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MULTIPLE SYMMETRIC POSITIVE SOLUTIONS FOR SYSTEMS OF HIGHER ORDER BOUNDARY-VALUE PROBLEMS ON TIME SCALES

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Abstract. In this article, we find multiple symmetric positive solutions for a system of higher order two-point boundary-value problems on time scales by determining growth conditions and applying a fixed point theorem in cones under suitable conditions.

1. Introduction

Symmetry creates beauty in nature and in nature every thing is almost symmetric. One can observe that symmetry in the structure of fruits, the structure of human body, the revolution of planets and the structure of atoms. Due to the importance of symmetric properties in both theory and applications, the study of existence of symmetric solutions of boundary value problems gained momentum.

In this paper, we address the question of the existence of at least three symmetric positive solutions for the system of dynamical equations on symmetric time scales,

$$
\begin{aligned}
(-1)^n y_1^{(\Delta\nabla)^n} &= f_1(t, y_1, y_2), \quad t \in [a,b]_{\mathbb{T}} \\
(-1)^m y_2^{(\Delta\nabla)^m} &= f_2(t, y_1, y_2), \quad t \in [a,b]_{\mathbb{T}}
\end{aligned}
\tag{1.1}
$$

subject to the two-point boundary conditions

$$
\begin{aligned}
y_1^{(\Delta\nabla)^i}(a) = 0 = y_1^{(\Delta\nabla)^i}(b), \quad i = 0,1,2,\dots,n-1, \\
y_2^{(\Delta\nabla)^j}(a) = 0 = y_2^{(\Delta\nabla)^j}(b), \quad j = 0,1,2,\dots,m-1,
\end{aligned}
\tag{1.2}
$$

where $f_i : [a,b]_{\mathbb{T}} \times \mathbb{R}^2 \to [0,\infty)$ are continuous and $f_i(t, y_1, y_2) = f_i(a+b-t, y_1, y_2)$ for $i = 1, 2$, $a \in \mathbb{T}_k$, $b \in \mathbb{T}^k$ for a time scale $\mathbb{T}$, and also $\sigma(a) < \rho(b)$.

By an interval time scale, we mean the intersection of a real interval with a given time scale; i.e.,

$$[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}.$$

For time scale calculus, we refer the reader to Bohner and Peterson [8, 9].

An interval time scale $\mathbb{T} = [a,b]_{\mathbb{T}}$ is said to be a symmetric time scale if $t \in \mathbb{T} \Leftrightarrow a+b-t \in \mathbb{T}$.

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If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}, (h > 0)$ then the symmetry definition is always satisfied. In addition to, the interval time scale $\mathbb{T} = [1, 2] \cup \{3, 4, 5\} \cup [6, 7] \cup \{8\} \cup [9, 10] \cup \{11, 12, 13\} \cup [14, 15]$ has the symmetrical property. But the time scale $\mathbb{T} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is not a symmetric time scale.

By a *symmetric solution* (y_1, y_2) of the system of boundary value problem (1.1)-(1.2), we mean (y_1, y_2) is a solution of (1.1)-(1.2) and satisfies

$$y_1(t) = y_1(b + a - t) \text{ and } y_2(t) = y_2(b + a - t), \quad t \in [a, b]_{\mathbb{T}}.$$

The development of the theory has gained attention by many researchers; To mention a few, we list some papers, Erbe and Wang [15], Eloe and Henderson [12, 13], Eloe, Henderson and Sheng [14], Henderson and Thompson [20], Avery and Henderson [4, 5, 6], Avery, Davis and Henderson [7], Davis and Henderson [10], Davis, Henderson and Wong [11], Anderson [2], Henderson and Wong [19], and Henderson, Murali and Prasad [18].

This article is organized as follows. In Section 2, we establish certain lemmas and inequalities on Green's function which are needed later. In Section 3, by using the cone theory techniques, we establish the existence of at least three symmetric positive solutions to (1.1)-(1.2). The main tool in this paper is an application of the Avery's generalization of the Leggett-Williams fixed point theorem for operator leaving a Banach space cone invariant.

2. Green's function and bounds

In this section, we construct the Green's function for the homogeneous SBVP corresponding to (1.1)-(1.2). We estimate bounds of the Green's function, and establish some lemmas, in which we prove some inequalities on the Green's function, which are needed in our main result.

Let us denote the Green's function of the problem

$$\begin{gathered} -y^{\Delta\nabla}(t) = 0, \quad t \in [a, b]_{\mathbb{T}}, \\ y(a) = 0 = y(b), \end{gathered}$$

as $G_1(t, s)$, and it is given by

$$G_1(t, s) = \begin{cases} \frac{(b-s)(t-a)}{(b-a)}, & t \le s \\ \frac{(b-t)(s-a)}{(b-a)}, & s \le t \end{cases}$$

for all $t, s \in [a, b]_{\mathbb{T}}$. Then, we can recursively define

$$G_j(t, s) = \int_a^b G_{j-1}(t, r) G_1(r, s) \nabla r, \quad \text{for all } t, s \in [a, b]_{\mathbb{T}}, \tag{2.1}$$

for $j = 2, 3, \dots, p$, and $p = \max\{m,\ n\}$, where $G_j(t, s)$ is the Green's function for the problem

$$\begin{gathered} (-1)^j y^{(\Delta\nabla)^j}(t) = 0, \quad t \in [a, b]_{\mathbb{T}}, \\ y^{(\Delta\nabla)^i}(a) = y^{(\Delta\nabla)^i}(b) = 0, \quad i = 0, 1, 2, \dots, j - 1, \end{gathered}$$

and $G_j(t, s) \ge 0$ for all $t, s \in [a, b]_{\mathbb{T}}$. For details we refer to [2, 18].

The following lemmas are needed to establish our main result.

Lemma 2.1. *Let $l \in [\frac{b-a}{8}, \frac{b-a}{2}]_{\mathbb{T}}$ and $(t,s) \in [a+l, b-l]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$,*

$$|G_j(t,s)| \geq L_l^j \phi_l^{j-1} \frac{(b-s)(s-a)}{b-a}, \quad \text{for } j = 1,2,\dots,p, \tag{2.2}$$

where p is maximum of $\{m,n\}$, $L_l = \frac{l}{b-a}$ and $\phi_l = \int_{a+l}^{b-l} \frac{(b-s)(s-a)}{b-a} \nabla s$.

Proof. For $j = 1$ the inequality (2.2) holds provided that $L_l = \frac{l}{b-a}$. Next for fixed j, assuming that (2.2) is true, from (2.1) we have for $(t,s) \in [a+l, b-l]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$,

$$\begin{aligned}
|G_{j+1}(t,s)| &= |\int_a^b G_j(t,r) G_1(r,s) \nabla r| \\
&\geq |\int_{a+l}^{b-l} G_j(t,r) G_1(r,s) \nabla r| \\
&\geq \int_{a+l}^{b-l} L_l^j \phi_l^{j-1} \frac{(b-r)(r-a)}{b-a} \times L_l \frac{(b-s)(s-a)}{b-a} \nabla r \\
&= L_l^{j+1} \phi_l^j \frac{(b-s)(s-a)}{b-a}.
\end{aligned}$$

Hence, by induction the result is true for all $j \leq p-1$. □

Lemma 2.2. *For $(t,s) \in [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$,*

$$|G_j(t,s)| \leq \phi_0^{j-1} \frac{(b-s)(s-a)}{b-a}, \quad \text{for } j = 1,2,\dots,p, \tag{2.3}$$

where $\phi_0 = \int_a^b \frac{(b-s)(s-a)}{b-a} \nabla s$.

Proof. For $j = 1$ the inequality (2.3) is obvious. Next for fixed j, assume that (2.3) is true, then from (2.1) we have

$$\begin{aligned}
|G_{j+1}(t,s)| &= |\int_a^b G_j(t,r) G_1(r,s) \nabla r| \\
&\leq \int_a^b \phi_0^{j-1} \frac{(b-r)(r-a)}{b-a} \times \frac{(b-s)(s-a)}{b-a} \nabla r \\
&= \phi_0^j \frac{(b-s)(s-a)}{b-a}.
\end{aligned}$$

Hence, by induction the result is true for all $j \leq p-1$. □

Lemma 2.3. *Let $t_k = \frac{b+a}{2}$ and $t_i \in [a, \frac{b+a}{2}]_{\mathbb{T}}$, $1 \leq i \leq 3$ with $t_1 \leq t_2$. For $s \in [a,b]_{\mathbb{T}}$,*

$$\frac{G_1(t_1,s)}{G_1(t_2,s)} \geq \frac{t_1-a}{t_2-a} \quad \text{and} \quad \frac{G_1(t_k,s)}{G_1(t_3,s)} \leq \frac{t_k-a}{t_3-a}.$$

Proof. For $t \leq s$, we have $\frac{G_1(t_1,s)}{G_1(t_2,s)} = \frac{t_1-a}{t_2-a}$. And for $s \leq t$, we have $\frac{G_1(t_1,s)}{G_1(t_2,s)} = \frac{b-t_1}{b-t_2}$. Since $t_1 \leq t_2$, we get $\frac{b-t_1}{b-t_2} \geq \frac{t_1-a}{t_2-a}$.

Similarly, for $t \leq s$, we have $\frac{G_1(t_k,s)}{G_1(t_3,s)} = \frac{t_k-a}{t_3-a}$. And for $s \leq t$, we have $\frac{G_1(t_k,s)}{G_1(t_3,s)} = \frac{b-t_k}{b-t_3}$. Since $t_1 \leq t_2$, we get $\frac{b-t_k}{b-t_3} \leq \frac{t_k-a}{t_3-a}$. □

Lemma 2.4. *For $t,s \in [a,b]_{\mathbb{T}}$, the Green's function $G_j(t,s)$ satisfies the symmetric property,*

$$G_j(t,s) = G_j(b+a-t,\ b+a-s), \quad \text{for } j = 1,2,\dots,p. \tag{2.4}$$

Proof. By the definition of $G_j(t,s)$, $(j=2,3,\dots,p-1)$,

$$G_j(t,s)=\int_a^b G_{j-1}(t,r)G_1(r,s)\nabla r,\quad \text{for all } t,s\in[a,b]_{\mathbb{T}}.$$

Clearly, $G_1(t,s)=G_1(a+b-t,a+b-s)$. Now, the proof is by induction. For $j=2$ the inequality (2.4) is obvious. Next, assume that (2.4) is true, for fixed j $(j=1,2,\dots,p-1)$, then from (2.1) we have

$$\begin{aligned}
G_{j+1}(t,s)&=\int_a^b G_j(t,r)G_1(r,s)\nabla r\\
&=\int_a^b G_j(a+b-t,a+b-r)G_1(a+b-r,a+b-s)\nabla r\\
&=\int_a^b G_j(a+b-t,r_1)G_1(r_1,a+b-s)\nabla r_1\\
&=G_{j+1}(a+b-t,a+b-s),
\end{aligned}$$

by using a transformation $r_1=a+b-r$. $\square$

Let $D=\{v \mid v:[a,b]_{\mathbb{T}}\to\mathbb{R}\text{is continuous function}\}$. We define the operator $F_j:D\to D$ by

$$(F_jv)(t)=\int_a^b G_j(t,s)v(s)\nabla s,\quad t\in[a,b]_{\mathbb{T}},\ \text{for } j=1,2,\dots,p-1.$$

By the construction of F_j and properties of $G_j(t,s)$, it is clear that

$$(-1)^j(F_jv)^{(\Delta\nabla)^j}(t)=v(t),\quad t\in[a,b]_{\mathbb{T}},$$
$$(F_jv)^{(\Delta\nabla)^i}(a)=(F_jv)^{(\Delta\nabla)^i}(b)=0,\quad i=0,1,\dots,j-1.$$

Lemma 2.5. *For $t\in[a,b]_{\mathbb{T}}$, the operator F_j satisfies the symmetric property*

$$F_jy(t)=F_jy(b+a-t)\quad \textit{for } j=1,2,\dots,p-1.$$

Proof. By definition of F_j, and using the transformation $s_1=b+a-s$,

$$\begin{aligned}
F_jy(t)&=\int_a^b G_j(t,s)v(s)\nabla s\\
&=\int_a^b G_j(a+b-t,a+b-s)v(s)\nabla s\\
&=\int_a^b G_j(a+b-t,s_1)v(s_1)\nabla s_1\\
&=F_jy(b+a-t),
\end{aligned}$$

from Lemma 2.4). $\square$

By using the above transformations and lemmas, we can reduce the SBVP (1.1),(1.2) into SBVP (2.5)-(2.6) and vice-versa.

Hence, we see that SBVP (1.1)-(1.2) has a solution if and only if the following problem has a solution:

$$\begin{gathered}
v_1^{\Delta\nabla}+f_1(t,F_{n-1}v_1,F_{m-1}v_2)=0,\quad t\in[a,b]_{\mathbb{T}}\\
v_2^{\Delta\nabla}+f_2(t,F_{n-1}v_1,F_{m-1}v_2)=0,\quad t\in[a,b]_{\mathbb{T}},
\end{gathered}\tag{2.5}$$

with boundary conditions

$$v_1(a) = 0 = v_1(b), \quad {}_2(a) = 0 = v_2(b). \tag{2.6}$$

Indeed, if (y_1, y_2) is a solution of (1.1)-(1.2), then $(v_1 = y_1^{(\Delta\nabla)^{(n-1)}}, v_2 = y_2^{(\Delta\nabla)^{(m-1)}})$ is a solution of (2.5)-(2.6). Conversely, if (v_1, v_2) is a solution of (2.5)-(2.6), then $(y_1 = F_{n-1}v_1, y_2 = F_{m-1}v_2)$ is a solution of (1.1)-(1.2). In fact, we have the representation

$$y_1(t) = \int_a^b G_{n-1}(t,s)v_1(s)\nabla s, \quad y_2(t) = \int_a^b G_{m-1}(t,s)v_2(s)\nabla s,$$

where

$$\begin{aligned} v_1(s) &= \int_a^b G_1(s,\tau)f_1(\tau, F_{n-1}v_1, F_{m-1}v_2)\nabla\tau, \\ v_2(s) &= \int_a^b G_1(s,\tau)f_2(\tau, F_{n-1}v_1, F_{m-1}v_2)\nabla\tau. \end{aligned}$$

It is also noted that a solution (v_1, v_2) of (2.5)-(2.6) is symmetric; i. e.,

$$v_1(t) = v_1(b + a - t) \quad \text{and} \quad v_2(t) = v_2(b + a - t), \quad t \in [a, b]_{\mathbb{T}},$$

and it gives rise to a symmetric solution (y_1, y_2) of (1.1)-(1.2).

3. Existence of multiple symmetric positive solutions

In this section, we establish the existence of at least three symmetric positive solutions for (1.1)-(1.2), by using Avery's generalization of the Leggett-Williams fixed point theorem. Let B be a real Banach space with cone P. We consider the nonnegative continuous convex functionals γ, β, θ and nonnegative continuous concave functionals α, ψ on P, for nonnegative numbers a', b', c', d' and h', we define the following sets

$$\begin{aligned} P(\gamma, c') &= \{y \in P : \gamma(y) < c'\}, \\ P(\gamma, \alpha, a', c') &= \{y \in P : a' \le \alpha(y),\ \gamma(y) \le c'\}, \\ Q(\gamma, \beta, d', c') &= \{y \in P : \beta(y) \le d',\ \gamma(y) \le c'\}, \\ P(\gamma, \theta, \alpha, a', b', c') &= \{y \in P : a' \le \alpha(y),\ \theta(y) \le b',\ \gamma(y) \le c'\}, \\ Q(\gamma, \beta, \psi, h', d', c') &= \{y \in P | h' \le \psi(y),\ \beta(y) \le d',\ \gamma(y) \le c'\}. \end{aligned}$$

For obtaining multiple symmetric positive solutions of (1.1)-(1.2), we state the following fundamental theorem the so called Five Functionals Fixed Point Theorem [3].

Theorem 3.1. *Let P be a cone in a real Banach space E. Suppose α and ψ are nonnegative continuous concave functionals on P and γ, β and θ are nonnegative continuous convex functionals on P such that, for some positive numbers c' and g',*

$$\alpha(y) \le \beta(y) \quad \text{and} \quad \|y\| \le g'\gamma(y) \quad \text{for all } y \in \overline{P(\gamma, c')}.$$

Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants $h', d', a', b' \ge 0$ with $0 < d' < a'$ such that each of the following is satisfied.

- (B1) $\{y \in P(\gamma, \theta, \alpha, a', b', c') | \alpha(y) > a'\} \neq \emptyset$ *and* $\alpha(Ty) > a'$ *for* $y \in P(\gamma, \theta, \alpha, a', b', c')$, (B2) $\{y \in Q(\gamma, \beta, \psi, h', d', c') | \beta(y) < d'\} \neq \emptyset$ *and* $\beta(Ty) < d'$ *for* $y \in Q(\gamma, \beta, \psi, h', d', c')$,

(B3) $\alpha(Ty) > a'$ *provided* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$,

(B4) $\beta(Ty) < d'$ *provided* $y \in Q(\gamma, \beta, d', c')$ *with* $\psi(Ty) < h'$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that $\beta(y_1) < d'$, $a < \alpha(y_2)$, and $d' < \beta(y_3)$ with $\alpha(y_3) < a'$.

To apply the fixed point theorem for our problem we need the space

$$C_0 = \{(v_1, v_2) | v_1, v_2 : [a, b]_{\mathbb{T}} \to \mathbb{R} \text{ are continuous functions } \}$$

equipped with the norm

$$\|(v_1, v_2)\| = \|v_1\|_0 + \|v_2\|_0$$

where $\|v\|_0 = \max_{t \in [a,b]_{\mathbb{T}}} |v(t)|$. For a fixed $k_0 \in [\frac{b-a}{8}, \frac{b-a}{2}]_{\mathbb{T}}$, define the cone $P \subset C_0$ by

$$\begin{aligned} P = \{(v_1, v_2) \in C_0 | v_1(t), v_2(t) &\text{ are nonnegative convex symmetric functions for} \\ t \in [a, b]_{\mathbb{T}} \text{ and } &\min_{t \in [a+k_0, b-k_0]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) \geq \frac{k_0}{t_k - a} \|(v_1, v_2)\|\}, \end{aligned}$$

where $t_k = \frac{b+a}{2}$. Let $k_i \in [\frac{b-a}{8}, \frac{b-a}{2}]_{\mathbb{T}}$, $1 \leq i \leq 3$, be fixed and $k_1 < k_2$. Also let $t_i = a + k_i$, $0 \leq i \leq 3$. Clearly, $t_1 < t_2$ and $t_i \leq t_k$, $i = 0, 1, 2, 3$. Define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P by

$$\begin{aligned} \gamma(v_1, v_2) &= \max_{t \in [a, a+k_0]_{\mathbb{T}} \cup [b-k_0, b]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_0)| + |v_2(t_0)|, \\ \psi(v_1, v_2) &= \min_{t \in [a+k_3, b-k_3]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_3)| + |v_2(t_3)|, \\ \beta(v_1, v_2) &= \max_{t \in [a+k_3, b-k_3]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_k)| + |v_2(t_k)|, \\ \alpha(v_1, v_2) &= \min_{t \in [a+k_1, a+k_2]_{\mathbb{T}} \cup [b-k_2, b-k_1]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_1)| + |v_2(t_1)|, \\ \theta(v_1, v_2) &= \max_{t \in [a+k_1, a+k_2]_{\mathbb{T}} \cup [b-k_2, b-k_1]_{\mathbb{T}}} (|v_1(t)| + |v_2(t)|) = |v_1(t_2)| + |v_2(t_2)|. \end{aligned}$$

We observe that for any $(v_1, v_2) \in P$,

$$\alpha(v_1, v_2) = |v_1(t_1)| + |v_2(t_1)| \leq |v_1(t_k)| + |v_2(t_k)| = \beta(v_1, v_2), \tag{3.1}$$

and

$$\|(v_1, v_2)\| = |v_1(t_k)| + |v_2(t_k)| \leq \frac{t_k - a}{t_0 - a} (|v_1(t_0)| + |v_2(t_0)|) = \frac{t_k - a}{t_0 - a} \gamma(v_1, v_2). \tag{3.2}$$

Let us denote

$$\overline{\phi_z} = \phi_0 - \phi_z = \int_{s \in [a,b]_{\mathbb{T}} \setminus [a+z, b-z]_{\mathbb{T}}} \frac{(b-s)(s-a)}{b-a} \nabla s.$$

We are now ready to present the main theorem of the paper.

Theorem 3.2. *Suppose there exist $0 < a' < b' < \frac{(t_2 - a)}{t_1 - a} b' \leq c'$ such that f_1 and f_2 satisfy the following conditions:*

(A1) $|f_i(t,u_{n-1},w_{m-1})| < (\frac{a'-c'(k_3^2-k_3)[(t_0-a)(b-t_0)]^{-1}}{(t_k-a)(b-t_k)+k_3-k_3^2})$ *for all* $(t,|u_{n-1}|,|w_{m-1}|)$ *in*

$$[a,b]_{\mathbb{T}}\times[\frac{a'(t_3-a)}{t_k-a}L_{k_3+1}^{n-1}\phi_{k_3+1}^{n-1},a'\phi_0^{n-2}\phi_{k_3+1}+\frac{c'(t_k-a)}{t_0-a}\phi_0^{n-2}\overline{\phi}_{k_3+1}]$$
$$\times[\frac{a'(t_3-a)}{t_k-a}L_{k_3+1}^{m-1}\phi_{k_3+1}^{m-1},a'\phi_0^{m-2}\phi_{k_3+1}+\frac{c'(t_k-a)}{t_0-a}\phi_0^{m-2}\overline{\phi}_{k_3+1}],\quad i=1,2.$$

(A2) $|f_i(t,u_{n-1},w_{m-1})| > \frac{b'}{k_1(k_2+1-k_1)}$ *for all* $(t,|u_{n-1}|,|w_{m-1}|)$ *in*

$$[a+l,b-l]_{\mathbb{T}}\times[b'L_{k_1+1}^{n-1}\phi_{k_1+1}^{n-2}(\phi_{k_1+1}-\phi_{k_2+2}),\frac{b'(t_2-a)}{t_1-a}\phi_0^{n-2}(\phi_{k_1+1}-\phi_{k_2+2})$$
$$+\frac{c'(t_k-a)}{t_0-a}\phi_0^{n-2}(\overline{\phi}_{k_1+1}+\phi_{k_2+2})]\times[b'L_{k_1+1}^{m-1}\phi_{k_1+1}^{m-2}(\phi_{k_1+1}-\phi_{k_2+2}),$$
$$\frac{b'(t_2-a)}{t_1-a}\phi_0^{m-2}(\phi_{k_1+1}-\phi_{k_2+2})+\frac{c'(t_k-a)}{t_0-a}\phi_0^{m-2}(\overline{\phi}_{k_1+1}+\phi_{k_2+2})],$$

either $i=1$ *or* $i=2$.

(A3) $|f_i(t,u_{n-1},w_{m-1})| < \frac{c'}{(t_0-a)(b-t_0)}$ *for all* $(t,|u_{n-1}|,|w_{m-1}|)$ *in*

$$[a,b]_{\mathbb{T}}\times[0,\frac{c'(t_k-a)}{t_0-a}\phi_0^{n-1}]\times[0,\frac{c'(t_k-a)}{t_0-a}\phi_0^{m-1}],\quad i=1,2.$$

Then (1.1)-(1.2) *has at least three symmetric positive solutions.*

Proof. Define a completely continuous operator $T:C_0\to C_0$ by
$$T(v_1,v_2):=(T_1(v_1,v_2),T_2(v_1,v_2)),\tag{3.3}$$
where
$$T_i(v_1,v_2):=\int_a^b G_1(t,s)f_i(s,F_{n-1}v_1,F_{m-1}v_2)\nabla s,\quad \text{for } i=1,2.$$
It is obvious that a fixed point of T is a solution of (2.5)-(2.6). We seek three fixed points $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in P$ of T. First, we show that T is self map on P. Let $(v_1,v_2)\in P$, then $T_1(v_1,v_2)(t)\ge 0$, $T_2(v_1,v_2)(t)\ge 0$ for $t\in[a,b]_{\mathbb{T}}$, and $T_1^{\Delta\nabla}(v_1,v_2)(t)\le 0$, $T_2^{\Delta\nabla}(v_1,v_2)(t)\le 0$ for $t\in[a,b]_{\mathbb{T}}$. Further $G_1(t,s)$ is symmetric, it follows that $T_1(v_1,v_2)(t)=T_1(v_1,v_2)(b+a-t)$, $T_2(v_1,v_2)(t)=T_2(v_1,v_2)(b+a-t)$, for $t\in[a,b]_{\mathbb{T}}$. Also, noting that $T_1(v_1,v_2)(a)=0=T_1(v_1,v_2)(b)$, $T_2(v_1,v_2)(a)=0=T_2(v_1,v_2)(b)$ and $\|T(v_1,v_2)\|=|T_1(v_1,v_2)(t_k)|+|T_2(v_1,v_2)(t_k)|$, we have
$$\begin{aligned}
&\min_{t\in[a+k_0,b-k_0]_{\mathbb{T}}}(|T_1(v_1,v_2)(t)|+|T_2(v_1,v_2)(t)|)\\
&=\min_{t\in[a+k_0,t_k]_{\mathbb{T}}}(|T_1(v_1,v_2)(t)|+|T_2(v_1,v_2)(t)|)\\
&\ge\min_{t\in[a+k_0,t_k]_{\mathbb{T}}}\frac{t-a}{t_k-a}\|T(v_1,v_2)\|\\
&=\frac{k_0}{t_k-a}\|T(v_1,v_2)\|.
\end{aligned}$$
Thus $T:P\to P$. Next, for all $(v_1,v_2)\in P$, and using (3.1),(3.2), $\alpha(v_1,v_2)\le\beta(v_1,v_2)$ and $\|(v_1,v_2)\|\le\frac{t_k-a}{t_0-a}\gamma(v_1,v_2)$. To show that $T:\overline{P(\gamma,c')}\to\overline{P(\gamma,c')}$,

let $(v_1, v_2) \in \overline{P(\gamma, c')}$ and hence $\|(v_1, v_2)\| \le \frac{t_k-a}{t_0-a}c'$. Using Lemma 2.2 and for $t \in [a, b]_{\mathbb{T}}$,

$$\begin{aligned}
|F_{n-1}v_1(t)| &= |\int_a^b G_{n-1}(t,s)v_1(s)\nabla s| \\
&\le \frac{c'(t_k-a)}{t_0-a}\int_a^b |G_{n-1}(t,s)|\nabla s \\
&\le \frac{c'(t_k-a)}{t_0-a}\phi_0^{n-2}\int_a^b \frac{(b-s)(s-a)}{(b-a)}\nabla s \\
&= \frac{c'(t_k-a)}{t_0-a}\phi_0^{n-1}.
\end{aligned}$$

Similarly, for $t \in [a, b]_{\mathbb{T}}$, we have

$$|F_{m-1}v_2(t)| \le \frac{c'(t_k-a)}{t_0-a}\phi_0^{m-1}.$$

By condition (A3),

$$\gamma(T_1(v_1, v_2), T_2(v_1, v_2)) = |T_1(v_1, v_2)(t_0)| + |T_2(v_1, v_2)(t_0)|.$$

and

$$\begin{aligned}
|T_1(v_1, v_2)(t_0)| &= |\int_a^b G_1(t_0,s)f_1(s, F_{n-1}v_1, F_{m-1}v_2)\nabla s| \\
&< \frac{c'}{(t_0-a)(b-t_0)}\int_a^b |G_1(t_0,s)|\nabla s = \frac{c'}{2}.
\end{aligned}$$

Similarly, $|T_2(v_1, v_2)(t_0)| < c'/2$, and hence $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. It is obvious that

$$\{(v_1, v_2) \in P(\gamma, \theta, \alpha, b', \frac{b'(t_2-a)}{t_1-a}, c') | \alpha(v_1, v_2) > b'\} \ne \emptyset.$$

Next, let $(v_1, v_2) \in P(\gamma, \theta, \alpha, b', \frac{b'(t_2-a)}{t_1-a}, c')$, denote the set $D_1 = [a + k_1, a + k_2]_{\mathbb{T}} \cup [b - k_2, \sigma(b) - k_1]_{\mathbb{T}}$. It follows that

$$|v_1(s)|, |v_2(s)| \in [b', \frac{b'(t_2-a)}{t_1-a}], \quad s \in D_1, \tag{3.4}$$

$$|v_1(s)|, |v_2(s)| \in [0, \frac{c'(t_k-a)}{t_0-a}], \quad s \in [a, b]_{\mathbb{T}} \setminus D_1. \tag{3.5}$$

Using (3.4), (3.5), Lemma 2.1 Lemma 2.2,

$$|F_{n-1}v_1(s)| = |\int_{\tau\in D_1} G_{n-1}(s,\tau)v_1(\tau)\nabla\tau + \int_{\tau\in[a,b]_{\mathbb{T}}\setminus D_1} G_{n-1}(s,\tau)v_1(\tau)\nabla\tau|$$

and

$$|F_{m-1}v_2(s)| = |\int_{\tau\in D_1} G_{m-1}(s,\tau)v_2(\tau)\nabla\tau + \int_{\tau\in[a,b]_{\mathbb{T}}\setminus D_1} |G_{m-1}(s,\tau)v_2(\tau)\nabla\tau|,$$

for $s \in D_1$, we have

$$\begin{aligned}
&|F_{n-1}v_1(s)| \\
&\le \frac{b'(t_2-a)}{t_1-a}\int_{\tau\in D_1} |G_{n-1}(s,\tau)|\nabla\tau + \frac{c'(t_k-a)}{t_0-a}\int_{\tau\in[a,b]_{\mathbb{T}}\setminus D_1} |G_{n-1}(s,\tau)|\nabla\tau,
\end{aligned}$$

$$\le \frac{b'(t_2-a)}{t_1-a}\phi_0^{n-2}(\phi_{k_1+1}-\phi_{k_2+2})+\frac{c'(t_k-a)}{t_0-a}\phi_0^{n-2}(\overline{\phi}_{k_1+1}+\phi_{k_2+2})$$

and

$$\begin{aligned}|F_{n-1}v_1(s)| &\ge \int_{\tau\in D_1}|G_{n-1}(s,\tau)v_1(\tau)|\nabla\tau\\ &\ge b'\int_{\tau\in D_1}|G_{n-1}(s,\tau)|\nabla\tau\\ &\ge b'L_{k_1+1}^{n-2}\phi_{k_1+1}^{n-2}(\phi_{k_1+1}-\phi_{k_2+2}).\end{aligned}$$

Similarly,

$$|F_{m-1}v_2(s)|\le \frac{b'(t_2-a)}{t_1-a}\phi_0^{m-2}(\phi_{k_1+1}-\phi_{k_2+2})+\frac{c'(t_k-a)}{t_0-a}\phi_0^{m-2}(\overline{\phi}_{k_1+1}+\phi_{k_2+2})$$

and

$$|F_{m-1}v_2(s)|\ge b'L_{k_1+1}^{m-2}\phi_{k_1+1}^{m-2}(\phi_{k_1+1}-\phi_{k_2+2}),\ for\ s\in D_1.$$

Applying (A2) we obtain

$$\begin{aligned}\alpha(T_1(v_1,v_2),&T_2(v_1,v_2))=|T_1(v_1,v_2)(t_1)|+|T_2(v_1,v_2)(t_1)|\ge|T_1(v_1,v_2)(t_1)|\\ &=|\int_a^b G_1(t_1,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)\nabla s|\\ &\ge\int_{s\in D_1}|G_1(t_1,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s\\ &>\frac{b'}{k_1(k_2+1-k_1)}\int_{s\in D_1}|G_1(t_1,s)|\nabla s=b'.\end{aligned}$$

Similarly, $\alpha(T_1(v_1,v_2),T_2(v_1,v_2))\ge|T_2(v_1,v_2)(t_1)|$ and from (A2) we have

$$\alpha(T_1(v_1,v_2),T_2(v_1,v_2))\ge b'.$$

Clearly,

$$\{(v_1,v_2)\in Q(\gamma,\beta,\psi,\frac{a'(t_3-a)}{t_k-a},a',c')|\beta(v_1,v_2)<a'\}\ne\emptyset.$$

Let $(v_1,v_2)\in Q(\gamma,\beta,\psi,\frac{a'(t_3-a)}{t_k-a},a',c')$, and define the set $E_1=[a+k_3,b-k_3]_{\mathbb{T}}$, then

$$|v_1(s)|,|v_2(s)|\in[\frac{a'(t_3-a)}{t_k-a},a'],\quad s\in E_1,\tag{3.6}$$

$$|v_1(s)|,|v_2(s)|\in[0,\frac{c'(t_k-a)}{t_0-a}],\ s\in[a,b]_{\mathbb{T}}\setminus E_1.\tag{3.7}$$

Then

$$\begin{aligned}|F_{n-1}v_1(s)|&=|\int_{\tau\in E_1}G_{n-1}(s,\tau)v_1(\tau)\nabla\tau+\int_{\tau\in[a,b]_{\mathbb{T}}\setminus E_1}G_{n-1}(s,\tau)v_1(\tau)\nabla\tau|,\\ |F_{m-1}v_2(s)|&=|\int_{\tau\in E_1}G_{m-1}(s,\tau)v_2(\tau)\nabla\tau+\int_{\tau\in[a,b]\setminus E_1}G_{m-1}(s,\tau)v_2(\tau)\nabla\tau|.\end{aligned}$$

Also using (3.6),(3.7), Lemma 2.1, and Lemma 2.2, we see that for $s\in E_1$,

$$|F_{n-1}v_1(s)|\le a'\int_{\tau\in E_1}|G_{n-1}(s,\tau)|\nabla\tau+\frac{c'(t_k-a)}{t_0-a}\int_{\tau\in[a,b]_{\mathbb{T}}\setminus E_1}|G_{n-1}(s,\tau)|\nabla\tau,$$

$$\le a'\phi_0^{n-2}\phi_{k_3+1} + \frac{c'(t_k-a)}{t_0-a}\phi_0^{n-2}\overline{\phi}_{k_3+1}$$

and

$$\begin{aligned}
|F_{n-1}v_1(s)| &\ge \int_{\tau\in E_1} |G_{n-1}(s,\tau)v(\tau)|\nabla\tau \\
&\ge \frac{a'(t_3-a)}{t_k-a}\int_{\tau\in E_1}|G_{n-1}(s,\tau)|\nabla\tau \\
&\ge \frac{a'(t_3-a)}{t_k-a}L_{k_3+1}^{n-2}\phi_{k_3+1}^{n-1}.
\end{aligned}$$

Similarly, for $s\in E_1$, we obtain

$$|F_{n-1}v_2(s)| \le a'\phi_0^{n-2}\phi_{k_3+1} + \frac{c'(t_k-a)}{t_0-a}\phi_0^{n-2}\overline{\phi}_{k_3+1},$$

$$|F_{n-1}v_2(s)| \ge \frac{a'(t_3-a)}{t_k-a}L_{k_3+1}^{n-2}\phi_{k_3+1}^{n-1}.$$

Thus, by (A1) and (A2), we obtain

$$\begin{aligned}
&\beta((T_1(v_1,v_2),T_2(v_1,v_2)) \\
&= |T_1(v_1,v_2)(t_k)| + |T_2(v_1,v_2)(t_k)| \\
&= \int_a^b |G_1(t_k,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s \\
&\quad + \int_a^b |G_1(t_k,s)f_2(s,F_{n-1}v_1,F_{m-1}v_2)\nabla s \\
&= \int_{s\in E_1} |G_1(t_k,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s \\
&\quad + \int_{s\in[a,b]_{\mathbb{T}}\setminus E_1} |G_1(t_k,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s \\
&\quad + \int_{s\in E_1} |G_1(t_k,s)f_2(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s \\
&\quad + \int_{s\in[a,b]_{\mathbb{T}}\setminus E_1} |G_1(t_k,s)f_2(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s \\
&< 2[a' - \frac{c'(k_3^2-k_3)}{(t_0-a)(b-t_0)}][(t_k-a)(b-t_k)+k_3-k_3^2]^{-1} \\
&\quad \times \int_{s\in E_1}|G_1(t_k,s)|\nabla s + \frac{2c'}{(t_0-a)(b-t_0)}\int_{s\in[a,b]_{\mathbb{T}}\setminus E_1}|G_1(t_k,s)|\nabla s = a'.
\end{aligned}$$

Let $(v_1,v_2)\in P(\gamma,\alpha,b',c')$ with $\theta(T_1(v_1,v_2),T_2(v_1,v_2)) > \frac{b'(t_2-a)}{t_1-a}$. Using Lemma 2.3, we obtain

$$\begin{aligned}
&\alpha(T_1(v_1,v_2),T_2(v_1,v_2)) \\
&= |T_1(v_1,v_2)(t_1)| + |T_2(v_1,v_2)(t_1)| \\
&= \int_a^b |\frac{G_1(t_1,s)}{G_1(t_2,s)}G_1(t_2,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s \\
&\quad + \int_a^b |\frac{G_1(t_1,s)}{G_1(t_2,s)}G_1(t_2,s)f_2(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{t_1-a}{t_2-a}\int_a^b |G_1(t_2,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s\\
&\quad+\frac{t_1-a}{t_2-a}\int_a^b |G_1(t_2,s)f_2(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s\\
&=\frac{t_1-a}{t_2-a}\theta(T_1(v_1,v_2),T_2(v_1,v_2))>b'.
\end{aligned}
$$

Finally, we show that (B4) holds. Let $(v_1,v_2)\in Q(\gamma,\beta,a',c')$ with

$$
\psi(T_1(v_1,v_2),T_2(v_1,v_2))<\frac{a'(t_3-a)}{t_k-a}.
$$

In view of Lemma 2.3, we have

$$
\begin{aligned}
\beta(T_1(v_1,v_2),T_2(v_1,v_2)) &= |T_1(v_1,v_2)(t_k)|+|T_2(v_1,v_2)(t_k)|\\
&=\int_a^b |\frac{G_1(t_k,s)}{G_1(t_3,s)}G_1(t_3,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s\\
&\quad+\int_a^b |\frac{G_1(t_k,s)}{G_1(t_3,s)}G_1(t_3,s)f_2(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s\\
&\leq \frac{t_k-a}{t_3-a}\int_a^b |G_1(t_3,s)f_1(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s\\
&\quad+\frac{t_k-a}{t_3-a}\int_a^b |G_1(t_3,s)f_2(s,F_{n-1}v_1,F_{m-1}v_2)|\nabla s\\
&=\frac{t_k-a}{t_3-a}\psi(T_1(v_1,v_2),T_2(v_1,v_2))<a'.
\end{aligned}
$$

We have thus proved that all the conditions of Theorem 3.1 are satisfied and so there exist at least three symmetric positive solutions for (1.1)-(1.2). □

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