How to Squander Your Endowment: Pitfalls and Remedies*

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Abstract

University donors choose to contribute to endowment if they want to make a permanent contribution to the university. It is consequently viewed as a responsibility of the university to preserve capital when choosing the investment policy and the spending rule. Practitioners commonly model the preservation-of-capital constraint by requiring the expected real rate of return to be greater than the spending rate, which is the condition for a unit to increase in real value on average. Unfortunately, this criterion does not imply that capital grows eventually because the law of large numbers applies to sums, not products. The measure can be corrected by requiring the log of the real value of a unit to increase on average, which reduces permitted spending by approximately half the variance of returns if period returns are not too volatile. Even if the correct target spending rule is applied, the common practice of smoothing spending using a partial adjustment model for spending makes spending unstable in bad times, and in fact the probability of eventual ruin is one. However, we show that a simple modification to the traditional smoothing rule does preserve capital.

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1 Introduction

Donors who wish to contribute to universities have a number of options depending on when they want their giving to have an impact. For example, donors wanting to have an immediate impact can contribute through annual giving, donors who want to have an impact for an intermediate time frame can give funds for a building, and donors who want to have a permanent impact can contribute to endowment. Since contributions to endowment are supposed to have a permanent impact, the university has a responsibility to make sure that the spending rule and investment strategy for endowment, taken together, preserve capital. In other words, preservation of capital is viewed as a constraint on universities' choice of policy. This paper takes a look at preservation of capital with a focus on existing practice. We find that the usual criterion (spending rate less than expected real return on investment) for preservation of capital is incorrect and actually it is consistent with polices of the form commonly used in practice for which wealth always tends to zero over time. The traditional rule says that the real value of a unit[1] of endowment increases on average; a corrected rule says that log of the real value of the endowment increases on average, and this can be significantly different. We also show that a stylized version of the practice of smoothing spending implies that the endowment never preserves capital with risky investment, and we show how to modify the smoothing rule to preserve capital.

A spending rate less than the expected return on assets, calculated in real terms, has long been used as a criterion for whether an endowment preserves capital. This criterion is based on the intuition of the law of large numbers, since it means that on average the expected return on investment in the endowment should cover spending, or equivalently what is left in the endowment grows on average. However, this intuition is implicitly based on a mis-application of the law of large numbers: the law of large

[1]The notion of unitization we are using here is similar to the unitization commonly used in measuring the performance of a portfolio manager. For performance measurement, the manager does not get credit for increases in value due to inflows and is not charged for spending out of the portfolio. For preservation of capital, we do not get credit for inflows, but we are charged for spending.

numbers applies to sums not products, but wealth grows as a product over time of one plus the return less spending. Here is a simple example to illustrate that the traditional criterion of spending at a rate less than the expected rate of return on assets does not necessarily preserve capital.

Example 1 (Destroying capital but satisfying the traditional criterion): Assume an endowment has a spending rate of 0% and an investment which has equal chances of tripling and going to zero each period:

$$1 \quad \begin{array}{l} \nearrow \quad 3 \quad \text{probability } 1/2 \\ \searrow \quad 0 \quad \text{probability } 1/2 \end{array}$$

The expected rate of return is (50%) which is greater than the spending rate (0%). According to the traditional criterion, capital should be preserved. However, in each year there is a 50% probability the endowment will be wiped out and the probability of surviving for T years is 2^{-T} which approaches 0 rapidly as T increases. Having no endowment at all with probability close to one certainly does not preserve capital but it satisfies the traditional rule. Moreover, having the possibility of the portfolio value dropping to zero is not critical in this example, as we will see in Example 2 in the text.

So far, we have ducked the question of how to define preservation of capital. In Example 1, the definition is not very critical, because soon having zero capital with probability close to one cannot reasonably be viewed as preserving capital. We say a policy preserves (resp. destroys) capital if the value of a unit of the endowment in real terms goes to infinity (resp. zero) over time in probability. These definitions are motivated by the intuition of the traditional criterion, and they are good for our purpose. Our definitions incorporate two reasonable features normally used in practice: 1) we use real "inflation-adjusted" returns since capital must be preserved in terms of spending power and not just nominal value, and 2) we look at the value of a unit of endowment and do not include future contributions but we do subtract spending. If we spend the entire contribution this year and replace it by someone else's contribution next year, we do not consider that to be making a permanent contribution.

Although the traditional criterion does not ensure that capital is preserved, we provide a simple alternative criterion that does. Taking logarithms converts products into sums, and capital is preserved if the expected log return net of spending, defined as the expected log of one plus the return less the spending rate, is positive. This criterion implies preservation of capital. In Subsection 2.2, we provide a reasonable example in which changing to the correct criterion reduces the admissible spending rate by 1%, which implies that endowments may need to reduce spending by 20% if they currently spend about 5% of their capital.

Besides looking at the basic spending criterion, we also look at the common practice of overlaying smoothing on the basic spending rule. Smoothing of spending is supposed to prevent the damage done by large fluctuations in spending. This is a reasonable idea: sudden decreases in spending are disruptive, and sudden increases may be used carelessly. Unfortunately, the usual partial adjustment rule of moving only a fixed fraction of the way toward the target spending level *never* preserves capital in the endowment if the target spending rate is positive (even if very small) and the portfolio is risky with i.i.d. returns. This result is based on a continuous-time model in which that portfolio returns are randomly drawn from the same distribution and are independent over time. Intuitively, random fluctuations imply that sooner or later we will have bad luck in the risky investment making the spending rate very large. When the spending rate is very large, capital is depleted relatively more quickly than the smoothing reduces spending, and as a result the endowment ends up sooner or later in a "death spiral" plunging to zero.

Since smoothing is a good idea and the traditional smoothing rule does not preserve capital, we have proposed a possible solution, a simple modified smoothing rule that includes a new term that changes spending to compensate for the expected change in spending rate given the excess of current spending over the expected return of assets. For this rule, we have a characterization of the parameter values for which capital is preserved. Moreover, an interest rate environment like the current one in which inflation exceeds the nominal rate is a special challenge, but there is a simple result: given some stationarity, expected log return net of spending does not have to be positive every period, and only has to be positive on average.

This paper investigates the conditions under which conventional dividend and spending policies or variants preserve capital. The focus is on the necessary conditions endowments need to meet, i.e., preserving capital as they promise to the donors when donating money. This contrasts to the usual optimal investment approach taken by academics which maximizes a utility function subject to constraints (see for example, Dybvig (1999) or Gilbert and Hrdlicka (2015)). In general, practitioners find optimization models less useful than academics would hope, since it is difficult to incorporate all the considerations that are important in practice. Nonetheless, optimization models are useful benchmarks for thinking about new rules. Although we do not solve any optimization models in this paper, we look at some implications of incorporating preservation of capital in these models. In particular, our results suggest that the plausible definition of preservation of capital we are using, which is fine for the sorts of policies traditionally considered, will have to be refined for use in optimization models. This plausible definition can be manipulated (and the optimization model will find the "optimal" manipulation to sidestep the rule), implying the constraint will either be irrelevant in an optimization problem or there will be no solution. In particular, we prove that any utility level that can be obtained without the constraint on preservation of capital can also be approached arbitrarily closely with the constraint. Intuitively, this is because the constraint only imposes a condition in the limit as time increases, for which compounding can obtain a large value from a trivial investment. For example, the current college president may choose to spend all but two cents worth of the endowment before the end of his term of service, with a plan of modest spending afterwards. Theoretically, the two cents will grow without limit over time to satisfy the constraint on preservation of capital, without having any material effect on the current president's plans.

The rest of the paper is arranged as follows. Section 2 documents the problem with the traditional criterion for preserving capital and provides the new correct criterion. Section 3 shows that traditional smoothing implies capital is not preserved. We provide a modified smooth spending rule that preserves capital. Section 4 comes up with the condition for preserving capital with temporarily negative risk-free rate. Section 5 discusses optimization model of spending and investment that preserve capital with smooth spending. Section 6 closes the paper.

2 Spending Rate Less Than Expected Return

In the following subsection, we present a reasonable definition of preservation of capital that will be used in most of the paper. As we show in Section 5, this definition would have to be strengthened to be used in an optimization model.

2.1 Definition of Preservation of Capital

To characterize preservation of capital, we require a formal definition of what this means. Fortunately, most of our results will be robust to a range of reasonable choices for how we define preservation of capital. We study the management of a unit of endowment, with a proportional change equaling the investment return less the spending rate, but not including any new contributions. Looking at a unit without credits for subsequent contributions is standard in practice for endowments and it is important because we are looking for a contribution to have a permanent impact. It is annual giving, not a permanent contribution to endowment, if we spend the entire contribution this year and replace it using future contributions. Including future contributions would be important for writing down optimization problems and spending from future contributions should be included in the objective function. However, in this paper we are focusing on the preservation-of-capital constraint rather than the objective function.

We let W_t be the real (inflation-adjusted) value of wealth in the unit at time t with spending S_t. We will consider both continuous and discrete time. In discrete time, we model wealth dynamics as $W_t = W_{t-1}(1+r_t-s_t)$, where r_t is the real rate of return and s_t is the spending rate (as a fraction of W_{t-1}) at time t.[2] We will not concern ourselves with valuation issues such as what price index to use or how to value illiquid assets, so that given the investment and spending policy for the endowment, the processes W_t, r_t, and s_t are well-defined. We also abstract from parameter uncertainty about the distribution of returns.

For most of the paper, we will use the following definitions:

Definition 1 *Endowment wealth is said to be* preserved *if the real value of a unit W_t becomes arbitrarily large over time:* $\text{plim}_{t\to\infty} W_t = \infty$.[3]

Definition 2 *Endowment wealth is said to be* destroyed *if the real value of a unit W_t vanishes over time:* $\text{plim}_{t\to\infty} W_t = 0$.

The forms of these definitions look the same in both continuous and discrete time although the implicit set of possible times is different. We think of the definition of destroying capital as relatively conservative, since no reasonable rule for preserving capital would say we are preserving capital if wealth is almost always close to 0 when t is large. This is what we need for our main results that the traditional rules are not sufficient to preserve capital. This is a good definition for the main purpose of our paper, which is to evaluate current practice, but it should refined for use in an optimization model, as discussed in Section 5.

[2] This convention amounts to having spending S_t taking place at the end of the period just before W_t is measured. It is straightforward to change our results for other conventions. For example, if spending S_t takes place at the beginning of the period just after W_{t-1} is measured, we would define $s_t \equiv S_t/W_{t-1}$ and then $W_t = W_{t-1}(1-s_t)(1+r_t)$ with obvious changes in the statements of our results.

[3] As is conventional, plim indicates convergence in probability. By definition, $\text{plim}_{t\uparrow\infty} W_t = \infty$ if for all $X > 0$, $\text{prob}(W_t > X) \to 1$ as $t \to \infty$.

2.2 Preserving Capital in Discrete Time

One traditional criterion says that a spending rate of no more than the average return on the endowment will preserve its value. This traditional criterion is widely adopted and clearly stated in the spending policy statements of many university endowments. For example, the spending policy statement of UCSD Foundation (2014) states that its objective is to "achieve an average total annual net return equivalent to the endowment spending rate adjusted for inflation." Moreover, the endowment of Henderson State University (2014) even employs a concrete example to illustrate its objective of achieving an inflation-adjusted average return equal to the spending rate: "Total return objective 7.00%, spending rate 4.00%, administration fee 1.50%, and inflation rate 1.50%." This criterion is also mentioned by Rice, Dimeo, and Porter (2012), which gives as a hypothetical example: "the primary objective of the Great State University Endowment fund is to preserve the purchasing power of the endowment after spending. This means that the Great State University Endowment must achieve, on average, an annual total rate of return equal to inflation plus actual spending." Despite its wide use, the traditional criterion is not sufficient to guarantee preservation of capital.

Absent risk, this criterion makes perfect sense. Suppose the real portfolio return $r_t = r$ and the spending rate $s_t = s$ are both riskless and constant over time. The traditional criterion says that the spending is less than the return on the portfolio, that is, $s < r$, then capital is preserved. We have that

$$
\begin{aligned}
W_t &= W_{t-1}(1 + r - s) && (1) \\
&= W_0 (1 + r - s)^t . && (2)
\end{aligned}
$$

In this riskless case, spending less than the return on the endowment implies the endowment increases without bound, so we have preservation of capital, while spending more than the return on the endowment implies the endowment decreases to zero over time, so we have destruction of capital. So far so good. In the traditional criterion, the next step says we can use the same analysis an uncertain world, "you know, because of the law of large numbers." However, the application of law of the large numbers is fallacious because the law of large numbers applies to sums, not products. Now that the return is random, (1) becomes

$$
\begin{aligned}
W_t &= W_{t-1}(1 + r_t - s_t) &(3)\\
&= W_0 \prod_{i=1}^{t} (1 + r_i - s_i). &(4)
\end{aligned}
$$

As was shown in Example 1 in the Introduction, even if $1 + r_i - s_i$ has mean larger than 1 and is i.i.d. over time, the wealth (4) does not necessarily grow over time and indeed capital may be destroyed.

Example 1 may seem extreme because the wealth can actually reach 0; the following example shows that the traditional criterion is consistent with destruction of capital even if wealth is always positive:

Example 2 (Destroying capital but satisfying the traditional criterion): Assume an endowment has a spending rate of 0% and an investment that triples or is reduced by a factor 1/9 with equal probabilities:

$$
1 \quad \begin{array}{l} \nearrow \; 3 \quad \text{probability } 1/2 \\ \searrow \; 1/9 \quad \text{probability } 1/2 \end{array}
$$

The expected rate of return 5/9 is greater than the spending rate 0%, but the endowment still vanishes over time, so the traditional criterion fails. To prove this, note that

$$
\begin{aligned}
W_t &= W_0 \prod\nolimits_{i=1}^{t} (1 + r_i - s_i) &(5)\\
&= W_0 \exp\left(\sum\nolimits_{i=1}^{t} \log\left(1 + r_i - s_i\right) \right). &(6)
\end{aligned}
$$

Moreover

$$\mathrm{E}\left[\log\left(1+r_i-s_i\right)\right]=\frac{1}{2}\log 3+\frac{1}{2}\log\left(\frac{1}{9}\right)=\left(\frac{1}{2}+\frac{1}{2}\times(-2)\right)\log 3=-\frac{1}{2}\log 3<0.$$

Therefore, by the law of large numbers, $\text{plim}\sum_{i=1}^{t}\log(1+r_i-s_i)=-\infty$ and by (5) $\text{plim}W_t=0$. ■

To correct the traditional criterion, we can to first convert the multiplication to a sum by taking logarithms:

$$\log(W_t)=\log(W_0)+\sum\nolimits_{i=1}^{t}\log(1+r_i-s_i),$$

and now we can use the law of averages (i.e., the law of large numbers or the central limit theorem) if we assume the appropriate regularity. This leads to the following theorem.

Theorem 1 *Recall that W_t is the value of a unit of endowment at time t, r_t is the endowment's rate of return from $t-1$ to t, and s_t is the spending rate at t as a fraction of wealth at $t-1$, so that $W_t/W_{t-1}=1+r_t-s_t$. Assume that $W_0>0$, that W_t/W_{t-1} is i.i.d. over time, and that $\log(W_t/W_{t-1})$ has finite mean and variance. Then 1) endowment capital is preserved according to Definition 1 if and only if $E[\log(W_t/W_{t-1})]=E[\log(1+r_t-s_t)]>0$ and 2) endowment capital is destroyed according to Definition 2 if and only if $E[\log(W_t/W_{t-1})]=E[\log(1+r_t-s_t)<0$.*

Moreover, by Jensen's inequality and concavity of the logarithm, we have

$$\mathrm{E}[\log(W_t/W_{t-1})]\leq\log(\mathrm{E}[W_t/W_{t-1}]),\tag{7}$$

with equality if and only if W_t/W_{t-1} is riskless. This demonstrates that the corrected criterion $\mathrm{E}\left[\log(W_t/W_{t-1})\right]=E[\log(1+r_t-s_t)]>0$ is stricter than the traditional criterion $E[r_t]>E[s_t]$, which is equivalent to $\mathrm{E}\left[W_t/W_{t-1}\right]=E[1+r_t-s_t]>1$.

Switching to the correct criterion can be economically significant. Suppose our portfolio has a mean return of 5% and a standard deviation of 15%. The traditional rule says the mean spending rate must be less than 5%. By the Taylor series expansion, we have

$$\begin{aligned} \mathrm{E}[\log(1+r-s)] &\approx \mathrm{E}[r-s] - \left(\frac{1}{2}\right) \mathrm{Var}[r-s] \\ &= 5\% - s - \frac{1}{2}(.15)^2 \\ &= 3.875\% - s, \end{aligned}$$

which means spending must be less than about 4% instead of less than 5% We will see that this rule that the spending must be less than the mean return less half the variance becomes exact in the usual continuous-time model.

Moving to the corrected (log) criterion fixes one unreasonable feature of the traditional rule. Consider investing in a portfolio putting part of wealth in a riskless asset with mean return r and part in a risky asset with a mean return $\mu_P > r$ that might underperform the riskless asset. Then if we put a proportion θ in the stock (θ could be larger than one for a levered position), the traditional criterion says we preserve capital if $r + \theta(\mu_P - r) > s$. However, this implies that we can spend at as high a rate s as we want, so long as we take on enough risk by choosing θ to be high enough! This is absurd on its face, and due entirely to the fallacy of the traditional criterion. However, the corrected criterion does not have this problem: the curvature of the logarithm implies that given s, $\mathrm{E}[\log(1 + r + \theta(\mu_P - r) - s)] < 0$ for θ large enough, so that taking on more risk eventually constrains spending more.

As mentioned briefly before, a couple of qualifications are in order for the positive result for the riskless case and are also relevant for the risky case. First, we should work with real returns, that is, returns in excess of inflation. This adjustment is normally done correctly in practice when using the traditional criterion: we are not preserving capital if the dollar value of the endowment increases by 2%/year but inflation is 5%/year. The second qualification says that we should be careful about the timing of the cash flows. The assumption in (1) is that spending takes place at the end of the period, so the wealth relative $W_t/W_{t-1} = 1+r_t-s_t$. However, the actual timing depends on the local convention. For example, if budgeted spending for the year is taken out of the endowment and placed in a separate account at the beginning of the year, the wealth relative would be $(1-s_t)(1+r_t)$ and the criterion for preservation of capital becomes $E[\log((1-s_t)(1+r_t))] > 0$. Calculations given other convention are straightforward but can be messy. For example, if the spending $S_t = s_t W_{t-1}$, is computed at the beginning of the year but taken out in two parts, half at the start of the year and half in the middle, the wealth relative is $W_t/W_{t-1} = (1-s_t/2)(1+r_t^{H1}-s_t/2)(1+r_t^{H2})$, where r_t^{H1} is the return on the assets in the first half of the year and r_t^{H2} is the return in the second half. In general, the corrected criterion is $E[\log(W_{t+1}/W_t)] > 0$, where the real value of a unit W_t is assessed for any spending but not credited for new contributions.

It is also implicit in our analysis that there is a degree of integrity in the endowment accounting process. For example, it would be improper for the university to borrow from the endowment and count the loan as an asset. This misrepresents the value of the endowment and could be used to circumvent entirely any requirements for preservation of capital. Just spend whatever you want out of endowment, record the spending as a ten-year bullet loan, and when the loan matures roll it over into a new ten-year bullet loan. Using this device, we could spend the entire endowment without recording any spending at all. In our view a university borrowing from its own endowment seems fraudulent since it misrepresents the value of the endowment, but we do not know how the law would view this.

2.3 Preserving Capital in Continuous Time

In continuous time, the approximate criterion $s < \mu - \sigma^2/2$ becomes exact. We can write a fairly general wealth dynamic as

$$dW_t = W_t\left(\mu_t dt + \sigma_t dZ\right) - S_t dt = \left(W_t\mu_t - S_t\right)dt + W_t\sigma_t dZ. \tag{8}$$

which implies that

$$W_t = W_0 \exp\left[\int_{v=0}^{t} \left(\mu_v - \frac{1}{2}\sigma_v^2 - s_v\right) dv + \int_{v=0}^{t} \sigma_v dZ_v\right]. \tag{9}$$

If, μ, σ, and s are constant, as we have been assuming, then $\log(W_t) \sim N(\log(W_0) + (\mu - \sigma^2/2)t, \sigma^2 t)$. For any fixed $X > 0$,

$$\begin{aligned} \text{prob}(W_t \le X) &= \text{prob}(\log(W_t) \le \log(X)) \\ &= N\left(\frac{\log(X) - \log(W_0) - (\mu - \sigma^2/2 - s)t}{\sigma\sqrt{t}}\right). \end{aligned}$$

Therefore,[4]

$$\lim_{t\uparrow\infty} \text{prob}(W_t \le X) \quad \begin{cases} 0 & \text{if } s < \mu - \sigma^2/2 \\ 1/2 & \text{if } s = \mu - \sigma^2/2 \\ 1 & \text{if } s > \mu - \sigma^2/2 \end{cases},$$

Therefore, the policy preserves capital if and only if $s < \mu - \sigma^2/2$, while it destroys capital if and only if $s > \mu - \sigma^2/2$.[5]

With a little structure on the stochastic processes for returns and spending, it is sufficient for spending to be less than the expected log return *on average*. This general result allows for time-varying parameters and also implies that we can keep spending in bad times with low interest rates and risk premium providing we also do not spend too much in good times. To get started, we the definition of a covariance stationary process in continuous and a corresponding basic ergodic theorem from Shalizi and Kontorovich (2010).

[4] Note that as $t \uparrow \infty$, the argument of $N(\cdot)$ tends to $-\infty$ if $s < \mu - \sigma^2/2$, to 0 if $s = \mu - \sigma^2/2$, or to $+\infty$ if $s > \mu - \sigma^2/2$. The next expression follows because $\lim_{q\uparrow\infty} N(q) = 0$, $\lim_{q\to 0} N(q) = 1/2$, and $\lim_{q\downarrow -\infty} N(q) = 1$.

[5] In the knife-edge case when the expected log return equals the expected spending rate, i.e., $s = \mu - \sigma^2/2$, if $\sigma > 0$ then $\log W$ is a random walk and takes on arbitrarily large and small real values over time, returning with probability one to $\log W_0$ again and again, so capital is not preserved or destroyed according to Definitions 1 and 2. Since we never really know the parameters exactly, understanding the knife-edge case is a mere mathematical curiosum rather than important for practice.

Definition 3 *A stochastic process* Y_t *for* $t \in (-\infty, \infty)$ *is said to be covariance stationary (also called weakly stationary) if its mean is constant over time and autocovariances depend only on time difference. In notation,* Y_t *is stationary if there exists a mean* M *and autocovariance function* $\Gamma(\tau)$ *such that, for all* t *and* τ, $E[Y_t] = M$ *and* $\mathrm{Cov}(Y_t, Y_{t+\tau}) = \Gamma(\tau)$.

Theorem 2 *(covariance stationary ergodic theorem) A covariance stationary process* Y_t *with covariance function* $\Gamma(\tau)$ *and mean* M *has a sample mean*

$$\frac{1}{T}\int_{t=0}^{T} Y_t dt$$

that tends to M *in* L^2 *as* T *increases if and only if*

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_{\tau=0}^{T} \Gamma(\tau) d\tau = 0.$$

Proof: Immediate, from applying Shalizi and Kontorovich (2010), Theorem 298, to $X(t) \equiv Y_t - M$. ■

Note that the expectations in the statement of the theorem are unconditional expectations. If we condition on the history at some time, then the asymptotic expectation is the same as the unconditional expectation in the theorem: $E[Y_t] = \lim_{T\to\infty} E[Y_T|Y_0, Y_{-1}, ...]$.

Here is the formal result that gives conditions under which it suffices for capital to be preserved on average.

Theorem 3 *Let the wealth process be generated by (8), where* μ_t, σ_t^2, *and* s_t *satisfy* $\sigma_t > 0$ *and* $s_t > 0$. *If* μ_t, σ_t^2, *and* s_t *are all covariance stationary and ergodic in the sense of Theorem 2, then capital is preserved if* $E[s_t] < E[\mu_t - \sigma_t^2/2]$ *and capital is destroyed if* $E[s_t] < E[\mu_t - \sigma_t^2/2]$ *(where the expectations do not depend on* t *because of stationarity).*

Proof. See Subsection A.1 in the Appendix.

We have used one of the simplest ergodic theorems to derive this result, but there are many obvious extensions. For example, instead of assuming the stationarity and ergodicity of μ_t, σ_t^2, and s_t, and assuming $E[\mu_t - \sigma_t^2/2] - E[s_t] \gtrless 0$, we could assume directly the implication

$$\operatorname*{plim}_{T\to\infty} \int_{t=0}^{T} \left(\mu_t - \frac{1}{2}\sigma_t^2 - s_t\right) dt + \int_{t=0}^{T} \sigma_t dZ_t \gtrless 0,$$

which does not imply any stationarity or ergodicity. Or, we could use a assume strict ergodicity in L^1 (without our assumption that variances exist), as in Shalizi and Kontorovich (2010, Chapters 22 and 23), or we could even have conditions where the limit does not exist but the $\limsup$ is positive. Our simple version is probably adequate for most practical cases, but we can keep in mind that there are implicitly many different results mirroring the catalog of different asymptotic results in probability theory.

3 Preserving Capital with Smooth Spending

Instead of making spending strictly proportional to the size of the endowment, it is common to smooth spending using a moving-average (partial adjustment) rule to move from current spending towards a spending target. There is some economic sense to smoothing spending, since a sudden decrease in a budget can cause distress, while a sudden increase can invite waste. As a result, many endowments use some kind of smooth spending formulas. For instance, several universities in the UC system use smooth spending policy (Mercer Investment Consulting (2015)): UC Berkeley, UC Irvine, and UC Santa Cruz plan to spend about 4.5% of a twelve-quarter (three year) moving average market value of the endowment pool. Another example: Grinnell College Endowment (2014) states that endowment distribution is calculated as 4.0% of the 12-quarter moving average endowment market value determined annually as of the December 31 immediately prior to the beginning of the fiscal year. According to the NACUBO and Commonfund (2017) study of endowments, a "clear majority," 73%

in FY2017, of respondents in their study "reported that they compute their spending by applying their policy spending rate to a moving average of endowment value." See also Acharya and Dimson (2007), Chapter 4, page 112.

However, the moving average rule tends to destabilize the endowment. We illustrate this with a riskless example for which an initial high spending rate sends the fund into a "death spiral" with the wealth going to zero for sure at a known finite time. Then we give a result for risky i.i.d. returns. When risky investment returns are bad, wealth goes down, but spending is slow to adjust so the spending rate goes up. At some point the fall in wealth becomes unstable because the adjustment is not fast enough to keep the spending rate from getting large as wealth (in the denominator) falls. In a risky investment environment, over time this scenario will play out sooner or later, and capital is always destroyed.

3.1 Traditional Moving Average Spending Rule: Riskless Case

To simplify our analysis without changing the economics, we will model the moving average rule used by practitioners as a partial adjustment model. The partial adjustment model is equivalent to a moving average model with different weights, instead of the moving average rule's constant weights back a few years and zero weights before that. Furthermore, although smoothing is usually done for nominal spending rather than real spending, this also does not affect the economics and we will model smoothing in real terms:[6]

$$dS_t = \kappa \left(\tau W_t - S_t \right) dt, \tag{10}$$

where τ is the target spending rate, and κ captures the adjustment speed. If the endowment only invests in a riskless bond with constant risk-free rate r, then the

[6]For example, if inflation is always at a rate $\iota > 0$, (10) should be replaced by $dS_t = \kappa \left(\tau W_t - S_t \right) dt - \iota S_t dt$, where the new final term gives the rate of reduction of real spending due to inflation. However, this expression is identical to (10) if we redefine κ to be $\kappa + \iota$ and we redefine τ to be $\tau\kappa/(\kappa + \iota)$. This redefinition preserves $\kappa > 0$ and $\tau > 0$, so the same analysis applies.

wealth process is given as

$$dW_t = rW_t dt - S_t dt. \tag{11}$$

We assume that if W_t reaches zero, then the endowment is shut down and W_t and S_t are both zero forever afterwards if wealth reaches zero. We will also assume $\tau < r$, which implies that spending at the target rate would preserve capital, so our policy has a fighting chance. We have the following result.

Theorem 4 *When the endowment only invests in a riskless asset, the moving average spending rule (10) does not preserve capital when the initial spending rate S_0/W_0 is sufficiently high. Specifically, given the dynamic (10) and (11), wealth W_t reaches 0, if S_0/W_0 is large enough, in finite time t^*, and*

$$t^* = \frac{1}{\lambda_1 - \lambda_2} \log\left(-\frac{K_2}{K_1}\right),$$

where

$$K_1 = \frac{W_0\,(r-\lambda_2) - S_0}{\lambda_1 - \lambda_2} \text{ and } K_2 = \frac{W_0\,(\lambda_1 - r) + S_0}{\lambda_1 - \lambda_2},$$

and $\lambda_2 < 0 < \lambda_1$ is given by

$$\lambda_1 = \frac{r - \kappa + \sqrt{(\kappa - r)^2 - 4\kappa\,(\tau - r)}}{2} \text{ and } \lambda_2 = \frac{r - \kappa - \sqrt{(\kappa + r)^2 - 4\kappa\tau}}{2}.$$

Proof. See subsection A.2 in the Appendix.

If the endowment starts with high spending under the moving average rule, capital will be wiped out quickly. Given a high initial spending rate, the value of a unit declines proportionately more (due to the shortfall of interest covering spending) than spending (due to the moving average rule). As the ratio of wealth to spending falls, this effect accelerates and wealth converges to zero in a "death spiral."

3.2 Traditional Moving Average Spending Rule: Risky Case

We have just seen that if the initial spending rate is high enough, an endowment making a riskless investment and smoothing towards any positive target spending rate will destroy capital. In this section, we show that an endowment smoothing towards a target spending rate and risky portfolio strategy will destroy capital for any initial spending rate. The intuition is that the random portfolio returns will lead us sooner or later into a situation with high spending that will deplete the portfolio.

To model this, we have to make an assumption about the portfolio returns. The portfolio choices of endowments in practice are not usually linked dynamically to the current spending rate.[7] Usually, the percentage allocations to different asset classes have fixed target values or ranges. As a result, it is a reasonable approximation (and will give us the correct qualitative results) to model the endowment returns as i.i.d. Given the moving average spending rule (10), if the endowment has return with constant mean and volatility, then the wealth process is given as

$$\begin{aligned} dW_t &= W_t\left(\mu dt + \sigma dZ\right) - S_t dt \\ &= \left(W_t \mu - S_t\right) dt + W_t \sigma dZ, \end{aligned} \tag{12}$$

so long as wealth stays positive. Also assume that zero is an absorbing barrier for wealth, that is, if W_t reaches zero, then the endowment is shut down and W_t and S_t are both zero forever afterwards if wealth ever reaches zero. We have the following result.

Theorem 5 *When the endowment uses the moving average spending rule (10) with positive target spending rate τ, no matter how small, and the i.i.d. investment process (12), the value of a unit hits zero in finite time (almost surely) and therefore capital is always destroyed according to Definition 2.*

[7] Arguably, this is not ideal, see Dybvig (1999), but in this paper we are focusing on typical current practice.

Sketch of proof: Given the joint dynamics of wealth and spending, we can write the dynamics of wealth over spending (which is Markov). Then find a function F of the variable W_t/S_t such that $F(W_t/S_t)$ is a local martingale (by deriving the dynamics of $F(W_t/S_t)$ using Itô's Lemma, and set the drift term equal to zero). Note that $F(0)$ is finite and $F(\infty) = \infty$. Since $F(W_t/S_t)$ is a continuous local martingale, we can change time to a Wiener process with constant variance per unit time. We use the known properties of the first-hitting problem with constant variance and the properties of the time change (using the local variance of $F(W_t/S_t)$) to show that W_t/S_t hits zero in finite time, just like the Wiener process we get from the state change (using $F(\cdot)$) and the time change.

See subsection A.3 in the Appendix for the detailed proof.

Recall that in the riskless case, wealth goes in a death spiral to zero if initial spending is high enough, since the proportional decrease in spending does not keep up with the proportional decrease in wealth. In the stochastic case, sufficiently bad luck in investments over a short time depletes the wealth, increasing the spending rate to a high level, starting a death spiral. Subsequent good luck can save the endowment, but sooner or later the endowment will have sufficiently bad luck starting a death spiral the endowment does not recover from.

3.3 A Smooth Spending Rule that Preserves Capital

The problem with the moving average rule is that when spending is high, the rate of reduction in spending from the moving average rule is less than the rate of reduction in wealth from spending more than the geometric average return on assets. Specifically, $S_t/W_t - (\mu - \sigma^2/2)$ is the drift of S_t/W_t we would have if S_t were constant. This motivates changing the original smoothing rule (10) to the alternative smoothing rule

$$dS_t = S_t \left(\underbrace{\kappa \left(\log \tau - \log \left(\frac{S_t}{W_t} \right) \right)}_{\text{Smooth spending with target } \tau} + \underbrace{\mu - \sigma^2/2 - \frac{S_t}{W_t}}_{\text{Adjusting for over-spending}} \right) dt, \tag{13}$$

where the wealth process still follows (12). This is a smooth spending rule (S_t is a differentiable function), unlike a fixed spending proportion (for which S_t is not differentiable because W_t is not). Unlike the original smoothing rule (10), this rule does not always destroy capital. As with the fixed spending rule, whether this preserves capital depends on the parameters. Intuitively, when κ is large and σ is small, spending moves quickly towards the target level τ, and the condition for preservation of capital is similar to that for fixed spending at the rate $S_t/W_t = \tau$, but if κ is small and σ is large, only smaller target spending τ can be supported.

With the proposed spending rule (13), we can prove the following theorem.

Theorem 6 *When the endowment invests in risky assets and the wealth process follows (12), the smooth spending rule given by (13) preserves capital in the sense of Definition 1 if and only if the parameters satisfy the following condition:*

$$\mu - \frac{\sigma^2}{2} - \exp\left[\log \tau + \frac{\sigma^2}{4\kappa}\right] > 0, \tag{14}$$

while capital is destroyed if and only if the inequality is reversed.

Sketch of proof: Given the spending and wealth dynamics, $\log(S_t/W_t)$ is a stationary Gaussian process and it can be derived that S_t/W_t is a covariance-stationary process satisfying the condition of the ergodic theorem. Then the result follows by Corollary **??** for covariance-stationary processes of the general Theorem 4 for stationary processes provided in Section 4. See subsection A.4 in the Appendix for the proof. ■

The condition (14) means that the log growth rate of the risky asset have to be larger than the long-term average spending rate $\mathrm{E}[S_t/W_t] = \exp\left(\log \tau + \sigma^2/\left(4\kappa\right)\right)$.

We can compute the long-term average because the spending rate is stationary and lognormally distributed. When the speed κ of mean-reversion is very large, then the spending rate is usually very close to the target spending rate τ, which is why this converges as κ increases to the formula $\mu - \sigma^2/2 > \tau$ for a fixed spending rate τ.

4 Real and Nominal Rates

So far, we have been assuming that the interest rates are expressed in real terms. In this section we consider some examples in which we explicitly separate real and nominal interest rates.

4.1 Preserving Capital with Temporarily Negative Real Risk-Free Rate

These calculations by practitioners are done in real terms (as they should be). An interest rate environment like the current one where inflation exceeds the nominal rate is a special challenge. The endowment never preserves capital if the expected real risk-free rate of return is always negative. For example, if investments in real riskless bonds are available but the local expectations hypotheses holds, then given a little regularity, no strategy with non-negative spending and investments in only bonds will preserve capital if the long-term expected short real interest rate is negative. However, under some conditions, capital can be persevered even if the real expected rate of return is temporarily negative. This subsection models temporarily negative real rate and provides the conditions needed for preserving capital by employing the results of Theorem 3.

Let the nominal interest rate r_t be modeled by some diffusion processes. Hence, the stock price follows a diffusion process as

$$\frac{dP_t}{P_t} = (r_t - \iota + \pi)\, dt + \sigma dZ_t, \tag{15}$$

where ι is a constant inflation rate and π is a constant risk premium. With a fixed proportion θ in stock, the wealth process follows,

$$\begin{aligned} dW_t &= (r_t - \iota)\, W_t dt + W_t \theta \left(\sigma dZ_t + \pi dt\right) - S_t dt, \\ &= W_t \left(\left(r_t - \iota + \theta\pi\right) dt + \theta\sigma dZ_t\right) - S_t dt. \end{aligned}$$

Employing the results in Theorem 3, we can obtain the following theorem:

Theorem 7 *Assume the stock price process follows (15), and the endowment has a constant proportion θ in stock, and the spending rate s_t is covariance-stationary process. Then the endowment preserves capital if and only if*

$$\mathrm{E}\left[r_t - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2}\right] > \mathrm{E}\left[s_t\right]. \tag{16}$$

By Theorem 7, we can cannot gain a high expected log rate of return after taking return volatility into account, which is different from the implausible implications of the traditional rule in Subsection 2.2. Since the quadratic function with a negative coefficient of the second order term is capped over the choices of portfolio.

Now we can provide examples of spending rule with negative real interest rate, both rules preserving capital and rules not preserving capital.

Example 4 (Successful preservation of capital with temporarily negative real rate): Let the nominal interest rate follows a CIR model, i.e.,

$$dr_t = a_0 \left(b - r_t\right) dt + \sigma\sqrt{r_t} dZ_t, \tag{17}$$

where a_0 is a constant adjustment speed, and b is the long-term mean of the nominal interest rate. Let the spending rule be a modified moving average rule which potentially preserves capital, following the form of spending (13) as

$$dS_t = S_t \left(\kappa \left(\log\tau - \log\left(\frac{S_t}{W_t}\right)\right) + r_t - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} - \frac{S_t}{W_t}\right) dt,$$

which, by the results in Theorem 6, implies that $\mathrm{E}[s_t] = \exp\left[\log\tau + \sigma^2/(4\kappa)\right]$.

Given $\iota = 4\%$, $b = 4\%$, $\pi = 5\%$, $\sigma = 15\%$, $\theta = 0.8$, $\tau = 3\%$ and $\kappa = 1$, then the expected real interest rate is zero, just quite similar to real rate in the current financial market. However, the spending still can be covered by a high enough risk premium. Consequently, in a long horizon, the capital can be preserved. For instance, suppose at a point of time, the inflation rate is 4% and the real rate is -4%, then given the risk premium is 5% and the endowment cannot cover a positive spending rate with a negative return at this point. However, capital is still preserved since when during a good time, say, real interest rate is 8% and, thus, the expected return of portfolio is 13%. If the endowment still has the target spending rate, then capital is preserved. To sum up, the point is that preservation of capital is not about a point of time, it is about the evolution of the underlying processes over time. Finally, by applying Theorem 7, it is easy to see condition (16) is satisfied, since

$$b - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} - \exp\left[\log\tau + \frac{\sigma^2}{4\kappa}\right] = 0.0024 > 0,$$

hence, capital is preserved.

Example 5 (Unsuccessful preservation of capital with temporarily negative real rate): Given $\iota = 6\%$, $\mathrm{E}[r_t] = 0$, $\pi = 5\%$, and $\sigma = 15\%$, then no choice of a fixed portfolio θ and nonnegative spending rate s can preserve capital locally. Since even the portfolio which maximizes the growth rate of log wealth, i.e., $\theta = \pi/\sigma^2$ maximizing $\theta\pi - \theta^2\sigma^2/2$, cannot preserve capital. Note according to (16) in Theorem 7, we can calculate the expected log turn with highest growth rate:

$$\mathrm{E}\left[r_t\right] - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} = \mathrm{E}\left[r_t\right] - \iota + \frac{\pi^2}{2\sigma^2} = -0.0044 < 0,$$

which is a negative number. However, expected spending cannot be negative. Hence, (16) is not satisfied, and capital is not preserved due to a too high expected inflation and a too low expected nominal interest rate.

5 Optimization Models

We have been emphasizing preservation of capital as a constraint facing by the universities. The traditional practice by endowments postulates a candidate portfolio strategy and spending rule, followed by a check of what parameter values, e.g., spending rate target and portfolio weights, are consistent with preservation of capital. Alternatively, we can impose preservation of capital as a constraint in an optimization problem. Unfortunately, defining preservation of capital as in Definition 1 is not up to this task. We investigate this using the following Problem 1.

Problem 1 *Given the initial wealth W_0, choose adapted portfolio process $\{\theta_t\}_{t=0}^{\infty}$, adapted spending process $\{S_t\}_{t=0}^{\infty}$ and wealth process $\{W_t\}_{t=0}^{\infty}$ to maximize the expected utility,*

$$\begin{aligned} &\sup_{\theta,S} \mathrm{E}\left[\int_{t=0}^{\infty} D_t u\left(S_t\right) dt\right] \\ &s.t.\ dW_t = rW_t dt + \theta_t\left(\left(\mu - r\right) dt + \sigma dZ_t\right) - S_t dt, \\ &\forall t,\ W_t \geq 0, \\ &\operatorname*{plim}_{t\to\infty} W_t = \infty. \end{aligned} \tag{18}$$

where the utility function $u : \Re_+ \to \Re$ is concave and increasing. It is assumed that $\mu - r$, σ, and r are all positive and the utility discount factor $D_t \geq 0$, and

$$0 < \int_{s=0}^{\infty} D_s ds < \infty.$$

The constraint (18) is preservation of capital according to Definition 1. The functional form of the objective function is flexible enough to accommodate the short-term orientation of a college president who does not value spending beyond the end of his term. For example, if the president is confident of retiring by time T, then perhaps $D_t = 0$ for all $t > T$.

The weakness of the constraint is that it concerns only the infinite limit, but does not restrict what happens at intermediate dates. And, due to the miracle of compounding, it only takes a small amount of money set aside to satisfy the condition that a unit grows without limit over time. Intuitively, we can put almost 100% of the endowment in our favorite strategy absent the constraint on preservation of capital and two cents in a strategy that preserves capital, to achieve almost the same utility as our favorite strategy. In this way, we can make the impact of the constraint on both our strategy and our utility negligible. Here is the formal statement:

Theorem 8 *Let S_t^*, θ_t^*, and W_t^* be the feasible spending, risky asset portfolio and wealth processes with finite value for Problem 1 without the preservation-of-capital constraint (18). Then, if $r > 0$, the supremum in Problem 1 is at least the value of following this strategy. Specifically, there exists a sequence $\left(\theta_t^k, S_t^k\right)$ of feasible policies such that*

$$\lim_{k\to\infty} \mathrm{E}\left[\int_{t=0}^{\infty} D_t u\left(S_t^k\right) dt\right] \geq \mathrm{E}\left[\int_{t=0}^{\infty} D_t u\left(S_t^*\right) dt\right].$$

Proof. See subsection A.5 in the Appendix.

Note: as should be clear from the proof, most of the special structure of Problem 1 is not needed. The important thing is that there is some asset or investment strategy that can (through the miracle of compounding) convert a trivial amount of capital today into an unbounded sum over time.

Theorem 6 implies that the traditional definition of preserving capital does not have teeth when included in an optimization model as a constraint. In the following subsection, we discuss some alternative and stricter definitions of preserving capital and their implications.

5.1 Preservation of Capital in Optimization Models

Preservation of capital and smoothed spending are two desirable features of an optimization models for endowments. To make the wealth constraint more effective in preservation of capital, we can impose the drawdown constraint introduced by Grossman and Zhou (1993), which requires that wealth can never fall below a certain percentage of the previous maximum of wealth, i.e., for some given $\beta \in (0, 1)$ and for all times t,

$$W_t \geq \beta \sup_{s \leq t} W_s.$$

The drawdown constraint carries a strong sense of preservation of capital, since it adds requirements on intermediate wealth. This forms a contrast to the implications of the traditional definition of preserving capital that the wealth converges to infinity approximately for sure, which sounds pretty conservative but actually is not. Elie and Touzi (2006) treat an optimization problem with a drawdown constraint; Rogers (2013) gives a concise exposition of their main results. The solution is given in the dual and is analytical up to some constants determined numerically. To apply their model to endowment management, we should probably modify it to consider the benefits of smoothing and add other practical considerations. However, even without additional features, considering both the drawdown constraint the value of smoothing is complex, since already we have three state variables, spending, wealth, and the previous maximum wealth, and, depending on how smoothing is modeled, a subtle boundary problem. With the property of homogeneity of power utility function, we can reduce the number of state variables to two, but the solution will be difficult.

Formulating and solving a problem incorporating preference for smoothed spending seems to be difficult even without an effective preservation-of-capital constraint. A natural way to model the desirability of smoothing spending is to incorporate a cost of changing spending, either in the felicity function or in the budget constraint. Moreover, a quadratic cost term can capture the idea that a larger rate of change in spending leads to a higher adjustment cost. However, we do not know how to solve this problem, stated below, except numerically.

Consider the portfolio problem faced by an endowment choosing to allocate wealth between a riskless asset and a single risky investment (presumably a portfolio of equities) whose price process evolves according to

$$\frac{dP_t}{P_t} = \mu_P dt + \sigma_P dZ_t.$$

The instantaneous riskless rate is r. To simplify interpretation later, we assume without loss of generality that $\mu_P > r$, so that the risky asset is an attractive investment. Assume the endowment has incentive to smooth spending, the problem of the endowment can be described as follows.

Problem 2 *Given the initial wealth W_0 and initial spending S_0, choose an adapted portfolio process $\{\theta_t\}_{t=0}^{\infty}$ and an adapted rate-of-change-of-spending process $\{\delta_t = S_t'\}_{t=0}^{\infty}$ to maximize expected utility,*

$$\begin{aligned}
&\max_{\theta,\delta} \mathrm{E}\left[\int_{t=0}^{\infty} e^{-\rho t}\frac{{S_t}^{1-R}}{1-R}dt\right] \\
s.t.\ &dW_t = rW_t dt + \theta_t\left(\left(\mu_P - r\right)dt + \sigma_P dZ_t\right) - S_t dt - k\frac{\delta_t^2}{S_t}, \\
&dS_t = \delta_t dt. \\
&\forall t,\ W_t \geq 0.
\end{aligned}$$

where ρ is the pure rate of time preference, and R is the constant relative risk aversion. It is assumed that $\mu_P - r$, ρ, σ_P, k, and r are all positive constants.

Denote the value function of the endowment as V. The HJB equation is given by

$$u\left(S_s\right) - \rho V + V_W\left(rW + \theta\left(\mu_P - r\right) - S_t - k\frac{\delta_t^2}{S_t}\right) + \delta_t V_S + \frac{\sigma_P^2\theta^2}{2}V_{WW} = 0.$$

By the first-order condition, the optimal choice of change of spending is given as

$$\delta_t = \frac{S_t V_S}{2kV_W}.$$

Substitute the optimal change in spending into the HJB equation, we have

$$u\left(S_s\right)-\rho V+V_W\left(rW+\theta\left(\mu_P-r\right)-S_t\right)+\frac{S_tV_S^2}{4kV_W}+\frac{\sigma_P^2\theta^2}{2}V_{WW}=0.$$

We can simplify it by let $x\equiv W/S$, and $\Theta\equiv\theta/S$, and conjecture $V\left(S,W\right)=S^{1-R}v\left(x\right)$. As a result, we have

$$V_W\left(W,S\right)=S^{-R}v_x,\quad V_{WW}\left(W,S\right)=S^{-R-1}v_{xx},\ \text{and}\ V_S=\left(1-R\right)S^{-R}v\left(x\right)-S^{-R}xv_x.$$

The HJB equation is thus simplified and transferred into

$$\frac{\sigma_P^2\Theta^2}{2}v_{xx}+\frac{\left(\left(1-R\right)v-xv_x\right)^2}{4kv_x}+v_x\left(rx+\Theta\left(\mu_P-r\right)-1\right)-\rho v+\frac{1}{1-R}=0. \tag{19}$$

Again by first-order condition, we have the optimal scaled portfolio in stock given as

$$\Theta=-\frac{v_x\left(\mu_P-r\right)}{\sigma_P^2v_{xx}},$$

and substitute it into (19) we have,

$$-\frac{v_x^2\kappa^2}{2v_{xx}}+\frac{\left(\left(1-R\right)v-xv_x\right)^2}{4kv_x}+\left(rx-1\right)v_x-\rho v+\frac{1}{1-R}=0.$$

We do not know how to solve this ODE analytically in the primal or the dual.

6 Conclusion

Two commonly used rules of thumb used for managing endowments that are supposed to preserve capital actually do not preserve capital. Having a spending rate less than the expected return on assets is not strong enough and is based on a fallacious application of the law of large numbers. A correct analogous criterion would take logs. We can think of an approximate criterion (correct for a lognormal world) that the spending rate has to be less than the mean return on the portfolio minus half the variance.

The second rule of thumb that has problems is the use of a moving average rule to smooth spending. This type of rule never preserves capital in a model where returns are random and i.i.d. We provide alternative rules that smooth spending but in a way that preserves capital for appropriate choice of parameter values.

In this paper, the focus was from the pracitioner's lens and not based on optimization, which is less useful for practitioners than we would hope. Nonetheless, solving optimization models may be more useful than most practitioners think, since they can suggest rules of thumb that are useful in practice. We showed that even the corrected form the traditional criterion for preservation of capital is not suitable for an optimization model, since an optimization model can exploit a weakness in the criterion and bypass the requirement entirely. We included some discussion of what sort of reasonable modified criterion could be used to nontrivial effect in an optimization problem.

We hope our results will help universities to do a better job managing their endowments.

References

[1] Acharya, Shanta and Elroy Dimson (2007). Endowment asset management: investment strategies in Oxford and Cambridge. Publisher: Oxford University Press.

[2] NACUBO and Commonfund (2017). NACUBO-Commonfund Study of Endowments (NCSE), 2017.

[3] Dybvig, Philip H. (1995). Duesenberry's racheting of consumption: Optimal dynamic consumption and investment given intolerance for any decline in standard of living. The Review of Economic Studies 62(2), 287–13.

[4] Dybvig, Philip H. (1999). Using Asset Allocation to Protect Spending. Financial Analysts Journal, January-February, 49–62.

[5] Elie, R. and N. Touzi (2006). Optimal lifetime consumption and investment under a drawdown constraint. Finance and Stochastics 12, 299–330.

[6] Grossman, Sanford, and Zhongquan Zhou (1993). Optimal Investment Strategies for Controlling Drawdowns. Mathematical Finance 3, 241–276

[7] Thomas Gilbert, T. and C. Hrdlicka (2015). Why Are University Endowments Large and Risky? Review of Financial Studies 28(9), 2643–2686.

[8] Grinnell College Endowment (2014). Grinnell College Endowment Spending Policy. Approved by the Foundation Board of Trustees in February 2014.

[9] Henderson State University Foundation (2014). Henderson State University Foundation Investment Policy Statement. Approved by the Foundation Board of Directors on December 18th, 2013; Amended on March 12th, 2014.

[10] UC San Diego Foundation (2014). UC San Diego Foundation Endowment Investment And Spending Policy. Approved by the Board of Trustees, Effective December 12th, 2014.

[11] Halmos, P. R. (1956). Lectures on ergodic theory. Math. Soc. Japan.

[12] Mercer Investment Consulting, Inc. (2015). UC Annual Endowment Report Fiscal Year 2014-2015. Mercer Investment Consulting, Inc.

[13] von Neumann, J. (1932). Proof of the quasi-ergodic hypothesis. Proceedings of the National Academy of Sciences, USA 18, 70-2.

[14] Rice, M., R. A. DiMeo, and M. Porter (2012). Nonprofit asset management: effective investment strategies and oversight. Publisher: John Wiley Sons.

[15] Rogers, L. C. G. (2013). Optimal Investment. Publisher: SpringerBriefs in Quantitative Finance.

[16] Shalizi, Cosma Rohilla and Aryeh Kontorovich (2010). Almost None of the Theory of Stochastic Processes. Manuscript, Carnegie Mellon, downloaded November 26, 2018 from http://www.stat.cmu.edu/

A Appendix

A.1 Proof of Theorem 3

First we provide a lemma that will be used in the proof.

Lemma 1 *Suppose that σ_t^2 is a covariance stationary process that is ergodic in the sense of Theorem 2. Then*

$$\frac{1}{T}\int_{t=0}^{T} \sigma_t dZ_t$$

converges to 0 *in* L^2 *as* $T \to \infty$*.*

Proof: We first derive a bound on the variance of the integral of σ_t^2 from its ergodic property. By Theorem 2, $(1/T)\int_{t=0^T} \sigma_t^2 dt$ converges in L^2 to $E[\sigma^2]$, which implies that for all $\varepsilon > 0$, there exists T^* such that for all $T > T^*$, $(1/T)\int_{t=0}^{T} \sigma_t^2 dt < E[\sigma_t^2] + \varepsilon$, or equivalently, $\int_{t=0}^{T} \sigma_t^2 dt < (E[\sigma_t^2] + \varepsilon)T$. Fix any $\varepsilon > 0$. Then for all T larger than the corresponding T^*,

$$\begin{aligned}
E\left[\left(\frac{1}{T}\int_{t=0}^{T} \sigma_t dZ_t\right)^2\right] &= \mathrm{Var}\left(\frac{1}{T}\int_{t=0}^{T} \sigma_t dZ_t\right) + \left(E\left[\frac{1}{T}\int_{t=0}^{T} \sigma_t dZ_t\right]\right)^2 \\
&= \frac{1}{T^2}\int_{t=0}^{T} \sigma_t^2 dt \\
&< \frac{1}{T^2}(E[\sigma_t^2] + \varepsilon)T \\
&\to 0 \quad \text{as } T \to \infty,
\end{aligned}$$

where the second and third steps follow because a driftless Itô integral with an L^2 integrand has mean zero and variance the squared L^2 norm of the integrand (see Arnold [1973, Theorem 4.4.14(e)]. ■

Proof of Theorem 3: From (9), we have

$$\log\left(\frac{W_T}{W_0}\right) = \int_{t=0}^{T}\left(\mu_t - \frac{1}{2}\sigma_t^2 - s_t\right)dt + \int_{t=0}^{T}\sigma_t dZ_t$$

Applying Theorem 2 to each of μ_t, σ_t^2, and s_t and applying Lemma 1, we conclude that

$$\begin{aligned}\frac{1}{T}\log\left(\frac{W_T}{W_0}\right) &= \frac{1}{T}\int_{t=0}^{T}\mu_t dt - \frac{1}{T}\int_{t=0}^{T}\frac{\sigma_t^2}{2}dt - \frac{1}{T}\int_{t=0}^{T}s_t dt + \frac{1}{T}\int_{t=0}^{T}\sigma_t dZ_t \\ &\to E[\mu_t] - E[\sigma_t^2/2] - E[s_t]\end{aligned}$$

in L^2 and therefore in probability. Consequently, $\mathrm{plim}_{T\to\infty} W_T = +\infty$ (preserving capital) if $E[\mu_t] - E[\sigma_t^2/2] - E[s_t] > 0$ and $= -\infty$ (destroying capital) if $E[\mu_t] - E[\sigma_t^2/2] - E[s_t] < 0$. ■

A.2 Proof of Theorem 4

Proof. We can rewrite (10) and (11) as

$$d\binom{W_t}{S_t} = A\binom{W_t}{S_t}dt,$$

where

$$A = \begin{pmatrix} r & -1 \\ \kappa\tau & -\kappa \end{pmatrix}.$$

This ODE can be solved by using an eigenvalue-eigenvector decomposition of A. The eigenvalues of A are the two roots of the eigenvalue equation $\det(A - \lambda I) = 0$, given by

$$\lambda = \frac{r - \kappa \pm \sqrt{(\kappa - r)^2 - 4\kappa(\tau - r)}}{2} = \frac{r - \kappa \pm \sqrt{(\kappa + r)^2 - 4\kappa\tau}}{2}.$$

We will label the eigenvalues so that $\lambda_2 < 0 < \lambda_1$. The corresponding eigenvectors are given by $\phi_i = (1, r - \lambda_i)^\intercal$. The solution of the ODE is:

$$\begin{pmatrix} W_t \\ S_t \end{pmatrix} = K_1 e^{\lambda_1 t} \phi_1 + K_2 e^{\lambda_2 t} \phi_2.$$

The constants K_1 and K_2 can be determined by the initial conditions as $K_1 = (W_0\,(r - \lambda_2) - S_0)/(\lambda_1 - \lambda_2)$ and $K_2 = (W_0\,(\lambda_1 - r) + S_0)/(\lambda_1 - \lambda_2)$. Note that $0 < r - \lambda_1 < r - \lambda_2$, so that if $S_0/W_0 > r - \lambda_2$, then $K_2 > W_0$ and $K_1 = W_0 - K_2 < 0$, so wealth goes to zero in finite time and, thus, capital is not preserved in this case. Specifically, let the time that wealth reaches zero be t^*, then we have

$$W_t = K_1 e^{\lambda_1 t^*} + K_2 e^{\lambda_2 t^*} = 0 \Longleftrightarrow e^{(\lambda_1 - \lambda_2) t^*} = -\frac{K_2}{K_1} \Longleftrightarrow t^* = \frac{1}{\lambda_1 - \lambda_2} \log\left(-\frac{K_2}{K_1}\right),$$

where $\log(-K_2/K_1) > 0$ since $K_1 < 0 < K_2$ and $-K_1 = K_2 - W_0 < K_2$. ■

A.3 Proof of Theorem 5

We want to show that wealth in a unit of endowment hits zero in finite time with probability 1. The dynamics of spending and wealth are given as

$$dS_t = \kappa\,(\tau W_t - S_t)\,dt, \tag{20}$$

$$dW_t = W_t\,(\mu dt + \sigma dZ_t) - S_t dt, \tag{21}$$

until (and unless) we reach the absorbing barrier $W_t = 0$, in which case $W_t = S_t = 0$ forever afterwards. Define

$$U_t \equiv \begin{cases} 0 & \text{if } W_t = 0 \\ W_t/S_t & \text{otherwise} \end{cases}$$

By Itô's lemma,

$$dU_t = \begin{cases} (-1 + (\mu + \kappa)U_t - \kappa\tau U_t^2)\, dt + U_t\sigma dZ_t & \text{if } U_t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that spending S_t remains positive so long as wealth is positive, and tends towards a positive number when W_t first reaches 0. This implies that U_t is continuous when it hits 0, and therefore for all time.

We are going to use a martingale sample-path approach to proving our result; see Rogers and Williams [1989, IV.44-51].[8] Before providing the proof, we offer the following outline: works:

1. Find an increasing function F such that $F(U_t)$ is a local martingale (has zero drift).

2. Find a change of time to convert $F(U_t)$ into a standard Wiener process. This is possible because $F(U_t)$ is a continuous local martingale.

3. For a standard Wiener process starting at a positive value, we know it will hit zero in finite time, and we know a lot about the sample path on the way to reaching zero the first time. For example, we know the sample path is continuous and therefore almost surely bounded, and that the occupancy measure (local time) between now in the hitting time is almost surely continuous. We also know the functional form of the expected occupancy measure.

4. The occupancy measure for $F(U_t)$ can be computed as a derivative of the change of time (which depends only on $F(U_t)$, not on t) times the occupancy measure for time-changed process (the Wiener process). We can use this expression plus the known facts about the sample path of the Wiener process before hitting zero to prove U_t hits zero in finite time.

[8] In essence, we are proving the "if" side their Theorem V.51.2(ii) using the same approach but slightly different details. We cannot apply their result directly because we require a lot of notation and some preliminary results to use their result as stated.

To start with, we want to find a $\mathcal{C}^2$ function $F : \Re_{++} \to \Re$ such that $F(U_t)$ is a local martingale, i.e. has no drift. By Itô's lemma, we have

$$dF(U_t) = \begin{cases} F'(U_t)\left[\left(-1+(\mu+\kappa)U_t-\kappa\tau U_t^2\right)dt+U_t\sigma dZ_t\right] & \\ \qquad +\frac{1}{2}F''(U_t)(\sigma U_t)^2\,dt & \text{if } U_t>0 \\ 0 & \text{otherwise} \end{cases}.$$

The drift of $F(U_t)$ is always 0 if and only if F satisfies

$$F'(u)\left(-1+(\mu+\kappa)u-\kappa\tau u^2\right)+\frac{1}{2}F''(u)(\sigma u)^2=0 \tag{22}$$

One solution is

$$F(U)=\int_{u=0}^{U}\exp\left(-\frac{2(\mu+\kappa)\log(u)}{\sigma^2}-\frac{2}{\sigma^2 u}+\frac{2\kappa\tau u}{\sigma^2}\right)du.$$

We will show momentarily that the integral exists, and given that existence, the condition (22) can be verified by direct calculation. For existence of the integral, first note that in the argument to the exponential, the term $-2/(\sigma^2 u)$ dominates when $u \downarrow 0$ (so the integrand tends to 0), and the term $2\kappa\tau u/\sigma^2$ dominates when u tends to infinity, so the integrand tends to infinity. Therefore, the integrand is finite, positive, and continuous everywhere, and the integral exists. Furthermore, since the integrand is always positive, $F'(u) > 0$ and since the integrand increases without bound as u increases, $\lim_{u\uparrow\infty} F(u) = \infty$. Furthermore, $F(0) = 0$ is finite.

Since $Q_t \equiv F(U_t)$ is a continuous local martingale, it is a time-changed Wiener process (perhaps on an augmented probability space). Specifically, there exists a Wiener process B_s starting at $B_0 = Q_0$ with variance one per unit time and a continuous and increasing time change $t = v(s)$ with $v(0) = 0$, such that $Q_{v(s)} = B_s$ up to the first time $Q_{v(s)}$ hits zero. Matching the cumulative variance, the time change can be computed implicitly as $\Sigma(v(s)) = s$ where the random process $\Sigma(t) = \int_{z=0}^{t} \mathrm{var}(dQ_z)$, the increasing cumulative variance (quadratic variation) process for Q_t in the original time frame. Applying Itô's Lemma to $Q_t = F(U_t)$, we have

$$dQ_t = F'(U_t)\sigma U_t dZ_t \tag{23}$$

and therefore

$$\Sigma(dQ_t) = \int_{\tau=0}^{t} (F'(U_\tau))^2 \sigma^2 U_\tau^2 d\tau. \tag{24}$$

Since F is one-to-one, $U_t = I(Q_t)$ where $I(\cdot)$ is the inverse function of F such that $I(F(u)) = u$, and therefore the rate of time change is a function of Q_t. This allows a characterization of whether the boundary $U_t = 0$ is hit in finite time.

In the time-changed version, B_s is a standard Wiener process, so B_s first hits zero at a random but finite time s, call it H_0. Therefore, Q_t hits zero in finite time if $v(H_0)$ is finite. Now, the spatial density of occupation for any location q over the time interval $[0, s]$ is given by the local time l_s^q of the process B_s, and the passage of time is a factor of $dt/ds = v'(s) = g(B_s)$ faster in the original version, where

$$g(q) \equiv \frac{1}{(F'(I(q))\sigma I(q))^2}. \tag{25}$$

Therefore, the time until wealth hits zero (which we want to show to be finite) is given by

$$\begin{aligned} \int_{s=0}^{H_0} g(B_s) ds &= \int_{q=0}^{\infty} l_{H_0}^q g(q) dq \\ &= \int_{q=0}^{Q_0} l_{H_0}^q g(q) dq + \int_{q=Q_0}^{\infty} l_{H_0}^q g(q) dq, \end{aligned}$$

Now the second term is finite a.s., since B_s is continuous and therefore bounded on the compact interval $[0, H_0]$, and $l_{H_0}^q$ is continuous and equals zero outside of $\max\{B_s | s \in [0, H_0]\}$. By Rogers and Williams [1987], V.51.1(i), for all $y > 0$, $E[l_{H_0}^q] = \min(q, Q_0)$, which implies that the expectation of the first term is $\int_{q=0}^{Q_0} q g(q) dq$. It suffices to show this expectation of the first term is finite, since that implies the first term is finite almost surely. Now,

$$\begin{aligned}
\int_{q=0}^{Q_0} q g(q) dq &= \int_{q=0}^{Q_0} \frac{1}{\left(F'(\Sigma(q))\sigma\Sigma(q)\right)^2} q dq \\
&= \int_{u=0}^{U_0} \frac{1}{(F'(u)\sigma u)^2} F(u)F'(u) du \\
&= \int_{u=0}^{U_0} \frac{F(u)}{F'(u)\sigma^2 u^2} du.
\end{aligned}$$

All we have left to show is that this integral is finite. The integrand is continuous on $(0, U_0]$, so it suffices to show is that it has a finite limit at 0. Using L'Hôpital's rule and (22), we have

$$\begin{aligned}
\lim_{u\downarrow 0} \frac{F(u)}{F'(u)\sigma^2 u^2} &= \lim_{u\downarrow 0} \frac{F'(u)}{F''(u)\sigma^2 u^2 + 2F'(u)\sigma^2 u} \\
&= \lim_{u\downarrow 0} \left(\sigma^2 u^2 F''(u)/F'(u) + 2\sigma^2 u\right)^{-1} \\
&= \lim_{u\downarrow 0} \left(\sigma^2 \left(-\frac{2(\mu+k)}{\sigma^2} u + \frac{2}{\sigma^2} - \frac{2\kappa\tau}{\sigma^2} u^2\right) + 2\sigma^2 u\right)^{-1} \\
&= \left(\sigma^2(-0 + \frac{2}{\sigma^2} - 0) + 0\right)^{-1} \\
&= \frac{1}{2}
\end{aligned}$$

which is finite. ■

A.4 Proof of Theorem 6

The spending and wealth dynamics are

$$dS_t = \left(S_t \left(\kappa \left(\log\tau - \log\left(\frac{S_t}{W_t}\right)\right) - \frac{S_t}{W_t} + \mu - \frac{\sigma^2}{2}\right)\right) dt$$

and

$$dW_t = W_t \left(\mu dt + \sigma dZ\right) - S_t dt.$$

Then, by Itô's lemma, $\log(S_t/W_t)$ is an Ornstein-Uhlenbeck velocity process

$$d\log\left(\frac{S_t}{W_t}\right) = \kappa\left(\log\tau - \log\left(\frac{S_t}{W_t}\right)\right)dt - \sigma dZ_t,$$

which has the moving average representation

$$\log\left(\frac{S_0}{W_0}\right) = \log\tau - \sigma\int_{-\infty}^{0} e^{\kappa t}dZ_v,$$

hence, the process of $\log\left(S_t/W_t\right)$ is stationary with constant mean, variance, and autocovariance

$$\begin{aligned}
\mathrm{E}\left[\log\left(\frac{S_t}{W_t}\right)\right] &= \log\tau, \\
\mathrm{Var}\left[\log\left(\frac{S_t}{W_t}\right)\right] &= \frac{\sigma^2}{2\kappa}, \\
\mathrm{Cov}\left[\log\left(\frac{S_v}{W_v}\right), \log\left(\frac{S_t}{W_t}\right)\right] &= \frac{\sigma^2}{2\kappa}e^{-\kappa|t-v|}.
\end{aligned}$$

As a result, S_t/W_t is log-normal distributed with mean, variance, and autocorrelation

$$\begin{aligned}
\mathrm{E}\left[\frac{S_t}{W_t}\right] &= \exp\left(\log\tau + \frac{\sigma^2}{4\kappa}\right), \\
\mathrm{Var}\left[\frac{S_t}{W_t}\right] &= \left(\exp\left(\frac{\sigma^2}{2\kappa}\right) - 1\right)\exp\left(2\log\tau + \frac{\sigma^2}{2\kappa}\right), \\
\mathrm{Cov}\left[\frac{S_v}{W_v}, \frac{S_t}{W_t}\right] &= \left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa|t-v|}\right) - 1\right)\exp\left(2\log\tau + \frac{\sigma^2}{2\kappa}\right).
\end{aligned}$$

Note that the autocovariance depends only on the lag $|t-v|$ and not on time t. Therefore, S_t/W_t is also covariance stationary.

We now prove it is a mean-square ergodic process. Note the *integral time scale* of the stationary random process S_t/W_t is given as

$$\begin{aligned}\Upsilon_{int} &= \frac{1}{\left(\exp\left(\frac{\sigma^2}{2\kappa}\right)-1\right)\exp\left(2\log\tau+\frac{\sigma^2}{2\kappa}\right)}\int_0^\infty\left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi}\right)-1\right)\exp\left(2\log\tau+\frac{\sigma^2}{2\kappa}\right)d\varphi\\ &= \frac{1}{\exp\left(\frac{\sigma^2}{2\kappa}\right)-1}\int_0^\infty\left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi}\right)-1\right)d\varphi.\end{aligned}$$

Let

$$\begin{aligned}&u=\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi}\Longrightarrow\frac{2\kappa}{\sigma^2}u=e^{-\kappa\varphi}\Longrightarrow-\kappa\varphi=\log\left(\frac{2\kappa}{\sigma^2}u\right)\Longrightarrow-\kappa d\varphi=d\log\left(\frac{2\kappa}{\sigma^2}u\right)\\ &\quad\Longrightarrow-\kappa d\varphi=\frac{2\kappa}{\sigma^2}\frac{\sigma^2}{2\kappa u}du\Longrightarrow-\kappa d\varphi=\frac{1}{u}du\Longrightarrow d\varphi=\frac{1}{-\kappa u}du,\end{aligned}$$

hence, we have

$$\int_0^\infty\left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi}\right)-1\right)d\varphi=-\int_{\frac{\sigma^2}{2\kappa}}^0\frac{e^u-1}{\kappa u}du=\frac{1}{\kappa}\int_0^{\frac{\sigma^2}{2\kappa}}\frac{e^u-1}{u}du.$$

Note

$$\lim_{u\to0}\frac{e^u-1}{u}=\lim_{u\to0}\frac{e^u}{1}=1,$$

and $(e^u-1)/u$ strictly increases in u, hence,

$$1\le\frac{e^u-1}{u}\le\frac{2\kappa}{\sigma^2}\left(e^{\frac{\sigma^2}{2\kappa}}-1\right),\text{ where }0\le u\le\frac{\sigma^2}{2\kappa}.$$

Therefore,

$$\int_0^{\frac{\sigma^2}{2\kappa}}\frac{e^u-1}{u}du<\infty\Longrightarrow\Upsilon_{int}<\infty.$$

Hence, based on the Mean-Square Ergodic Theorem (Finite Autocovariance Time), [9]

[9] The original proof of the ergodic theorem was in von Neumann (1932). It is based on the spectral decomposition of unitary operators. Later a number of other proofs were published. The simplest is due to F. Riesz, see Halmos (1956).

we have that the process S_t/W_t is mean-square ergodic in the first moment, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\int_{v=0}^{t} \frac{S_v}{W_v} dv = \exp\left[\log\tau + \frac{\sigma^2}{4\kappa}\right],$$

the average converges in squared mean over time. According to the properties of mean-square ergodic convergence, we have

$$\lim_{t\to\infty} \mathrm{E}\left[\frac{1}{t}\int_{v=0}^{t} \frac{S_v}{W_v} dv\right] = \exp\left[\log\tau + \frac{\sigma^2}{4\kappa}\right], \tag{26}$$

$$\lim_{t\to\infty} \mathrm{Var}\left[\frac{1}{t}\int_{v=0}^{t} \frac{S_v}{W_v} dv\right] = 0. \tag{27}$$

By the definition of preservation of capital, to prove the spending rule preserves capital, we need to prove

$$\operatorname*{plim}_{t\to\infty} \log\frac{W_t}{W_0} = \infty.$$

Note

$$W_t = W_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t - \sigma Z_t - \int_{v=0}^{t} \frac{S_v}{W_v} dv\right],$$

hence, we have

$$\begin{aligned} \log\frac{W_t}{W_0} &= \left(\mu - \frac{\sigma^2}{2} - \frac{1}{t}\int_{v=0}^{t} \frac{S_v}{W_v} dv\right) t - \sigma Z_t, \\ \implies \frac{1}{t}\log\frac{W_t}{W_0} &= \mu - \frac{\sigma^2}{2} - \frac{1}{t}\int_{v=0}^{t} \frac{S_v}{W_v} dv - \frac{\sigma}{t} Z_t. \end{aligned}$$

According to the Chebyshev's inequality, we have for $\forall\epsilon > 0$,

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0} - \mathrm{E}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)\right| \geq \epsilon\right) \leq \frac{\mathrm{Var}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)}{\epsilon^2}. \tag{28}$$

Moreover, note

$$Z_t \sim N(0,t), \quad \text{and} \quad -\frac{\sigma}{t} Z_t \sim N\left(0, \frac{\sigma^2}{t}\right),$$

and

$$\frac{1}{t}\int_{v=0}^{t}\frac{S_v}{W_v}dv \xrightarrow{L^2} \exp\left[\log\tau+\frac{\sigma^2}{4\kappa}\right],$$

hence, based on the results of (26) and (27), we have as $t \to \infty$,

$$\mathrm{E}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)=\mu-\frac{\sigma^2}{2}-\exp\left[\log\tau+\frac{\sigma^2}{4\kappa}\right],\ \text{and}\ \mathrm{Var}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)=\frac{\sigma^2}{t}.$$

Then according (28), we have as $t \to \infty$,

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0}-\left(\mu-\frac{\sigma^2}{2}-\exp\left[\log\tau+\frac{\sigma^2}{4\kappa}\right]\right)\right|\geq\epsilon\right)\leq 0.$$

Since probability cannot be negative, hence, we have as $t \to \infty$, for $\forall\epsilon > 0$

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0}-\left(\mu-\frac{\sigma^2}{2}-\exp\left[\log\tau+\frac{\sigma^2}{4\kappa}\right]\right)\right|\geq\epsilon\right)= 0.$$

Therefore, according to the definition of convergence in probability, we have

$$\operatorname*{plim}_{t\to\infty}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)=\mu-\frac{\sigma^2}{2}-\exp\left[\log\tau+\frac{\sigma^2}{4\kappa}\right].$$

By the condition (14)

$$\mu-\frac{\sigma^2}{2}-\exp\left[\log\tau+\frac{\sigma^2}{4\kappa}\right]>0,$$

hence, we have

$$\operatorname*{plim}_{t\to\infty}\left(\log\frac{W_t}{W_0}\right)=\infty \Longrightarrow \lim_{t\to\infty}\Pr\left(W_t<W_0\right)=0.$$

Given

$$\mu-\frac{\sigma^2}{2}-\exp\left[\log\tau+\frac{\sigma^2}{4\kappa}\right]<0,$$

we have

$$\operatorname*{plim}_{t\to\infty}\left(\log\frac{W_t}{W_0}\right)=-\infty \Longrightarrow \lim_{t\to\infty}\Pr\left(W_t<W_0\right)=1,$$

which completes the proof. ■

A.5 Proof of Theorem 8

Let S_t^*, θ_t^*, and W_t^* be the feasible spending, investment, and wealth whose value we want to match in the limit. Consider the alternative safe strategy

$$S_t^{\text{safe}} \equiv rW_0/2, \theta_t^{\text{safe}} \equiv 0, \quad \text{and} \quad W_t^{\text{safe}} \equiv (1 + e^{rt})W_0/2.$$

Then we will let

$$\begin{aligned} S_t^k &= (1 - 1/(k+1))S_t^* + (1/(k+1))S_t^{\text{safe}}, \\ \theta_t^k &= (1 - 1/(k+1))\theta_t^* + (1/(k+1))\theta_t^{\text{safe}}, \\ W_t^k &= (1 - 1/(k+1))W_t^* + (1/(k+1))W_t^{\text{safe}}. \end{aligned}$$

It is easy to check that this is feasible for every k. Let M be the product of probability measure (across states) and Lebesgue measure (for positive times). Then, noting that probability measure integrates to one, we can write the expected utility of the safe strategy, S_t^{safe}, as

$$\int D_t u(S_t^{\text{safe}})dM = u(rW_0/2) \int_{t=0}^{\infty} D_t dt,$$

which is finite because $\int_{t=0}^{\infty} D_t dt$ and $u(rW_0/2)$ are both finite. In other words, $D_t u(S_t^{\text{safe}}) \in \mathcal{L}^1(M)$. Since the strategy (θ^*, S^*, W^*) has finite value, we also know that $D_t u(S_t^*) \in \mathcal{L}^1(M)$. It also follows that

$$z_{min} \equiv \min(D_t u(S_t^{\text{safe}}), D_t u(S_t^*)) \in \mathcal{L}^1(M),$$

and

$$z_{max} \equiv \max \min(D_t u(S_t^{\text{safe}}), D_t u(S_t^*)) \in \mathcal{L}^1(M).$$

Since $(\forall k) z_{min} \leq D_t u(S_t^k) \leq z_{max}$ and $D_t u(S_t^k)$ converges almost-surely to $D_t u(S_t^*)$, then

$$\int D_t u(S_t^k) dM \to \int D_t u(S_t^*) dM, \text{ as } k \to \infty,$$

which is another way of stating the required convergence of expected utility. ■