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# Abū al-Wafā' Latinus? A Study of Method

Dominique Raynaud<sup>a,b</sup>

<sup>a</sup>PLC, Université de Grenoble, France

<sup>b</sup>GEMASS, CNRS / Université de Paris Sorbonne, France

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## Abstract

This article studies the legacy in the West of Abū al-Wafā's *Book on those Geometric Constructions which are Necessary for Craftsmen*. Although two-thirds of the geometric constructions in the text also appear in Renaissance works, a joint analysis of original solutions, diagram lettering and probability leads to a robust finding of independent discovery. The analysis shows that there is little chance that the similarities between the contents of Abū al-Wafā's Book and the works of Tartaglia, Marolois and Schwenter owe anything to historical transmission. The commentary written by Kamāl al-Dīn Ibn Yūnus seems to have had no Latin legacy either.

## Résumé

Cet article étudie la descendance européenne du *Livre sur les constructions géométriques nécessaires aux artisans* d'Abū al-Wafā'. Bien que deux-tiers des constructions géométriques exposées dans ce livre apparaissent dans des œuvres de la Renaissance, l'analyse des solutions originales, du lettrage des figures et des probabilités montre qu'il y a peu de chance que les similitudes observées entre le livre d'Abū al-Wafā' et les œuvres de Tartaglia, Marolois et Schwenter résultent d'une transmission historique. Le commentaire rédigé par Kamāl al-Dīn Ibn Yūnus ne semble pas avoir eu davantage de descendance latine.

*Keywords:* Geometric constructions, East-West diffusion, rediscoveries.  
*2010 MSC:* 01A30, 01A35, 01A60.

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## 1. Introduction

The present article is a study on East–West mathematical borrowings. The overall focus is methodological. Results about the dependence of geometrical works are sought through a threefold method, including an analysis of differences, a study of diagram lettering, and an index of independence. This method is applied to Abū al-Wafā's putative legacy in Europe.

A mathematician and astronomer from Khorāsān, Abū al-Wafā' Muḥammad ibn Muḥammad ibn Yaḥyā al-Būzjānī (1st Ramaḍān 328–387 H./10 June 940–997 or 998) is known for a collection of geometric problems entitled *Kitāb fī mā yaḥtāju al-ṣāni' min al-a'māl al-handasiyya* (*Book on those Geometric Constructions which are Necessary for Craftsmen*). The treatise, known in five Arabic MSS<sup>1</sup>, has been the subject of various studies (Woepcke, 1855; Suter,

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<sup>1</sup>Istanbul, Ayasofya MS 2753; Cairo, Dār al-Kutub al-Miṣriyya MS 31024, MS 44795; Milan, Ambrosiana MS &68  
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1922; Youshkevich, 1981; Hashemipour, 2007) and editions (Būzjānī, 1966, 1979, 1990, 1997, 2010). From information contained in the preface it is possible to date it to between 993 and 1008.<sup>2</sup> This treatise is a collection of 171 problems of geometry, divided into eleven chapters.<sup>3</sup> It includes 150 problems of plane geometry, two-thirds of which are echoed in Western geometric treatises. If we compare the solutions, the impression is that almost all of them were known in the Latin World (see Appendix C). Does this fact provide evidence of a diffusion of Abū al-Wafā's collection in Latin Europe?

## 2. Abū al-Wafā's Putative Legacy

Focusing on the problem of transmission, the choice of Abū al-Wafā's work is exciting because a conclusion about its Latin legacy has not yet been reached. As far as I know, the treatise has not given rise to any full or partial translation throughout history.<sup>4</sup> However the history of geometric constructions is poorly documented in the Latin Middle Ages. This uncertain situation has given rise to opposing assumptions about the sources of practical geometry as well as about Abū al-Wafā's legacy. The thesis of historical transmission is counterpoised against the thesis of independent discoveries.

### 2.1. Woepcke's hypothesis

Scholars have long admitted that Renaissance geometry was Arabic dependent. Identifying the geometric constructions of Abū al-Wafā' with those of European geometers would provide

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sup; Uppsala, Universitetsbibliotek MS Tornberg 324. The latter is entitled *Kitāb al-ḥiyāl al-rūḥāniyya wa-al-asrār al-tabī'yya fī daqā'iq al-ashkāl al-handasiyya* (*Book of Skilful Spiritual Devices and Natural Secrets on the Refinements of Geometrical Figures*). It has been ascribed to Abū Nasr al-Fārābī (872–950) (Kubesov and Rosenfeld, 1969; Kubesov et al., 1972; Sezgin, 1974). However, it has been convincingly argued that only the first and last pages are by al-Fārābī, the rest of the treatise being a simple copy of Abū al-Wafā's work. See Hogendijk (1993, 145), Özdural (2000, 193).

<sup>2</sup>There one can read: "I have accomplished what my Master, the Lord, shāhanshāh (King of the Kings), the victorious ruler Bahā' al-Dawla wa-Diyā' al-Milla (Light of the Community) wa-Ghiyāth al-Umma (Asylum of the Nation)—may God preserve his family, his power and his reign—ordered about the geometric constructions most used by artisans that were discussed before him" Būzjānī 1966, 56; 1997, 1), Ayasofya, MS 2753, fol. 2:1:

هذا كتاب ابي الوفاء محمد بن محمد البوزجاني فيما يحتاج اليه الصانع من اعمال الهندسية. قد اشلت ما رسمه مولانا الملك شاهنشاه السيد الاجلي المنصور بهاء الدولة وضيء الملة وغيث الامة، اطال اسر بقاءه وادام علاه وقدرته وسلطانه، من اثبات المعاني التي كان تذاكر بحضرتة العالمة من الاعمال الهندسية التي كثر استعمالها عند الصنّاع محرزًا من العلل والبراهين، يسهل على الصنّاع ناوله ويعرف عليهم طريقته.

The Būyid ruler was invested 10 Jumādā II 379 H./15 September 989. The second *laqab* Ghiyāth al-Umma was granted to the ruler by caliph al-Qādir in 381 H./992. He did not yet have the third *laqab* Qiwwam al-Dīn appearing on dirhams of 399H./1008. It has been argued that the old Sassānid title shāhanshāh proves the treatise to be written by a disciple after Abū al-Wafā's death in 388 H./998 (Özdural, 2000, 172). Admittedly, Bahā' al-Dawla ruled over Fārs and Kirmān only after the death of his brother Ṣamṣām al-Dawla in Dhū al-Hijja 388 H./December 998. However, as this title was always used in the context of military conquests, it could be that Bahā' al-Dawla used it from the time he began to direct troops upon Fārs, that is, from 383 H./993 (Donohue, 2003, 98). At that time, Abū al-Wafā' was still alive. Should that be the case, his treatise of geometry was written in the span of years 993–998.

<sup>3</sup>The introduction gives 13. Two chapters are missing: "On the division of scalene figures" and "On tangent circles."

<sup>4</sup>Since latinized versions of Arabic names similar to that of Abū al-Wafā' al-Būzjānī survive, it is likely that any existing translation would have been recognized by now. Ptolemy's *Almagest* is often preceded by a prologue entitled *Bocados de Oro*, written by Abū al-Wafā' Mubāshshir ibn Fātiq. His name is transcribed "Albugaefe" in the incipit: "Quidam princeps nomine Albugaefe in libro suo [...]". We find similar mentions in Florence, BNCF, MS. Conv. Soppr. J. III. 24, c. 1300; Toledo, Biblioteca Catedral, MS. 98-15, XIIIth c., Madrid, Biblioteca Nacional, MS. 10113, XIIIth c. See Björnbo (1912, 104), Millás Vallicrosa (1942, 148).

a further argument in favour of the diffusion of geometry across East–West borders. It would be a sign of borrowing from Arabic mathematics. This is the thesis that Woepcke defends while discussing constructions made with one opening of the compass:

The Renaissance geometers Cardan, Tartaglia and especially Benedetti, dealt with such problems by imposing precisely the same condition that we find in Abū al-Wafā's treatise [...] I am very inclined to believe that the very idea of treating this question could well have been inspired by traditions coming from the East.<sup>5</sup>

The same opinion is voiced in Abū al-Wafā's biography, where it is said that “these constructions were widely circulated in Renaissance Europe” (Youshkevich, 1981, 42). Woepcke's and Youshkevich's assessment is that, even if a historical tradition appears distorted because of multiple reworkings, similarities ought to be interpreted as survivals of ancient treatises. This hypothesis is supported by the many works that were available in the Middle Ages but have disappeared since: Books V–VII of Apollonius' *Conics* are extant in Arabic only; the Latin version of the *Elements* which Adelard of Bath had access to is also lost, etc. It is not unreasonable to think that the medievals had access to numerous other texts which are no longer at our disposal.

## 2.2. Henry's hypothesis

On the other hand, we must pay attention to what are referred to as multiple rediscoveries: “Sometimes the discoveries are simultaneous or almost so; sometimes a scientist will make a new discovery which, unknown to him, somebody else has made years before” (Merton, 1973, 371). This is a widespread phenomenon in science and mathematics (Coolidge, 1940, 122).

As regards Abū al-Wafā's geometric constructions, Charles Henry is the first to have supported the multiple rediscoveries thesis, while speaking of “problems that, by their very nature, come to every civilization” (Henry, 1883, 514). Some historians of science have agreed with this assessment on factual grounds: “The works of al-Khwārizmī, Thābit ibn Qurra and Ibn al-Haytham were far from being all translated into Latin, and Medieval Europe knew nothing of the work of al-Bīrūnī. European scientists were also unaware of most geometric constructions by al-Fārābī and by Abū al-Wafā” (Rashed, 1997, II, 162). Since geometrical problems start from rational grounds, investigators are able to solve them independently in any region of the world, provided they are sufficiently trained. Contrary to Woepcke's opinion, constructions to be made with one opening of the compass are found in several works prior to Cardan, Tartaglia and Benedetti. For example, Leonardo Da Vinci gives instructions to proceed with “one (or a given) opening of the compass”: “un solo aprire di sesto” (MS A, fol. 15v, 16v), “una data apertura di sesto” (Codex Atlanticus, fol. 551r) (e-Leo, 2010, s.v.). One possible explanation for this resurgence is that using a fixed opening served a rational purpose, such as the need for precision. This is why Abū al-Wafā himself recommends abandoning the compass and using fixed opening callipers instead:

If there is a defect in one part, the movement varies when opening or closing the compass [...] What we have said refers to the accuracy of the compass, when the

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<sup>5</sup>“Des géomètres de la Renaissance, Cardan, Tartaglia et surtout Benedetti, se sont occupés de ce genre de problèmes en s'imposant précisément la même condition que nous trouvons énoncée dans le traité d'Abou' Wafā [...] Je serais très-porté à croire que l'idée même de traiter cette question pouvait bien leur avoir été inspirée par des traditions venues d'Orient” (Woepcke, 1855, 225-226).

drawing is small and the opening is less than two cubits. If we exceed this size, this kind of compass is unfaithful during the construction. This is why we must speak of the callipers, that is, a compass whose wheels are mounted on a rule.<sup>6</sup>

So then, the cross-cultural identity of the geometric constructions performed with the fixed-opening compass could simply be the result of a universal constraint. From this point of view, the conclusion to be drawn is as simple as it is harsh: in geometry, striking resemblances are possibly fortuitous. When two mathematicians faced with the same problem come up with the same solution, they may have done so independently, unless there is evidence to the contrary.

### 2.3. *The puzzle still unsolved*

From the time of Woepcke's and Henry's opinions, no progress has been made on the issue of whether or not Renaissance geometric constructions depend on Abū al-Wafā's treatise. In his critical edition of Chuquet's *Géométrie*, L'Huilier does not consider the ten textual parallels to Abū al-Wafā' as "borrowings" (1979, 310-335, 387-399). Several years later, however, he endorses the dependence thesis:

There is room to suppose that [practical geometry] was brought to the West by Arab intermediaries (a point that seems to merit deeper studies) while becoming richer with time. In particular, there are similarities between certain passages in Western works and Arabic tradition known as *misāḥa*. But further similarities may be revealed between the works of Abū al-Wafā' and the major treatises of this stream (L'Huilier, 2003, 188).

When a problem admits exact solutions, it is easy to subscribe to Henry's hypothesis. This is no longer the case when the problem has many approximate solutions (such as the construction of the regular heptagon). How can one explain that geometers, separated in space and time, tackling mathematical problems with different resources, picked out exactly the same solution? The case is now in favour of Woepcke's thesis.

Nevertheless, both opinions are poorly supported when judged by the standards of historical scholarship. They are, at best, intimate convictions.

### 3. **Devising a Test of These Hypotheses**

The present article aims to provide objective criteria to test diffusion hypotheses by describing a method applicable whenever no historical transmission is visible. To date, there have been three main approaches to deducing a borrowing from historical sources.

The first way is to scrutinize the procedures which act as equivalents of textual parallels. While discussing the construction of the regular heptagon by means of conic sections, Jan Hogendijk says:

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<sup>6</sup>Ayasofya, MS 2753, fol. 4:  
فان الخلل اذا وقع في واحدٍ من هن تغير حركات عند الفتح والضم ( ... ) فهذا الذي ذكرناه نعرف به صحة البركار، الذي رسم به دوائر صغار وما كان يقارب ذراعين. وما دونها فاما ما زاد على هذا المقدار فان هذا النوع من البركار يضطرب فيه عند العمل. فلاجل ذلك أحتجنا ان نذكر البركار الدولابية وهي ما كان مركبًا على مساطر.

In Manisa manuscript there is a geometrical construction of the root  $x$  of the cubic equation  $x^3 + p = qx^2 + rx$ ,  $p, q, r > 0$  by Kamāl al-Dīn that resembles 21 [al-Sijzī's construction]. Kamāl al-Dīn discusses the construction [...] mentioned by al-Sijzī, and he even draws Figure 32. It is therefore extremely implausible that the similarity between this construction and the construction of  $x^3 + p = qx^2 + rx$  by Kamāl al-Dīn is mere coincidence (Hogendijk, 1984, 240-1).

Other historians base their conclusions on simplicity considerations. Jens Høyrup notes that the scheme for the construction of the regular octagon given by Hero, *De Mensuris*, 206, continuously survived from Abū al-Wafā' to Roriczer (1484) and Serlio (1584). He writes:

It is difficult to believe that anyone would get the idea to draw this diagram if the construction was not known already; and indeed, a much more intuitive diagram can be drawn [...] It appears that the construction of the octagon [W78<sup>7</sup>] was known in Classical Antiquity and by late medieval Gothic master-builders; it is near at hand to assume some kind of continuity (Høyrup, 2006, 6).

Some other scholars think it more conclusive to base a judgment of dependence on similarities restricted to transcription errors. This is the method used by Wilbur Knorr in his study of al-Sijzī's trisection of an angle:

Al-Sijzī commits odd slips in his synthesis. For instance, he incorrectly terms as the *latus rectum* ("right side") what is in fact the diameter (or "inclining side") of the hyperbola [...] Such errors might ordinarily be lodged against the scribe. But in the present case al-Sijzī himself is the scribe. This indicates that al-Sijzī has copied his method from a source without detecting these errors (Knorr, 1989, 287).

These strategies are not always implementable. Copying errors are the scribe's affair. The length of a geometric construction (Hartshorne, 2000, 21) does not necessarily prevent transmission. For example, the method to  $n$ -sect the line<sup>8</sup> survived despite its 19 steps against the 14 steps for *Elements*, VI, 9, etc. Other tests of diffusion hypotheses are conceivable.

As we are facing a problem sensitive to nationalist passions—so that today's Arabs can claim for themselves the origin of these constructions, while Europeans may deny this heritage, having some political agenda in mind—we must tackle the problem with impartiality and independence. It seems suitable to rely on "robustness" considerations (Wimsatt, 1981). I shall consider the concept of robustness in relation to testability, where it means the multiple determination of truth. A result is robust if it remains the same while the method to get it is replaced by another. In sum, a result must be accepted if it is established by different routes. In the present case, I shall be using three different tests of historical diffusion.

<sup>7</sup>W78 is an abridged form of "Abū al-Wafā's solution 78." The solutions of Ibn Yūnus, who provided a Commentary on Abū al-Wafā's treatise (see Section 8), are referred to by the letter Y, keeping a continuous numbering so as to insert each construction at the right place in Abū al-Wafā's list. Solutions not mentioned by Ibn Yūnus are referred to by an asterisk, like W47\*. A complete list of statements is given in Appendix A. Diagrams relating to selected problems are included in Appendix B. In the tables of the main text, titles are abridged into initials, viz. PG for *Practica Geometriae*, SDL for *De Superficierum Divisionibus Liber*, and the like.

<sup>8</sup>See Appendix A, construction W3.

### 3.1. *Defining the data*

A geometric construction is basically the solution to a given problem. The available data are the statements, procedures, demonstrations and diagrams, which illustrate how the procedure is instantiated in a particular case. We shall ignore demonstrations since Abū al-Wafā' omits them all. Most geometric problems can lead to multiple types of solution because each style of geometry (with the straightedge and the compass, with the straightedge vs. the compass alone, with a variable opening vs. one opening of the compass, etc.) defines a special *modus operandi*.

Recent articles have tried to improve the algorithmic description of mathematical procedures (Imhausen, 2002; Ritter, 2004; Høyrup, 2008). Although promising, I have not taken this path, because geometric constructions provide opportunities that allow us to meet the issue of reliability differently. Geometric constructions are basically procedures.

The main factor that leads to overinterpreting a text is scale. By choosing too broad a scale of description, differences between two texts disappear, thus leading to the conclusion that they are identical. Since Abū al-Wafā's constructions are made with the ruler and compass, the choice of the right scale is easy. I have taken as a unit step: "draw one straight line" or "draw one circle", basing myself again on Hartshorne (2000). Next, the original text is transcribed in unit steps, whatever its verbal formulation.

### 3.2. *Solutions echoed in the West (Test 1)*

We now turn to the comparison of Abū al-Wafā's collection to the geometrical works of the Renaissance. With regard to the corpus covered, ancient works available either on paper or in digital form were examined with particular attention being paid to practical geometries, which are key sources for geometric constructions.<sup>9</sup>

The corpus being clarified, the first task is to determine the ratio of the number of Abū al-Wafā's constructions echoed in the Latin West to the total number of original constructions to be found in his collection—having first removed those that do not allow for any conclusion to be drawn because, for example, they could have arrived in Europe by other channels. The first test (Section 4) consists in sifting the problems to identify the exact matching solutions. The higher the number of geometric constructions echoed, the more probable the borrowing. Criticism of this test will be discussed in Section 10.

The other two tests are run on the group of constructions identified by Test 1, without departing from robustness requisites.<sup>10</sup>

### 3.3. *Diagram lettering (Test 2)*

A second way to investigate geometrical legacy is to focus on the diagram lettering (Knorr, 1989; Netz, 1999). It should be noted that when a geometric construction is copied, the letters are usually below the threshold of attention, and they are reproduced without difference. It is unlikely that the scribe would take the trouble to change the letters, because he would thus increase the risk of making an error. We consider the ratio of the number of identical letters to

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<sup>9</sup>Digital libraries, such as the ones of the IMSS or the MPIWG, provide a rich set of over 300 ancient works of geometry. A list of practical geometries published in Western languages until the late XVIIth century has been drawn upon for the present study. Between 998 and 1600, available texts represent 45 works out of 49 (0.92); between 998 and 1650, they represent 65 works out of 74 (0.88); between 998 and 1700, they represent 78 works out of 119 (0.66).

<sup>10</sup>Test 2 and Test 3 are different, and yet dependent on Test 1: it makes no sense to compare the lettering, or calculate a probability, on constructions that do not match each other.

the total number of letters used in both diagrams. The higher the ratio, the more probable the transmission.<sup>11</sup>

One case of quite well established transmission (setting aside the uniqueness of the diagram) is the reappearance of Tūsī's couple—a planetary model based on a circle rotating inside a larger circle—in Copernicus' work. One of the major arguments for diffusion is precisely the identity of the letters used in Tūsī's *Tadhkira fī 'ilm al-hay'a* (*Memoir on Astronomy*) (Istanbul, MS. Laleli 2116, fol. 38b) and in Copernicus' *De Revolutionibus Orbium Coelestium*, fol. 67r (Copernicus, 1543; Hartner, 1973). Five letters out of six are identical and, with regard to the single remaining difference, letters Z/F are very similar in Arabic script Saliba (2007, 200),<sup>12</sup>

There is a need for defining a stricter test on diagrams with only a few letters. Since A, B, G are used first in the lettering of geometric diagrams in both Greek and Arabic, we shall assume that the matches are significant only from the threshold of three letters. Test 2 is implemented in Section 6. Criticisms of this test will be discussed in Section 10.

### 3.4. Index of independence (Test 3)

A third way to estimate Abū al-Wafā's legacy in the West is to apply probability theory to historical borrowings. *Source-author* hereinafter refers to the mathematician whose constructions give rise to a possible legacy. *Target-author* is the one who is presumed to have borrowed from the source. *Solution* is the name given to any triplet (statement, diagram, procedure). *Available* is the term used here to refer to a construction attested at the time the target-treatise was composed. The number of available solutions is estimated by the number of solutions that existed prior to time  $t$ . *Identical* is the name given to geometric constructions that use the same procedure described in unit steps (Section 3.1). Furthermore, I assume that any author is free to either reproduce an available solution or invent a new one.

The overall idea is that probability theory can be applied to the study of transmission whenever the facts are equiprobable events. Obviously, this is a drastic simplification that can hold only under special conditions that need to be carefully stipulated. I shall discuss this issue further in Section 10.

Suppose the target-author picks one solution among  $n$  available geometric constructions. The probability of drawing the solution at random is equal to  $1/n$ . If the target-author solves a set of problems, and if we can consider each problem as an independent event, then the probabilities multiply each other and after a certain number of correspondences, the result will come below a likelihood threshold. For example, if the target-author solves three problems, each admitting ten solutions, and if he gives the solutions mentioned by the source, the chance of an independent discovery is  $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{1000}$ . The more numerous the solutions, the more possible it is to decide on the solution's legacy.

Suppose again a problem with ten solutions. Put the solutions in a bag and make successive draws with replacement. For one solution to appear almost certainly, we must make 28 draws from the bag (see Section 7). If the number of draws before the solution appears is well below

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<sup>11</sup>I have kept the original lettering of diagrams. As regards Arabic treatises, I have adopted system ALA-LC (1997) throughout the article, except in procedures, where Ṭ, Ḥ, D, Š, Ġ are given in DIN-31635 (Arabica).

<sup>12</sup>The six letters are {A, B, G, D, H, Z}. The ratio  $r$  is  $5/6 = 0.833$ ,  $r \in [0, 1]$ . Since this discovery, other diffusion arguments have been used, including Arabic/Byzantine terminology, historical contexts and intercultural contacts, see Ragep (2007), Saliba (2007, 193-232). There is as yet no conclusive evidence regarding the actual channel of diffusion to Europe (Guessoum, 2008). The most promising route is Gregory Chioniades.



that number, there is a negligible chance that the solution occurred by chance. Hence, the solution is a case for historical transmission.

To carry out this analysis we must count: the solutions available to the target ( $n$ ), the solutions given by the target ( $m$ ), the original solutions given by the source ( $k$ ), and the solutions common to the source and target ( $\ell$ ). The index of independence will be calculated by using the quadruplets ( $n, m, k, \ell$ ) (see Section 5).

#### 4. Solutions Echoed in the West (Test 1)

In this section, the first test is applied in order to identify the entire set of Abū al-Wafā's original constructions echoed in the Latin West.

##### 4.1. Obvious solutions

Some problems must be cast aside, because they have obvious solutions. Abū al-Wafā's treatise contains eleven problems of this kind, plus three problems whose solutions are a combination of solutions given elsewhere.<sup>13</sup>

- WE. Check the right angle (identical to WC)
- WF. Check the right angle: Method 2: Egyptian triangle
- W9. Draw a parallel to a line through a given point: Method 2
- W22\*. Trisect an arc (equivalent to W19+W10)
- W29. Describe a regular hexagon
- W49. Inscribe a regular enneagon in the circle (equivalent to W36+W20)
- W56. Circumscribe a circle to a regular hexagon
- W80. Circumscribe a square to a regular octagon
- W81. Divide a triangle in two parts by a line passing through the vertex
- W88. Double or triple the area of a triangle by a line passing through the vertex
- W125. Make a square of nine squares
- W126. Make a square of four squares
- W127. Make a square of sixteen squares
- W128. Make a square of two squares

Four out of sixteen problems which deal with the composition and decomposition of squares lead to obvious solutions. Only one problem, W134: Split a square into ten squares<sup>14</sup> is echoed in a European work, namely (Ozanam, 1694, 297). Clavius (1591, 342) treated problems W125-139 but using an algebraic approach that owes nothing to Abū al-Wafā'.

##### 4.2. Solutions independently transmitted to the West

Twenty-nine geometric constructions in Abū al-Wafā's collection were taken from earlier works, and consequently could have been known to Renaissance geometers from independent sources. In these cases, there is no need to consider the work of Abū al-Wafā's in the transmission process.<sup>15</sup> The twenty-nine constructions are distributed between four ancient authors as follows: Euclid (21), Hero of Alexandria (3), Pappus (3) and Ptolemy (2):

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<sup>13</sup>See for instance Appendix B: W29, W56.

<sup>14</sup>See Appendix B: W134 for a diagram.

<sup>15</sup>See Appendix B: W13, W23A, W79.

- WA. Make a right angle (Euclid, *Elements*, I, 11)
- WB. Make a right angle: Method 2 (Hero, *Geometrica*, 439)
- W1A. Bisect a line (Euclid, *Elements*, I, 10)
- W1B. Bisect an arc (Euclid, *Elements*, III, 30)
- W4. Bisect an angle (Euclid, *Elements*, I, 9)
- W5. Draw a perpendicular to a given line from an outside point (Euclid, *Elements*, I, 12)
- W6. Draw a perpendicular to a plane from an outside point (Euclid, *Elements*, XI, 11)
- W7. Describe an angle equal to a given angle (Euclid, *Elements*, I, 23)
- W10. Find the missing center of a given circle (Euclid, *Elements*, III, 1)
- W11. Find the missing center of a given circle: Method 2 (Hero, *Geometrica*, 439)
- W13. Draw a tangent to a circle by an outside point (Euclid, *Elements*, III, 17)
- W14. Draw a tangent to a circle by a point on circumference (Euclid, *Elements*, III, 19)
- W18. Describe a triangle equal to a given triangle (Euclid, *Elements*, I, 22)
- W19. Trisect a right angle (Pappus, *Collectio*, IV, 32)
- W21. Trisect a right angle: Method 2 (Pappus, *Collectio*, IV, 38)
- W23. Duplicate the cube (Hero's solution; Pappus, *Collectio*, III, 9)
- W26. Describe an equilateral triangle whose side is given (Euclid, *Elements*, I, 1)
- W27. Describe a square whose side is given (Euclid, *Elements*, I, 46)
- W36. Inscribe an equilateral triangle in the circle (Euclid, *Elements*, IV, 2)
- W37. Circumscribe an equilateral triangle to the circle (Euclid, *Elements*, IV, 3)
- W38. Inscribe a square in the circle (Euclid, *Elements*, IV, 6)
- W43. Inscribe a regular pentagon in the circle (Ptolemy, *Almagest*, I, 10)
- W46. Inscribe a regular hexagon in the circle (Euclid, *Elements*, IV, 15)
- W48. Inscribe a regular octagon in the circle (Euclid, *Elements*, XII, 12)
- W51. Inscribe a regular decagon in the circle: Method 2 (Ptolemy, *Almagest*, I, 10)
- W52. Circumscribe a circle around a scalene triangle (Euclid, *Elements*, IV, 5)
- W54. Circumscribe a circle around a square (Euclid, *Elements*, IV, 9)
- W57. Inscribe a circle in any given triangle (Euclid, *Elements*, IV, 4)
- W79. Inscribe a regular octagon in a square: Method 2 (Hero, *De Mensuris*, 206)

Furthermore, Abū al-Wafā' transmits two Arabic constructions that could have been known through multiple sources, because some of the earlier constructions on which he comments dif-fused separately in the West. The method for  $n$ -secting the line (W3), derived from al-Nayrīzī's *Commentary on Euclid's Elements*, was available through Gerard of Cremona's translation, as well as through Albertus Magnus' and Bacon's commentaries (Tummers, 1984; Busard, 1974). The construction of the parabola (W25) given by Abū al-Wafā' is a borrowing from Ibrāhīm ibn Sinān's *Maqāla fī rasm al-qutū' al-thalātha* (*Epistle on the Drawing of the Three Sections*) (Neugebauer and Rashed, 1999; Rashed and Bellost, 2000). Many European authors have reproduced this construction: Werner, *Libellus super Elementis Conicis*, prop. XI; Cavalieri, *Specchio Ustorio*, fol. 9r; Milliet de Chales, *Cursus seu Mundus Mathematicus*, I, 297; Orsini, *Geometria Practica*, XX, 9, etc.

W3.  $n$ -sect a straight line (al-Nayrīzī, CEE, I, 31)

W25. Draw the pattern of a parabolic mirror: Method 2 (Ibn Sinān, EDTS, 268)

#### 4.3. Solutions recovered through intermediate sources

Another set of solutions consists of constructions that Abū al-Wafā' borrowed from ancient works, which are now lost but whose contents have been preserved through intermediate sources.

This is the case of Euclid's lost *Book on the Division of Figures* (Hogendijk, 1993), which was known through multiple sources. Renaissance geometers could access Euclid's constructions through Abraham bar Ḥiyya's *Sefer ha-Meshihah ve-ha-Tishboret*, translated into Latin by Plato of Tivoli in 1145, Muḥammad al-Baghdādī's *De Superficierum Divisionibus Liber*, presumably translated by Gerard of Cremona,<sup>16</sup> Leonardo Fibonacci's *Practica Geometriae*, composed in 1220–1, and its Italian adaptation by Cristofano di Gherardo di Dino (Arrighi, 1966). A few constructions could also have been spread through Jordanus Nemorarius' *Liber de Triangulis* (Clagett, 1984) and John of Muris' *De Arte Mensurandi* (Busard, 1998).<sup>17</sup> In total, they consist of twenty seven geometric constructions:

- W82. Divide a triangle in two parts by a side point,  $BD=BJ/2$  (Fibonacci, PG, 112)
- W84. Divide a triangle in two parts by a parallel to a given side (PG, 119)
- W85. Divide a triangle in three parts by two parallels to a given side (PG, 122)
- W90. Divide a parallelogram in two parts through a vertex (PG, 122)
- W91. Divide a quadrilateral in two parts through a vertex (PG, 138)
- W92. Divide a quadrilateral in two parts through a side point, with  $HZ//BJ$  (PG, 138)
- W93. Case 2:  $HZ$  not parallel to  $BJ$  (PG, 138)
- W94. Case 3:  $BH$  outside the quadrilateral (bar Ḥiyya, LE, 148)
- W95. Divide a trapezium in two parts by a parallel to its base (Fibonacci, PG, 125)
- W96. Divide a parallelogram in two parts through a side point (PG, 123)
- W97. Cut off one third of a parallelogram through a side point,  $AH=AD/3$  (PG, 124)
- W98. Case 2:  $AH<AD/3$  (PG, 124)
- W99. Case 3:  $AH>AD/3$  (PG, 124)
- W100. Case 4:  $HH<BR$  (PG, 125)
- W102. Divide a trapezium in two parts through a side point,  $AH=HD$  (PG, 126)
- W103. Case 2:  $AH\neq HD$  (PG, 127)
- W104. Divide a parallelogram in two parts through an outside point (PG, 124)
- W106. Cut off a given part of a trapezium through a side point,  $AH=AD/3$  (PG, 135)
- W107. Case 2:  $AH\neq AD/3$  (PG, 133)
- W108. Case 3:  $AH<AZ$  (PG, 136)
- W109. Divide a trapezium in two parts through a point outside the figure (PG, 129)
- W111. Cut off one third of a trapezium, with  $BH=BD/3$  (PG, 140)
- W112. Case 2:  $BH\neq BD/3$  (PG, 141)
- W113. Cut off one third of a quadrilateral by a side point,  $HZ//BD$  (PG, 140)
- W114. Case 2:  $HZ$  not parallel to  $BD$  (PG, 140)
- W118. Draw two parallels cutting off one third of the circle (PG, 146)
- W119. Divide a circular sector in two parts (PG, 148)

#### 4.4. Solutions with no following in the West

Solutions with no following consist of a set of fifty-one geometric constructions, among which are simple solutions, such as WD: Raise a perpendicular at the endpoint of a line, redrawn by Ibn Yūnus in a somewhat shaky diagram.<sup>18</sup>

<sup>16</sup>MS. British Museum, Cotton Tiberius B.IX, was afterwards copied by John Dee in 1559 and published by Federico Commandino in 1570 (Rose, 1972).

<sup>17</sup>Appendix B provides W82, W85, W103, W114.

<sup>18</sup>See Appendix B: WD.

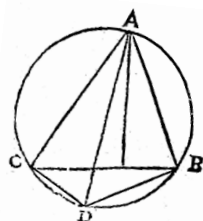
- WD. Raise a perpendicular at the endpoint of a line: Method 2  
 WG. Check the right angle: Method 3  
 W16. Draw a parallel DH to the basis BJ of a triangle, equal to HB  
 W17. Draw a parallel DH to the basis BJ of a triangle, with  $DH=BH+BZ$   
 W24. Draw the pattern of a parabolic mirror  
 W28B. Describe a regular pentagon whose side is given: Method 2  
 W30, 32. Describe a regular heptagon whose side is given  
 W33. Describe a regular enneagon whose side is given  
 W34, 35. Describe a regular hexagon whose side is given  
 W39, 40, 41, 42. Inscribe a square in the circle: Methods 2–5  
 W44, 45. Inscribe a regular pentagon in the circle: Methods 2–3  
 W49. Inscribe a regular enneagon in the circle  
 W58, 59, 61. Inscribe an equilateral triangle in a square  
 W65, 66, 67. Circumscribe a square around a scalene triangle  
 W74. Circumscribe an equilateral triangle around a regular pentagon  
 W75. Inscribe a square in a regular pentagon  
 W76. Circumscribe a square around a regular pentagon  
 W77. Inscribe a regular pentagon in a square  
 W86. Divide a triangle in three parts by two parallels to a given side: Method 2  
 W87. Double or triple the area of a triangle by a parallel to its side  
 W89. Draw a half or a third triangle inside a given triangle  
 W101. Cut off one third of a parallelogram through a side point,  $HH>BZ$   
 W110. Cut off a given part of a trapezium through a point outside the figure<sup>19</sup>  
 W115. Cut off one third of a quadrilateral by a side point,  $BH$  outside the quadrilateral  
 W120. Divide a square in two parts, putting aside a strip of width  $DH$   
 W121. Divide a square in three parts, putting aside a strip of width  $MN$   
 W122. Divide a triangle in two parts, putting aside a strip of width  $DJ$   
 W123. Divide a triangle in three parts, putting aside a strip of width  $DJ$   
 W124. Divide a trapezium in two parts, putting aside a strip of width  $DH$   
 W129. Make a square of eight squares  
 W130. Make a square of thirteen squares  
 W131. Make a square of ten squares  
 W132. Split a square in eight squares  
 W133. Split a square in eighteen squares  
 W134. Split a square in ten squares  
 W135. Split a square in twenty squares  
 W136ABC, 137. Make a square of three squares  
 W138. Make a square of any given number of squares  
 W139. Divide a square whose side is given in two squares

#### 4.5. False resemblances

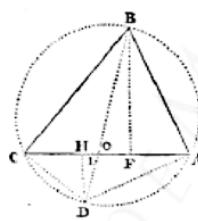
In recent decades, scholars have paid increasing attention to scientific diagrams. We have now a better idea of the synoptic, mnemonic and explanatory functions of scientific illustrations.

<sup>19</sup>W110 was known to Fibonacci, but he did not develop it: “Nec non et diuidemus ipsum quadrilaterum ab omni puncto dato super aliquod laterum ipsius, et etiam ab omni puncto dato infra, uel extra,” that is: “Also we will divide the quadrilateral from any point on whichever side, and even from any point within or outside [the figure]” (Boncompagni, 1862, 134).

This is particularly true in mathematics (Netz, 1999; De Young, 2005; Saito, 2006). The latter two authors have found in manuscript diagrams a way to trace the different Euclidean traditions. However, this applies only to a strictly delimited corpus. No conclusion can be reached from figures only, because similar geometric diagrams may be used for different problems. For example, consider the following diagrams by Pico (1597, 28) and Marolois (1616, 70).



Pico



Marolois

The similarity between Pico's and Marolois' diagrams does not prove a borrowing from Pico, for the diagrams actually serve very different purposes. Pico asks about the chord of the arc BDC, given the circle ABDC and diameter AD, while Marolois aims to show that rectangle  $AB \times BC$  is equal to rectangle  $BF \times BD$ . Circle ABDC is to be drawn during the construction. The example of the Pico and Marolois diagrams shows that, for each problem, a careful examination of the statement, procedure and diagram is needed. There are eight problems leading to false resemblances.<sup>20</sup> They must be considered as problems without a following in the West:

- W2. Bisect the line: Method 2 (unlike Schwenter, GPN, 410)
- W15. Draw a parallel  $ZH$  to triangle's basis BJ,  $ZH=BH$  (unlike Euclid, E, VI, 4)
- W31. Describe a regular octagon whose side is given (unlike Schwenter, GPN, 203)
- W53\*. Circumscribe a circle around a scalene triangle: Method 2 (unlike Pico, TM, 28)
- W60. Inscribe an equilateral triangle in a square: Method 3 (unlike Fiorentino, TGP, 160)
- W63. Circumscribe a triangle around a square (unlike Fibonacci, PG, 223)
- W64. Circumscribe a square around an equilateral triangle (unlike Marolois, OM, 48)
- W71. Inscribe an equilateral triangle in a scalene triangle (unlike Huygens, TM, 24)

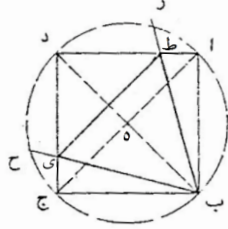
#### 4.6. Matching solutions

The removal of the solutions described in Sections 4.1 to 4.5 from the list of all solutions given by Abū al-Wafā', reveals which of Abū al-Wafā's geometric constructions were echoed in Renaissance works. The latter constructions all provide an exact match of statement, procedure and diagram with constructions produced by Abū al-Wafā'. These geometric constructions are to be found in a wide range of treatises, from ancient works, such as those of Leonardo Fibonacci, Jordanus Nemorarius and Campanus of Novara, to modern practical geometries, such as those of Tartaglia and Marolois, very popular in the seventeenth century.

Consider the problem W62: *Inscribe an equilateral triangle in a square*, and compare Abū al-Wafā's solution (Būzjānī, 1966, 86; 1997, 63) with the one given by Pacioli (1494). Statements, procedures and diagrams are exactly the same:

The description of W62 in unit steps (see Section 3.1) is as follows:

<sup>20</sup>See Appendix B: W2, W15, W53\*, W60.



Abū al-Wafā's W62: On how to inscribe an [equilateral] triangle in the square [...] If we want to draw this figure, we circumscribe around the square ABJD a circle corresponding to the diameters BD, AJ, cutting at point H. Take D as a center. By drawing an arc of radius DH up to the points H and Z, we get the chords BZ and BH, cutting AD and DJ at points T and Y. These two points will be on the equilateral triangle BTY, which is inscribed in the square ABJD.

في عمل المثلث في المربع - اذا اردنا ها ان نعمل ذلك  
عملنا على مربع ا ب ج د دائرة و اخرجنا قطري ب د ا ج  
يقاطعان على نقطة ه وجعلنا نقطة د مركزا وبعد ده  
علاماتي ح ز ووصلنا خيطي ب ز ب ح يقطعان حطي  
اددج على نقطتي ط ي فيكون مثلث ب ط ي متساوي  
الاضلاع وقد عمل في مربع ا ب ج د ( add. ب ) وهذه  
صورة. (Ayasofya 2753, fol. 29:7)



Pacioli: Let ABCD be a square, in which there are four sides. Draw the greatest equilateral triangle fitting in it [...] Otherwise. Around the said square, circumscribe a circle and draw the two diameters AB and CD. Then, from the point A, extend a half-diameter. It will reach the point E [on the one side] and the point F on the other side. I say that the greatest triangle in the circle is BEF and his sides cut the sides of the square at points G H, which make up the triangle.

Eglie il qdrato ABCD qle e 4 faccia mettoui dētro el maggiore triangolo che vi capa eq̄latero [...] Aliter p<sup>a</sup> al ditto quadro circunscriui vn cerchio e tira li 2 diametri AB e CD. Poi dal ponto A stendi mezzo diametro. Finira in ponto E e dal altro canto in ponto F. Dico el maggior triangolo nel cerchio esser BEF e li soi lati taglian li lati del quadro in ponti G H quali fan il triangolo” (Pacioli, 1494, fol. 62r).

- (0) Square ABJD [given]
- (1) Line BD
- (2) Line AJ  
→ point H
- (3) Circle H, HA
- (4) Circle D, DH  
→ points Z, H
- (5) Line BZ
- (6) Line BH  
→ points T, Y
- (7) Line BT
- (8) Line BY
- (9) Line TY  
→ triangle BTY □

Abū al-Wafā's possible influence in the West is deducible only from such solutions, i.e. those solutions in which the statements, diagrams and procedures that appear in Abū al-Wafā's text have identical counterparts in a Renaissance text. These constructions make up a set of twenty-one (out of one hundred and fifty), viz. nineteen solutions, plus two variants of construction W12, intervening in constructions W22\* and W52B\*.<sup>21</sup>

- WC. Raise a perpendicular at the endpoint of a line
- W8\*. Draw a parallel to a line through a given point
- W12. Find the missing center of a given circle<sup>22</sup>
- W20. Trisect an acute angle<sup>23</sup>
- W22\*. Find the missing center of a given circle
- W28A. Describe a regular pentagon whose side is given
- W47\*. Inscribe a regular heptagon in the circle<sup>24</sup>
- W50\*. Inscribe a regular decagon in the circle
- W52B\*. Find the missing center of a given circle
- W55. Circumscribe a circle around a regular pentagon<sup>25</sup>
- W62. Inscribe an equilateral triangle in a square: Method 5
- W68, 69. Inscribe a square in a scalene triangle
- W70. Inscribe a square in an equilateral triangle
- W72. Circumscribe an equilateral triangle around a scalene triangle
- W73. Inscribe an equilateral triangle in a regular pentagon
- W78. Inscribe a regular octagon in a square<sup>26</sup>
- W83. Divide a triangle in n parts by a line through a side point
- W105. Cut off one third of a parallelogram through a point outside the figure.
- W116. Describe a double square around a given square<sup>27</sup>
- W117. Describe a half square within a given square<sup>28</sup>

The conclusion of the first test is that only 23 out of 82 original solutions by Abū al-Wafā were echoed in the Latin world. The ratio is low (0.28).

<sup>21</sup>See Appendix B: W12, W20, W47\*, W52B\*, W72, W78.

<sup>22</sup>W12 is identical, in figure and procedure, to Leonardo Fibonacci's method: "Si in circulo trigonum describatur, cuius tres anguli periferiam cinguli contingant, possibile est per notitiam ipsius trigoni laterum dyametrum inuenire," that is: "If a triangle is inscribed in a circle so that its three vertices touch the circumference, then the diameter of the circle can be found by the lengths of the sides of the triangle" (Boncompagni, 1862, 102).

<sup>23</sup>W20 is a variant based on Archimedes' *Book of Lemmas*, 8. Diagram and operations are identical to those given by Jordanus Nemorarius' *De Triangulis* (Curze, 1887; Clagett, 1984, IV, 20) and Campanus' *Preclarissimus Liber Elementorum Euclidis* (Campanus, 1482, IV, 16).

<sup>24</sup>W47\* is a variation on the construction of the regular heptagon by Hero, *Metrica* (Hultsch, 1878, 155). Draw the diameter ADJ. Draw an arc of center A and radius AD, cutting the circle at B and E. Chord BE will cut the diameter AJ at point Z. From B as a center and BZ as radius, draw point H. BH is the seventh part of the circle. When BH is carried along the circumference, it will form the heptagon BHTIKLM.

<sup>25</sup>W55 provides an original solution. Euclid (2000, IV, 14), finds the center of the pentagon by bisecting the angles, not the sides of the pentagon, as Abū al-Wafā does.

<sup>26</sup>Despite the fact that diagram W78 has been placed within the Heronian tradition (Høytrup, 2006), the procedure of *De Mensuris* corresponds to Abū al-Wafā's solution W79, not W78.

<sup>27</sup>Though deriving from Euclid's *Elements*, II, 14, W116 leads to an original solution. Since Abū al-Wafā contents himself with rough indications concerning how to make the square concentric, both Tartaglia's and Schwenter's solutions are admitted.

<sup>28</sup>See note 27.

## 5. Common Part to Tests 2 and 3

In Section 4, we wondered if Abū al-Wafā’s solutions were known in the Latin world, without specifying to whom they were known. If his solutions were actually transmitted, we would have expected several European authors to have reproduced many of them. In fact, only a few Renaissance works contain a significant part of the geometric constructions devised by Abū al-Wafā’. Abū al-Wafā’s constructions are not in the works of Cardano (1545) and Benedetti (1553), who give only a few identical constructions to his. As far as I know, the maximum number of constructions is to be found in Niccolò Tartaglia’s *Quinta Parte del General Trattato* (1560), Samuel Marolois’ *Opera Mathematica* (1616) and Daniel Schwenter’s *Geometria Practica Nova* (1618). I shall now limit myself to these three works.

This choice has the effect of eliminating solutions W12, W20, W47\*, W72, W78 found in other works. What remains makes up a set of sixteen specific solutions:

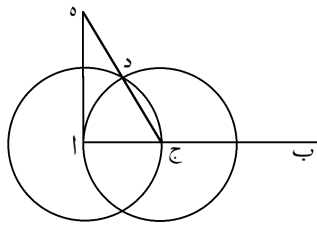
Abū al-Wafā’s Constructions	Tartaglia	Marolois	Schwenter
WC. Raise a perpendicular at the endpoint of a line		•	•
W8*. Draw a parallel to a line through a given point		•	•
W22*. Find the missing center of a circle	•	•	•
W28A. Describe a regular pentagon whose side is given			•
W50*. Inscribe a regular decagon in the circle		•	
W52B*. Find the missing center of a circle	•		•
W55. Circumscribe a circle around a regular pentagon		•	•
W62. Inscribe an equilateral triangle in a square		•	
W68. Inscribe a square in a scalene triangle		•	
W69. Inscribe a square in a scalene triangle: Method 2	•		
W70. Inscribe a square in an equilateral triangle	•		
W73. Inscribe an equilateral triangle in a regular pentagon		•	
W83. N-sect a triangle by a line through a side point	•		•
W87. Double the area of a triangle by a parallel to its side	•		
W105. Cut off $\frac{1}{3}$ of a parallelogram by an outside point	•		
W116. Describe a double square around the given one	•		•
W117. Describe a half square within the given one	•		•

These sixteen remaining geometric constructions will now be described, and all relevant information collected in order to work out the diagram lettering (Test 2) and index of independence (Test 3). These tests will be critically discussed in Section 10. For each construction:

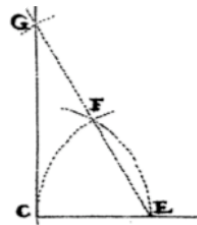
- (1) I first provide the statement of the problem;
- (2) Then, I give the quadruplet  $(n, m, k, \ell)$ , which provides the number of solutions available to the target ( $n$ ), the number of solutions given by the target ( $m$ ), the number of solutions devised by the source ( $k$ ), and the number of solutions that belong to both the source and target ( $\ell$ ). These data are necessary to calculate the index of independence;<sup>29</sup>
- (3) Finally, I describe the geometric constructions common to the source and target in unit steps, as explained in Sections 3.1 and 4.6.

<sup>29</sup>The numbers  $n, k, m, \ell$  are counted throughout the corpus defined in Section 3.2.



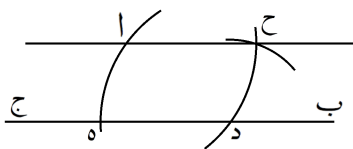


Abū al-Wafā' WC

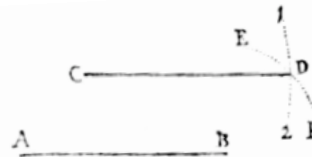


Bachot, GP, n.p.

WC. *Raise a perpendicular at the endpoint of a given line.* Marolois (9, 2, 2, 1), Schwenter (9, 3, 2, 1). This basic problem attracted nine different solutions prior to 1616: WC; WD; Egyptian triangle; Triangle 5-12-13; Hero, *Geometrica* (Hultsch, 1878, 435); al-Nayrīzī, *Elements*, I, 11; Cremona, *Elements*, I, 11; Boulenger (1690, 41); Clavius (1591, 33). Abū al-Wafā' draws two circles of centers A and J, that cut each other at the point D. Extend line JD beyond H. Make DH equal to DJ. Draw line HA, it will be perpendicular to line AJ, as required.

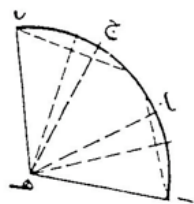


Abū al-Wafā' W8\*

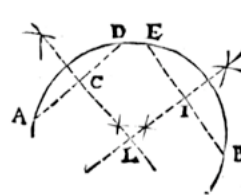


Marolois, OM, 14

W8\*. *Draw a parallel to a line through a given point.* Marolois (7, 2, 1, 1), Schwenter (7, 4, 1, 1). We know of some fifteen different solutions, including seven constructions prior to 1616-18: W8\*; Aristotle, *Posterior An.* I, V, 74a; Euclid (2000, I, 31); Pappus (1878, VII, 106); Tartaglia (1560, fol. 3v, 4r); Ryff (1600, 61). Abū al-Wafā's construction W8\* is as follows: From point D of BJ, describe the arc AH. From A taken as a center and with the same opening, draw the arc DH. Make AH equal to DH. Then the line HA will be parallel to BJ.

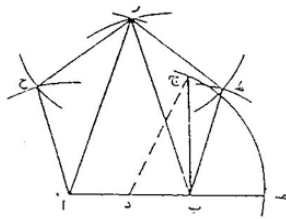


Abū al-Wafā' W22\*

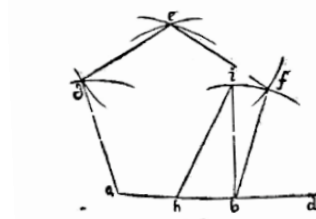


Ardüser, GTP, 26v

W22\*. *Find the center of a circle.* Tartaglia (7, 4, 3, 2), Marolois (7, 1, 3, 1), Schwenter (7, 4, 3, 2). They were seven solutions prior to Tartaglia's *Quinta Parte*: W12; W22\*; W52B\*; Euclid (2000, III, 1); Hero, *Geometrica* (Hultsch, 1878, 435); Pappus (1878, VII, 96); Chuquet, *Géométrie*, fol. 256r (L'Huillier, 1979). Both Abū al-Wafā's solutions have been used in Europe. Solution W22\* is as follows: Draw on the circle the chords AB and JD. Raise two perpendiculars at their bisection points. These lines will cut each other at a point H, the center of the circle.

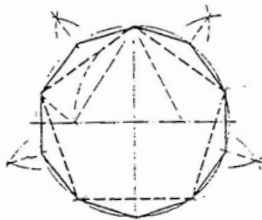


Abū al-Wafā' W28A

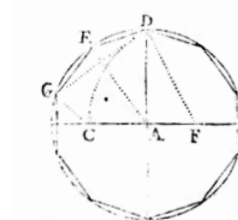


Schwenter, GPN, 197

W28A. *Describe a regular pentagon whose side is given.* Schwenter (10, 3, 2, 1). This problem gave rise to ten solutions prior to Schwenter's books: W28A; W28B; Abū Bakr (Rodet, 1883, four solutions); Hösch (1844, 96); Bovelles (1547, fol. 20r); Bachot (1598, two solutions). Solution W28 is as follows: At point B, draw a perpendicular line BJ equal to AB. Bisect AB at the point D. Make DH equal to DJ. Point Z, belonging to the pentagon's triangle ABZ, is such that  $AZ=BZ=AH$ . With the opening AB, draw the arcs  $A\dot{H}$  and  $Z\dot{H}$ , cutting each other at  $\dot{H}$ , draw the arcs  $B\dot{T}$  and  $Z\dot{T}$ , cutting each other at  $\dot{T}$ .  $AB\dot{T}Z\dot{H}$  is the pentagon we require.

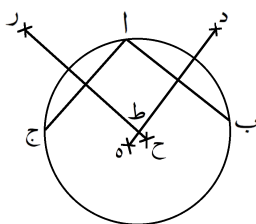


Abū al-Wafā' W50\*

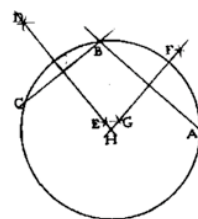


Marolois, OM, 42

W50\*. *Inscribe a regular decagon in the circle.* Marolois (3, 1, 1, 1). Once the constructions of the decagon with a given side are eliminated, as well as the algebraic solutions, we are left with three solutions prior to 1616–18: W50\*; Ptolemy, *Almagest*, I, 9; Bachot (1598). W50\* simply consists of inscribing a regular pentagon in the circle, as in W43, then bisecting the sides of the pentagon to get a regular decagon.

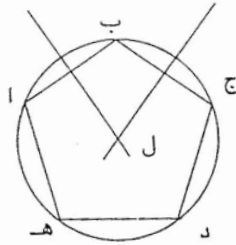


Abū al-Wafā' W52B\*

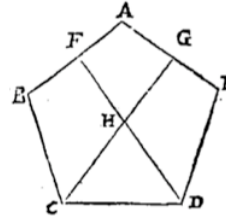


Galli Bibiena, AC, 10

W52B\*. *Find the missing center of a circle.* Tartaglia (7, 4, 3, 2), Schwenter (7, 4, 3, 2). The problem gave rise to seven constructions, all before 1560: W12; W22\*; W52B\*; Euclid (2000, III, 1); Hero, *Geometrica* (Hultsch, 1878, 435); Pappus (1878, VII, 96); Chuquet, *Géométrie*, fol. 256r (L'Huillier, 1979). W52B\* is as follows: From A and B, taken as centers, draw two circles cutting each other at the points D and H. Similarly, from centers A and J, draw two other circles intersecting at Z and  $\dot{H}$ . Lines DH and  $Z\dot{H}$  will cut at point  $\dot{T}$ , the center of the circle.

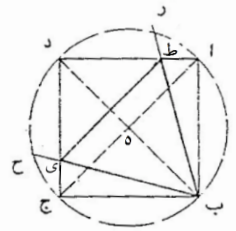


Abū al-Wafā' W55



Peletier, UG, 39

W55. *Circumscribe a circle around a regular pentagon.* Marolois (2, 1, 1, 1), Schwenter (2, 1, 1, 1). Only two geometric constructions have been described prior to 1616–18: Euclid (2000, IV, 14) and Abū al-Wafā', W55. The latter is as follows: From points A and B taken as centers, draw two arcs cutting at Z and H. From points B and D taken as centers, draw two arcs intersecting at Y and K. Lines ZH and YK cut each other at the point L, which is the center of the circle circumscribed around the regular pentagon ABJDH.

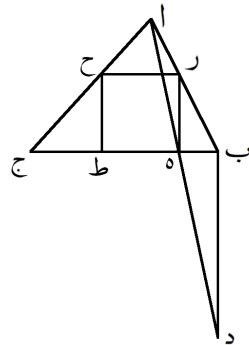


Abū al-Wafā' W62

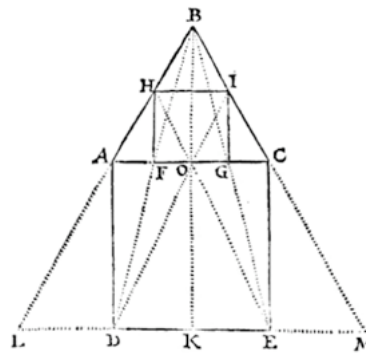


Pacioli, S, 62r

W62. *Inscribe an equilateral triangle in a square.* Marolois (5, 1, 5, 1). We know about eight different solutions for this problem, including five prior to Marolois' and Schwenter's treatises: W58; W59; W60; W61 and W62. Solution W62 is as follows: first circumscribe a circle around the square ABJD: draw the diameters BD and AJ cutting at H, which is the center of the circle. Then, with an opening DH from D taken as a center, draw an arc that will cut the circle at points Z and H. Draw BZ and BH, intersecting the sides of the square at T and Y. Draw the line TY. BTY is the triangle required.

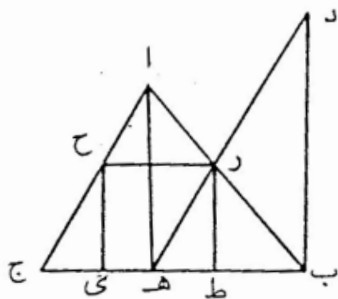


Abū al-Wafā' W68

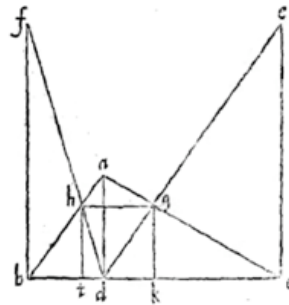


Marolois, OM, 46

W68. *Inscribe a square in a scalene triangle.* Marolois (6, 2, 2, 1). This problem led to six known solutions prior to 1616–18: W68; W69; Ibn Yūnus, *Sharḥ*, fol. 44v (two solutions); Tartaglia (1560, fol. 17v, two solutions). Solution W68 is as follows: At the endpoint B, drop the line BD perpendicular and equal to BJ. Join AD, cutting BJ at H. Draw HZ perpendicular to HB cutting AB at the point Z. Extend ZH parallel to BJ. Draw HṬ perpendicular to BJ. The square HZHT is inscribed. Marolois draws a slight variant, in which he applies the same procedure to an equilateral triangle. Nevertheless, he does not take advantage of symmetry to simplify the diagram.

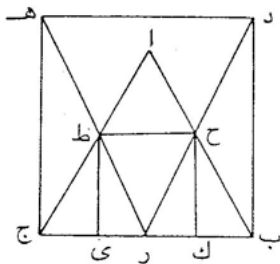


Abū al-Wafā' W69

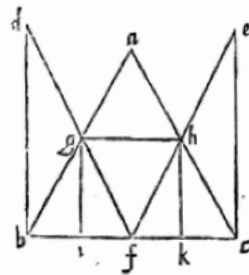


Tartaglia, QP, 18r

W69. *Inscribe a square in a scalene triangle.* Tartaglia (4, 1, 2, 1). This problem, which is identical to W68, gave rise to four documented solutions prior to 1560: W68; W69; Ibn Yūnus, *Sharḥ*, fol. 44v (two solutions). Solution W69 is as follows: At the endpoint B, raise the line BD perpendicular and equal to BJ. From the vertex A, drop the line AH perpendicular to BJ. Join DH, cutting AB at a point Z. Draw ZṬ perpendicular to BJ, and ZH parallel to BJ. The square ZHYT is inscribed in the scalene triangle ABJ, as required.

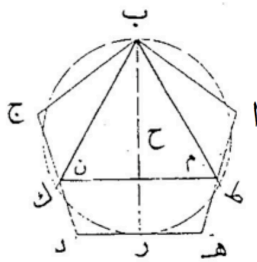


Abū al-Wafā' W70

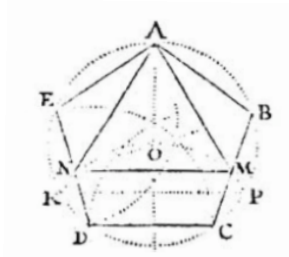


Tartaglia, QP, 16v

W70. *Inscribe a square in an equilateral triangle.* Tartaglia (3, 1, 1, 1). This problem possessed three solutions prior to 1560: W70; Ibn Yūnus, *Sharḥ*, fol. 44v and 45r. Abū al-Wafā's solution W70 is as follows: describe the square BDHJ on the base BJ. Then bisect the base BJ at point Z. Draw the lines ZD and ZH, cutting the sides of the triangle at the points Ṭ and Ḥ. Join ṬḤ. Draw the perpendiculars ḤK and ṬY. As a result, the square ḤṬYK is inscribed in the equilateral triangle ABJ, as required.

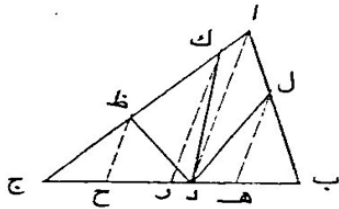


Abū al-Wafā' W73

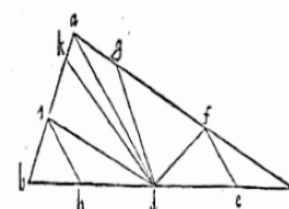


Marolois, OM, 49

W73. *Inscribe an equilateral triangle in a regular pentagon.* Marolois (1, 1, 1, 1). There was only one solution prior to 1616. Solution W73 is as follows: From the vertex B, draw the perpendicular BZ to the base DH. Bisect BZ at H. Draw the circle of center H and radius HB. From Z as center and with the same opening, draw an arc cutting the pentagon at the points T and K. Draw BT and BK, cutting AH at M and JD at N. Join MN. The triangle BMN is inscribed in the pentagon ABJDH, as required.

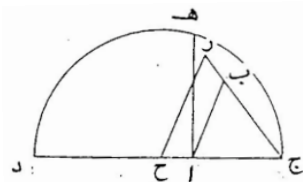


Abū al-Wafā' W83

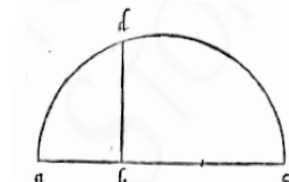


Tartaglia, QP, 26r

W83. *Divide a triangle in  $n$  equal parts by a line through a side point.* Tartaglia (1, 1, 1, 1), Schwenter (2, 1, 1, 1). This problem has only two solutions: one prior to 1560: W83, the other in 1599: Pomodoro (1624, XXI, 5). Solution W83 is as follows: Join A to point D. Divide BJ in  $n$  equal parts, viz. at H, Z, and H. Draw parallels to AD through H, Z, H. They will cut the sides of the triangle at L, K, and Z. The four triangles DBL, DLK, DKZ, DZJ have the same area. Abū al-Wafā' concludes: "We will have the same construction if we want to divide the triangle in three, in five, or in any equal parts" (Būzjānī, 1966, 95; 1997, 79).

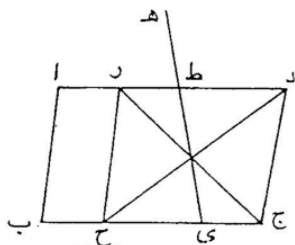


Abū al-Wafā' W87

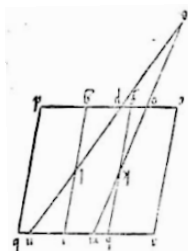


Tartaglia, QP, 6r

W87. *Double or triple the area of a triangle by a line parallel to one side.* Tartaglia (2, 1, 1, 1). Only two different solutions are documented: W87 and Clavius (1591, 343). Abū al-Wafā' duplicates the triangle as follows: extend JA of length AD equal to 2JA. Describe the semicircle JHD on JD. Raise the perpendicular AH to JD at A, that will cut the semicircle at H. Make JH equal to AH. Through H, draw HZ parallel to AB. Extend JB up to the point of intersection Z. Thus, the triangle HZJ is twice the triangle ABJ.

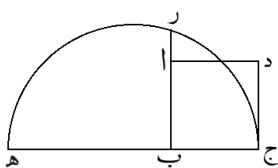


Abū al-Wafā' W105

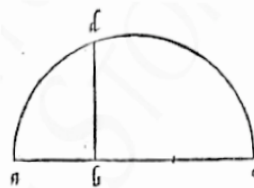


sim. Tartaglia, QP, 33v

W105. *Cut off one third of a parallelogram by a line passing through a point outside the figure.* Tartaglia (1, 1, 1, 1). This problem of Euclidean origin had little following, being omitted by Fibonacci. There is no other solution than Abū al-Wafā's construction W105: On the base of the parallelogram, make  $BH$  equal to  $BJ/3$ . At point  $H$ , raise  $HZ$  parallel to  $AB$ , cutting off one third of the figure. Draw the diagonals  $ZJ$  and  $DH$ , cutting each other at point  $[S]$ , the center of the parallelogram. From the outside point  $H$ , extend line  $H\bar{T}SY$  to the point  $Y$  of the base. Then  $DJY\bar{T}$  will cut off one third of  $ABJD$ . The only difference in Tartaglia's construction is that he determines the center  $S$  by bisecting the parallel to  $ZH$  joining the midpoints of lines  $DZ$  and  $JH$  (Tartaglia, 1560, fol. 33v).

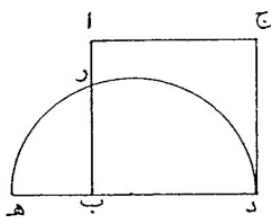


Abū al-Wafā' W116

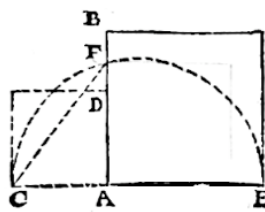


Tartaglia, QP, 6r.

W116. *Describe a double square around a given square.* Tartaglia (2, 2, 1, 1), Schwenter (2, 1, 1, 1). Two solutions were devised before the sixteenth century: W116 and Villard, *Carnet*, fol. 39r (Bechmann, 1991). W116 is as follows: extend the base  $JB$  up to  $H$ , with  $BH=2JB$ . Describe, on the diameter  $JH$ , a semicircle  $JZH$ . Extend  $BA$  up to  $Z$ . Add to the square's sides a width equal to  $AZ/2$ . The resulting square will be the double of the square  $ABJD$ .



Abū al-Wafā' W117



Ardüser, GTP, 130r

W117. *Describe a half square within a given square.* Tartaglia (2, 2, 1, 1), Schwenter (2, 1, 1, 1). This problem gave rise to two solutions: W117 and Villard, *Carnet*, fol. 39r (Bechmann, 1991). W117 is as follows: extend the base  $BD$  of  $BH=BD/2$ . Describe on  $DH$  the semicircle  $DZH$ , cutting the side  $AB$  at  $Z$ . Remove from each side of the square a width equal to  $AZ/2$ . The resulting square will be half of the square  $ABJD$ . Tartaglia has an overall view of the

problem: “Similarly, from any given equilateral triangle, we may draw another one equal to the half of that, and so wanting [...] the fourth, or fifth, etc.”<sup>30</sup>

## 6. Diagram Lettering (Test 2)

In this section, Abū al-Wafā’s lettering is compared to the one used by European geometers. Two hypotheses are worked out separately. In case of phonetical matching [PHON] between Latin and Arabic letters, multiple correspondence is admissible for the letters *jīm* {G, J}, *hā’* {E, H}, *wāw* {U, W} and *yā’* {I, Y}. In case of numerical matching [NUM], the letters follow the Levantine alphabet called *abjad*, except for one or two irregularities and the letter *wāw*, which is unused by Abū al-Wafā’. The series is ‘*alif* 1 {A}, *bā’* 2 {B}, *jīm* 3 {C}, *dāl* 4 {D}, etc. (Table 1).

Cstr.	Tartaglia (1560)		Marolois (1616)		Schwenter (1618)	
	PHON	NUM	PHON	NUM	PHON	NUM
WC			1/5	1/5	0/5	0/5
W8*			1/8	1/8	1/6	2/6
W22*	1/9	1/9	0/10	1/10	3/8	4/8
W28A					2/8	2/8
W50*			0/6	0/6		
W52B*	1/8	2/8			0/14	2/14
W55			0/10	0/10	0/10	0/10
W62			2/9	1/9		
W68			1/13	2/13		
W69	3/10	4/10				
W70	5/10	8/10				
W73			4/11	2/11		
W83	4/10	1/10			6/12	5/12
W87	0/7	0/7				
W105	1/15	1/15				
W116	1/6	1/6			0/6	0/6
W117	1/6	1/6			0/6	0/6
Ratio	0.209	0.234	0.125	0.111	0.160	0.200

TABLE 1. Abū al-Wafā’s Diagram Lettering

The number of matching letters is small in each case, the best matching being in the case of Tartaglia who reproduces an average of c. 2.1 letters out of 9 per diagram (NUM). Thus the second test does not provide evidence of transmission.

## 7. Index of Independence (Test 3)

In this section, an attempt is made to apply probability theory to the study of borrowings vs. multiple discoveries. For the sake of simplicity, I assume geometric constructions to be equiprobable and independent events—a condition which is not always fulfilled in the real

<sup>30</sup>“Similmente di ogni dato triangolo equilatero potremmo designarne vn’altro eguale alla mita di quello et cosi volendo [...] il quarto, ouero il quinto, ecc.” (Tartaglia, 1560, fol. 6v-7r).

world.<sup>31</sup> With this assumption, I then define an index of independence for the constructions presenting the most striking resemblances to those of Abū al-Wafā'. Such a test is usable only if the number of matches is high, which is precisely the case here, for many geometric constructions by Abū al-Wafā' were echoed in the Latin West.

#### Random draws

Consider the problem of finding the center of a circle, a construction which is necessary for solutions W22\*-52B\*. We know  $n = 7$  solutions prior to 1560 (they are described in Section 5). Niccolò Tartaglia mentions  $m = 4$ , Abū al-Wafā' gives  $k = 3$ , among which  $\ell = 2$  are common to Tartaglia and Abū al-Wafā'. Put the seven solutions in a bag. Then pick four solutions (the same number mentioned by Tartaglia) at random. What is the chance that the draw contains at least the  $\ell = 2$  solutions by Abū al-Wafā'?

There are  $\binom{n}{m} = \binom{7}{4} = 35$  ways to pick  $m = 4$  geometric constructions out of seven at random. In addition, there are  $\binom{n-k}{m-2} = \binom{4}{2} = 6$  ways to pick  $m = 4$  solutions including the two solutions given by Abū al-Wafā'. Since the conclusion would be the same if we had picked any other pair of solutions given by the source, we multiply this number by  $\binom{k}{\ell} = \binom{3}{2} = 3$ , and proceed similarly with three solutions. The chance that the draw contains at least the two solutions given by Abū al-Wafā' is  $\left(\binom{3}{2}\binom{4}{2} + \binom{3}{3}\binom{4}{1}\right) / \binom{7}{4} = \frac{22}{35}$ .

The probabilities that Tartaglia picks out Abū al-Wafā's solutions for the other problems are calculated similarly,  $p(I)_{69} = \frac{1}{2}$ ,  $p(I)_{70} = \frac{1}{3}$ ,  $p(I)_{83} = 1$ ,  $p(I)_{87} = \frac{1}{2}$ ,  $p(I)_{105} = 1$ ,  $p(I)_{116} = 1$ ,  $p(I)_{117} = 1$ .<sup>32</sup> Then, according to the multiplication theorem, the probability of drawing the  $N$  solutions  $\{W22^* \dots W117\}$  all together is, for all  $j \leq m$ :

$$p(I)_{X|Y} = \prod_{i=1}^N \left\{ \sum_{j=\ell}^k \left[ \binom{k_i}{j_i} \binom{n_i - k_i}{m_i - j_i} \right] / \binom{n_i}{m_i} \right\} \quad (1)$$

That is, in Tartaglia's case:

$$p(I)_{W|T} = \frac{22}{35} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot 1 \approx \frac{1}{19} \quad (2)$$

The chance of an independent reconstruction by Marolois is calculated in the same manner (using the data from Section 5). Since Marolois (1616, 13) mentions the  $n$ -section of the straight line by Tartaglia (1560, fol. 22rv), he had access to the *Quinta Parte*. W22\* is thus removed and the calculation is as follows:

$$p(I)_{W|M} = \frac{5}{12} \cdot \frac{2}{7} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \cdot \frac{3}{5} \cdot 1 = \frac{1}{84} \quad (3)$$

We proceed in the same manner with Schwenter. Since Tartaglia's  $n$ -section is quoted again by Schwenter (1618, 73), we must remove W22\*, W52B\*, W83, W116, W117 to give:

$$p(I)_{W|S} = \frac{7}{12} \cdot \frac{4}{7} \cdot \frac{8}{15} \cdot \frac{1}{2} \approx \frac{1}{11} \quad (4)$$

<sup>31</sup>For discussion, see Section 10.

<sup>32</sup>We read  $p(I)_{69}$  as the probability that Tartaglia picks solution W69 at random.



### Significance

To determine whether these probabilities are significant, we must compare them to the number of draws  $\alpha_{X/Y}$  that could be made between the source  $X$  and target  $Y$ .

We can assume a “draw” is equivalent to either a treatise or an author. I choose the second option, for the solutions given by one author are stable—for example, most of Tartaglia’s geometric constructions appear in his *Quinta Parte*. We can therefore equate a draw to a geometer.

Since basic problems have a small number of solutions (see Section 5), the more a problem is studied, the more the solution given by  $X$  has a chance to be randomly rediscovered by  $Y$ . As noted in Section 3.4, the number of draws needed for a given solution to appear with certainty in a series of draws with replacement is known. Suppose there exists a bag containing  $n$  solutions. On any draw, the chance of not drawing a given solution is  $(n - 1)/n$ . After  $q$  draws, this number becomes  $p = ((n - 1)/n)^q$ . If we want to be almost certain ( $p = 0.95$ ) that this event will appear in successive draws, then we need  $q = \ln(0.05)/\ln(n - 1/n)$  draws. In order to move away from the critical zone, I reject the hypothesis of an independent reconstruction if the number of draws  $\alpha$  is well below  $q$ —say the same order of magnitude as  $n$ . The question thus comes down to comparing the index of independence  $p(I)_{X/Y}$  to the number of geometers  $\alpha_{X/Y}$  who have existed between the source  $X$  and target  $Y$ , and who therefore could have drawn a solution:

1.  $p(I)_{X/Y} < 1/\alpha_{X/Y}$  means that there were not enough draws between the source  $X$  and target  $Y$  to rediscover the solution by chance. In this case, identical solutions advocate for a historical transmission.
2.  $p(I)_{X/Y} \geq 1/\alpha_{X/Y}$  means that there were a lot of draws between the source  $X$  and target  $Y$  so that the geometers could have rediscovered the solution by themselves at random. In this case, an independent reconstruction cannot be rejected.

The number of geometers active between the tenth and the sixteenth century can be estimated from biographical records. Geometric constructions appear in treatises of either pure or practical geometry. To establish the list of geometers active in this given span of time, we can use the convenient *Chronological List of Mathematicians* (Joyce, 1995), which compiles previous data from W.W. Rouse Ball, C.C. Gillispie, R.S. Westfall, J. O’Connor and E.F. Robertson. From the *Chronological List* we first remove the names of those who have not contributed to geometry (arithmetic, algebra, trigonometry, etc.) and then add the names of translators and commentators of the *Elements*, as well as the names of all authors who have composed practical geometry treatises.

Next, we look for the number of geometers that may have informed Renaissance mathematics. In other words, we look for all Arabic-, Hebrew- and Latin-speaking geometers. By reclassifying the authors by the date of their first geometric work, we get a fairly accurate picture of the number of “draws”. We count in the said list:  $\alpha_{W/T} = 104$ ,  $\alpha_{W/M} = 165$ ,  $\alpha_{W/S} = 168$ . Hence:

$$p(I)_{W/T} = \frac{1}{19} \gg \frac{1}{104} \quad p(I)_{W/M} = \frac{1}{84} \gg \frac{1}{165} \quad p(I)_{W/S} = \frac{1}{11} \gg \frac{1}{168} \quad (5)$$

The probability of drawing Abū al-Wafā’s solutions at random is greater than  $1/\alpha$  in each case. Therefore, the conclusion of the third test is that it is likely that European geometers reached the solutions independently. Finally, since the partial findings of Tests 1 and 2 also yield negative results, the three tests taken *all together* dismiss Woepcke’s diffusion hypothesis.

## 8. Ibn Yūnus' *Commentary*

Kamāl al-Dīn ibn Yūnus (5th Šafar 551–14th Sha‘bān 640 H./30 March 1156–17 Feb. 1242) is known for his *Sharḥ al-a‘māl al-handasiyya li Abū al-Wafā’* (*Commentary on the Geometric Constructions by Abū al-Wafā’*), henceforth called his *Commentary*. A professor of mathematics in Mosul, Ibn Yūnus was in contact with the Latin world through his answers to the scientific questions posed by Emperor Frederick II to Arabic scholars,<sup>33</sup> as well as through several students, such as ‘Alam al-Dīn or Theodore of Antioch, who afterwards attended the cosmopolitan court of Sicily (Ibn Khallikān, 1944; Kedar and Kohlberg, 1995; Burnett, 1995; Raynaud, 2007). Insofar as Abū al-Wafā’s collection of problems was unknown in Latin Europe, as we have just seen, and since many of his constructions appear in Ibn Yūnus’ commentary, this latter work would seem to provide a possible mechanism for the diffusion of Arabic geometric constructions to the West.

### 8.1. *Solutions echoed (Test 1)*

Ibn Yūnus’ *Commentary* is very similar to Abū al-Wafā’s except in two respects: (1) Ibn Yūnus provides proofs for every construction, whereas Abū al-Wafā’, in order to make his work more fitting for craftsmen, omits them:<sup>34</sup>; (2) Ibn Yūnus has a keen interest in conic sections. He uses them for the trisection of the angle (Y25GH), for the construction of the regular heptagon (Y47BCD), and stresses the fact that Abū al-Wafā’ should have given exact solutions, based on conic sections, rather than approximate solutions. Nevertheless, for the most part, the problems studied by Ibn Yūnus are identical to the ones solved by Abū al-Wafā’, with few constructions added or removed. As before, we proceed by considering the differences between the two texts. There are six specific constructions by Abū al-Wafā’ not treated by Ibn Yūnus:

- W8\*. Draw a parallel to a line through a given point
- W22\*. Find the center of a circle and trisect an arc
- W47\*. Inscribe a regular heptagon in the circle
- W50\*. Inscribe a regular decagon in the circle
- W52B\*. Circumscribe a circle around an isosceles triangle
- W53\*. Circumscribe a circle around an isosceles triangle: Method 2

We must remove thirteen propositions that do not involve constructions in plane geometry: Y21CD, Y23DEHIJK, Y25BJKL (properties of conics), Y70D (similar triangles), Y124D (polyhedra). Two further propositions are illegible in Mashhad’s MS:<sup>35</sup> Y23G (conics), Y124C (division of areas). Ibn Yūnus’ collection includes two constructions of Greek origin:<sup>36</sup>

<sup>33</sup>The questions solved by Ibn Yūnus are discussed by, among others, Ibn Khallikān, who writes: “In the year 633 H./1236, when I was in Damascus, a number of questions on arithmetic, algebra and geometry were posed to a man of this city, expert in mathematics. Unable to solve them, he copied them all on a roll of parchment and sent them [to Kamāl al-Dīn ibn Yūnus], then in Mosul. A month later, he received a response in which all the obscurities were clarified and all the difficulties were explained” (Ibn Khallikān, 1944, 471).

<sup>34</sup>“I left out all the motives and demonstrations. This will make [the constructions] easier for craftsmen and will pave the way to them.” Istanbul, Ayasofya MS 2753, fol. 2:

(. . .) محذراً من العلل والبراهين، يسهل على الصنّاع ناوله ويعرف عليهم طريقته.

<sup>35</sup>Since statements of problems are rubricated, many of them now appear in a very faint color. Some figures are almost entirely erased. The text and the figures are too faded for propositions Y23G and Y124C to be readable.

<sup>36</sup>See Appendix B: Y25I.

- Y23F. Duplicate the cube: Method 2 (Eutocius, CA, II, 9)  
 Y25I. Trisect an acute angle: Method 5 (Pappus, CM, IV, 36)

We must also remove from Ibn Yūnus' collection several constructions that could have been known in the West by intermediate sources: Y23C, dealing with the construction of the parabola, which is a slight reworking of Ibn Sinān's *Epistle on the Drawing of the Three Sections* (Rashed and Bellosta, 2000), Y47BCD, dealing with the construction of the regular heptagon by means of conics, inherited from either Abū al-Jūd's or al-Sijzī's works (Hogendijk, 1984).<sup>37</sup>

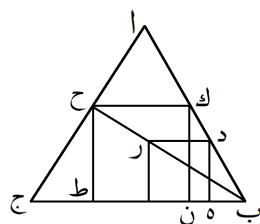
- Y23C. Describe a parabolic mirror: a variant of W25 (Ibn Sinān, EDTS, 268)  
 Y47B. Inscribe a regular heptagon in the circle (Abū al-Jūd, BCHC, Lem. 1)  
 Y47C. Continuation: draw the triangle of the heptagon (Abū al-Jūd, BCHC, Lem. 2)  
 Y47D. Continuation: draw the regular heptagon (Abū al-Jūd, BCHC)

Twenty constructions had no following in Western geometrical treatises, including three false resemblances: Y23DE, Y45B, which must be considered as problems without a following.<sup>38</sup>

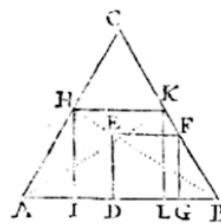
- Y21B. Trisect an acute angle: Method 3  
 Y23DE. Properties of the parabola (diff. Nemorarius, DT, I.12)  
 Y25CDEF. An instrument to draw the hyperbola  
 Y25GH. Trisect an acute angle by means of the hyperbola  
 Y28CDE. Describe a regular pentagon whose side is given: Method 3  
 Y45B. Inscribe a regular pentagon in the circle: Method 4 (diff. Mydorge, PG, 221)  
 Y70B. Inscribe a square in an equilateral triangle: Method 2  
 Y70C. Circumscribe a scalene triangle around an equilateral triangle  
 Y71B. Inscribe an equilateral triangle in a scalene triangle: Method 2  
 Y95B. Divide a trapezium in two parts by a parallel to its base: Method 2  
 Y122B. Divide a triangle in two parts, putting a strip aside: Method 2  
 Y123B. Divide a triangle in  $n$  parts, putting a strip aside  
 Y124B. Divide a trapezium in two parts, putting a widening strip aside

Consequently, Ibn Yūnus' specific solutions, which are identical to Latin geometric constructions, are limited to the following two:

- Y69B. Inscribe a square in a scalene triangle: Method 3 (Marolois, OM, 46)  
 Y69C. Inscribe a square in a scalene triangle: Method 4 (Tartaglia, QID, 202)



Ibn Yūnus Y69B

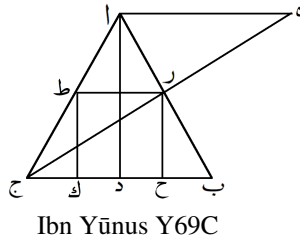


Marolois, OM, 46

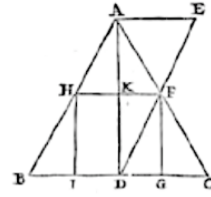
<sup>37</sup>See Appendix B: Y47C.

<sup>38</sup>Appendix B includes a reproduction of diagram Y23D.

Y69B. *Inscribe a square in a scalene triangle: Method 3.* Marolois (6, 1, 2, 1). There were six solutions prior to 1616. Samuel Marolois gives one. Ibn Yūnus gives two. Solution Y69B was adapted from al-Sijzī's *Anthology of Problems* (Crozet, 2010, 61-62) but European geometers were unaware of his work. Solution Y69B is as follows: from any point D taken on AB, draw a perpendicular DH. Make DZ parallel to BJ, with DZ=DH. Join the points B and Z, and extend the line BZ until it cuts AJ at point H. Draw HK parallel to BJ, then trace HT and KL perpendicular to HK. Thus, the square KLTH is inscribed in the triangle ABJ. <sup>39</sup> Both Ibn Yūnus and Marolois study the problem on a diagram that overspecifies the scalene triangle in an equilateral triangle.



Ibn Yūnus Y69C



sim. Tartaglia, QID, 202

Y69C. *Inscribe a square in a scalene triangle: Method 4.* Tartaglia (4, 2, 2, 1). There were four solutions available prior to 1560. Tartaglia mentions two solutions. Ibn Yūnus gives two. Ibn Yūnus' Y69C is as follows: in triangle ABJ, draw AD perpendicular to BJ. Draw AH parallel to BJ, with AH=AD. Draw a line JH that will cut the side AB at the point Z. Draw ZT parallel to BJ, then ZH and TK perpendicular to BJ. Square HZTK is inscribed in the triangle ABJ as required. <sup>40</sup> Tartaglia uses a variant in which he takes  $AE=AD/2$  and thus he draws the oblique line to the midpoint D instead of the point B.

Considering the work of both Abāl-Wafā' and Ibn Yūnus, the conclusion of the first test is that only 18 geometric constructions out of 82 were echoed in the Latin world, that is, 16 from Abū al-Wafā' and 2 by Ibn Yūnus.

<sup>39</sup>Mashhad, MS 5357, fol. 44v:3:

(... ) الاولى منهما ليكن ا ب ج و لنعلم على ضلع من اضلاعه نقطة كيف اتفقت كنقطة د و نخرج من نقطة د عمود ده و نخرج دز موازيا ل ب ج و نفضله مثل ده و نصل بز و نخرجه على استقامته حتى يلقا ا ج على نقطة ح و نخرج من نقطة ح عمود ح ط على ب ج و منها خط ح ك موازيا ل ب ج و من نقطة ك عمودا على ب ج و ليكن كل يكون ح ك ل ط مربعا يرهان ذلك ان نسبة كل الى ده كنسبة ك ب الى ب د و نسبة ك ب الى ب د كنسبة ك ح ( corr. ك ج ) الى دز فنسبة كل الى ده كنسبة ك ح الى دز فاذا بدلنا كانت نسبة كل الى ك ح كنسبة ده الى دز و ده مثل دز ف كل مثل ك ح ف كل ح ط مربع و ذلك ما اردنا ان نبين.

<sup>40</sup>Mashhad, MS 5357, fol. 44v:10:

الطريقة الثانية نرجع الى التي قبلها و ليكن مثلث ا ب ج ( add. ج ) و نخرج من نقطة ا على ب ح عمود اد و نقيم على نقطة ا من اد عمود اه مثل اد و نصل ه ج يقطع اب على نقطة ز فنخرج من نقطة ز عمود ز ح على ب ح و من نقطة ز خط طز موازيا ل ب ج و من نقطة ط عمود ط ك على ب ج فيكون ز ح ط ك مربعا نرهانه ان نسبة ه ج ( corr. ال ) الى ز ح كنسبة اب الى ب ج ( corr. ب ج ) و نسبة اب الى ب ج كنسبة ا ج الى ج ط و نسبة ا ج الى ج ط كنسبة اه الى ط ز فنسبة اد الى ز ح كنسبة اه الى ط ز فاذا بدلنا كانت نسبة اد الى اه كنسبة ز ح الى ط ز و اد مثل اه ف ز ح مثل ز ط ف ز ح ط ك مربع و ذلك ما اردنا بيانه.

### 8.2. Diagram lettering (Test 2)

A second way to estimate Ibn Yūnus' legacy in Latin Europe is to compare the diagram lettering of Ibn Yūnus' *Commentary* to the ones used in European treatises. I use the conventions defined in Section 6. As we can see from Table 2, the number of letters which are the same is small in each case. The best matching is again in the case of Tartaglia, who reproduces less than 2.9 letters out of 10 per diagram (NUM). Accordingly, the second test yields a negative result.

Cstr.	Tartaglia (1560)		Marolois (1616)		Schwenter (1618)	
	PHON	NUM	PHON	NUM	PHON	NUM
WC			1/5	1/5	0/5	0/5
W55			0/10	0/10	0/10	0/10
W62			0/9	1/9		
W68			1/13	2/13		
W69	4/10	4/10				
Y69B			3/11	1/11		
Y69C	2/9	7/9				
W70	2/10	8/10				
W73			4/11	2/11		
W83	2/10	1/10			6/12	1/12
W87	0/7	0/7				
W105	1/15	1/15				
W116	1/7	1/7			0/7	0/7
W117	1/9	1/9			0/9	0/9
Ratio	0.168	0.298	0.152	0.118	0.133	0.022

TABLE 2. Ibn Yūnus' Diagram Lettering

### 8.3. Index of independence (Test 3)

Since Abū al-Wafā's treatise was unknown in Europe, and Ibn Yūnus reproduces most solutions of this collection, the calculus of probability just consists in removing from Tartaglia's index of independence (Eq. 2) constructions W22\*-W52B\*, not mentioned by Ibn Yūnus, and in multiplying this number by the probability of Ibn Yūnus' unpublished construction Y69C.

$$p(I)_{Y/T} = \frac{1}{2} \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot 1 \cdot \frac{5}{6} \approx \frac{1}{14} \quad (6)$$

We proceed in the same manner with Samuel Marolois, by removing the two constructions W8\* and W50\*, not mentioned by Ibn Yūnus, from Marolois' index of independence (Eq. 3), and by multiplying this number by the probability of Ibn Yūnus' construction Y69B. We get:

$$p(I)_{Y/M} = \frac{5}{12} \cdot \frac{1}{2} \cdot 1 \cdot \frac{3}{5} \cdot 1 \cdot \frac{1}{3} = \frac{1}{24} \quad (7)$$

We interpret these numbers by referring to the *Chronological List of Mathematicians*, in the same way as we did in Section 7. We find  $\alpha_{Y/T} = 73$  and  $\alpha_{Y/M} = 134$ . Thus:

$$p(I)_{Y/T} = \frac{1}{14} \gg \frac{1}{73} \quad p(I)_{Y/M} = \frac{1}{24} \gg \frac{1}{134} \quad (8)$$

The third test tells us that European geometers could have found the solutions of these problems independently. Finally, since Tests 1 and 2 yield negative results, the three tests provide adjusted findings. They suggest rejecting the conclusion that Ibn Yūnus' *Commentary* intervened in the history of East–West diffusion of Abū al-Wafā's geometric constructions. Therefore, surprising as it may seem, Ibn Yūnus' treatise probably had no more influence than Abū al-Wafā's.

## 9. A Counter-Proof: Fibonacci's Legacy

To establish the reliability of the threefold method we have applied to Abū al-Wafā's legacy, we must check that the tests do not yield negative results in all cases. Let us proceed to a counter-test on Tartaglia's dependence on Fibonacci's *Practica Geometriae* (Boncompagni, 1862; Hughes, 2008). We choose the part of Tartaglia's work dealing with the division of triangles and quadrilaterals. This choice is interesting because, in the time of Tartaglia, Euclid's *Division of Figures* had been long lost and Fibonacci's solutions can be seen as original by substitution.

### *Test 1: Solutions echoed*

First put aside the obvious solution W90: Divide a parallelogram in two parts through one of its vertices. Fibonacci reproduced twenty five constructions from Euclid's *Division of Figures*.<sup>41</sup> Tartaglia provided sixteen constructions identical to Fibonacci's. The ratio of transmitted solutions is thus  $16/25 = 0.64$ .

- W82. Divide a triangle in two parts by a line through a side point (PG, 112; QP, 25r)
- W84. Divide a triangle in two parts by a parallel to a given side (PG, 119; QP, 23v)
- W88. Double or triple the area of a triangle by a line passing through the vertex (PG, 110; QP, 23v)
- W91. Divide a quadrilateral in two parts through a vertex (PG, 138; QP, 35v)
- W95. Divide a trapezium in two parts by a parallel to its base (PG, 125; QP, 24r)
- W96. Divide a parallelogram in two parts through a side point (PG, 123; QP, 32r)
- W97. Cut off one third of a parallelogram through a side point, Case 1 (PG, 124; QP, 32r)
- W98. Case 2 (PG, 124; QP, 32v)
- W102. Divide a trapezium in two parts through a side point, Case 1 (PG, 126; QP, 34r)
- W103. Case 2 (PG, 127; QP, 34r)
- W104. Divide a parallelogram in two parts through a point outside the figure (PG, 124; QP, 33r)
- W106. Cut off a given part of a trapezium through a side point, Case 1 (PG, 135; QP, 34v)
- W107. Case 2 (PG, 136; QP, 34v)
- W108. Case 3 (PG, 136; QP, 35r)
- W114. Cut off one third of a quadrilateral through a side point, Case 2 (PG, 140; QP, 39r)
- W118. Draw two parallels cutting off a part of the circle, e.g. the third (PG, 146; QP, 43v)

### *Test 2: Diagram lettering*

Compare now the letters used by Fibonacci and Tartaglia to mark the diagrams W82 to W118. Phonetical and numerical hypotheses are studied separately. In the case of numerical matching, except for one or two errors, Greek letters appear in the following order:  $A\ 1 = \{A\}$ ,  $B\ 2 = \{B\}$ ,  $\Gamma\ 3 = \{C\}$ ,  $\Delta\ 4 = \{D\}$ ,  $E\ 5 = \{E\}$ ,  $Z\ 6 = \{F\}$ , etc. Each diagram is investigated, then the ratio of identical letters from all diagrams is calculated (Table 3).

<sup>41</sup>Fibonacci's geometric constructions of Euclidean origin are: W82, W84, W88, W90–W93, W95–W100, W102–W104, W106–W109, W111–W114, W118, W119.

Cstr.	Tartaglia (1560)		Cstr.	Tartaglia (1560)	
	PHON	NUM		PHON	NUM
W82	4/6	6/6	W84	2/5	4/5
W88	3/4	4/4	W91	2/7	4/7
W95	1/7	4/7	W96	3/9	3/9
W97	4/8	4/8	W98	4/8	4/8
W102	1/6	3/6	W103	1/8	5/8
W104	3/8	3/8	W106	2/8	4/8
W107	1/10	2/10	W108	1/8	3/8
W114	0/7	1/7	W118	2/7	2/7
			Ratio	0.293	0.483

TABLE 3. Fibonacci’s Diagram Lettering

The numerical hypothesis applied to Fibonacci yields a result twice higher than in the case of Abū al-Wafā’s (NUM). Tartaglia has c. 3.5 letters out of 7 per diagram, even though he systematically changes the lettering of the vertices of all quadrilaterals. While Fibonacci marks the letters ABCD counterclockwise from the top left-hand corner of the diagram, Tartaglia marks AB on the top side and CD on the bottom side of the quadrilateral, from left to right. Despite this choice—which removes exactly three letters from the diagram—the similarity ratio is high. Thus, even assuming that personal choice might interfere with the original lettering of diagrams, the invariance of several letters seems to be a good indication that a transmission occurred.

### Test 3: Index of independence

To calculate the index of independence, we need the quadruplet  $(n, k, m, \ell)$  for the constructions echoed by Tartaglia. We proceed as in Sections 7 and 8.3.<sup>42</sup> Hence:

$$p(I)_{F/T} = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{1536} = 6.5 \times 10^{-4} \quad (9)$$

It remains to estimate the number of geometers active from 1220 to 1560. According to the *Chronological List of Mathematicians*, there were  $\alpha_{F/T} = 42$ . Assuming that Fibonacci faithfully reproduced Euclid’s constructions, I also count all the geometers active between –300 and 1560. They were  $\alpha_{E/T} = 157$ . Thus:

$$p(I)_{F/T} = 6.5 \times 10^{-4} \ll \frac{1}{42} \quad p(I)_{E/T} = 6.5 \times 10^{-4} \ll \frac{1}{157} \quad (10)$$

To sum up, Tartaglia borrowed many solutions from Fibonacci ( $16/25 = 0.640$ ) and reproduced c. 3.5 letters out of 7 per diagram ( $56/116 = 0.483$ ). The most discriminating criterion is Test 3, which produces a chance of independent discovery ( $1/1536 = 6.5 \times 10^{-4}$ ) much smaller than that resulting from the works written from Fibonacci—if not from Euclid—to Tartaglia.

<sup>42</sup> W82 (3, 1, 1, 1). The other solutions are Hero’s *Metrica* Γ3, 146, and al-Baghdādī’s DSL, II. – W84 (2, 1, 1, 1). The other solution is the Anonimo Fiorentino, TGP, 123. – W88 (1, 1, 1, 1). – W91 (2, 1, 1, 1). There is another solution by al-Baghdādī, DSL, XVI. – W95 (2, 1, 1, 1). al-Baghdādī, DSL, XII. – W96 (2, 1, 1, 1). The other (underspecified) solution is al-Baghdādī, DSL, VIII. – W97 (2, 1, 1, 1). al-Baghdādī, DSL, XI. – W98 (2, 1, 1, 1). There is another (underspecified) solution by al-Baghdādī, DSL, VIII. – W102 (1, 1, 1, 1). – W103 (2, 1, 1, 1). al-Baghdādī, DSL, VIII. – W104 (1, 1, 1, 1). – W106 (1, 1, 1, 1). – W107 (1, 1, 1, 1). – W108 (1, 1, 1, 1). – W114 (2, 1, 1, 1). al-Baghdādī, DSL, IX. – W118 (1, 1, 1, 1).

Since the three tests walk side by side, the result is robust: Tartaglia's *Quinta Parte* is based on Fibonacci and, after two millenia, still appears dependent on Euclid.<sup>43</sup>

## 10. Can We Be Confident in the Method?

As with any other method, the strategy of inquiry presented in these guidelines may be subject to criticism. In this Section, I will examine one by one possible objections and provide replies.

Test 1. Some scholars think the number of problems echoed is a decisive criterion; others not. According to the method developed in this article, the number of problems reproduced is only the target's affair. In any case, however, the criterion should not be used alone. This is because it would contradict a rather well established fact at the East–West crossroads, namely that Copernicus inherited Tūsi's couple (Section 3.3).

Test 2. Some scholars believe that the lettering of diagrams cannot serve as a criterion because geometric constructions were used by craftsmen, and thus were subject to an oral transmission that did not preserve the diagram lettering. If we accept the objection, we must admit likewise that only basic constructions were orally transmitted. But this is false. Some of the constructions studied in this article are complex ones: W20, W28A, W47\*, W68, W69, W83, W116, W117. Had they been subject to oral teaching only, they would have disappeared.

Test 3. Some scholars might be reluctant to accept the application of probability theory to the study of cultural transmission, because such seemingly clinical methods are thought to be insensitive to the subtlety of human affairs. Clearly, any transmission includes the appropriation and subsequent transformation of older material. Concepts are acclimated, and methods are adapted to new environments. Nevertheless, it is clearly the case that mathematical contents are not completely distorted by appropriation. Otherwise sources would be unrecognizable and all attempts to restore lost works, such as those of Apollonius (Hogendijk, 1986), Euclid (Archibald, 1915; Hogendijk, 1993) or al-Hajjāj (De Young, 1991), would be doomed to failure.

Being new, Test 3 requires a more detailed examination. To better understand the following discussion, let us recall the four relations that can bias the result in favor of transmission:

$$\begin{array}{ll} (1) & n \uparrow \Leftrightarrow p(I)_{X/Y} \downarrow \\ (2) & m \downarrow \Leftrightarrow p(I)_{X/Y} \downarrow \\ (3) & k \downarrow \Leftrightarrow p(I)_{X/Y} \downarrow \\ (4) & \alpha \downarrow \Leftrightarrow 1/\alpha \uparrow \end{array}$$

Insofar as the extant works are only a part of what existed in book or manuscript form, any corpus is partial. As a result, some solutions may not be detected. The number of solutions is always a lower limit. Suppose one finds new authors. According to relation (4), a higher  $\alpha$  makes it more difficult to establish transmission. On the other hand, finding new authors increases the chance of detecting new solutions. According to relation (1), when  $n$  increases, the chance of an independent rediscovery reduces and it is easier to prove transmission. Therefore, any increase in the number of authors has opposing effects: it favors both the thesis of transmission and the thesis of independent rediscovery. The two relations balance one another.

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<sup>43</sup>We can also test Fibonacci's dependence on Abū al-Wafā' in a similar manner. The *Practica Geometriae* (hereafter PG) has thirty constructions identical to constructions in Abū al-Wafā's collection. Test 1 tells us that eight solutions from Euclid's *Division of Figures* (hereafter DF) were taken up by Fibonacci. These constructions are as follows: DF, 17: PG, 140; DF, 19: PG, 115; DF, 20: PG, 121; DF, 26: PG, 116; DF, 27: PG, 121; DF, 32: PG, 131; DF, 33: PG, 137; DF, 34: PG, 140. Since these constructions were omitted by Abū al-Wafā', there is no need to proceed any further: this means Abū al-Wafā's collection was unknown to Fibonacci, who had access to Euclid's *Division of Figures* through another (unknown) Arabic intermediary source.



The solutions are not necessarily independent events in the real world. We may guess that, once the target-author has taken one solution from a given source, the chance of picking others from the same source is increased. This would be the case if the solutions were an undifferentiated set. In practice, however, they answer separate problems. Suppose Tartaglia had access to two treatises  $L, G$ , each providing different solutions to three problems  $A, B, C$ , say,  $L = \{a, b, c\}$  and  $G = \{\alpha, \beta, \gamma\}$ . If these events are independent,  $P(a) = P(b) = \frac{1}{2}$ , therefore  $P(a \cap b) = P(a)P(b) = \frac{1}{4}$ . Suppose the events are dependent. Knowing that Tartaglia picked  $a$  from  $L$ , what is the chance he also picks  $b$ ? The reduced set is  $\{ab, a\beta\}$ , there is one favorable case  $\{ab\}$ , thus  $P(b|a) = \frac{1}{2}$  and  $P(a \cap b) = P(a)P(b|a) = \frac{1}{4}$ . Since  $P(a)P(b|a) = P(a)P(b)$  is the definition of independence, geometric constructions can be seen as independent events.

However, it could be that the same solution comes into play in different problems. This is another case of dependence. Suppose a basic micro-construction—such as to draw a perpendicular, to bisect an angle, etc.—is used in the course of a macro-construction. Abū al-Wafā's collection is quite special in this respect, because micro-constructions are explicitly described only once. After that, they are just foreshadowed. For instance, six different methods to raise a perpendicular are described, but Abū al-Wafā' does not mention which one he uses in the subsequent macro-constructions W28A, W68, W69, W70, W73, W87. Similarly, the method of drawing parallel lines is not clearly referred to in macro-constructions W83, W87, W105. Consequently, one can pick any solution ensuring that constructions are independent events at all times.

Available solutions are not always equiprobable events. Some solutions are more accessible, or attractive, than others. In such circumstances, probability calculations are inapplicable. I maintain, however, that we can proceed with the simplification presented in Section 7. To account for the attractiveness of certain solutions, we would need to operate at the level of all known geometrical works. It would then be possible to know how many times a given solution was reproduced. However, such rewriting would result in restricting the number of solutions.<sup>44</sup> According to relation (1), when  $n$  reduces, the probability  $p(I)_{X/Y}$  increases, thus strengthening the case for a transmission. Since the bias favors the theory of transmission, the present method is more reliable when it tends to deny a legacy than when it aims to establish one. Therefore, in the context of the study of Abū al-Wafā's legacy, Test 3 produces an acceptable result.<sup>45</sup>

## 11. Conclusions

1. The fact that two-thirds of Abū al-Wafā's constructions were echoed in the West suggests that his collection was known—especially the constructions to be made with one opening of the compass. But, unlike many other Arabic treatises that were known in the Latin Middle Ages

<sup>44</sup> Although problem W3 gave rise to a dozen solutions from antiquity to the late classical period, nobody has reproduced John of Murs' solutions to  $n$ -sect the line (Busard, 1998, 147-148). If we were to take into account the attractiveness of geometric constructions, then we should remove these two solutions, and  $n$  would be reduced accordingly.

<sup>45</sup> There is a more detailed test of the conclusion that Renaissance geometers could have found Arabic constructions by themselves. Suppose one challenges the thesis of independence vis-à-vis Abū al-Wafā'. On the one hand, one might have doubts about the number of solutions. Take  $p(I)_{W/T} = 1/104$  instead of  $1/19$  and reintroduce this number in Eq. 1. The minimum number of solutions needed for a borrowing is:  $5_{W83}$  (Tartaglia),  $2_{W73}$  (Marolois),  $16_{W55}$  (Schwenter). The only way to reject the conclusion would be to find at least  $4-1 = 3$  (Tartaglia),  $2-1 = 1$  (Marolois) and  $16-2 = 14$  (Schwenter) new solutions. On the other hand, one might consider that too many authors in geometry were included in the list. Accordingly, it would be necessary to remove  $104 - 19 = 85$  (Tartaglia),  $165 - 84 = 81$  (Marolois) and  $168 - 15 = 157$  (Schwenter) geometers from the list to reject the conclusion. This task is out of reach.

through direct appropriation or translation,<sup>46</sup> Abū al-Wafā's collection seems not to have had the same destiny. The best candidate for reviving this legacy—i.e. Marolois—provides disappointing results: he has only a few solutions identical to Abū al-Wafā's, his diagram lettering is different, and the index of independence is too high.

2. The impression that Abū al-Wafā left a legacy is based on several factors. Resemblances exist in many Renaissance works, but when a single author is picked out, the number of identical problems reduces to eight or nine at best. Before the Renaissance these problems did not give rise to many solutions ( $n$ ); European geometers provided many solutions ( $m$ ); and Abū al-Wafā gave only a few original solutions ( $k$ ).

3. As to the way the approach described in this article can contribute to the methods in history and sociology of science, it is noteworthy that the three tests can be applied with no knowledge whatsoever of the historical process of diffusion. The approach is especially useful in the case of unattested relationships. If the tests yield a positive result, it is worthwhile searching for material evidence of the transmission. Otherwise, there is no need to engage in further investigation.

4. The index of independence enables us to distinguish between cases which, at face value, appear similar. Despite Marolois and Schwenter having the same number of constructions in common with Abū al-Wafā—and despite Schwenter being professor of Arabic at the University of Altdorf—Marolois' index (1/84) is much more discriminating than that of Schwenter (1/11). This counter-intuitive result appears because the index of independence is not based on similarity alone—which, regrettably, is the only element available to qualitative inspection.

The exploratory method described in this article requires improvements. However, several factors suggest that the general methodology is valid. First, it achieves robust results, which can be waived only by rejecting outright all three tests together. Second, the results are unambiguously tied to specific works. It is not claimed that the results can be extrapolated to a geographical area or period, and other works can still be subjected to the tests.

In short, the present findings urge sociologists and historians of science not to rely too heavily on appearances when drawing conclusions about the diffusion of mathematics.

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<sup>46</sup>Ishāq-Thābit and Hajjāj versions of the *Elements* spurred the growth of Western geometry (De Young, 1984, 1991; Folkerts, 1989). Latin geometry inherited many other texts of Arabic origin, such as John of Palermo's *De duabus lineis semper approximantibus sibi invicem et nunquam concurrentibus* (*On two lines always approaching each other but never meeting*)(Clagett, 1954), Abū Bakr's *Liber mensurationum*, Muḥammad al-Baghdādī's *De Superficierum Divisionibus Liber*, Sa'īd Abū Uthmān's *Liber Saydi Abuothmi*, 'Abd al-Rahmān's *Liber Aderameti* (Busard, 1969), etc.

## AppendixA. A List of Abū al-Wafā's problems in plane geometry

### *Chap. I. Introduction*<sup>47</sup>

- WA. Make a right angle (I.v)
- WB. Method 2 (I.vi)
- WC. Raise a perpendicular at the endpoint of a line (I.vii)
- WD. Method 2 (I.viii, var. AJ drawn)
- WE. Check the right angle (I.ix, var. one more line)
- WF. Method 2 (I.x, var. one more line)
- WG. Method 3 (I.xi, var. one more line)

### *Chap. II. Basic constructions*

- W1A. Bisect a line (II.ia)
- W1B. Bisect an arc (II.ib)
- W2. Bisect a line: Method 2 (II.ii)
- W3. N-sect a straight line (II.iii)
- W4. Bisect an angle (II.iv)
- W5. Draw a perpendicular to a given line from an outside point (II.v)
- W6. Draw a perpendicular to a plane from an outside point in the space (II.vi)
- W7. Describe an angle equal to a given angle (II.vii)
- W8\*. Draw a parallel to a line through a given point (II.viii)
- W9. Method 2 (II.ix)
- W10. Find the missing center of a given circle (II.xa)
- W11. Method 2 (II.xb)
- W12. Method 3 (II.xi)
- W13. Draw a tangent to a circle by an outside point (II.xii)
- W14. Draw a tangent to a circle by a point of the circumference (II.xiii)
- W15. Draw a parallel ZH to the basis BJ of a triangle, equal to BH (II.xiv, two figs.)
- W16. Draw a parallel HD to the basis BJ of a triangle, equal to BH (II.xv)
- W17. Draw a parallel HD to the basis BJ of a triangle, with  $HD=BZ+BH$  (II.xvi)
- W18. Describe a triangle equal to a given triangle (II.xvii)
- W19. Trisect a right angle (II.xviii)
- W20. Trisect an acute angle (II.xix)
- W21. Method 2 (II.xx)
  - Y21B. Trisect an acute angle: Method 3
- W22\*. Trisect an arc (II.xxi)
- W23. Duplicate the cube (II.xxii)
  - Y23C. Draw the pattern of a parabolic mirror: variant of W25
  - Y23F. Duplicate the cube: Method 2
  - Y23G. [Illegible diagram]
- W24. Draw the pattern of a parabolic mirror (II.xxiii)
- W25. Method 2 (II.xxiv)
  - Y25C. An instrument to draw the hyperbola
  - Y25D. Continuation
  - Y25E. Continuation

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<sup>47</sup>Krasnova's numbering in Roman numerals with variant, if any, is given at the end of the line. (Krasnova, 1966).

- Y25F. Draw the hyperbola
- Y25G. Trisect an acute angle: Method 4
- Y25H. Continuation
- Y25I. Trisect an acute angle: Method 5

*Chap. III. Construction of polygons*

- W26. Describe an equilateral triangle whose side is given (III.i)
- W27. Describe a square whose side is given (III.ii)
- W28A. Describe a regular pentagon whose side is given (III.iii)
- W28B. Method 2 (III.iv, var. BDJHY downside up)
  - Y28C. Method 3
  - Y28D. Continuation
  - Y28E. Continuation
- W29. Describe a regular hexagon whose side is given (III.v)
- W30. Describe a regular heptagon whose side is given (III.vi)
- W31. Describe a regular octagon whose side is given (III.vii)
- W32. Method 2 (III.viii)
- W33. Describe a regular enneagon whose side is given (III.ix)
- W34. Describe a regular hexagon whose side is given (III.x, var. BJD upside down)
- W35. Method 2 (III.xi)

*Chap. IV. Inscription of polygons in the circle*

- W36. Inscribe an equilateral triangle in the circle (IV.i)
- W37. Circumscribe an equilateral triangle to the circle (IV.ii)
- W38. Inscribe a square in the circle (IV.iii)
- W39. Method 2 (IV.iv)
- W40. Method 3 (IV.vi)
- W41. Method 4 (IV.v)
- W42. Method 5 (IV.vii)
- W43. Inscribe a regular pentagon in the circle (IV.viii)
- W44. Method 2 (IV.ix)
- W45. Method 3 (IV.x)
  - Y45B. Method 4
- W46. Inscribe a regular hexagon in the circle (IV.xi)
- W47\*. Inscribe a regular heptagon in the circle (IV.xii, var. BH missing)
  - Y47B. Method 2: find the side of the heptagon by means of conics
  - Y47C. Continuation: draw the triangle of the heptagon
  - Y47D. Continuation: draw the regular heptagon
- W48. Inscribe a regular octagon in the circle (IV.xiii)
- W49. Inscribe a regular enneagon in the circle (IV.xiv)
- W50\*. Inscribe a regular decagon in the circle (IV.xva)
- W51. Method 2 (IV.xvb)

*Chap. V. Circumscription of the circle around polygons*

- W52A. Circumscribe a circle around a scalene triangle (V.i)
- W52B\*. Circumscribe a circle around an isosceles triangle
- W53\*. Method 2 (V.ii)

- W54. Circumscribe a circle around a square (V.iii)
- W55. Circumscribe a circle around a regular pentagon (V.iv)
- W56. Circumscribe a circle around a regular hexagon (V.v)

*Chap. VI. Inscription of the circle in polygons*

- W57. Inscribe a circle in any given triangle (VI.i)

*Chap. VII. Inscription of polygons with each other*

- W58. Inscribe an equilateral triangle in a square (VII.i, different fig.)
- W59. Method 2 (VII.ii)
- W60. Method 3 (VII.iii)
- W61. Method 4 (VII.iv)
- W62. Method 5 (VII.v)
- W63. Circumscribe a triangle around a square (VII.vi)
- W64. Circumscribe a square around an equilateral triangle (VII.vii)
- W65. Circumscribe a square around a scalene triangle (VII.viii)
- W66. Method 2 (VII.ix)
- W67. Method 3 (VII.x)
- W68. Inscribe a square in a scalene triangle (VII.xi)
- W69. Method 2 (VII.xii)
  - Y69B. Method 3
  - Y69C. Method 4
- W70. Inscribe a square in an equilateral triangle (VII.xiii)
  - Y70B. Method 2
  - Y70C. Circumscribe a scalene triangle around an equilateral triangle
- W71. Inscribe an equilateral triangle in a scalene triangle (VII.xiv)
  - Y71B. Method 2
- W72. Circumscribe an equilateral triangle around a scalene triangle (VII.xv)
- W73. Inscribe an equilateral triangle in a regular pentagon (VII.xvi)
- W74. Circumscribe an equilateral triangle around a regular pentagon (VII.xvii)
- W75. Inscribe a square in a regular pentagon (VII.xviii)
- W76. Circumscribe a square around a regular pentagon (VII.xix)
- W77. Inscribe a regular pentagon in a square (VII.xx, two figs.)
- W78. Inscribe a regular octagon in a square (VII.xxi, var.  $H\dot{H}$ , HS drawn)
- W79. Method 2 (VII.xxii, var.  $H\dot{H}$ , HS drawn)
- W80. Circumscribe a square around an octagon (VII.xxiii)

*Chap. VIII. Division of triangles*

- W81. Divide a triangle in two parts by a line passing through the vertex (VIII.i)
- W82. Divide a triangle in two parts by a line through a side point D,  $BD > BJ/2$  (VIII.ii)
- W83. Divide a triangle in  $n$  parts by a line through a side point (VIII.iii, var.  $A\dot{H}$ , AZ, AH drawn)
- W84. Divide a triangle in two parts by a parallel to a given side (VIII.iv, var. DH, HJ drawn)
- W85. Divide a triangle in three parts by two parallels to a given side (VIII.v)
- W86. Method 2 (VIII.v, var. circle DHJ downside up)
- W87. Double or triple the area of a triangle by a parallel to its side (VIII.vii, different fig.)
- W88. Double or triple the area of a triangle by a line passing through the vertex (VIII.viii)
- W89. Draw a half or a third triangle inside a given triangle (VIII.ix)

*Chap. IX. Division of quadrilaterals*

- W90. Divide a parallelogram in two parts through a vertex (IX.i)  
W91. Divide a quadrilateral in two parts through a vertex (IX.ii, var. DZ drawn)  
W92. Divide a quadrilateral in two parts through a side point, with  $HZ \parallel BJ$  (IX.iii, var.  $AD \parallel BJ$ )  
W93. Case 2:  $HZ$  not parallel to  $BJ$  (IX.iv, var.  $AB \parallel DJ$ )  
W94. Case 3:  $BH$  outside the quadrilateral (IX.v)  
W95. Divide a trapezium in two parts by a parallel to its base (IX.vi)  
    Y95B. Method 2  
W96. Divide a parallelogram in two parts through a side point (IX.vii)  
W97. Cut off one third of a parallelogram through a side point, with  $AH = AD/3$  (IX.viii)  
W98. Case 2:  $AH < AD/3$  (IX.ix, two figs.)  
W99. Case 3:  $AH > AD/3$  (IX.x, var.  $H=A$ )  
W100. Case 4:  $HH < BZ$  (IX.xi)  
W101. Case 5:  $HH > BZ$  (IX.xii, line  $HI$  missing)  
W102. Divide a trapezium in two parts through a side point, with  $AH = HD$  (IX.xiii, var.  $AB \parallel DJ$ )  
W103. Case 2:  $AH \neq HD$  (IX.xiv)  
W104. Divide a parallelogram in two parts through a point outside the figure (IX.xv)  
W105. Cut off one third of a parallelogram through a point outside the figure (IX.xvi, var. ZJ,  $DH$  missing)  
W106. Cut off a given part of a trapezium through a side point, with  $AH = AD/3$  (IX.xvii)  
W107. Case 2:  $AH \neq AD/3$  (IX.xviii)  
W108. Case 3:  $AH < AZ$   
W109. Divide a trapezium in two parts through a point outside the figure (IX.xix, var.  $JH$  missing)  
W110. Cut off a given part of a trapezium through a point outside the figure (IX.xx)  
W111. Cut off one third of a trapezium, with  $BE = BD/3$  (IX.xxi)  
W112. Case 2:  $BE \neq BD/3$  (IX.xxii, var. AZ, JZ drawn)  
W113. Cut off one third of a quadrilateral through a side point, with  $HZ \parallel BD$  (IX.xxiii, var.  $AB \parallel JD$ )  
W114. Case 2:  $HZ$  not parallel to  $BD$  (IX.xxiv, var.  $AB \parallel DJ$ )  
W115. Case 3:  $BH$  outside the quadrilateral (IX.xxv, var.  $AB \parallel JD$ )  
W116. Describe a double square around the given one (IX.xxvi)  
W117. Describe a half square within the given one (IX.xxvii)  
W118. Draw two parallels cutting off a given part of the circle, e.g. the third (IX.xxviii, var. Euclid's fig.)  
W119. Divide a circular sector in two parts (IX.xxix, two figs.)  
W120. Divide a square in two parts, putting aside a strip of width  $DH$  (IX.xxx)  
W121. Divide a square in three parts, putting aside a strip of width  $MN$  (IX.xxxi)  
W122. Divide a triangle in two parts, putting aside a strip of width  $DJ$  (IX.xxxii, var. line  $LQ$  missing)  
    Y122B. Method 2  
W123. Divide a triangle in two parts, putting aside a strip of width  $DJ$  (IX.xxxiii)  
    Y123B. Divide a triangle in  $n$  parts, putting aside a strip of width  $DJ$   
W124. Divide a trapezium in two parts, putting aside a strip of width  $DH$  (IX.xxxiv)  
    Y124B. Divide a trapezium in two parts, putting aside a widening strip  
    Y124C. [Illegible diagram]

*Chap. X. Division and composition of squares*

- W125. Make a square of nine squares (X.i)  
W126. Make a square of four squares (X.ii)  
W127. Make a square of sixteen squares (X.iii)

- W128. Make a square of two squares (X.iv)
- W129. Make a square of eight squares (X.v)
- W130. Make a square of thirteen squares (X.vi)
- W131. Make a square of ten squares (X.vii)
- W132. Split a square in eight squares (X.viii, ix)
- W133. Split a square in eighteen squares (X.x)
- W134. Split a square in ten squares (X.xi)
- W135. Split a square in twenty squares (X.xii)
- W136A. Make a square of three squares (X.xiii)
- W136B. Method 2 (X.xiv)
- W136C. Method 3 (X.xv)
- W137. Method 4 (X.xvi)
- W138. Make a square of any given number of squares (X.xvii)
- W139. Divide a square whose side is given in two squares (X.xviii)

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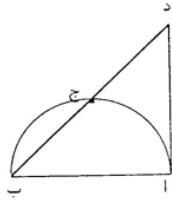
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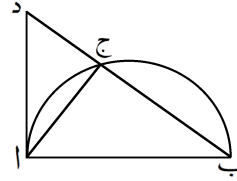
The diagrams appearing in this article are taken from the following sources: W-Diagrams (Büzjānī, 1997); W12 (Boncompagni, 1862); Y23D (Curze, 1887); W56 (Ravaisson Mollien, 1881); W60 (Simi, 1974); W21 (Commandino, 1588), W47\*, W52B\* (Galli Bibiena, 1711), W8\*, W29, W50\*, W53\*, W70, W68, W73, W85 (Marolois, 1616), W25 (Milliet de Chales, 1674), W13, W62 (Pacioli, 1494), W53\* (Pico, 1597), W23A, W69, W70, W72, W83, W103, W108, W114, W116 (Tartaglia, 1560) (IMSS Digital Library, Florence); WC (Bachot, 1598), W3 (Bovelles, 1547), W55 (Peletier, 1573), W47C (Viète, 1646) (Gallica-BNF, Paris); W20 (Campanus, 1482) (courtesy of the Posner Memorial Collection, Carnegie Mellon University Libraries, Pittsburgh); W22\*, W117 (Ardüser, 1646), W2, W28A (Schwenker, 1618) (courtesy of the Staats- und Universitätsbibliothek Dresden); W79 (Hirschvogel, 1543) (courtesy of the Bayerische Staatsbibliothek). All other diagrams by the author.

**Dominique Raynaud**, formerly trained as an architect, teaches the history and sociology of science at Grenoble University. He is a member of the PLC (Grenoble) and associate researcher at the GEMASS (CNRS UMR 8598, Paris). He has devoted several books and articles to the history of optics, perspective and architecture, among which is the next *Optics and the Rise of Perspective. A Study in Network Knowledge Diffusion* (Oxford, forthcoming).

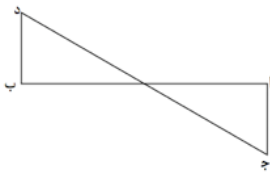
**Appendix B. Additional diagrams from Abū al-Wafā's and Western treatises**



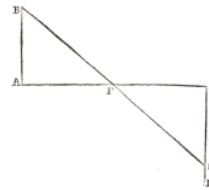
Abū al-Wafā' WD



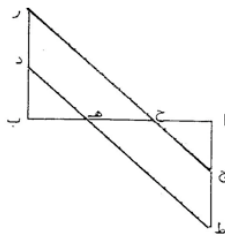
Ibn Yūnus, SAH, 3v



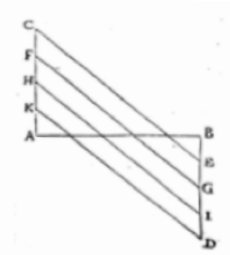
Abū al-Wafā' W2



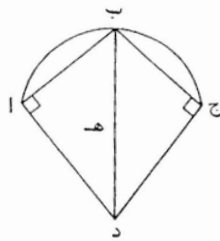
diff. Schwenter, GPN, 410



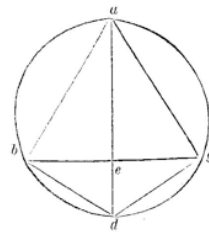
Abū al-Wafā' W3



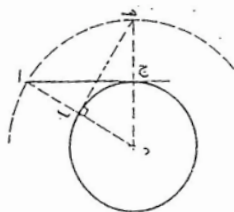
Bovelles, GP, 8r



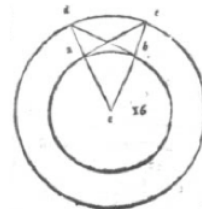
Abū al-Wafā' W12



Fibonacci, PG, 102

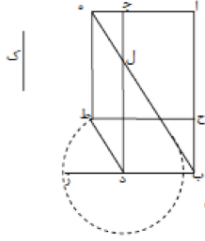


Abū al-Wafā' W13

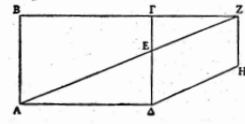


Pacioli, S, 27r

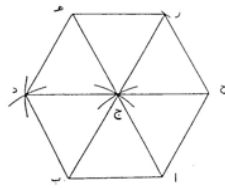




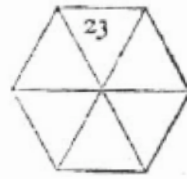
Ibn Yūnus Y25I



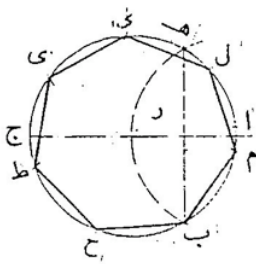
Commandino, PC, 61v



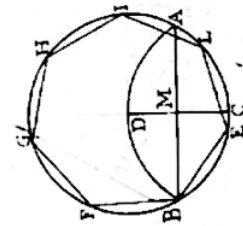
Abū al-Wafā' W29



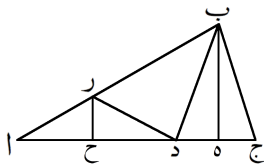
Marolois, OM, 6



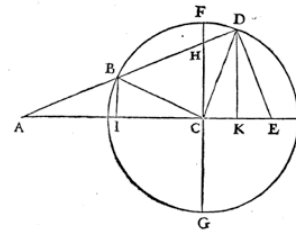
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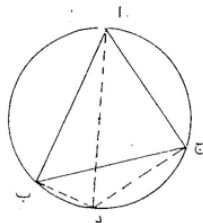
Galli Bibiena, AC, 10



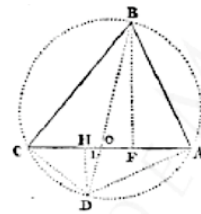
Ibn Yūnus Y47C



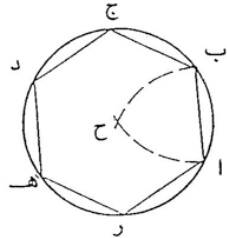
Viète, SG, 249



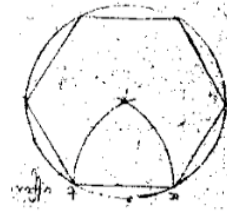
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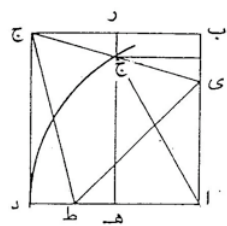
diff. Marolois, OM, 70



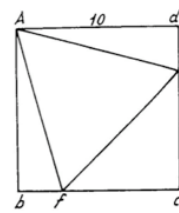
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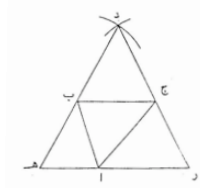
Vinci, MS A, 16v



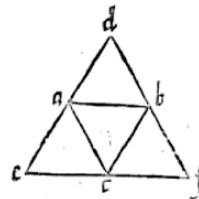
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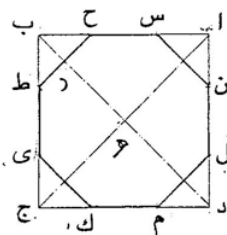
diff. Fiorentino, TGP, 160



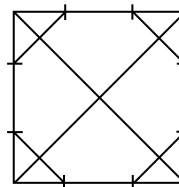
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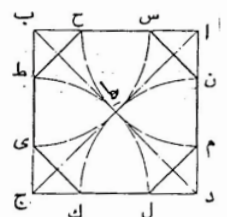
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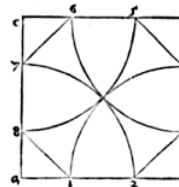
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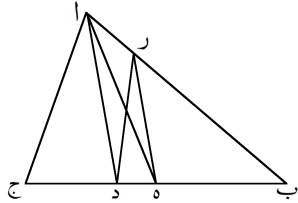
S. Nicolás, AUA, 144r



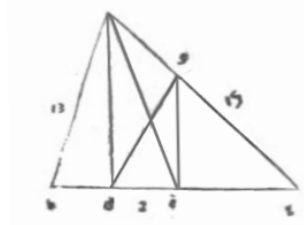
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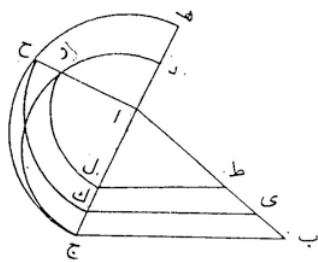
Hirschvogel, G, 18



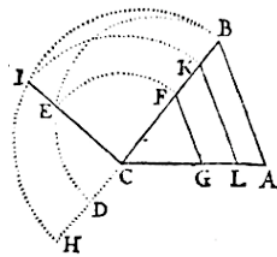
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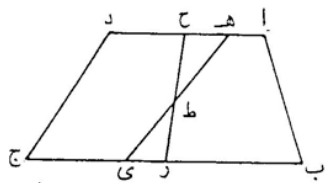
Pacioli, S, 36r



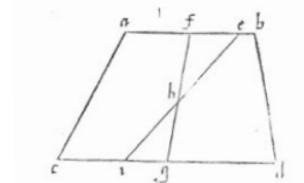
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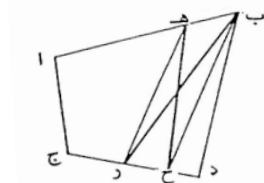
Marolois, OM, 52



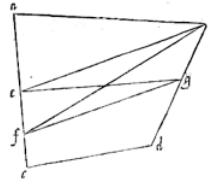
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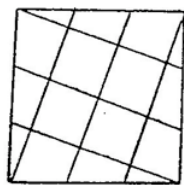
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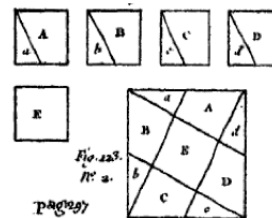
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Tartaglia, QP, 39r



Abū al-Wafā' W134



sim. Ozanam, RM, 297