Representations of Lorentz and Poincaré groups

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I. LORENTZ GROUP

We consider first the Lorentz group $\mathrm{O}(1,3)$ with infinitesimal generators $J^{\mu\nu}$ and the associated Lie algebra given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \tag{1}$$

In quantum field theory, we actually consider a subgroup of $\mathrm{O}(1,3)$, the proper orthochronous or restricted Lorentz group $\mathrm{SO}^+(1,3) = \{\Lambda \in \mathrm{O}(1,3) | \det\Lambda = 1 \text{ and } \Lambda^0{}_0 \geq 0\}$, which excludes parity and time-reversal transformations that are thus considered as separate, discrete operations P and T. A generic element Λ of the Lorentz group is given by exponentiating the generators together with the parameters[1] of the transformation, $\Lambda = \exp(-i\omega_{\mu\nu}J^{\mu\nu}/2)$.

The Lorentz group has both finite-dimensional and infinite-dimensional representations. However, it is non-compact, therefore its finite-dimensional representations are not unitary (the generators are not Hermitian). The generators of the infinite-dimensional representations can be chosen to be Hermitian.

A. Finite-dimensional representations

We first study the finite-dimensional representations of $\mathrm{SO}^+(1,3)$. These representations act on finite-dimensional vector spaces (the base space). Elements of these vector spaces are said to transform according to the given representation.

Trivial representation. In the trivial representation, we have the one-dimensional representation $J^{\mu\nu} = 0$. Hence any Lorentz transformation Λ is represented by 1. This representation acts on a one-dimensional vector space whose elements are 1-component objects called Lorentz scalars. One can thus say that the trivial representation implements a Lorentz transformation Λ on a scalar ϕ by the rule $\phi \xrightarrow{\Lambda} 1 \cdot \phi = \phi$. The trivial representation is denoted by $(0,0)$.

Vector representation. In the vector or 4-vector representation, each generator $J^{\mu\nu}$ is represented by a 4×4 matrix $(\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma}$, which acts on a four-dimensional vector space whose elements are 4-component objects called Lorentz four-vectors. The vector representation is thus four-dimensional. The explicit form of the matrices is $(\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}{}_{\sigma} - g^{\nu\rho}\delta^{\mu}{}_{\sigma})$. A Lorentz transformation Λ is now implemented on a 4-vector V^{ρ} by the rule $V^{\rho} \xrightarrow{\Lambda} (e^{-i\omega_{\mu\nu}\mathcal{J}^{\mu\nu}/2})^{\rho}{}_{\sigma}V^{\sigma}$ where the $\mathcal{J}^{\mu\nu}$ in the argument of the exponential is now a matrix, so that we exponentiate a matrix to get another matrix which multiplies the 4-vector to give the transformed 4-vector. Since the elements of the Lorentz group $\mathrm{SO}^{+}(1,3)$ are actually matrices[2], it is easily seen that the matrices of the vector representation are nothing but the $\mathrm{SO}^{+}(1,3)$ matrices themselves, i.e. $(e^{-i\omega_{\mu\nu}\mathcal{J}^{\mu\nu}/2})^{\rho}{}_{\sigma} = \Lambda^{\rho}{}_{\sigma}$ where $\Lambda^{\rho}{}_{\sigma}$ is the usual Lorentz transformation matrix. A representation of a matrix group which is given by the elements (matrices) of the group itself is called the *fundamental representation*. Hence the vector representation is the fundamental representation of the Lorentz group. The vector representation is denoted by $(\frac{1}{2},\frac{1}{2})$.

Tensor representations. Tensor representations are given by the direct (tensor) product of copies of the vector representation. They act on the set of tensors of a given rank, which is indeed a linear vector space. These tensors are called Lorentz 4-tensors. For example, consider $(2,0)$ tensors, that is, tensors with two contravariant (upper) indices $T^{\rho\sigma}$. An element of the Lorentz group Λ will be represented by a 16×16 matrix $\Lambda^{\rho\sigma}{}_{\rho'\sigma'} \equiv \Lambda^{\rho}{}_{\rho'}\Lambda^{\sigma}{}_{\sigma'}$, which is clearly seen to be the direct product of two 4×4 matrices of the vector representation. As a result, a tensor $T^{\rho\sigma}$ will transform as $T^{\rho\sigma} \xrightarrow{\Lambda} \Lambda^{\rho\sigma}{}_{\rho'\sigma'}T^{\rho'\sigma'} = \Lambda^{\rho}{}_{\rho'}\Lambda^{\sigma}{}_{\sigma'}T^{\rho'\sigma'}$, which reproduces the usual Lorentz transformation law for a tensor. Representations for higher-rank tensors are constructed in the same way, with additional copies of the vector representation in the direct product.

Adjoint representation. The Lie algebra of Eq. (1) can be written as $[J^{\mu\nu}, J^{\rho\sigma}] = if^{\mu\nu\rho\sigma}{}_{\alpha\beta}J^{\alpha\beta}$ where $f^{\mu\nu\rho\sigma}{}_{\alpha\beta}$ are the structure constants. If we define 16×16 matrices $(J^{\mu\nu})^{\rho\sigma}{}_{\alpha\beta} \equiv -if^{\mu\nu\rho\sigma}{}_{\alpha\beta}$, it is possible to show from the Jacobi identity $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ satisfied by any matrices A, B, C that the matrices $(J^{\mu\nu})^{\rho\sigma}{}_{\alpha\beta}$ satisfy the Lorentz algebra. We have thus constructed a representation of the generators of the Lorentz group from the structure constants of the group: this is called the *adjoint representation*. Its dimension is the number of generators of the group. However, since the generators $J^{\mu\nu}$ of the Lorentz group are antisymmetric $J^{\mu\nu} = -J^{\nu\mu}$, there are actually only

$4 \cdot (4-1)/2 = 6$ independent[3] generators (corresponding to the 3 rotations and 3 boosts) so that the adjoint representation of the Lorentz group is six-dimensional. In other words, we could write the Lie algebra as $[\tilde{J}^a, \tilde{J}^b] = i f^{ab}{}_c \tilde{J}^c$ where $a, b, c = 1, \ldots, 6$, by defining $(\tilde{J}^1, \tilde{J}^2, \tilde{J}^3) = \mathbf{J}$ where $J^i \equiv \frac{1}{2}\epsilon^{ijk} J^{jk}$ are the generators of rotations, and $(\tilde{J}^4, \tilde{J}^5, \tilde{J}^6) = \mathbf{K}$ where $K^i \equiv J^{0i}$ are the generators of boosts. With this notation, the adjoint representation is composed of 6×6 matrices $(\tilde{J}^a)^b{}_c \equiv -i f^{ab}{}_c$.

Spinorial representations. Spinorial representations of the Lie group $\mathrm{SO}(n, m)$ are given by representations of the double cover[4] of $\mathrm{SO}(n, m)$ called the spin group $\mathrm{Spin}(n, m)$. It is possible to show that the double cover of the restricted Lorentz group $\mathrm{SO}^+(1, 3)$ is $\mathrm{Spin}^+(1, 3) = \mathrm{SL}(2, \mathbb{C})$ where $\mathrm{SL}(2, \mathbb{C})$ is the set of complex 2×2 matrices with unit determinant. We can thus construct a first spinorial representation of $\mathrm{SO}^+(1, 3)$ by the matrices M of $\mathrm{SL}(2, \mathbb{C})$ themselves, i.e. the fundamental representation of $\mathrm{SL}(2, \mathbb{C})$ is a spinorial representation of $\mathrm{SO}^+(1, 3)$ called $(\frac{1}{2}, 0)$. The vector space upon which this representation acts is the set of two-component objects (complex 2×1 column vectors) called *spinors*, or more precisely *left-handed Weyl spinors* for the fundamental representation. If we take the complex conjugated matrices M^*, this defines another inequivalent representation of $\mathrm{SL}(2, \mathbb{C})$ called the *anti-fundamental* representation. We can thus construct a second spinorial representation of $\mathrm{SO}^+(1, 3)$ called $(0, \frac{1}{2})$. Objects that are acted upon in this representation are now called *right-handed Weyl spinors*. These two representations are thus seen to be 2-dimensional. By taking the direct sum $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the two representations, we obtain a 4-dimensional (reducible) representation of the Lorentz group which acts upon four-component objects called *Dirac spinors*.

In nonrelativistic quantum mechanics, invariance under Lorentz boosts is not required so that only SO(3), the rotational subgroup of $\mathrm{SO}^+(1, 3)$, is relevant. In this case, the double cover of SO(3) is $\mathrm{Spin}(3) = \mathrm{SU}(2)$ so that the relevant spinorial representations are representations of SU(2), leading to the usual spinors of quantum mechanics. Since SO(3) and SU(2) are compact[5] Lie groups, their generators can be chosen to be Hermitian.

Summary of finite-dimensional representations:

1. For a scalar ϕ, element of a 1-dimensional vector space $\mathbb{R}$, we have

$$\phi \xrightarrow{\Lambda} \Lambda_S \phi = \phi$$

where $\Lambda_S = \exp(-i\omega_{\mu\nu} J^{\mu\nu}/2) = 1$ with $J^{\mu\nu} = 0$ are trivial 1×1 matrices.

2. For a vector V^ρ, element of a 4-dimensional ($\rho = 0, 1, 2, 3$) vector space $\mathbb{R}^4$, we have

$$V^\rho \xrightarrow{\Lambda} (\Lambda_V)^\rho{}_\sigma V^\sigma$$

where $\Lambda_V = \exp(-i\omega_{\mu\nu}\mathcal{J}^{\mu\nu}/2)$ with $(\mathcal{J}^{\mu\nu})^\rho{}_\sigma = i(g^{\mu\rho}\delta^\nu{}_\sigma - g^{\nu\rho}\delta^\mu{}_\sigma)$ are 4×4 matrices.

3. For a left-handed Weyl spinor ψ_α, element of a 2-dimensional ($\alpha = 1, 2$) vector space $\mathbb{C}^2$, we have

$$\psi_\alpha \xrightarrow{\Lambda} (\Lambda_L)_{\alpha\beta}\psi_\beta$$

where $\Lambda_L = \exp(-i\omega_{\mu\nu}S^{\mu\nu}/2)$ with $S^{\mu\nu} = J^{\mu\nu}$ are 2×2 matrices such that $S^{ij} \equiv \frac{1}{2}\epsilon^{ijk}\sigma^k$ and $S^{0i} \equiv -\frac{i}{2}\sigma^i$.

4. For a right-handed Weyl spinor[6] ψ_α, we have similarly

$$\psi_\alpha \xrightarrow{\Lambda} (\Lambda_R)_{\alpha\beta}\psi_\beta$$

where $\Lambda_R = \exp(-i\omega_{\mu\nu}S^{\mu\nu}/2)$ with $S^{ij} \equiv \frac{1}{2}\epsilon^{ijk}\sigma^k$ again but now $S^{0i} \equiv \frac{i}{2}\sigma^i$.

5. For a Dirac spinor Ψ_a, element of a 4-dimensional ($a = 1, 2, 3, 4$) vector space $\mathbb{C}^4$, we have

$$\Psi_a \xrightarrow{\Lambda} (\Lambda_D)_{ab}\Psi_b \tag{2}$$

where $\Lambda_D = \exp(-i\omega_{\mu\nu}S^{\mu\nu}/2)$ but now the $S^{\mu\nu}$ are 4×4 matrices given by $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. They are actually the direct sum of the 2×2 $S^{\mu\nu}$ matrices for the left-handed and right-handed Weyl spinors:

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k\,; \quad S^{0i} = -\frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix},$$

so that a Dirac spinor (also called *bispinor*) transforms like $\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ where $\psi_{L/R}$ is a left/right-handed Weyl spinor.

B. Infinite-dimensional representations

Field representations. So far we have been dealing only with finite-dimensional representations, that acted on finite-dimensional vector spaces whose elements were scalars, vectors, tensors, spinors, giving us respectively the scalar, vector, tensor and spinorial representations. These objects are however only 'constants': in quantum field theory, we deal

with fields, which are functions of spacetime. Therefore, a generic multicomponent field Φ_a will not only transform as

$$\Phi_a \xrightarrow{\Lambda} M_{ab}(\Lambda)\Phi_b$$

where $M(\Lambda)$ corresponds to the $\Lambda_{S,V,L,R,D}$ matrices of the finite-dimensional representations, but since it is a function of coordinates $\Phi_a(x)$ and these coordinates (being Lorentz 4-vectors) are affected by the Lorentz transformations as $x^\mu \xrightarrow{\Lambda} \Lambda^\mu{}_\nu x^\nu$ where $\Lambda^\mu{}_\nu \equiv (\Lambda_V)^\mu{}_\nu$, then we will actually have

$$\Phi_a(x) \xrightarrow{\Lambda} M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x) \tag{3}$$

In other words, the contours of the function $\Phi_a(x)$ are 'boosted' as well by the Lorentz transformation[7]. It is not difficult (by considering an infinitesimal Lorentz transformation, for instance) to check that this transformation of coordinates in a generic field $\psi(x)$ can be implemented by

$$\psi(\Lambda^{-1}x) = e^{-i\omega_{\mu\nu}L^{\mu\nu}/2}\psi(x)$$

where now $L^{\mu\nu}$ is a differential operator defined as

$$L^{\mu\nu} \equiv i(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{4}$$

i.e. we implement the transformation on coordinates by a (exponential-resummed) Taylor expansion. It can be checked that the $L^{\mu\nu}$ satisfy the Lorentz algebra Eq. (1). Since $L^{\mu\nu}$ acts on a space of functions $\psi(x)$ which is an infinite-dimensional vector space, it corresponds to an infinite-dimensional representation of the Lorentz algebra.

So we can put together both contributions to the transformation of the field $\Phi_a(x)$ under a Lorentz transformation, and we obtain

$$\Phi_a(x) \xrightarrow{\Lambda} (e^{-i\omega_{\mu\nu}S^{\mu\nu}/2})_{ab}e^{-i\omega_{\mu\nu}L^{\mu\nu}/2}\Phi_b(x) \tag{5}$$

where for simplicity, we will henceforth denote generically by $S^{\mu\nu}$ all the finite-dimensional representations of the infinitesimal Lorentz generators $J^{\mu\nu}$ encountered in the previous section ($S^{\mu\nu}$, $\mathcal{J}^{\nu\nu}$, ...) such that

- for a scalar field, $S^{\mu\nu} = 0$;
- for a vector field, $(S^{\mu\nu})^\rho{}_\sigma \equiv (\mathcal{J}^{\mu\nu})^\rho{}_\sigma = i(g^{\mu\rho}\delta^\nu{}_\sigma - g^{\nu\rho}\delta^\mu{}_\sigma)$;
- for a spinor (Weyl or Dirac) field, $S^{\mu\nu}$ is given in 3.4.5. of the previous section.

Now, since all the $S^{\mu\nu}$ matrices are finite-dimensional and constant, we can put the two factors of Eq. (5) into a single exponential, and write

$$\Phi_a(x) \xrightarrow{\Lambda} (e^{-i\omega_{\mu\nu}J^{\mu\nu}/2})_{ab}\Phi_b(x)$$

where

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{6}$$

is now an *infinite*-dimensional representation of the Lie algebra of the Lorentz group. While the $S^{\mu\nu}$ have different forms corresponding to the different types of fields (scalars, vectors, spinors, ...), $L^{\mu\nu}$ always has the form given in Eq. (4). The representations of Eq. (6) act on the space of fields $\Phi_a(x)$ and are called the *field representations* for the generators of the Lorentz group.

Representations on 1-particle Hilbert space. So now we have representations of the Lorentz group on fields, which are multicomponent functions of spacetime $\Phi_a(x)$. However, there has actually been no mention of whether these fields are classical or quantum. In other words, we use the representation theory of the Lorentz group to construct a Lorentz-invariant Lagrangian for a given set of fields. However, this is independent of whether or not we quantize the resulting field theory. If we quantize the theory, then we can construct still other representations of the Lorentz group, formed by the set of unitary operators acting on the quantum states belonging to the 1-particle Hilbert space of our quantum field theory. Indeed, a famous theorem by Wigner, at the heart of the application of group theory to quantum mechanics, asserts that the symmetry group of the Hamiltonian (or the Lagrangian, better suited for a relativistic theory) can be represented by a group of unitary[8] symmetry operators acting on the Hilbert state space. Correspondingly, the algebra of infinitesimal generators (in the case of a continuous symmetry) is represented by an algebra of Hermitian operators. In the case of the Lorentz group, to each element of the Lorentz group Λ we assign a unitary operator $U(\Lambda)$ which implements this transformation on the 1-particle states of the free field theory. For example, consider a 1-fermion state $|\mathbf{p}, s\rangle \equiv \sqrt{2E_{\mathbf{p}}}a^{s\dagger}_{\mathbf{p}}|0\rangle$ with momentum $\mathbf{p}$ and spin s. We require that[9]

$$U(\Lambda)|\mathbf{p}, s\rangle \equiv |\Lambda\mathbf{p}, s\rangle = \sqrt{2E_{\Lambda\mathbf{p}}}a^{s\dagger}_{\Lambda\mathbf{p}}|0\rangle$$

Since the vacuum is Lorentz invariant $U(\Lambda)|0\rangle = |0\rangle$, this implies that the cre-

ation/annihilation operators transform as

$$U(\Lambda)a^{s\dagger}_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\mathbf{p}}}{E_{\mathbf{p}}}}a^{s\dagger}_{\Lambda\mathbf{p}}\,; \qquad U(\Lambda)a^{s}_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\mathbf{p}}}{E_{\mathbf{p}}}}a^{s}_{\Lambda\mathbf{p}}.$$

To write down an algebra of Hermitian operators to represent the abstract Lorentz algebra Eq. (1), it is easier to work with the equivalent six generators $\tilde{J}^a$ defined in the section on the adjoint representation, namely $J^i \equiv \frac{1}{2}\epsilon^{ijk}J^{jk}$ and $K^i \equiv J^{0i}$. The corresponding algebra of Hermitian operators is defined by the following commutation relations, which all follow directly from Eq. (1):

$$[J^i, J^j] = i\epsilon^{ijk}J^k\,; \qquad [J^i, K^k] = i\epsilon^{ijk}K^k\,; \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k.$$

If we define the combinations $\mathbf{J}_+ = \frac{1}{2}(\mathbf{J} + i\mathbf{K})$ and $\mathbf{J}_- = \frac{1}{2}(\mathbf{J} - i\mathbf{K})$, it is straightforward to show that both sets $(J^1_\pm, J^2_\pm, J^3_\pm)$ satisfy an independent $\mathfrak{su}(2)$ algebra,

$$[J^i_\pm, J^j_\pm] = i\epsilon^{ijk}J^k_\pm\,; \qquad [J^i_\pm, J^j_\mp] = 0, \tag{7}$$

so the Lie algebra of the Lorentz group is isomorphic to $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

The operators $\mathbf{J}^2_+$ and $\mathbf{J}^2_-$ commute with all the generators: such operators are called *Casimir operators* or *Casimir invariants*. As follows from the $\mathfrak{su}(2)$ commutation relations, they have eigenvalues $j_\pm(j_\pm+1)$ with $j_\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ If we now consider finite-dimensional representations of the two $\mathfrak{su}(2)$ algebras of Eq. (7), i.e. the usual spinorial representations of quantum mechanics, we see that we recover the finite-dimensional representations of the Lorentz algebra, and we also see that they can be labeled by the pair of integers/half-integers (j_+, j_-). This explains the notation we have been using: $(0,0)$ for scalars, $(\frac{1}{2}, \frac{1}{2})$ for vectors, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ for Weyl spinors, $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ for Dirac spinors.

II. POINCARÉ GROUP

In the Poincaré group, one adds spacetime translations to the set of (homogeneous) Lorentz transformations:

$$x^\mu \xrightarrow{(\Lambda,a)} \Lambda^\mu{}_\nu x^\nu + a^\mu$$

where $a^\mu \in \mathbb{R}^4$. The Poincaré group is thus also called the inhomogeneous Lorentz group. The proper orthochronous or restricted Poincaré group $\mathrm{ISO}^+(1,3)$ is a subgroup of the full

Poincaré group that contains only proper orthochronous Lorentz transformations. It is given by a semidirect product,

$$\mathrm{ISO}^+(1,3) \cong \mathrm{SO}^+(1,3) \rtimes \mathbb{R}^4$$

where the translation subgroup $\mathbb{R}^4$ is a normal subgroup[10]. To generate translations, one has to add a new generator P^μ to the Lorentz algebra Eq. (1) to form the Poincaré algebra which reads

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

$$[P^\mu, P^\nu] = 0$$

$$[P^\mu, J^{\rho\sigma}] = i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho)$$

A correct relativistic quantum field theory has to be not only Lorentz covariant, but Poincaré covariant as well.

Field representations. The field representations of the Poincaré algebra are

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) + S^{\mu\nu}$$

as previously, and

$$P^\mu = i\partial^\mu$$

for the generator of translations.

Representations on 1-particle Hilbert space. On 1-particle Hilbert space, the generators of the above Poincaré algebra are represented by Hermitian operators (hence the elements of the Poincaré group by unitary operators). In order to do that, it is easier to work with the equivalent set of generators J^i, K^i, P^i and P^0. As explained before, J^i and K^i have a clearer physical meaning than the abstract generators $J^{\mu\nu}$; J^i are the generators of rotations and K^i are the generators of boosts. In addition, P^i are the generators of space translations and P^0 is the generator of time translations. It follows that $\mathbf{P}$ will be represented by the total momentum operator $\mathbf{P}$, P^0 will be represented by the Hamiltonian operator H, and $\mathbf{J}$ will be represented by an angular momentum operator $\mathbf{J}$.

A convenient way to label representations consisting of operators acting on Hilbert space is by the eigenvalues of Casimir operators (or Casimir invariants), which are operators that commute with all the generators of the algebra. In the case of the Poincaré algebra, there are two Casimir operators $P^\mu P_\mu$ and $W^\mu W_\mu$ where

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$$

is the Pauli-Lubanski pseudovector. We now consider two cases, depending on whether the particle in the 1-particle state $|\mathbf{p}\rangle \sim a^\dagger_{\mathbf{p}}|0\rangle$ is massive or massless.

- *Massive representations*: on a massive 1-particle state, $P^\mu P_\mu = P^2$ has eigenvalue $p^2 = E^2_{\mathbf{p}} - |\mathbf{p}|^2 = m^2$ where m is the mass of the particle, and $W^\mu W_\mu$ has eigenvalue $-m^2 j(j+1)$ with j the spin of the particle. The 1-particle states are actually $|\mathbf{p}, j\rangle = \sqrt{2E_{\mathbf{p}}} a^{j\dagger}_{\mathbf{p}}|0\rangle$ in this case. The representations can be labeled by two labels (m, j) where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Another way to see this is that if we bring P^μ to the form $(m, \mathbf{0})$ by a Lorentz transformation, P^μ is unchanged by arbitrary spatial rotations. This subgroup of the Poincaré group is called the *little group*. In this case, it is the group of rotations SO(3); however, to allow for spinorial representations, we choose the little group to be the double cover SU(2). It is well-known that the irreducible representations of SU(2) are labeled by the spin quantum number j. Thus to completely specify a massive representation of the Poincaré group, we have to provide two labels: the mass m, and the spin j which indexes the representation of the little group SU(2), also called the *small representation*. Each spin-j representation is $(2j+1)$-dimensional; therefore a massive particle of spin j has $2j+1$ degrees of freedom.

- *Massless representations*: on a massless 1-particle state, obviously P^2 has eigenvalue 0. In this case, P^μ can be reduced to $P^\mu = (\omega, 0, 0, \omega)$ where ω is the eigenvalue of P^0. The transformations that leave this P^μ invariant are rotations in the (x, y) plane, that is, elements of SO(2). We choose the little group to be the double cover U(1) which has only one generator. Its 1-dimensional representations are labeled by a single index h which corresponds to the helicity of the particle. In principle, each massless particle has thus only one degree of freedom. However, for massless particles which participate only in parity-conserving interactions such as the photon ($h = 1$) and graviton ($h = 2$), we group the $+h$ and $-h$ representations together since parity transforms positive to negative helicity and vice-versa. Thus the photon and graviton have two degrees of freedom.

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[1] The antisymmetric tensor of parameters $\omega_{\mu\nu}$ is given from the usual rotation $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ and boost $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ parameters by $\omega_{ij} = \epsilon_{ijk}\theta_k$ and $\omega_{0i} = \beta_i$.

[2] The Lie groups $\mathrm{O}(n,m)$, $\mathrm{O}(n)$ and their various subgroups $\mathrm{SO}(n,m)$, $\mathrm{SO}(n)$ and others are actually all subgroups of a matrix group, the general linear group $\mathrm{GL}(n,\mathbb{R})$ which is nothing but the set of real invertible $n \times n$ matrices. For example, $\mathrm{O}(1,3)$ is a subgroup of $\mathrm{GL}(4,\mathbb{R})$.

[3] The number of independent components of a second-rank tensor is $n(n+1)/2$ for a symmetric tensor $S^{\mu\nu} = S^{\nu\mu}$ and $n(n-1)/2$ for an antisymmetric tensor $A^{\mu\nu} = -A^{\nu\mu}$, where n is the dimension, i.e. $\mu, \nu = 1, \ldots, n$.

[4] For $n > 2$, $\mathrm{Spin}(n,m)$ is simply connected so that it is also the universal covering group of $\mathrm{SO}(n,m)$.

[5] For example, $\mathrm{SU}(2)$ is homeomorphic to the 3-sphere $S^3 = \{(x,y,z,w) \in \mathbb{R}^4 | x^2+y^2+z^2+w^2 = 1\}$ which is a compact manifold; $\mathrm{SO}(3)$ is homeomorphic to the real projective space $\mathbb{R}P^3$ which is also a compact manifold (think of $\mathbb{R}P^3$ as a cube $[0,1]\times[0,1]\times[0,1]$ with antiperiodic boundary conditions in all three directions). In general, $\mathbb{R}P^n \cong S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is the cyclic group of order 2 and S^n is the double cover of $\mathbb{R}P^n$. One can thus write $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$ which makes clear the meaning of 'double cover': $\mathrm{SU}(2)$ 'covers' the operations of $\mathrm{SO}(3)$ twice, so that we have to remove the distinction between two *different* elements of $\mathrm{SU}(2)$ (such as $R(2\pi) = -1$ and $R(4\pi) = 1$) which give the *same* element of $\mathrm{SO}(3)$. To do that, we 'divide' by $\mathbb{Z}_2 = \{1, -1\}$. In the same spirit we can write $\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. In analogy with $\mathbb{R}P^n$, we see that viewed as a manifold, $\mathrm{SO}^+(1,3)$ is a projective space and we write $\mathrm{SO}^+(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$ where $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ is the projective special linear group.

[6] A technical detail following up on our discussion of the representations of $\mathrm{SL}(2,\mathbb{C})$. A left-handed Weyl spinor ψ_α transforms according to the matrices M of the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$ as $\psi_\alpha = M_\alpha{}^\beta \psi_\beta$. However, a right-handed Weyl spinor is usually rather denoted by $\bar{\chi}^{\dot{\alpha}}$ (dotted spinor) and will transform according to the complex conjugated matrices M^*, which belong to the (inequivalent) anti-fundamental representation of $\mathrm{SL}(2,\mathbb{C})$, as $\bar{\chi}^{\dot{\alpha}} = \bar{\chi}^{\dot{\beta}} (M^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}$.

[7] This is the usual Wigner's convention for symmetry operators in quantum mechanics: to each coordinate transformation $\mathbf{R}$ that acts on the coordinates as $\mathbf{x} \xrightarrow{\mathbf{R}} \mathbf{R}\mathbf{x}$, we assign a symmetry operator $P_{\mathbf{R}}$ that acts on quantum states as $|\psi\rangle \xrightarrow{\mathbf{R}} P_{\mathbf{R}}|\psi\rangle$ such that $\langle \mathbf{r}|P_{\mathbf{R}}|\psi\rangle = \langle \mathbf{R}^{-1}\mathbf{r}|\psi\rangle$, or equivalently on functions as $\psi(\mathbf{x}) \xrightarrow{\mathbf{R}} P_{\mathbf{R}}\psi(\mathbf{x}) \equiv \psi(\mathbf{R}^{-1}\mathbf{x})$.

[8] If we include time-reversal symmetry, then we must allow for antiunitary operators as well. However we restrict our discussion here to orthochronous Lorentz transformations as explained earlier.

[9] Note that we could have chosen $U(\Lambda)|\mathbf{p},s\rangle \equiv |\Lambda^{-1}\mathbf{p},s\rangle$ to stick blindly to our previous convention. However, our previous convention applied to fields which are functions $\Phi_a(x)$ of some coordinates x. While we can still consider a 1-particle state as an abstract function of some coordinates $\mathbf{p}$, it actually describes a physical particle with momentum $\mathbf{p}$ wandering around in vacuum. Thus applying a Lorentz transformation $U(\Lambda)$ to such a state means to rotate and boost this particle's momentum according to Λ: $\mathbf{p} \stackrel{\Lambda}{\to} \Lambda\mathbf{p}$. The resulting state thus has a particle with momentum $\Lambda\mathbf{p}$, i.e. $U(\Lambda)|\mathbf{p},s\rangle = |\Lambda\mathbf{p},s\rangle$. The only drawback of this approach is that when we *quantize* our field $\Psi_a(x)$, the associated field operator $\hat{\Psi}_a(x)$ that is expanded in creation and annihilation operators will transform in a way opposite to the field itself. For example, the operator for a free Dirac field will transform as $\hat{\Psi}_a(x) \stackrel{\Lambda}{\to} U(\Lambda)\hat{\Psi}_a(x)U^{-1}(\Lambda) = (\Lambda_D^{-1})_{ab}\hat{\Psi}_b(\Lambda x)$ whereas we know from Eqs. (2) and (3) that the field itself transforms as $\Psi_a(x) \stackrel{\Lambda}{\to} (\Lambda_D)_{ab}\Psi_b(\Lambda^{-1}x)$. However, the relative orientation of the spinor transformation Λ_D and the coordinate transformation Λ is correct.

[10] The translation subgroup $\{(0,a)\}$ is normal because $(\Lambda',a')^{-1}(0,a)(\Lambda',a') = (0,\Lambda'^{-1}a)$ is still a translation for any Poincaré transformation (Λ',a'). However, the Lorentz subgroup $\{(\Lambda,0)\}$ is not normal because $(\Lambda',a')^{-1}(\Lambda,0)(\Lambda',a') = (\Lambda'^{-1}\Lambda\Lambda', (\Lambda'^{-1}\Lambda - \Lambda'^{-1})a')$ which is not a homogeneous Lorentz transformation. When only one of the factors in the group product is normal, one speaks of a *semidirect* group product $\rtimes$ instead of a direct group product $\times$.