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ON THE DUCTILE ENLARGEMENT OF VOIDS IN TRIAXIAL STRESS FIELDS*

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SUMMARY

THE FRACTURE of ductile solids has frequently been observed to result from the large growth and coalescence of microscopic voids, a process enhanced by the superposition of hydrostatic tensile stresses on a plastic deformation field. The ductile growth of voids is treated here as a problem in continuum plasticity. First, a variational principle is established to characterize the flow field in an elastically rigid and incompressible plastic material containing an internal void or voids, and subjected to a remotely uniform stress and strain rate field. Then an approximate Rayleigh–Ritz procedure is developed and applied to the enlargement of an isolated spherical void in a non-hardening material. Growth is studied in some detail for the case of a remote tensile extension field with superposed hydrostatic stresses. The volume changing contribution to void growth is found to overwhelm the shape changing part when the mean remote normal stress is large, so that growth is essentially spherical. Further, it is found that for any remote strain rate field, the void enlargement rate is amplified over the remote strain rate by a factor rising exponentially with the ratio of mean normal stress to yield stress. Some related results are discussed, including the long cylindrical void considered by F. A. MCCLINTOCK (1968, *J. appl. Mech.* **35**, 363), and an approximate relation is given to describe growth of a spherical void in a general remote field. The results suggest a rapidly decreasing fracture ductility with increasing hydrostatic tension.

1. INTRODUCTION

THE FRACTURE of ductile solids has frequently been observed to be the result of the growth and coalescence of microscopic voids, both in nominally uniform stress fields (ROGERS, 1960; GURLAND and PLATEAU, 1963; BLUHM and MORRISSEY, 1966) and ahead of an extending crack (BEACHEM, 1963). ROSENFIELD (1968) has recently surveyed metallurgical aspects of this fracture mechanism. Rogers explains that the central portion of the cup and cone fracture which occurs at the neck of a specimen is produced by the coalescence of internal voids which grow by plastic deformation under the influence of the prevailing triaxial stress system. To begin development of a comprehensive fracture criterion, the relation between the growth of a void and imposed stress and strain must be found. MCCLINTOCK (1968) has presented a start on the problem through his analysis of the expansion of a long circular cylindrical cavity in a non-hardening material, pulled in the direction of

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its axis while subjected to transverse tensile stresses. The noteworthy result is that the relative void expansion per unit applied strain increment increases exponentially with the transverse stress. Our present work seeks to determine the relation between void growth and stress triaxiality for a more realistic model, i.e. an isolated spherical void in a remotely uniform stress and strain rate field. We treat the void growth problem as in the domain of continuum plasticity, in accord with the contemporary view of ductile fracture as described by Meakin and Petch (1963) which considers separation as a kinematical result of large but localized plastic deformations.

The non-linearity of field equations seems to exclude exact analyses for all but the one-dimensional case studied by McClintock. Thus we start by deriving a variational principle governing cavity expansion in an infinite rigid–plastic medium, and employ a Rayleigh–Ritz procedure for approximate solutions. The Budiansky and Vidensek (1955) variational methods for infinite regions could be adapted to our present purposes, but we find that the procedure used in Section 2 requires somewhat weaker assumptions on the behaviour of solutions at large distances. While the formulation applies either to hardening or non-hardening materials, we present applications to the latter type only.

2. Variational Principle for Void Growth in Rigid–Plastic Materials

Consider an infinite body of an incompressible rigid–plastic material (either perfectly plastic or strain hardening) containing an internal void or voids with bounding surface S_v. We assume that the prior deformation history is known so that the current void boundaries and flow stress at each point are specified. At the current instant the material is subjected to a uniform remote strain rate field $\dot{\epsilon}_{ij}^{\infty}$. This determines the remote deviatoric stress state s_{ij}^{∞} and, in addition, the current remote mean normal stress σ^{∞} is specified so that

$$\sigma_{ij}^{\infty} = s_{ij}^{\infty} + \sigma^{\infty}\, \delta_{ij}. \tag{1}$$

We seek to determine the current velocity field throughout the material and, in particular, the growth rate of the void(s). Consider any velocity field $\dot{u}_i$ satisfying incompressibility and agreeing with the remote strain rate:

$$\dot{\epsilon}_{ij} = \tfrac{1}{2}\,(\dot{u}_{i,j} + \dot{u}_{j,i}) \rightarrow \dot{\epsilon}_{ij}^{\infty} \quad \text{as} \quad x_i\, x_i \rightarrow \infty; \;\; \dot{\epsilon}_{ii} = 0. \tag{2}$$

The yield surface in stress space at each point of the material is assumed convex with normal strain rate increments so that a deviatoric stress state $s_{ij}(\dot{\boldsymbol{\epsilon}})$ can be associated with each $\dot{\epsilon}_{ij}$ in such a manner that $s_{ij}(\dot{\boldsymbol{\epsilon}})\,\dot{\epsilon}_{ij}$ is unique. Note that convexity and normality both follow from and imply the basic inequality of plasticity

$$[s_{ij}(\dot{\boldsymbol{\epsilon}}) - s_{ij}^*]\,\dot{\epsilon}_{ij} \geqslant 0. \tag{3}$$

where s_{ij}^* is any stress state within or on the current yield surface. The inequality follows from Hill's (1950) principle of maximum plastic work, from Drucker's (1951) stability postulate, or from other roughly equivalent starting assumptions in plasticity.

Now define a functional $Q(\dot{u})$ of any velocity field $\dot{u}_i$ satisfying (2) as

$$Q(\dot{u}) = \int_V [s_{ij}(\dot{\epsilon}) - s_{ij}^{\infty}]\, \dot{\epsilon}_{ij}\, dV - \sigma_{ij}^{\infty} \int_{S_v} n_i\, \dot{u}_j\, dS. \tag{4}$$

Here V denotes the infinite volume exterior to the void(s). In the integral over the void surface, n_i is a unit normal drawn into the material so that σ_{ij}^{∞} times the surface integral represents the work rate of the remote stress field on the distortion of the void interior. The following convergence assumptions are essential to the subsequent development. Let S_e denote some imaginary spherical surface drawn in the material exterior to the void(s). Then it is assumed that all fields $\dot{\epsilon}_{ij}$ considered, including the actual field, approach $\dot{\epsilon}_{ij}^{\infty}$ sufficiently rapidly so that, if the superscripts 1 and 2 refer to any two of these fields and the superscript A denotes the actual field, then

$$\left.\begin{aligned} &\lim_{S_e \to \infty} \int_{S_e} (\sigma_{ij}^{\mathrm{A}} - \sigma_{ij}^{\infty})\, n_i\, (\dot{u}_j^2 - \dot{u}_j^1)\, dS = 0 \\ &\text{and} \quad \int_V [s_{ij}(\dot{\epsilon}^1) - s_{ij}^{\infty}]\, (\dot{\epsilon}_{ij}^2 - \dot{\epsilon}_{ij}^{\infty})\, dV \quad \text{is bounded.} \end{aligned}\right\} \tag{5}$$

The second assumption is satisfied if $\dot{u}_i - \dot{\epsilon}_{ij}^{\infty} x_j$ falls off faster than $R^{-\frac{1}{2}-\delta}$ where R is radial distance and δ is any positive number. The first is certainly satisfied if $\dot{u}_i - \dot{\epsilon}_{ij}^{\infty} x_j$ falls off as R^{-2}, and is most likely satisfied under much less stringent conditions. The lack of knowledge of the actual field makes more precise statements impossible, but we note that corresponding convergence requirements in a two-dimensional case discussed subsequently, for which the solution is known, are satisfied within a wide margin. The latter convergence assumption is sufficient to show the existence of the volume integral of (4) for the perfectly plastic case and to show that if it diverges in the strain hardening case, it does so in an essentially trivial way. To see this, rearrange terms so that

$$\left.\begin{aligned} \int_V [s_{ij}(\dot{\epsilon}) - s_{ij}^{\infty}]\, \dot{\epsilon}_{ij}\, dV &= \int_V [s_{ij}(\dot{\epsilon}) - s_{ij}^{\infty}]\, (\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^{\infty})\, dV \\ + \int_V [s_{ij}(\dot{\epsilon}^{\infty}) - s_{ij}^{\infty}]\, \dot{\epsilon}_{ij}^{\infty}\, dV &+ \int_V [s_{ij}(\dot{\epsilon}) - s_{ij}(\dot{\epsilon}^{\infty})]\, \dot{\epsilon}_{ij}^{\infty}\, dV. \end{aligned}\right\} \tag{6}$$

The integral on the left is non-negative and the third integral on the right is non-positive by (3), and the first integral on the right is bounded by hypothesis. For perfect plasticity, $s_{ij}(\dot{\epsilon}^{\infty}) = s_{ij}^{\infty}$ so that the volume integral on the left cannot diverge. For strain hardening, convergence is tied to the second integral on the right. But its integrand is independent of the assumed field $\dot{\epsilon}_{ij}$ and, if divergent, one may show that a convergent integral results in (4) if one subtracts out the second integrand on the right in (6) above. This point is not pursued further, since it affects neither the computation of $Q(\dot{u}) - Q^{\mathrm{A}}$ nor the validity of the minimum principle.

Again letting the superscript A refer to the actual field,

$$Q(\dot{\boldsymbol{u}}) - Q^{\mathrm{A}} = \int_V \{[s_{ij}(\dot{\boldsymbol{\epsilon}}) - s_{ij}^{\infty}]\,\dot{\epsilon}_{ij} - [s_{ij}^{\mathrm{A}} - s_{ij}^{\infty}]\dot{\epsilon}_{ij}^{\mathrm{A}}\}\,dV$$
$$- \int_{S_v} (\sigma_{ij}^{\infty} - \sigma_{ij}^{\mathrm{A}})\,n_i\,(\dot{u}_j - \dot{u}_j^{\mathrm{A}})\,dS \tag{7}$$

since $\sigma_{ij}^{\mathrm{A}} n_i$ vanishes on the void surface. But by an application of the principle of virtual work to the infinite region, as justified by the first convergence assumption of (5),

$$- \int_{S_v} (\sigma_{ij}^{\infty} - \sigma_{ij}^{\mathrm{A}})\,n_i\,(\dot{u}_j - \dot{u}_j^{\mathrm{A}})\,dS = \int_V (s_{ij}^{\infty} - s_{ij}^{\mathrm{A}})\,(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^{\mathrm{A}})\,dV. \tag{8}$$

Upon combining terms above, one finds

$$Q(\dot{\boldsymbol{u}}) - Q^{\mathrm{A}} = \int_V [s_{ij}(\dot{\boldsymbol{\epsilon}}) - s_{ij}^{\mathrm{A}}]\,\dot{\epsilon}_{ij}\,dV \geqslant 0, \tag{9}$$

since the integrand is non-negative by the fundamental inequality of (3). The resulting minimum principle is that no assumed field can render the functional $Q(\dot{\boldsymbol{u}})$ smaller than its value for the actual flow field.

We shall employ the minimum principle as a basis for approximate solutions via the Rayleigh–Ritz method. In particular, an assumed flow field will be represented in the form

$$\dot{u}_i = \dot{\epsilon}_{ij}^{\infty} x_j + q_1\,\dot{u}_i^{(1)} + q_2\,\dot{u}_i^{(2)} + \ldots + q_n\,\dot{u}_i^{(n)} \tag{10}$$

where each $\dot{u}_i^{(k)}$ is a specified incompressible velocity field approaching zero at infinity so as to meet convergence requirements. The set of constants q_k giving the 'best' approximation are chosen to minimize the functional $Q(\dot{\boldsymbol{u}}) = Q(q_1, q_2, \ldots, q_n)$. Note that in computing derivatives for the minimization,

$$\frac{\partial s_{ij}(q_1, \ldots, q_n)}{\partial q_k}\,\dot{\epsilon}_{ij} = 0, \tag{11}$$

by normality, since the stress derivative is tangent to the yield surface. Thus the 'best' set is given by

$$\int_V [s_{ij}(q_1, q_2, \ldots, q_n) - s_{ij}^{\infty}]\,\dot{\epsilon}_{ij}^{(k)}\,dV = \sigma_{ij}^{\infty} \int_{S_v} n_i\,\dot{u}_j^{(k)}\,dS$$
$$\text{for } k = 1, 2, \ldots, n. \tag{12}$$

Now let q_1^*, q_2^*, ... denote a solution of these equations with s_{ij}^* being the corresponding deviatoric field. By virtual work, the surface integral is transformed to

$$\sigma_{ij}^{\infty} \int_{S_v} n_i\,\dot{u}_j^{(k)}\,dS = \int_{S_v} (\sigma_{ij}^{\infty} - \sigma_{ij}^{\mathrm{A}})\,n_i\,\dot{u}_j^{(k)}\,dS$$
$$= - \int_V (s_{ij}^{\infty} - s_{ij}^{\mathrm{A}})\,\dot{\epsilon}_{ij}^{(k)}\,dV. \tag{13}$$

Thus,

$$\int_V (s_{ij}^* - s_{ij}^{A})\, \dot{\epsilon}_{ij}^{(k)}\, dV = 0, \tag{14}$$

so that the difference between the actual and the approximate deviatoric fields is orthogonal, in a weighted sense, to each assumed field.

Uniqueness of the Rayleigh–Ritz approximation is of interest since solutions to (12) for the q's must generally be searched out numerically. Let superscripts * and ** denote two solutions. Both must satisfy (14) and thus

$$\int_V (s_{ij}^* - s_{ij}^{**})\, \dot{\epsilon}_{ij}^{(k)}\, dV = 0. \tag{15}$$

Multiplying by $q_k^* - q_k^{**}$ and summing on k,

$$\int_V (s_{ij}^* - s_{ij}^{**})\, (\dot{\epsilon}_{ij}^* - \dot{\epsilon}_{ij}^{**})\, dV = 0. \tag{16}$$

Now the integrand in this equation is non-negative by (3), so that it must vanish everywhere. Hence, for yield surfaces containing no corners, the direction of the strain rate is fully unique at every point. Since both strain rate fields are derivable from a velocity field in the form of (10), in general, for a finite number of terms, coincidence of direction can be achieved only if $q_k^* = q_k^{**}$. This is indeed the case for velocity fields employed subsequently, but no all-inclusive statement can be made since uniqueness of strain rate direction is the most that can be asserted even for exact solutions, without extra considerations.

3. Growth of a Spherical Void in a Uniaxial Tension Strain Rate Field

As a first application of the variational principle, consider a spherical void of radius R_0 as in Fig. 1, and suppose that the remote strain field consists of a tensile extension at the rate $\dot{\epsilon}$ in the x_3 direction, with contractions at the rate $\frac{1}{2}\dot{\epsilon}$ in the x_1 and x_2 directions (as required by incompressibility). The remote deviatoric stress state s_{ij}^{∞} then corresponds to that of a tensile test, and it is supposed that in addition the remote mean normal stress σ^{∞} is specified. Now for the assumed incompressible velocity field to be employed in the variational approximation, it is clear that any assumed field (as well as the actual field) can be split into three parts: (i) a velocity field resulting in a uniform strain rate field $\dot{\epsilon}_{ij}^{\infty}$, so as to meet remote boundary conditions, (ii) a spherically symmetric velocity field corresponding to a change in volume of the void but no change in shape, and (iii) a velocity field, decaying at remote distances, which changes the void shape but not its volume. Hence we write in the form of (10),

$$\dot{u}_i = \dot{\epsilon}_{ij}^{\infty}\, x_j + D\dot{u}_i^{D} + E\dot{u}_i^{E}, \tag{17}$$

where D and E, playing the role of the q's in our general development, are constants to be determined, $\dot{u}_i^{D}$ is a spherically symmetric volume changing field, and $\dot{u}_i^{E}$ is a shape changing field which preserves void volume.

Incompressibility and spherical symmetry require that the volume changing field be

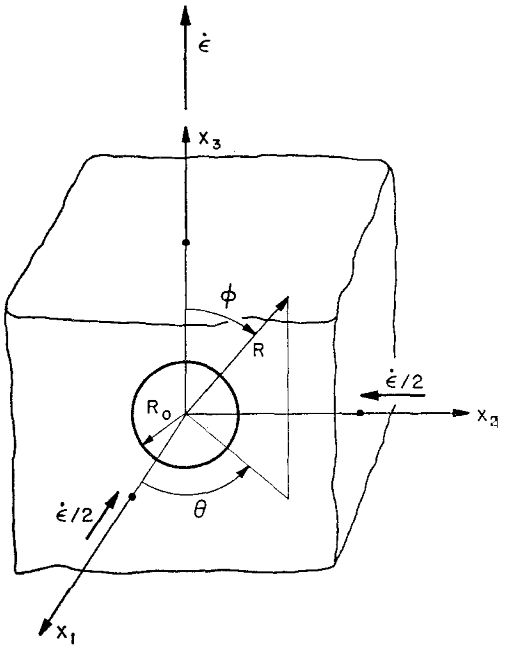

FIG. 1. Spherical void in a remote simple tension strain rate field. Later used for a void in a general remote field, with x_1 the greatest, x_2 the least, and x_3 the intermediate principal strain rate directions.

$$\dot{u}_i{}^D = \dot{\epsilon}\left(\frac{R_0}{R}\right)^3 x_i, \tag{18}$$

where we have chosen the constants in this expression so that D may be interpreted as the ratio of the average strain rate of sphere radii to the remotely imposed strain rate. Thus if we use the symbol $\dot{R}_0$ to represent the average radial velocity on the void boundary, D will equal $\dot{R}_0/\dot{\epsilon}R_0$ (note that the other velocity terms in (17) result in zero average radial velocity). The approximation is involved with choice of the shape changing field $\dot{u}_i{}^E$ and fortunately, as we shall see, results are not very sensitive to the particular choice. We assume that spherical surfaces concentric with the void are moved by this field so as to become axially symmetric ellipsoids of the same volume in a small time interval. This constraint is met by deriving $u_i{}^E$ from a stream potential,

$$\dot{u}_R{}^E = \frac{1}{R^2 \sin\phi}\frac{\partial\psi^E}{\partial\phi}, \quad \dot{u}_\phi{}^E = -\frac{1}{R\sin\phi}\frac{\partial\psi^E}{\partial R}, \tag{19}$$

having the form

$$\psi^E = \tfrac{1}{2}\,\dot{\epsilon}R_0{}^3\,F\,(R)\sin^2\phi\cos\phi, \quad \text{where } F\,(R_0) = 1, \tag{20}$$

with $F\,(R)$ otherwise arbitrary, but resulting in vanishing strain rates at infinity.

The choice of constants in (20) is such that the net radial velocities on the void boundary, in the direction of remote tensile extension and in the transverse direction, are computed from (17) as

$$\left.\begin{aligned} \dot{u}_R(R_0, 0) &= (D + 1 + E)\,\dot{\epsilon} R_0 \quad \text{and} \\ \dot{u}_R(R_0, \tfrac{1}{2}\pi) &= \left(D - \frac{1+E}{2}\right)\dot{\epsilon} R_0 \end{aligned}\right\} \tag{21}$$

respectively. Thus the assumed velocity field deforms the void interior with a mean dilatational strain rate $D\dot{\epsilon}$, on which is superimposed an incompressible extension strain rate $(1 + E)\,\dot{\epsilon}$ in the remote tensile direction. Several different choices of the function $F(R)$ of (20) were employed for the numerical calculations, as described subsequently.

The amplification factors D and E are determined so as to minimize the functional $Q(\boldsymbol{u}) = Q(D, E)$, and this leads to two equations in the form of (12):

$$\left.\begin{aligned} \int_V [s_{ij}(D, E) - s_{ij}^{\infty}]\,\dot{\epsilon}_{ij}^{D}\,dV &= \sigma^{\infty} \int_{S_v} n_i\,\dot{u}_i^{D}\,dS, \\ \int_V [s_{ij}(D, E) - s_{ij}^{\infty}]\,\dot{\epsilon}_{ij}^{E}\,dV &= s_{ij}^{\infty} \int_{S_v} n_i\,\dot{u}_j^{E}\,dS. \end{aligned}\right\} \tag{22}$$

Here, $s_{ij}(D, E)$ is the deviatoric stress field corresponding to the assumed strain rate field

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}(D, E) = \dot{\epsilon}_{ij}^{\infty} + D\dot{\epsilon}_{ij}^{D} + E\dot{\epsilon}_{ij}^{E}. \tag{23}$$

Also, we have simplified (12) by noting that the deviatoric remote stress does no work in the surface integral involving the D field and the mean remote stress does no work in the surface integral involving the E field.

Numerical solutions for D and E have been obtained in the case of a non-hardening Mises material, with yield stress τ_0 in shear, for which deviatoric stresses corresponding to a strain rate $\dot{\epsilon}_{ij}$ are

$$s_{ij} = \sqrt{2}\,\tau_0\,\dot{\epsilon}_{ij}/(\dot{\epsilon}_{kl}\,\dot{\epsilon}_{kl})^{\frac{1}{2}}. \tag{24}$$

Upon carrying out the surface integrals in (22), introducing spherical coordinates in the volume integrals, and dividing through by a few constant terms, the two equations to be solved are:

$$\frac{1}{\sqrt{2}\,\dot{\epsilon}\,R_0^3}\int_{R_0}^{\infty}\int_0^{\pi}\left[\frac{\dot{\epsilon}_{ij}\,\dot{\epsilon}_{ij}^{D}}{(\dot{\epsilon}_{kl}\,\dot{\epsilon}_{kl})^{\frac{1}{2}}} - \frac{\dot{\epsilon}_{ij}^{\infty}\,\dot{\epsilon}_{ij}^{D}}{(\dot{\epsilon}_{kl}^{\infty}\,\dot{\epsilon}_{kl}^{\infty})^{\frac{1}{2}}}\right] R^2 \sin\phi\,d\phi\,dR = \sigma^{\infty}/\tau_0, \tag{25}$$

$$\frac{\sqrt{3}}{\sqrt{2}\,\dot{\epsilon}R_0^3}\int_{R_0}^{\infty}\int_0^{\pi}\left[\frac{\dot{\epsilon}_{ij}\,\dot{\epsilon}_{ij}^{E}}{(\dot{\epsilon}_{kl}\,\dot{\epsilon}_{kl})^{\frac{1}{2}}} - \frac{\epsilon_{ij}^{\infty}\,\dot{\epsilon}_{ij}^{E}}{(\dot{\epsilon}_{kl}^{\infty}\,\dot{\epsilon}_{kl}^{\infty})^{\frac{1}{2}}}\right] R^2 \sin\phi\,d\phi\,dR = \frac{2 + R_0 F'(R_0)}{5}, \tag{26}$$

where $\dot{\epsilon}_{ij}$ depends on D and E as in (23). The integrands are quite complicated functions of the integration variables R and ϕ, depending on the former through the function $F(R)$, but their detailed forms need not be given here. We note simply that after multiplying by the factors in front, the integrals define dimensionless functions involving only pure numbers and the unknown factors D and E, to be set equal to the triaxiality ratio in the first case and to a pure number in the second. Completion of the solution relies on quite lengthy numerical integrations and search techniques. The most efficient procedure is to choose a value of D and then to search out the corresponding value of E so as to satisfy the second equation. Then this D, E set is inserted into the first equation to compute the ratio of remote mean stress to yield stress required to produce the chosen dilatational amplification D.

We have carried out detailed computations for a non-hardening Mises material, employing six different choices for the function $F(R)$ appearing in (20) for the E field. Remarkably, the value of D corresponding to a given σ^{∞}/τ_0 appears almost insensitive to the particular function $F(R)$ employed. Also, at large values of σ^{∞}/τ_0, D turns out to be large compared to E so that the volume changing part of the growth overwhelms the shape changing part.

The six different functions $F(R)$ examined are

$$\left.\begin{aligned} F_1(R) &= \tfrac{1}{2}\,[5 - 3\,(R_0/R)^2], \\ F_2(R) &= 2 - (R_0/R)^3, \\ F_3(R) &= 4 - 3\,(R_0/R)^2, \\ F_4(R) &= 7 - 9\,(R_0/R) + 3\,(R_0/R)^2, \\ F_5(R) &= 3\,(R_0/R)^3 - 2\,(R_0/R)^6, \\ F_6(R) &= \tfrac{1}{9}\,[13\,(R_0/R) - 4\,(R_0/R)^{10}]. \end{aligned}\right\} \qquad (27)$$

Some of those were chosen because they appear in similar incompressible flow problems for a linearly viscous material. For example, $F_1(R)$ provides a solution for a spherical inclusion bonded to a viscous matrix, with the inclusion given a uniform incompressible extension strain rate field. $F_2(R)$ satisfies the same boundary conditions, but is not admissible for a viscous material. $F_3(R)$ is the viscous field corresponding to a similar spherical inclusion undergoing a uniform incompressible extension, except that the inclusion is in smooth contact so that normal stress but no shear stress is transmitted to the matrix. Thus, this field makes $\dot{\epsilon}_{R\phi}$ vanish on the void surface, a proper boundary condition for the plastic case also. $F_4(R)$ simultaneously satisfies both the bonded inclusion and zero shear strain rate boundary conditions, and therefore cannot be a viscous solution. Finally, $F_5(R)$ and $F_6(R)$ were chosen simply because they represent somewhat unrealistic fields, and we wished to see if even this would significantly alter the essentially identical values of the dilatational factor D resulting for the other four fields. For example, these fields both result in a sign reversal of $\dot{u}_{\phi}$ at particular values of R. Of course, the best procedure would be to let the variational principle serve as a basis for choice of $F(R)$ through the associated Euler–Lagrange differential equation, rather than to examine a set of different choices with only the multiplying factor E as a free parameter. This is not as simple as it might seem. Not only is the resulting differential equation of fourth order and highly nonlinear, but terms in the equation depend on R, D, $F(R)$, $F'(R)$, and $F''(R)$ through difficult integrals on ϕ which cannot be evaluated in closed form.

Figure 2 gives the resulting values of the incompressible extension portion of the void enlargement rate per unit remotely imposed strain rate, $1 + E$, as a function of D, as computed from (26) for each $F(R)$. The first four functions $F(R)$ give roughly similar results for $1 + E$, with differences that are large in absolute terms but quite small compared to D over most of the range plotted. $F_5(R)$ and $F_6(R)$ give a quite erratic behaviour, as might have been expected, and actually

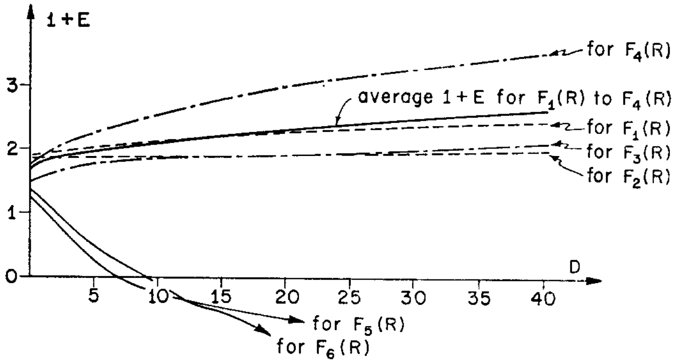

FIG. 2. Relation between incompressible extension factor $(1 + E)$ and dilatational factor (D), as computed from several assumed forms for the velocity field, for a spherical void in a remote tensile strain field.

predict that at larger values of D the void should flatten rather than elongate in the remote extension direction.

The dimensionless mean stress σ^{∞}/τ_0 is shown as a function of D in Fig. 3. The solid line represents results from all six incompressible extension fields. Differences from field to field are not great enough to appear on the graph, except for a slight broadening of the curve in the range of D between 0 and 3. For example, at

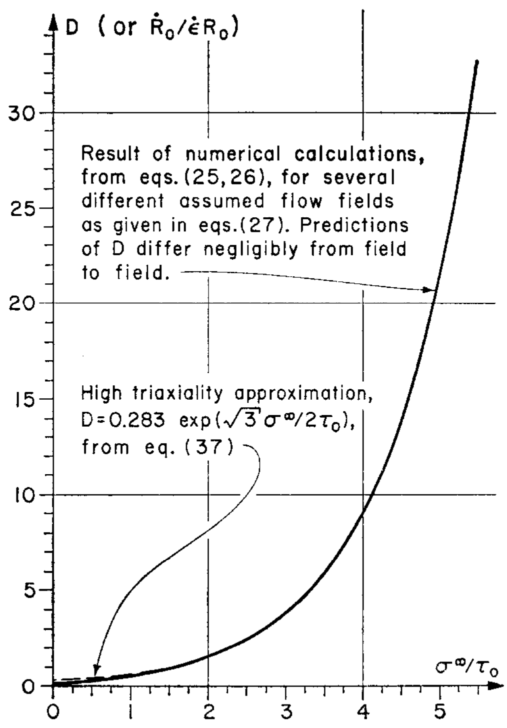

FIG. 3. Dilatational amplification factor D, as a function of mean normal stress σ^{∞}, for a spherical void in a remote tensile strain rate field.

$D = 0{\cdot}01$ the range of σ^∞/τ_0 values is from $-0{\cdot}015$ to $-0{\cdot}086$, but the curve is so steep that all points essentially fall on the line drawn. At $D = 1{\cdot}0$ the variation appears greatest, with a range of σ^∞/τ_0 from 1·40 to 1·60. At $D = 3{\cdot}0$, the range is from 2·70 to 2·76. The variation rapidly decreases at larger values of D, being in the neighbourhood of 0·5 per cent or less of the mean for D greater than 6. The dashed line represents the result of a large D calculation in Section 4.

4. Growth in a General Remote Strain Rate Field with High Stress Triaxiality

We have seen in Section 3 on a tensile remote field that the spherically symmetric volume changing part of void growth far overwhelms the shape changing part when the remote mean stress is large. A calculation is presented here of the relation between this dilatational amplification and the remote mean stress on the assumption that both are large. It will be seen that a simple exponential dependence on σ^∞/τ_0 results, and that this gives an excellent approximation to the detailed results of Fig. 3 even at low triaxiality.

Since the calculations are relatively straightforward, it is possible to consider the spherical void in a general remote strain rate field $\dot\epsilon_{ij}{}^\infty$, rather than just a tensile field. Anticipating that dilatational growth dominates, we choose an assumed velocity field involving only the contribution from the remote strain rate field and a spherically symmetric void expansion field:

$$\dot u_i = \dot\epsilon_{ij}{}^\infty x_j + D\dot u_i{}^D, \tag{28}$$

where

$$\dot u_i{}^D = (\tfrac{2}{3}\,\dot\epsilon_{ij}{}^\infty\,\dot\epsilon_{ij}{}^\infty)^{\frac{1}{2}}\left(\frac{R_0}{R}\right)^3 x_i. \tag{29}$$

Comparing with (18), we see that the tensile strain rate $\dot\epsilon$ has been replaced by the factor involving the square root sign, and this is simply the 'equivalent' tensile strain rate, which equals $\dot\epsilon$ when the remote field consists of a simple tensile extension. If $\dot R_0$ is the average radial velocity of the void surface, the physical interpretation of D is

$$\frac{\dot R_0}{R_0} = (\tfrac{2}{3}\,\dot\epsilon_{ij}{}^\infty\,\dot\epsilon_{ij}{}^\infty)^{\frac{1}{2}}\,D. \tag{30}$$

Of course, for large D the growth is nearly spherical and the radial velocity differs little from $\dot R_0$.

The dilatational amplification is chosen so as to minimize the functional Q, leading to

$$\int_V [s_{ij}(D) - s_{ij}{}^\infty]\,\dot\epsilon_{ij}{}^D\,dV = \sigma^\infty \int_{S_v} n_i\,\dot u_i{}^D\,dS. \tag{31}$$

For a non-hardening Mises material, the deviatoric stresses corresponding to the assumed field are given by (24) as

$$s_{ij}(D) = \frac{\sqrt{2}\,\tau_0\,(\dot\epsilon_{ij}{}^\infty + D\,\dot\epsilon_{ij}{}^D)}{(\dot\epsilon_{ij}{}^\infty\,\dot\epsilon_{ij}{}^\infty + 2D\,\dot\epsilon_{ij}{}^\infty\,\dot\epsilon_{ij}{}^D + D^2\,\dot\epsilon_{ij}{}^D\,\dot\epsilon_{ij}{}^D)^{\frac{1}{2}}}\,. \tag{32}$$

Upon computing the various terms involved and simplifying, equation (31) for D becomes

$$\frac{1}{\sqrt{3}\,\pi}\int\limits_{\Omega}\int\limits_{0}^{1}\left[\frac{D-\mu/2\lambda}{(4D^2\lambda^2-4D\mu\lambda+1)^{\frac{1}{2}}}+\mu/2\lambda\right]d\lambda d\Omega=\sigma^{\infty}/\tau_0, \tag{33}$$

where $\lambda=(R_0/R)^3$, $d\Omega$ denotes an increment of solid angle, the outer integration is over the unit sphere, and μ is a function of position on the unit sphere given by the ratio of the radial component of the remote strain rate to the equivalent strain rate,

$$\mu=\dot{\epsilon}_{RR}{}^{\infty}/(\tfrac{2}{3}\,\dot{\epsilon}_{ij}{}^{\infty}\,\dot{\epsilon}_{ij}{}^{\infty})^{\frac{1}{2}}. \tag{34}$$

Upon performing the inner integral,

$$\frac{1}{2\sqrt{3}\,\pi}\int\limits_{\Omega}\{\log\,[(4D^2-4D\mu+1)^{\frac{1}{2}}+2D-\mu]-\log\,(1-\mu) \\ +\mu\log\,[\tfrac{1}{2}\,(4D^2-4D\mu+1)^{\frac{1}{2}}+\tfrac{1}{2}-D\mu]\}\,d\Omega=\sigma^{\infty}/\tau_0. \tag{35}$$

Making the assumption that D is large, as consistent with choosing a dilatational field alone for the assumed field, we now expand the arguments of the logarithms in powers of D and drop all terms of order $1/D$. The terms involving D can be integrated at once over the unit sphere, and there results

$$\frac{2}{\sqrt{3}}\log\,(4D)-\frac{1}{2\sqrt{3}\,\pi}\int\limits_{\Omega}(1-\mu)\log\,(1-\mu)\,d\Omega=\sigma^{\infty}/\tau_0. \tag{36}$$

Solving for D, we have the high triaxiality exponential result

$$D=C\,(\nu)\exp\left(\frac{\sqrt{3}\,\sigma^{\infty}}{2\tau_0}\right), \tag{37}$$

where the constant $C\,(\nu)$ (this notation will be explained shortly) is given by

$$C\,(\nu)=\tfrac{1}{4}\exp\left[\frac{1}{4\pi}\int\limits_{\Omega}(1-\mu)\log\,(1-\mu)\,d\Omega\right]. \tag{38}$$

The constant depends on the ratios of the remotely imposed strain rate components, as somewhat different functional forms for μ result in (34) for different remote fields. We show this dependence through a Lode variable ν for the imposed strain rates, which we define as

$$\nu=-\frac{3\dot{\epsilon}_{\mathrm{II}}{}^{\infty}}{\dot{\epsilon}_{\mathrm{I}}{}^{\infty}-\dot{\epsilon}_{\mathrm{III}}{}^{\infty}}, \tag{39}$$

where $\dot{\epsilon}_{\mathrm{I}}{}^{\infty}\geqslant\dot{\epsilon}_{\mathrm{II}}{}^{\infty}\geqslant\dot{\epsilon}_{\mathrm{III}}{}^{\infty}$ are the principal components of the remote field. This variable lies between -1 and $+1$, with $\nu=+1$ for a remote simple extension (or biaxial compression), $\nu=0$ for a remote simple shear, and $\nu=-1$ for a remote simple compression (or biaxial extension). To evaluate $C\,(\nu)$, we choose the angles θ, ϕ of Fig. 1 to describe position on the unit sphere. Then if the x_1, x_3, and x_2 axes are chosen to agree with the greatest, intermediate, and least principal strain rate directions, respectively, one finds that

$$\mu=\frac{1}{2\,(3+\nu^2)^{\frac{1}{2}}}\,[\nu\,(1-3\cos^2\phi)+3\sin^2\phi\cos 2\theta]. \tag{40}$$

Making the substitution $\xi=\cos\,\phi$ and performing the integral on θ in the solid angle integration of (38), one finds for the constant,

$$C(\nu) = \tfrac{1}{4}\exp\left\{\int_0^1 \left[A\log\left(\frac{A+(A^2-B^2)^{\frac{1}{2}}}{2}\right) + A - (A^2-B^2)^{\frac{1}{2}}\right] d\xi\right\}$$

where $$A = 1 - \frac{\nu(1-3\xi^2)}{2(3+\nu^2)^{\frac{1}{2}}} \quad \text{and} \quad B = \frac{3(1-\xi^2)^{\frac{1}{2}}}{2(3+\nu^2)^{\frac{1}{2}}}. \tag{41}$$

Numerical calculations have been performed for the range of ν from -1 to $+1$. Remarkably, $C(\nu)$ is very nearly independent of ν, and the dependence on ν can be given to within an error never exceeding 0·2 per cent by the linear relation

$$C(\nu) \approx 0{\cdot}279 + 0{\cdot}004\,\nu. \tag{42}$$

Thus, to within about 1 per cent, the high triaxiality void growth rate depends on the remote strain rate only through the equivalent tensile rate.

For a simple tension remote field, as in Section 3, $C(\nu) = C(1)$ can be evaluated exactly as $1{\cdot}5\,e^{-5/3} = 0{\cdot}283$, and $D = 0{\cdot}283\exp(\sqrt{3}\,\sigma^{\infty}/2\tau_0)$. This result is shown by the dashed line in Fig. 3. It is indistinguishable from the solid line, representing detailed calculations, at values of D greater than unity (or σ^{∞}/τ_0 greater than 1·5) so that the high triaxiality result appears accurate even at low stress levels where the dilatational growth does not dominate the shape changing growth at all.

Finally, we note that a similar analysis to the above could be carried out for the case of a large negative remote mean stress. One then finds in analogy to (37) that

$$D = -C(-\nu)\exp\left(-\frac{\sqrt{3}\,\sigma^{\infty}}{2\tau_0}\right). \tag{43}$$

Since $C(\nu)$ is almost independent of ν, for a given remote strain rate the dilatational amplification is nearly the same under tensile and compressive mean stress.

Actually, one can obtain an even closer approximation to the detailed calculations of D in Fig. 3 by choosing a relation between D and σ^{∞}/τ_0 which reproduces both the correct high positive mean stress result (37) and the high negative result (43). It is natural to think of the two exponential tails as resulting from hyperbolic functions at large values of their arguments, and so to choose

$$D = [C(\nu) + C(-\nu)]\sinh(\sqrt{3}\,\sigma^{\infty}/2\tau_0) + [C(\nu) - C(-\nu)]\cosh(\sqrt{3}\,\sigma^{\infty}/2\tau_0). \tag{44}$$

With the close approximation to $C(\nu)$ in (42), this becomes

$$D = 0{\cdot}558\sinh(\sqrt{3}\,\sigma^{\infty}/2\tau_0) + 0{\cdot}008\,\nu\cosh(\sqrt{3}\,\sigma^{\infty}/2\tau_0). \tag{45}$$

When $\nu = 1$ for comparison with tensile calculations, this expression for D comes quite close to the solid curve in Fig. 3, and the value of $D = 0{\cdot}008$ for zero mean stress is a close approximation to the axis crossing of the solid line (about 0·01). Also, for a pure shear remote field, $\nu = 0$ and (45) predicts no dilatational growth when $\sigma^{\infty} = 0$. That this prediction is correct is easily seen from a symmetry argument, for reversing all quantities in the remote shear field must reverse the void growth rate, but the reversed remote shear field is indistinguishable from the original remote shear field except for changes in principal directions. These remarks suggest that (45) be viewed as a good approximation to the dilatational part of void growth for all values of the remote mean stress and for all values of ν (i.e. for all remote strain rate ratios).

5. Some Related Results

The exponential amplification of void growth rates by stress triaxiality, as found here for the spherical void, has also been found by McClintock (1968) in his study of the long cylindrical void (Fig. 4), stretched at a uniform rate $\dot{\epsilon}$ in the

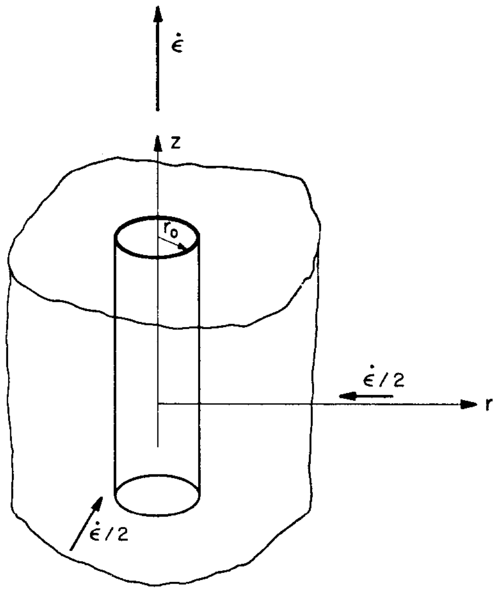

Fig. 4. Long cylindrical void extended in the direction of its axis.

direction of its axis while subjected to a remote transverse stress σ_{rr}^{∞}. This problem could be solved directly from our variational principle. In view of axial symmetry and incompressibility, the velocity field is

$$\dot{u}_z = \dot{\epsilon} z, \quad \dot{u}_r = -\tfrac{1}{2}\dot{\epsilon} r + (\dot{r}_0 + \tfrac{1}{2}\dot{\epsilon} r_0)\frac{r_0}{r}, \tag{46}$$

where $\dot{r}_0$ is the unknown transverse velocity of the cavity boundary. The corresponding two-dimensional version of the functional $Q(\dot{\boldsymbol{u}})$ in our general development is

$$Q(\dot{\boldsymbol{u}}) = \int_{r_0}^{\infty} [s_{ij}(\dot{\boldsymbol{\epsilon}}) - s_{ij}^{\infty}]\, \dot{\epsilon}_{ij}\, (2\pi r)\, dr - \sigma_{rr}^{\infty}\, \dot{r}_0\, (2\pi r_0), \tag{47}$$

where $\dot{\boldsymbol{\epsilon}}$ is the strain rate derived from the velocity field (46). Now, since the actual velocity field has the form of (46), we get the exact answer by minimizing Q with respect to $\dot{r}_0$, which leads to

$$\int_{r_0}^{\infty} [s_{\theta\theta}(\dot{\boldsymbol{\epsilon}}) - s_{rr}(\dot{\boldsymbol{\epsilon}})]\, \frac{dr}{r} = \sigma_{rr}^{\infty}. \tag{48}$$

This could have been written down through a conventional equilibrium approach by integrating the radial equation $\partial \sigma_{rr}/\partial r + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0$. McClintock's (1968) results from solving (48) for non-hardening Mises and Tresca materials are

$$\frac{\dot{r}_0}{r_0} = \frac{\sqrt{3}}{2}\left|\dot{\epsilon}\right| \sinh\left(\frac{\sigma_{rr}^{\infty}}{\tau_0}\right) - \tfrac{1}{2}\,\dot{\epsilon} \qquad \text{(Mises)}, \tag{49}$$

$$\frac{\dot{r}_0}{r_0} = \tfrac{1}{2}\left|\dot{\epsilon}\right| \operatorname{sgn}(\sigma_{rr}^{\infty}) \exp\left(\frac{\left|\sigma_{rr}^{\infty}\right|}{\tau_0}\right) - \tfrac{1}{2}\,\dot{\epsilon} \qquad \text{(Tresca)} \tag{50}$$

(here sgn (. . .) means 'sign of'). We note that the growth rate in a Tresca material is non-unique at $\sigma_{rr}^{\infty} = 0$, with any value between the limits $-(|\dot{\epsilon}| + \dot{\epsilon})/2$ and $+(|\dot{\epsilon}| - \dot{\epsilon})/2$ constituting a valid solution for $\dot{r}_0/r_0$.

Results for the spherical cavity in a Tresca material are somewhat difficult to obtain, but we have performed the high triaxiality computation when the remote field is a simple tensile extension at the rate $\dot{\epsilon}$, as in Fig. 1. Again one assumes a velocity field consisting of a term corresponding to the remote strain rate and a term representing spherically symmetric void growth, just as in (28, 29) with $(\frac{2}{3}\dot{\epsilon}_{ij}^{\infty}\,\dot{\epsilon}_{ij}^{\infty})^{\frac{1}{2}}$ replaced by $\dot{\epsilon}$. The dilatational growth factor D has the interpretation as in (30) and it is obtained as the solution to (31). The following steps are involved in computing deviatoric stresses corresponding to the assumed deformation field. First, principal extension rates are found, as is possible in view of the axial symmetry, and it turns out that the assumed deformation field splits into two parts, one having the intermediate rate $\dot{\epsilon}_{\mathrm{II}} > 0$ and the other having $\dot{\epsilon}_{\mathrm{II}} < 0$. The boundary between these two regions is the sphere $R^3 = 2DR_0^3$. The two regions correspond to corners on the Tresca yield surface and in the former $s_{\mathrm{I}} = s_{\mathrm{II}} = -s_{\mathrm{III}}/2 = 2\tau_0/3$, whereas in the latter $s_{\mathrm{I}} = -2s_{\mathrm{II}} = -2s_{\mathrm{III}} = 4\tau_0/3$. Once principal stresses and directions are found, the tensor product in (31) is computed, and with the substitutions $\lambda = (R_0/R)^3$ and $\xi = \cos\phi$, there results

$$\sigma^{\infty}/\tau_0 = \int_0^1 \left\{ \int_0^{(2D)^{-1}} \left[\frac{2D + (1-2\xi^2)/\lambda}{[4D^2\lambda^2 + 4D\lambda(1-2\xi^2) + 1]^{\frac{1}{2}}} - (1-2\xi^2)/\lambda \right] d\lambda + \int_{(2D)^{-1}}^{1} \left[\frac{2D + (1-2\xi^2)/\lambda}{[4D^2\lambda^2 + 4D\lambda(1-2\xi^2) + 1]^{\frac{1}{2}}} - (\tfrac{1}{3} - 2\xi^2)/\lambda \right] d\lambda \right\} d\xi. \tag{51}$$

The large D result is now obtained through steps similar to those following from (33). First one integrates over λ and then performs a series expansion which drops all terms vanishing when D is large, with the final result being

$$D = 2\,\mathrm{e}^{-5/3} \exp(3\sigma^{\infty}/4\tau_0) = 0{\cdot}376 \exp(3\sigma^{\infty}/4\tau_0) = 0{\cdot}376 \exp(3\sigma^{\infty}/2\sigma_0). \tag{52}$$

Here, in the final form, we have employed the tensile yield stress $\sigma_0 = 2\tau_0$. For comparison, the large D result for a Mises material with the same remote simple extension strain rate field is

$$D = \tfrac{3}{2}\,\mathrm{e}^{-5/3} \exp(\sqrt{3}\,\sigma^{\infty}/2\tau_0) = 0{\cdot}283 \exp(\sqrt{3}\,\sigma^{\infty}/2\tau_0) = 0{\cdot}283 \exp(3\sigma^{\infty}/2\sigma_0), \tag{53}$$

where the Mises tensile yield $\sigma_0 = \sqrt{3}\,\tau_0$ is employed in the last form. Examining (52, 53) together with (49, 50), we see that both idealizations (and perhaps all isotropic idealizations) lead to the same coefficients in the exponential terms when materials are matched in tension for the case of a spherical cavity, and in shear for a long cylindrical cavity. Apart from this, however, the coefficient multiplying the exponential for expansion of a sphere in a Tresca material is one-third greater than that for a Mises material.

So far, all of our examples have been for non-hardening materials. An indication of strain hardening effects for spherical voids is given by the extreme case of a rigid-plastic material exhibiting a linear relation between stress and 'true' strain in simple tension. Then for isotropic strain hardening with a Mises equivalent stress definition, stress–strain relations are

$$\dot{\epsilon}_{ij} = \frac{1}{2G\tau}\, s_{ij}\, \dot{\tau} \quad \text{where} \quad \tau = (s_{ij}\, s_{ij}/2)^{\frac{1}{2}}. \tag{54}$$

Here G is a constant and the formula applies only if, at the current instant, $\dot{\tau} > 0$ and τ equals the greatest value achieved in previous stressing. Otherwise, the material is rigid and $\dot{\epsilon}_{ij} = 0$. For small strains and proportional stress elevations, this results in the incompressible linear elastic form $\epsilon_{ij} = s_{ij}/2G$, with ϵ_{ij} being the infinitesimal strain tensor. Thus proportional stress elevations will result for small deformations around the spherical void if the remotely applied stresses are kept in constant ratio. The analogous linear elastic problem is readily solved for general remote strain fields. One may verify that the deformations transform initially spherical surfaces into infinitesimally neighbouring ellipsoids, as was assumed for the shape changing fields employed in the non-hardening analysis with a remote tensile field. In fact, the exact shape changing field in the present case is given by $F_1(R)$ of (27), for a remote tensile field. The results for void growth rates are most simply expressed in terms of remote principal extension rates $\dot{\epsilon}_{\mathrm{I}}{}^{\infty}$, $\dot{\epsilon}_{\mathrm{II}}{}^{\infty}$, $\dot{\epsilon}_{\mathrm{III}}{}^{\infty}$. Then if $\dot{R}_{0\mathrm{I}}$, $\dot{R}_{0\mathrm{II}}$, $\dot{R}_{0\mathrm{III}}$ denote radial velocities of the void boundary at points aligned with the remote principal directions, the linear elastic analogy leads to

$$\dot{R}_{0K} = \left[\tfrac{5}{3}\, \dot{\epsilon}_K{}^{\infty} + (\tfrac{2}{3}\, \dot{\epsilon}_L{}^{\infty}\, \dot{\epsilon}_L{}^{\infty})^{\frac{1}{2}}\, \frac{\sqrt{3}\, \sigma^{\infty}}{4\tau^{\infty}}\right] R_0, \quad K, L = \mathrm{I, II, III}, \tag{55}$$

where τ^{∞} is the remote value of the equivalent flow stress in shear.

6. Discussion and Summary

Our results here for a spherical void, as well as McClintock's (1968) for the long cylindrical void, show that growth rates are significantly elevated by the superposition of hydrostatic tension on a remotely uniform plastic deformation field. In both cases, moderate and high stress triaxiality leads to an amplification of relative void growth rates over imposed strain rates by a factor depending exponentially on the mean normal stress. In view of the complexity of detailed computations, a simple approximate formula is developed below for the computation of void growth rates in arbitrary remote fields.

A step has already been taken by (45), describing the dilatational contribution to growth of a spherical void in a non-hardening material. We now turn to approximating the shape changing part. In (55) for the strongly hardening material, the first term represents the shape change, and this involves a simple amplification of the remote strain rate field by a factor of 5/3. Now, if we put (21) for a remote tensile field into the same form, it is seen that the 5/3 factor is replaced by $1 + E$. While Fig. 2 for this shape change factor does not lead to a definitive result, it is seen that for D near zero (low triaxiality) all four 'reasonable' shape change fields give values around 5/3 or slightly higher. Then, over a substantial range of D (say, D greater than 2, corresponding to a triaxiality ratio σ^{∞}/τ_0 greater than 2·3 in Fig. 3) all predictions of $1 + E$ give a value in the neighbourhood of 2 as does the solid average line. Deviations are significant at large values of D, but not very important since D then overwhelms $1 + E$ in value. Assuming these observations to be appropriate also for other types of remote fields, we can now write a general approximate equation for growth rates in the form of (55) as

$$\dot{R}_{0K} = \left\{(\tfrac{5}{3} \text{ to } 2)\, \dot{\epsilon}_K{}^{\infty} + (\tfrac{2}{3}\, \dot{\epsilon}_L{}^{\infty}\, \dot{\epsilon}_L{}^{\infty})^{\frac{1}{2}}\, D\right\} R_0, \quad K, L = \text{I, II, III}. \tag{56}$$

Here D is given by (45) for a non-hardening material, and by the second term of (55) for the strong linear hardening. It is evidently influenced greatly by hardening. The shape changing part is not much influenced, with the 5/3 factor appropriate for strong hardening or very low triaxiality in a non-hardening material, and the 2 factor appropriate at higher triaxiality in the latter case. Thus the clarification of strain hardening effects on dilatational growth is an important area for further study. McCLINTOCK (1968) has suggested a simple empirical correction to the long cylinder results in terms of a hardening exponent. Also, TRACEY (1968) reports some results of detailed solutions for the same simple configuration in a companion paper. Our variational procedure could be applied for the more realistic spherical void model through a step by step procedure. In view of the relative insensitivity of the shape changing field to hardening, a simplification would result by letting the dilatational growth factor D be the only free parameter in the assumed velocity field, with an equation similar to (31) resulting at each step of the deformation.

To gain an appreciation for the numbers involved, consider contained non-hardening plastic deformation near a crack in plane strain (RICE, 1968). The mean normal stress directly ahead is $\sigma^{\infty} = (1 + \pi)\tau_0$, and since the state of deformation is pure shear, strain rates can be represented as $\dot{\epsilon}_{\text{I}}{}^{\infty} = \dot{\epsilon}$, $\dot{\epsilon}_{\text{II}}{}^{\infty} = 0$, $\dot{\epsilon}_{\text{III}}{}^{\infty} = -\dot{\epsilon}$ where $\dot{\epsilon}$ is the maximum extensional strain rate. Computing D from (45) and taking the factor of 2 for the shape changing field, velocities in principal directions on the void boundary are given by (56) as

$$\dot{R}_{0\text{I}} = 13{\cdot}6\, \dot{\epsilon} R_0, \quad \dot{R}_{0\text{II}} = 11{\cdot}6\, \dot{\epsilon} R_0, \quad \dot{R}_{0\text{III}} = 9{\cdot}6\, \dot{\epsilon} R_0. \tag{57}$$

The significance of the numerical factors becomes clearer after integration, which can be done so as to account for finite shape changes in an approximate way by identifying R_0 as the mean of the three principal radii. We note that ϵ is a 'true' strain, and if we let $\bar{\epsilon} = \exp(\epsilon) - 1$ be the associated 'engineering' strain, the results are

$$\frac{R_{0\text{I}}}{(R_0)_{\text{init.}}} = 1{\cdot}17\,(1 + \bar{\epsilon})^{11{\cdot}6} - 0{\cdot}17, \quad \frac{R_{0\text{II}}}{(R_0)_{\text{init.}}} = (1 + \bar{\epsilon})^{11{\cdot}6},$$

$$\frac{R_{0\text{III}}}{(R_0)_{\text{init.}}} = 0{\cdot}83\,(1 + \bar{\epsilon})^{11{\cdot}6} + 0{\cdot}17, \tag{58}$$

where $(R_0)_{\text{init.}}$ is the initial radius of the void. Thus, for a 10 per cent strain the three size ratios are 3·3, 3·0, and 2·7, whereas for a 50 per cent strain the ratios are 129, 110, and 91. These large numbers suggest that the 50 per cent dimension change, so readily achieved in a tension test of a ductile metal, would be unachievable over any reasonable size scale ahead of a crack, and that even the 10 per cent dimension change would be difficult to accommodate without large void spacings. Strain hardening no doubt reduces these ratios significantly for the same stress triaxiality, but the problem is further complicated by a very rapid increase of stress triaxiality ahead of a crack with increasing values of the hardening exponent (Rice and Rosengren, 1968).

Apart from hardening, the interaction and unstable coalescence of neighbouring voids are major features yet to be brought into the modeling of ductile fracture. While the isolated void analysis predicts ample growth for fracture with high stress triaxiality, quite the opposite is true in a simple tension stress field. Our results then indicate no transverse expansion, and McClintock (1968) found that actual tensile ductility, as recorded in extensive data on copper by Edelson and Baldwin (1962), was greatly overestimated by his long cylindrical void model even when stress triaxiality in necking was included. Tracey's (1968) study of the same model did, however, lead to a substantial reduction in predicted ductility with the approximate inclusion of interaction effects. We note that an analysis based on continuum models of void growth will necessarily lead to a tensile test ductility independent of the absolute void size or spacing, but dependent only on the volume fraction, as observed by Edelson and Baldwin. The same will not be true for fracture at a crack tip, or for other situations in which strain gradients over typical void spacings are large, so that size effects are to be expected.

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