 3^{n}", "unowned" and "disqualified", respectively.

We may, at this point, make various conventions about whether any combination of these three components would be allowed. Should we, for instance, accept a situation where no tickets are bought, two horses (one owned by each player) have already won, while three others are disqualified? The answer, we believe, is a matter of taste. In any case, it makes no difference in the final analysis, because as long as we take the proper initial situation as the one where all the tickets are unbought and all the horses are unowned, then the move rules restrict T to exclude all such nonsense states, since such states cannot be reached by any sequence of legal moves. This exemplifies the strong influence of the set T on all our discussions and explains why it plays such an import,ant role in our definitions.

Continuing with the example, let position $(t_{.},b_{0},m_{0})_{2}$ be a legal move from position $(t'_{1} ,b_{1},m_{1}^{2})$. Then t_{2} contains all the tickets in t_{1} except some specific ticket Mi. Also m_{1}=0 if m_{4}-1 and vice versa, b,, is obtained from b_{1} as follows: the ownership and the position of all the horses are the same in b_{1} and b,, except for those whose columns contains an X in the Mi row of the move table in Fig. 1. For these horses, their values are advanced one step if they are unowned or belong to the player on move. That is, if m2=0, then a value of P_{1} becomes P2, P_{2} becomes ^{2}P3 and UO becomes P1. If m_{2}=1 then E1 becomes E_{2} etc, and U0 becomes $E1^{2}$. If the horse belongs to the opponent of the player on move, it is disqualified, i.e., if m_{2}=0 then values of E1 and E2 become DQ and if m_{2}=1 then P1 and *V?* become DQ. Disqualified horses on the Mi list, remain disqualified.

A situation (t,b,m) is winning if and only if there is some horse h with a value P3 in b and m=1 . Similarly (t,b,m)L if and only if *m-0* and there is a horse whose value is E3.

To see that this game is reducible, one sets up one function f_i for each ticket, Mi as follows. The value of f_i (1<i<?) for a situation is 0 if and only if Mi is unbought, i.e., if Mi et. If Mi is not in t, the value of f has to be surmised from the values of the horse as given in b. If at least one horse appearing in the row of the ticket Mi has a value P1, P2 or P3, the value of f_i. is 1 . If at least one horse on the row o f Mi ft as a value E1 , E2 or E3, the value of f. is 2. (If the game is played legally from a starting state where all the tickets arc on the table and all the horses are unowned, then these two rules will never contradict one another). If all horses in the row Mi are disqualified, then f. has the value 3 for the state.

It is clear that if any winning situation is reached from the usual starting situation, then the table shows that there should be exactly three tickets be!onging to the winning player which has the winning horse's name on it. So there is a set of f's such that all their values are 1 at that situation. Thus one can isolate a class of sets of the f's such that a situation is a win if and only if all the members of some set in this class has the value 1. This shows that the wins and losses of a race are specified in the same way as the wins and losses of a reducible game.

To see how the move rules of race fit the above description of reducible games, let the value of f be 0 for some situation. From what has gone above it is clear that in this case Mi will be an unbought ticket. If this is a reachable situation which is neither a win nor a loss, then the player on move can buy this ticket. If all the horses on this ticket are either disqualified or belong to the opponent, then after buying this ticket all the horses on it will be disqualified and the value of the ticket for the new state will be 3. Otherwise the value of f will be determined by the identity of the player, which in turn, will be reflected by the ownerships of the horses in the next position. As far as all the other tickets are concerned, the unbought ones will remain unbought, and the others will either retain their values (since horses do not change hands, nor do 'the tickets bearing the horses names) or some tickets take on the value 3 if their horses get disqualified by the purchase.

3.3 A Homomorphism

In this section we define a relation h between positional and reducible games. Theorem 2 proves that this relation is an s_0-morphism. Our reason for doing this is that we have powerful methods for playing positional games. Since s -morphisms preserve winning strategies, we can now apply these same methods to reducible games.

Given G, i.e., given n functions, f_i for $1 \leq i \leq n$ and the subsets A and B construct a positional game G' defined by the triple (N, A, B) where $N = \{1, 2, \ldots, n\}$. Now construct $h \subseteq S \times S'$ as follows: For each $s \in S$, the element $s' = ((X, Y, \#), m)$ will be related to s (i.e., $s' \in h(s)$) if and only if

(1) m=1 if $s \in P$; else m=0.

(2) Every j such that $f_j(s) = 1$ is in X.

(3) Every j such that $f_j(s) = 2$ is in Y.

(4) $j \in \#$ if and only if $f_j(s) = 0$.

(5) If $f_j(s) = 3$ and $j \in p \in A \cup B$, then there is an i and a k in p such that $i \in X$ and $k \in Y$.

The intent of (1)-(4) is clear. (5) indicates that if f_j is inessential then so is "square" j in the positional game. A square is essential when all winning and losing paths containing it, contain a square occupied by the player and a square occupied by the opponent.

We now can prove:

<u>Theorem 2.</u> Given a s_o-reducible game G, a positional game, G', defined by (N, A, B), and a relation h that satisfied (1)-(5), h is a s_o-morphism from G to G'.

The proof of this theorem, together with the formal definition of a reducible game is given in Banerji and Ernst [4] and in Banerji [3].

It is worthwhile pointing out here that our invocation of theorem 2 above has established race only as homomorphic to a positional game and not necessarily to tic-tac-toe. The establishment of the isomorphism (i.e., a one-one onto homomorphism) between the resulting positional game and tic-tac-toe will need a more detailed examination of the horse-ticket table. However, this is really unnecessary as far as the mechanical playing of the game is concerned. The Koffman technique mentioned earlier in section 1 is designed to play any positional game of moderate size and hence the isomorphism with tic-tac-toe is a matter of human visual convenience only. For instance, the number scrabble game is clearly positional since each chip is either unowned or owned by one of the two players and so the partitioning of the chips at each game situation is obvious. Hence a good positional game playing program could play the game whether the chips were considered arranged in a magic square or not. In this sense the jam game also is obviously positional so that whether it is tic-tac-toe or not is somewhat immaterial. With regard to race, it could also have been remarked that it is obviously positional since each situation partitions the set of tickets into three subsets-unbought, owned by the first player and owned by the second player. If we do this then the board showing the position of the horses is a mere distraction. However, the way the Race game is described, the board appears as important.

as the tickets. Also, if the'image of a race position in tic-tac-toe is to be determined, it- has to be done, not in terms of the history of the game, but only in terras of the position itself. As we have remarked before, the position is adequately described in terms of the board and just the unbought tickets. An onlooker walking in at the middle of the game is at no disadvantage if he does not know who bought which ticket in the past. Our method of formulation reflects this fact. It also points out how theorem 2 is capable of exhibiting the homomorphism even in this formulation. Our surmise is that theorem 2 would be able to recognize a positional game no matter how it is formulated. The examples in this paper have used only the games in Fig. 1, However, more complicated games of race (i.e., more horses and tickets) would have throe dimensional tic-tac-toe, Bridgit or other positional games as their images. We have used the race in Fig. 1 to keep the examples simple.

4. Concluding Remarks

Of the various ways in which one can change the representation of games, we have considered the case where the change takes the game to a homomorphic image. Of course, the kind of homomorphism considered in this paper (to wit, the s -homomorphism), is not the only kind of homomorphism that can be studied. The various examples studied by Amarel [1] and the graph homomorphism studied by Banerji [ ], are examples of other kinds of homomorphisms. Whether all these can ultimately be unified to a general class of relations between representations or whether we will have to remain satisfied with a number of special cases, remains to be seen.

If is our belief that some of the results in this paper, together with other work on homomorphisms, is a first step towards the automatic change of representation. Theorem 1, for instance, <-ould lead to the automatic verification of a "hunch" that a given game may have e strategy similar to that of a previously known game. Given a repertoire of previously understood games, the theorem may lead to on exhaustive search for possible changes of representation.

Although much less general, theorem 2 lends greater strength to this search since jt allows one to establish the fact that a given game is homomorphic to one of a wide class of games (in this case the positional games), thus reducing the search space. If the search produces an affirmative answer, it, also allows one to construct a specific homomorphic image as opposed to searching for one.

Needless to say, such theorems do not obviate the need for heuristics. They merely add to our stock of applicable heuristics or render previously known heuristics more widely applicable. In the absence of a generally efficient problem solving technique (it is not hard to see that given any such technique, a problem can be devised which renders it inefficient), one can at present only hope for a larger arsenal of applicable ideas.

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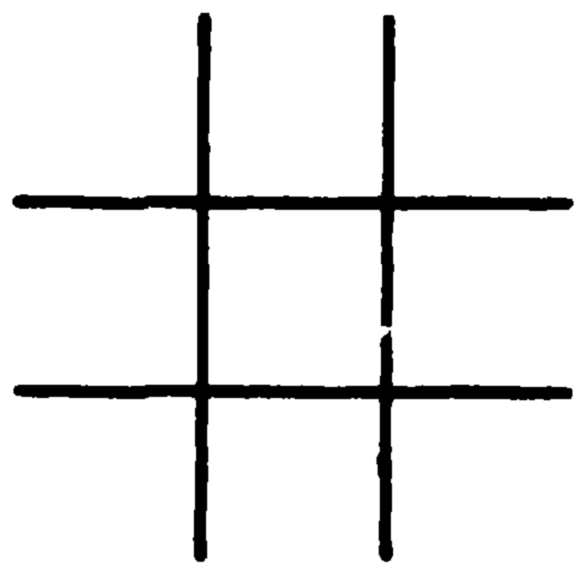
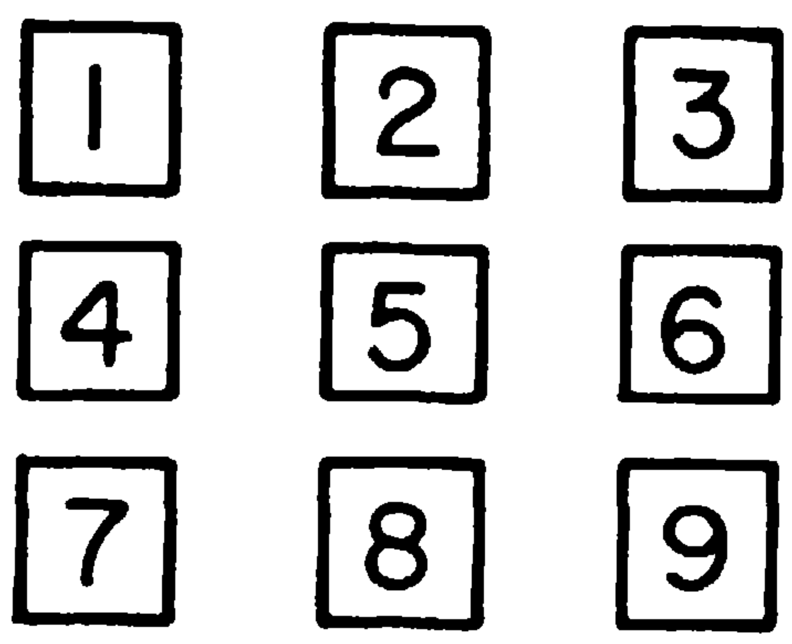
(a) Tic-tac-toe: Played on the following square board:



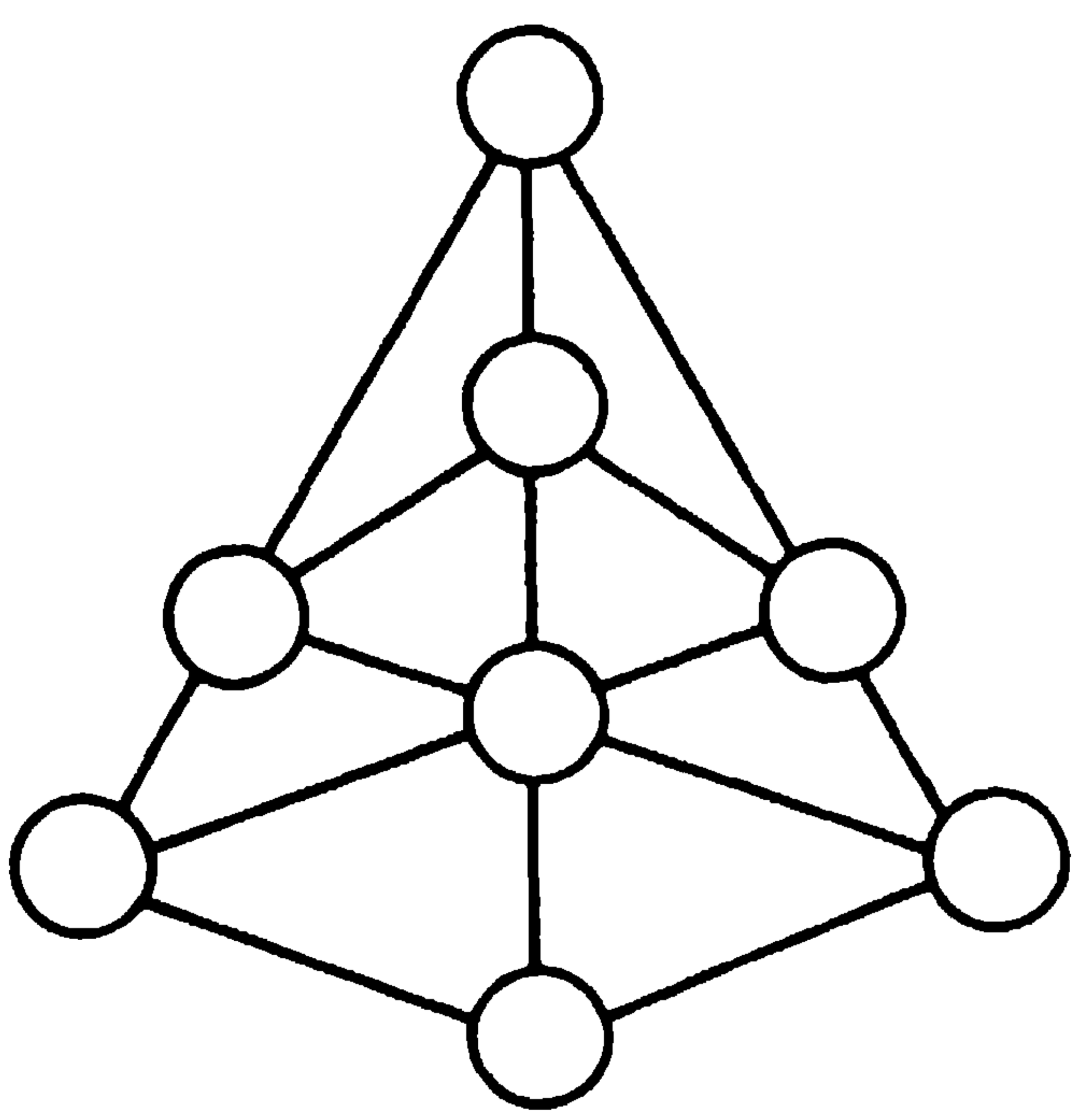
At his turn, each player puts a characteristic mark in any empty square, X for player 1 and O for player 2. The first player to mark an entire row, column or diagonal wins.

(b) Number Scrabble: The nine digits, 1, 2, ..., 9, are used to label a set of nine blocks, which constitute the initial pool as follows:



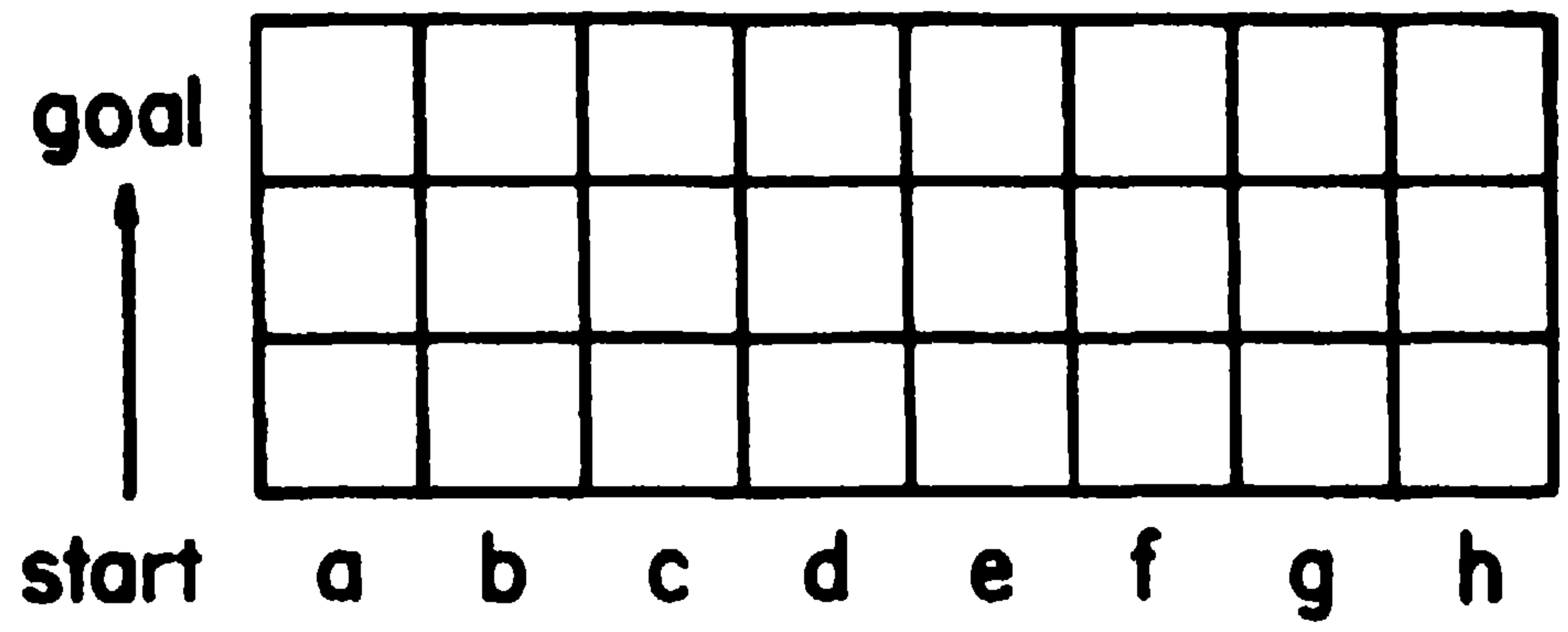
At his turn each player draws a block from the pool. The first player able to make a set of three blocks that sum to 15 is the winner.

(c) Jam: Play takes place upon a network of roads, as follows:



Each straight line constitutes a road, and each junction of roads is a town. At his turn each player can occupy a road (all of it) and thus jam (i.e., block) access to the towns on the road. Note that there are up to 4 towns on a single road. The first person who succeeds in isolating a town, in the sense of jamming all roads leading to the town, wins.

(d) Race: Played on the rectangular board:



There are eight race horses in the starting row, and the first player to get a horse in the goal row wins. At his turn each player buys a ticket. The players select from the same set of tickets but each ticket may only be purchased once. The effects of buying a ticket is shown in the matrix, below. An X means the purchase affects the horse; a blank means it doesn't. If there is an X one of three things occurs:

> if the horse belongs to the purchaser, then it is advanced one square toward the goal;
>
> if the horse belongs to the opponent, then it is disqualified from the race;
>
> if the horse doesn't belong to either player, then it becomes part of the purchaser's stable by moving it onto the bottom rank.

	a	b	c	d	e	f	g	h
M1			X		X			
M2	X			X			X	
M3		X				X		
M4	X					X		X
M5		X			X		X	X
M6			X	X				X
M7		X		X				
M8			X			X	X	
M9	X				X			

Figure 1. Four games that are homomorphic to one another. This figure is taken with minor modifications from Newell [8].

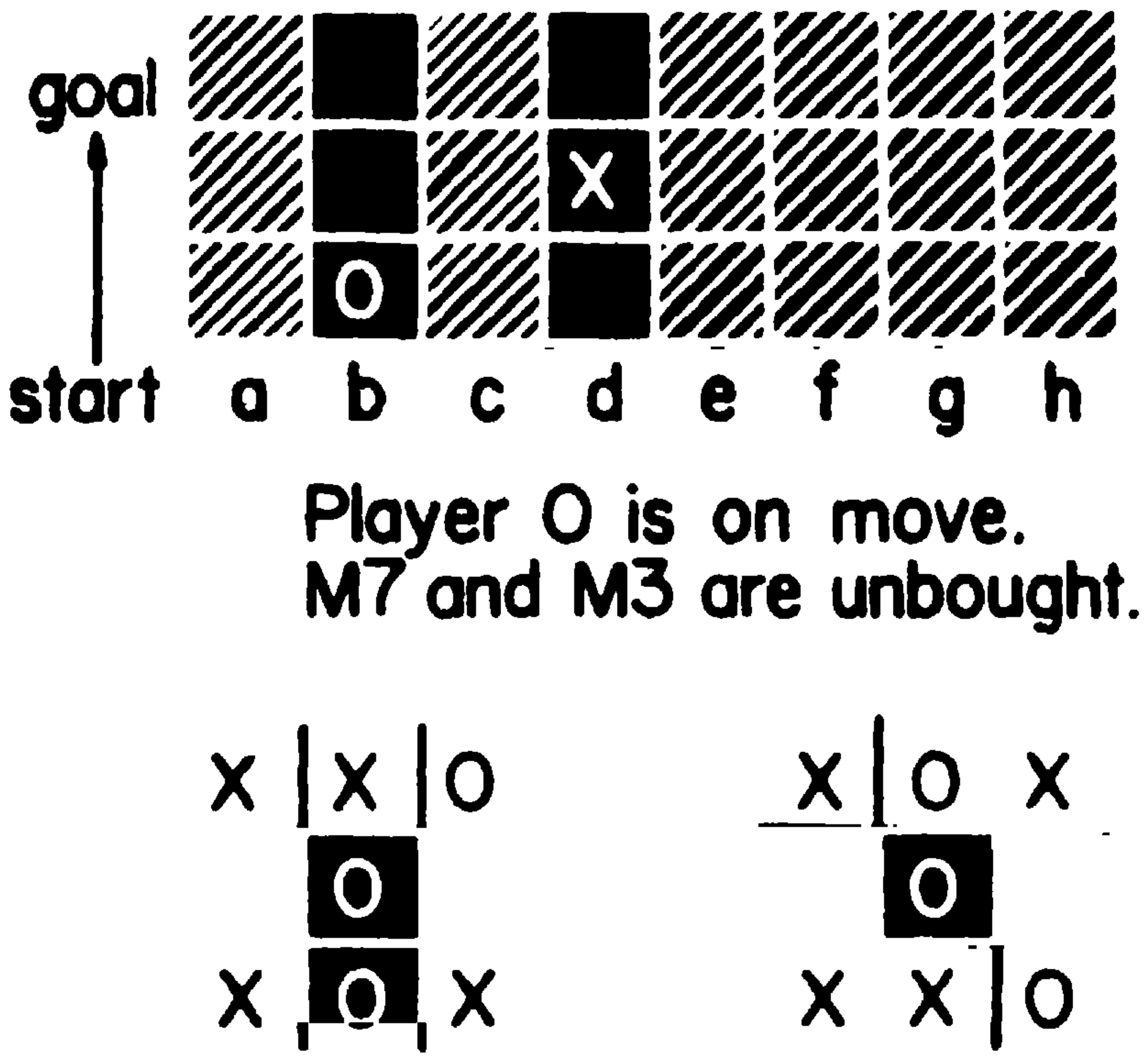


Figure 2. A race position and two of its tio-tac-toe images. In the rare position the shaded areas are disqualified horses; an X indicates the position of a horse belonging to player X; an 0 indicates the position of a horse belonging to player 0.