

離散最適化基礎論

第10回

妥当不等式の追加

岡本 吉央 (電気通信大学)

okamotoy@uec.ac.jp

2022年12月13日

最終更新：2022年12月14日 08:32

<準備>

1. 整数計画法と線形計画法 (10/4)
2. 線形計画法の復習 (1) : 線形不等式系と凸多面集合 (10/11)
- * 休み (体育祭) (10/18)
3. 線形計画法の復習 (2) : 単体法と双対定理 (10/25)
4. 線形計画緩和 (11/1)

<モデリング>

5. 整数計画モデリング (1) : 組合せ最適化問題 (11/8)
6. 整数計画モデリング (2) : より複雑な問題 (11/15)
7. 整数計画モデリング (3) : 離接計画 (11/22)

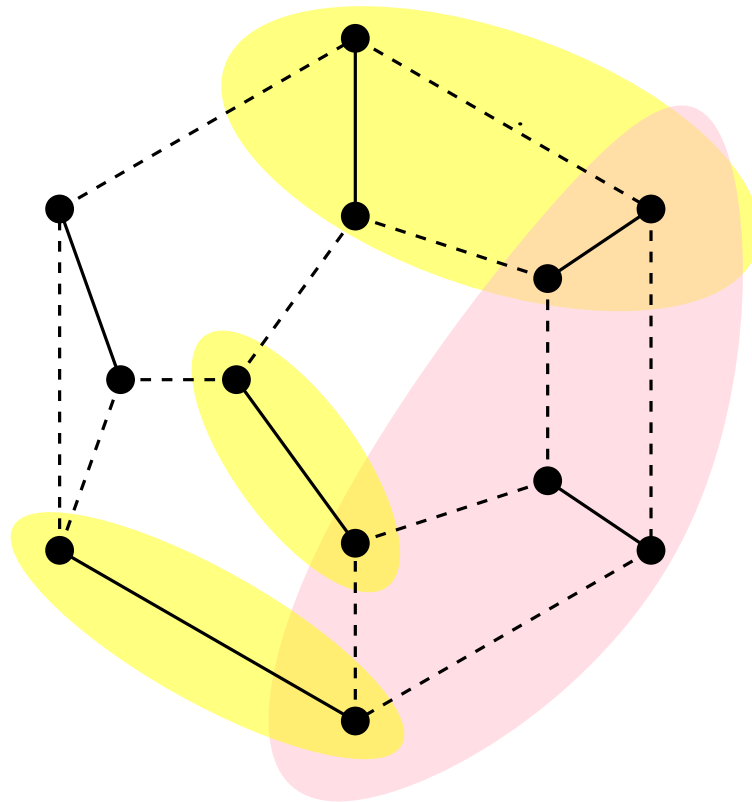
<アルゴリズム>

- | | |
|-------------------------------|---------|
| 8. 分枝限定法 | (11/29) |
| 9. 切除平面法 | (12/6) |
| 10. 妥当不等式の追加 | (12/13) |
| 11. 列生成法 | (12/20) |
| * 休み (国内出張) | (12/27) |
| * 休み (冬季休業) | (1/3) |
| 12. ラグランジュ緩和 (1) : 原理 | (1/10) |
| 13. ラグランジュ緩和 (2) : 最適ラグランジュ緩和 | (1/17) |

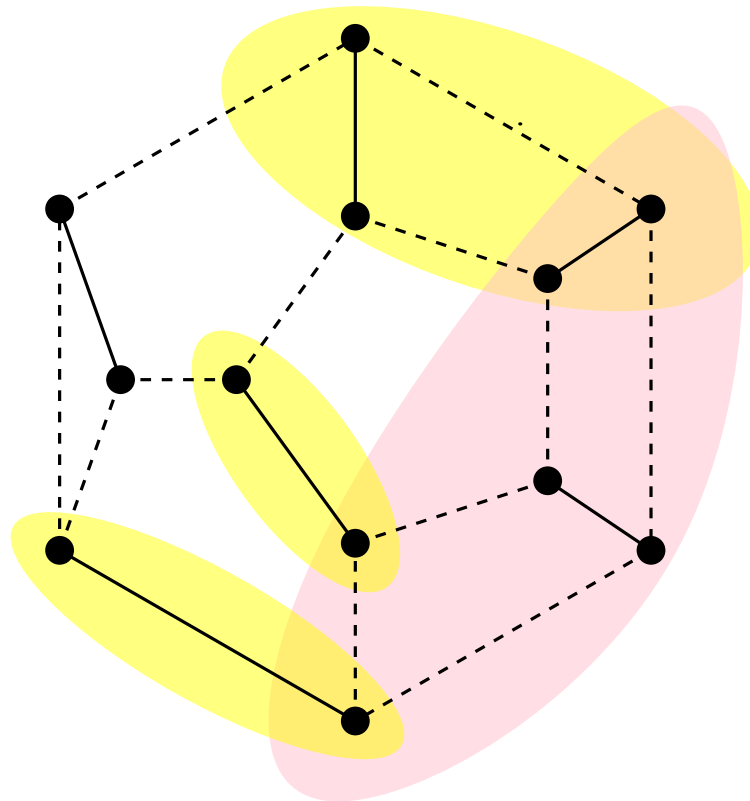
<まとめ・予備>

- | | |
|---------|--------|
| 14. まとめ | (1/24) |
| 15. 予備日 | (1/31) |

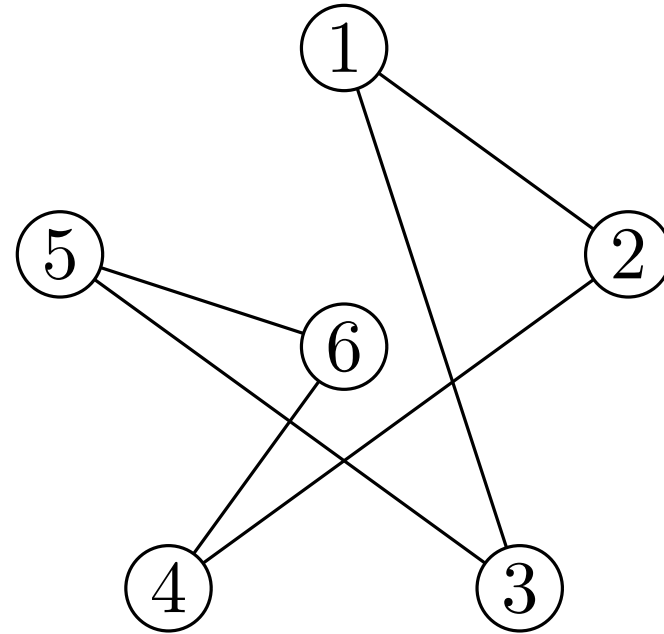
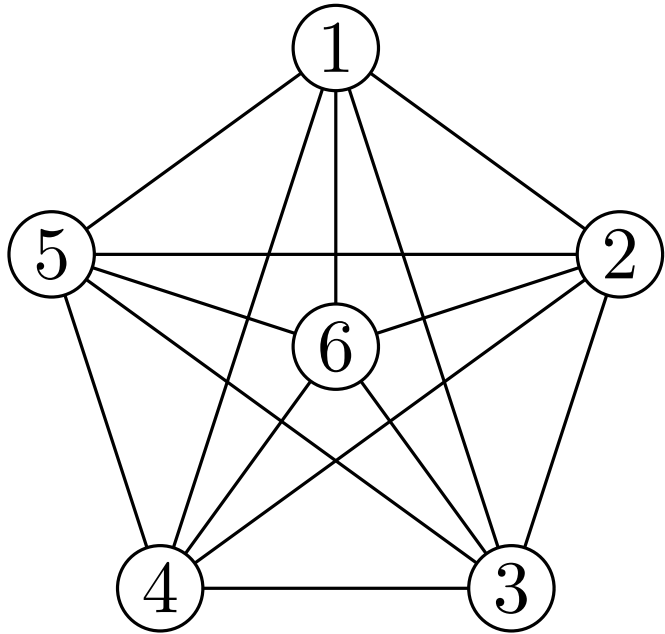
- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



対称巡回セールスマン問題もグラフの問題とみなす



無向グラフ $G = (V, E)$

$V =$ 都市集合

$E = \{\{i, j\} \mid i, j \in V, i \neq j\}$

すべての頂点を通る単一の閉路

ハミルトン閉路とも呼ばれる

$$\text{minimize } \sum_{e \in E} d(e)x_e$$

Dantzig, Fulkerson, Johnson ('54)

$$\text{subject to } \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \in V$$

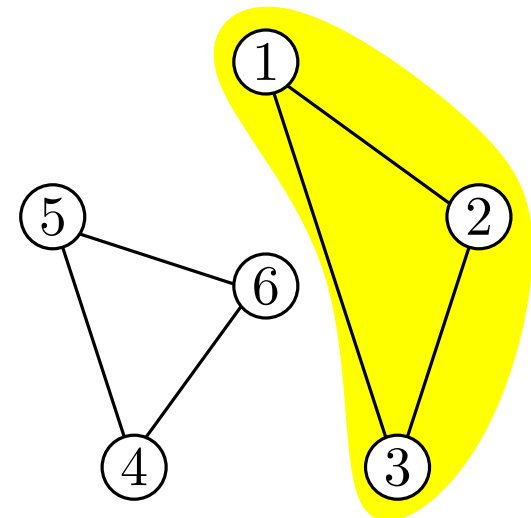
$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in 2^V - \{\emptyset, V\}$$

$$x_e \in \{0, 1\} \quad \forall e \in E$$

部分巡回路除去制約

$$\text{変数の総数} = |E| = \frac{1}{2} |V| (|V| - 1)$$

$$\text{制約の総数} = |V| + 2^{|V|} - 2$$



Dantzig, Fulkerson, Johnson ('54)

$$\text{minimize } \sum_{e \in E} d(e)x_e$$

$$\text{subject to } \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \in V$$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in 2^V - \{\emptyset, V\}$$

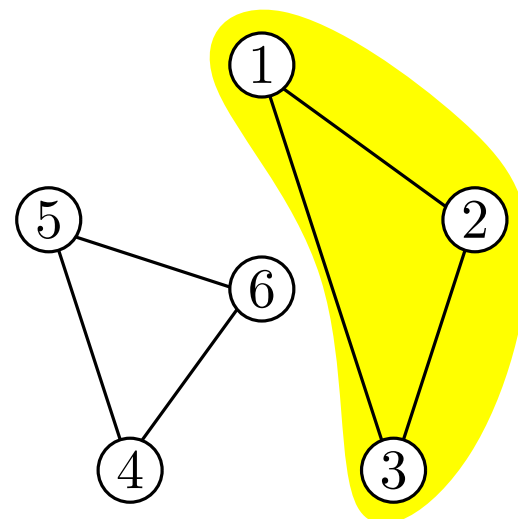
$$x_e \in \{0, 1\} \quad \forall e \in E$$

別の書き方

$$\sum_{e \in \delta(i)} x_e = 2$$

i を端点とする辺の集合

部分巡回路除去制約



課題

巡回セールスマン問題の整数計画モデルには制約が膨大に存在

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} d(e)x_e \\ \text{subject to} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \in V \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in 2^V - \{\emptyset, V\} \\ & x_e \in \{0, 1\} \quad \forall e \in E \end{aligned}$$

解決法

制約を **アルゴリズムの実行の中で** 追加していく

〜 第9回, 第10回講義の内容
(切除平面法)

解きたい問題 **P** $\xrightarrow{\text{緩和}}$ 線形計画緩和 **R**

$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & x_e \in \{0, 1\} \quad \forall e \end{aligned}$$

$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

まず、線形計画緩和 **R** を解きたい

しかし、**制約の数が多すぎて**、そのまま解くのは難しい

線形計画緩和 R

$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}} \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

小さな部分集合 $\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$ を考える

線形計画緩和 $R(\mathcal{F})$

$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}} \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

線形計画緩和 R



$$\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$$

線形計画緩和 $R(\mathcal{F})$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}}$$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}}$$

線形計画緩和 R



$$\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$$

線形計画緩和 $R(\mathcal{F})$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}}$$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}}$$

性質

x^* が $R(\mathcal{F})$ の最適解 かつ x^* が R の許容解
 $\Rightarrow x^*$ は R の最適解

線形計画緩和 R



$$\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$$

線形計画緩和 $R(\mathcal{F})$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}}$$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}}$$

性質

x^* が $R(\mathcal{F})$ の最適解 かつ x^* が R の許容解
 $\Rightarrow x^*$ は R の最適解

証明 : x^* が仮定を満たすとする

x を R の任意の許容解とすると, x は $R(\mathcal{F})$ の許容解でもある

$$\therefore \sum_{e \in E} d(e)x_e^* \leq \sum_{e \in E} d(e)x_e$$

□

線形計画緩和をどう解くか？ (3)

11/77

線形計画緩和 R

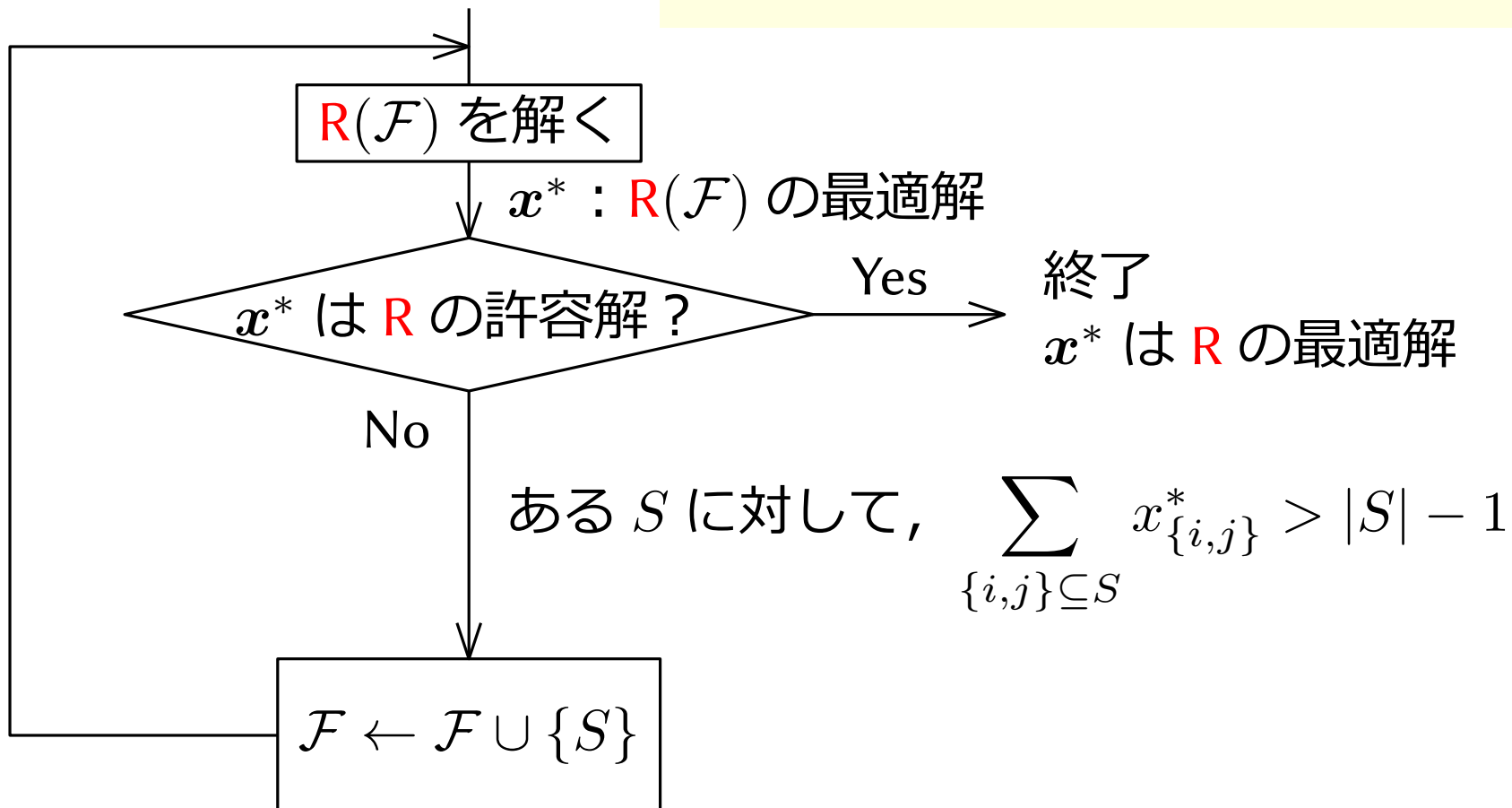


$$\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}}$$

線形計画緩和 $R(\mathcal{F})$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}}$$

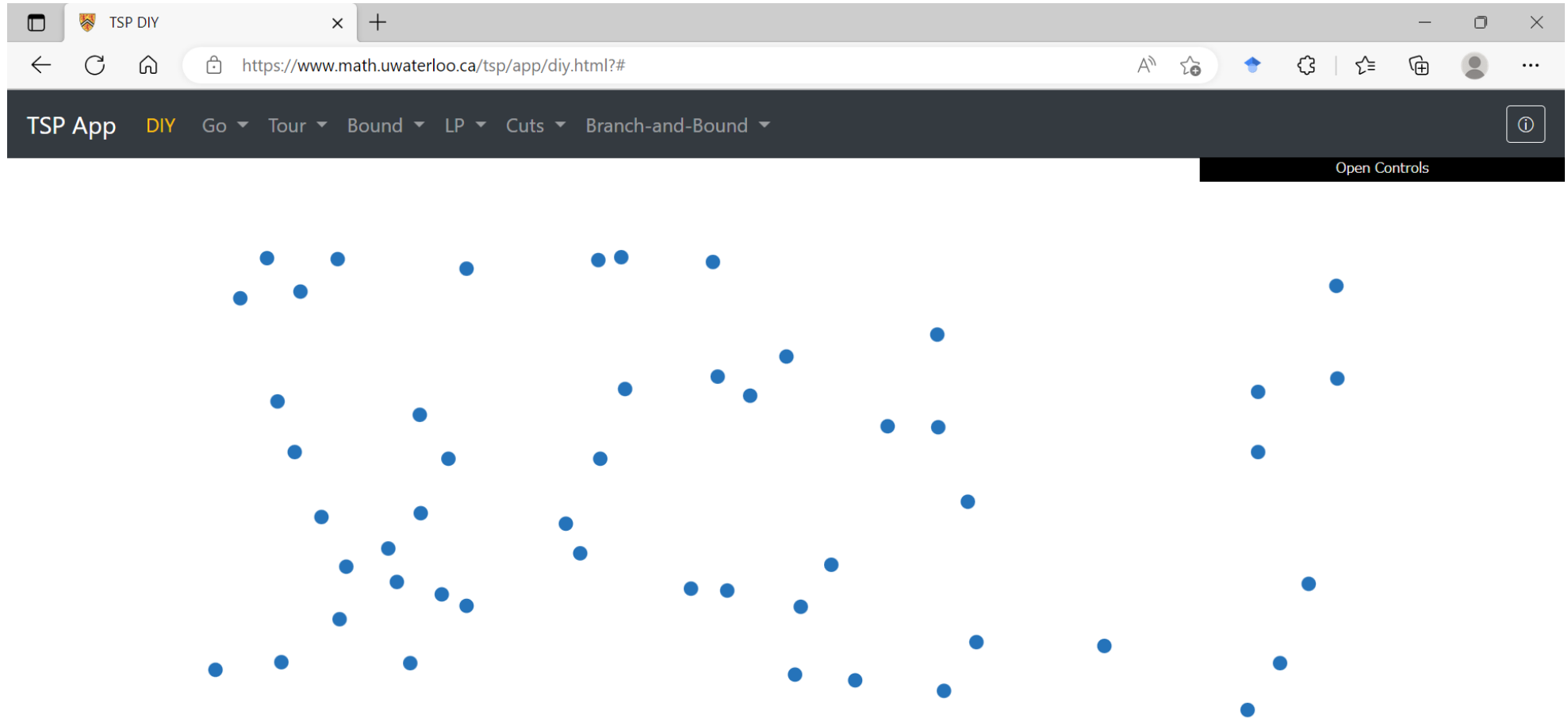


部分巡回路除去制約の追加：例 (1)

12/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

random seed = 3.14, # points = 50



TSP instance with 50 points.

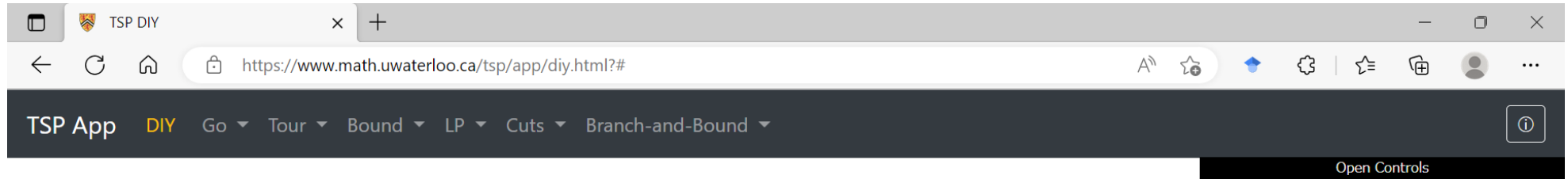
部分巡回路除去制約の追加：例 (2)

13/77

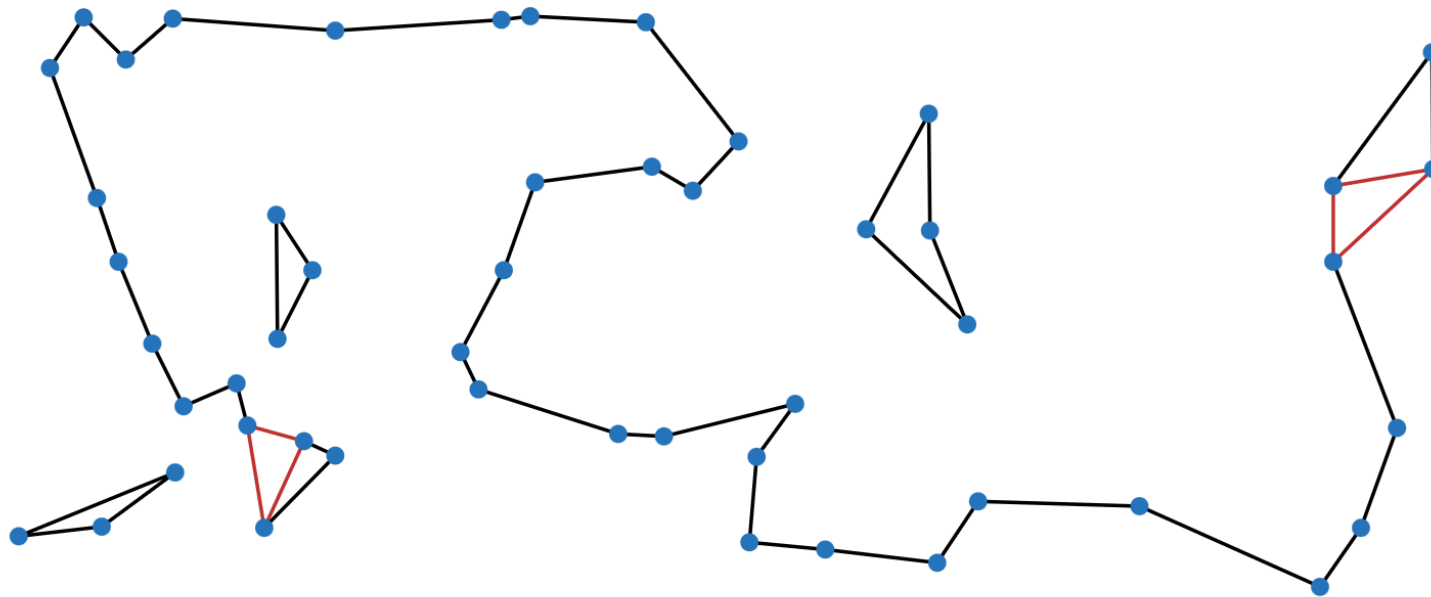
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$$|\mathcal{F}| = 0$$

random seed = 3.14, # points = 50



The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser window, the TSP App interface is shown. The top navigation bar includes "TSP App", "DIY", and several dropdown menus: "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A button labeled "Open Controls" is located in the bottom right corner of the interface.



Solution of degree LP relaxation. Bound = 41476.5.

部分巡回路除去制約の追加：例 (3)

14/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

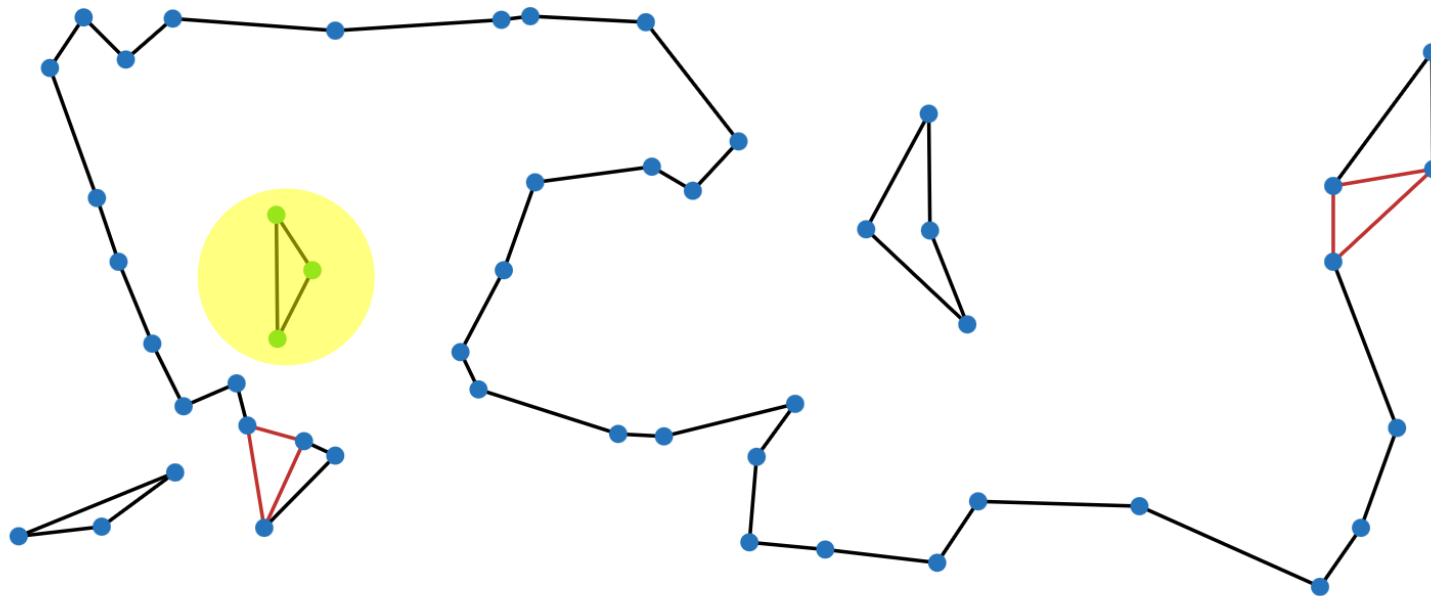
$$|\mathcal{F}| = 0$$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser window is the TSP App interface. The top navigation bar includes 'TSP App', 'DIY', and several dropdown menus: 'Go', 'Tour', 'Bound', 'LP', 'Cuts', and 'Branch-and-Bound'. A red 'Add Cut' button is located on the left. On the right, there is an 'Open Controls' button. The main area displays a tour of 50 points (blue dots) connected by black lines. A specific cut is highlighted in yellow, consisting of three green points forming a triangle. Other parts of the tour are highlighted in red.

Add Cut

Open Controls



Selected set has cut value 0. Violated!

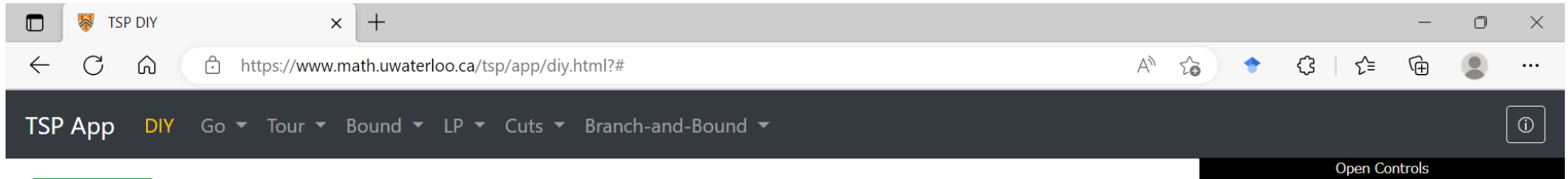
部分巡回路除去制約の追加：例 (4)

15/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 1$

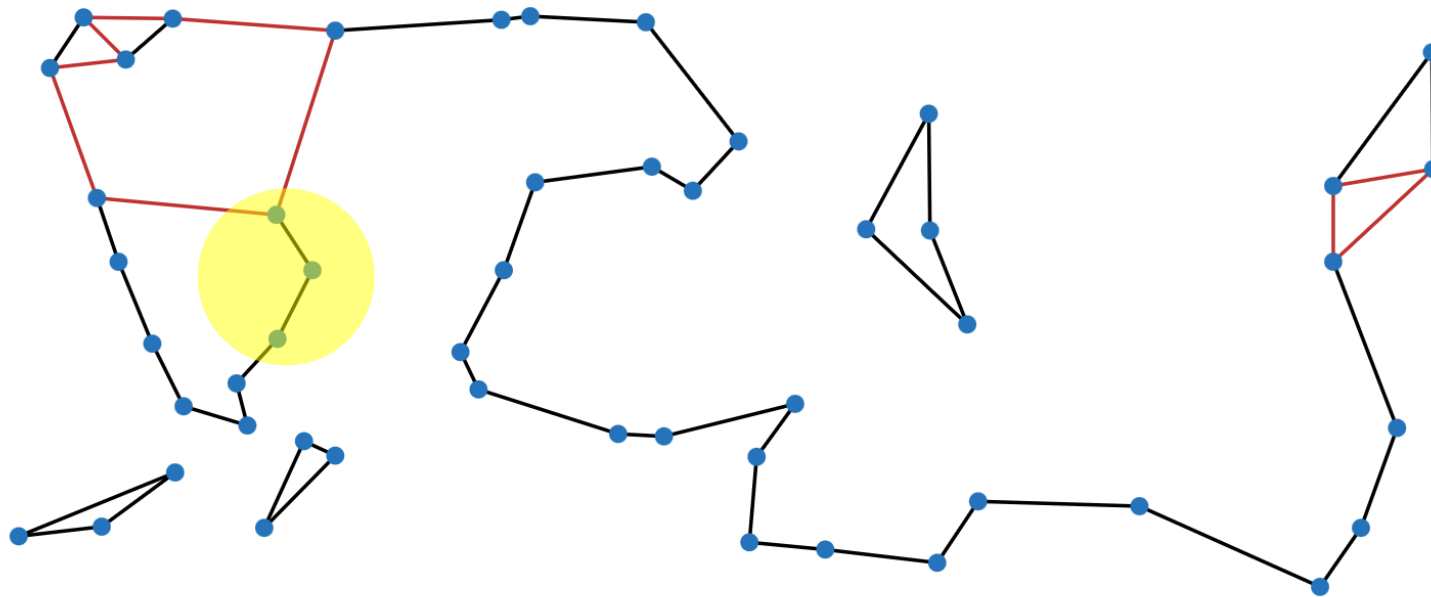
random seed = 3.14, # points = 50



TSP App **DIY** Go Tour Bound LP Cuts Branch-and-Bound

Open Controls

Build a Cut



Added a subtour cut. Bound = 41485.5 (+9).

部分巡回路除去制約の追加：例 (5)

16/77

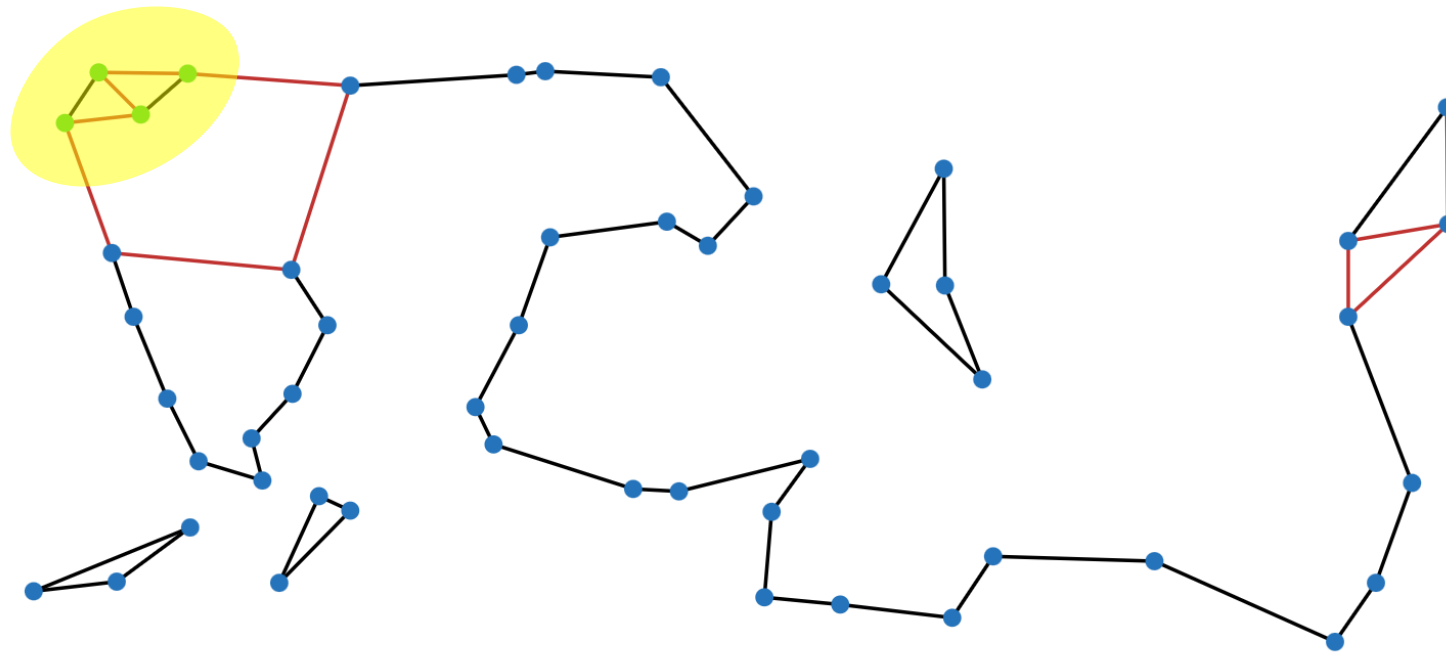
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 1$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser, there is a dark navigation bar with the text "TSP App" and several menu items: "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". On the right side of this bar is an "Open Controls" button. The main area of the browser displays a Traveling Salesman Problem (TSP) tour. The tour is represented by a series of blue dots (points) connected by black lines. A specific set of points, consisting of a small cluster of four points, is highlighted with a yellow oval. The edges connecting these four points are colored red, indicating they are part of the selected set. A red button labeled "Add Cut" is positioned to the left of the highlighted set.

Add Cut



Selected set has cut value 1. Violated!

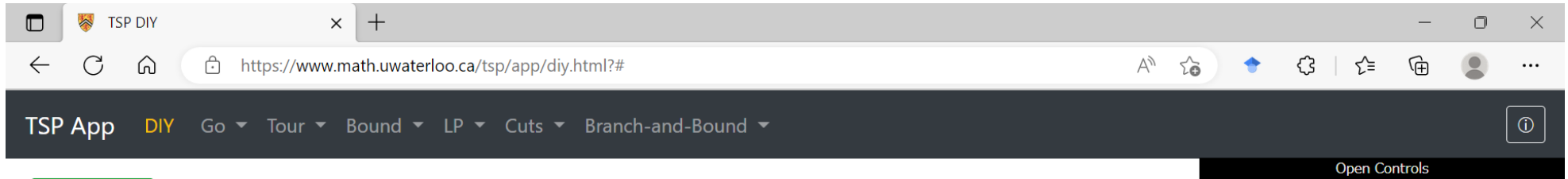
部分巡回路除去制約の追加：例 (6)

17/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 2$

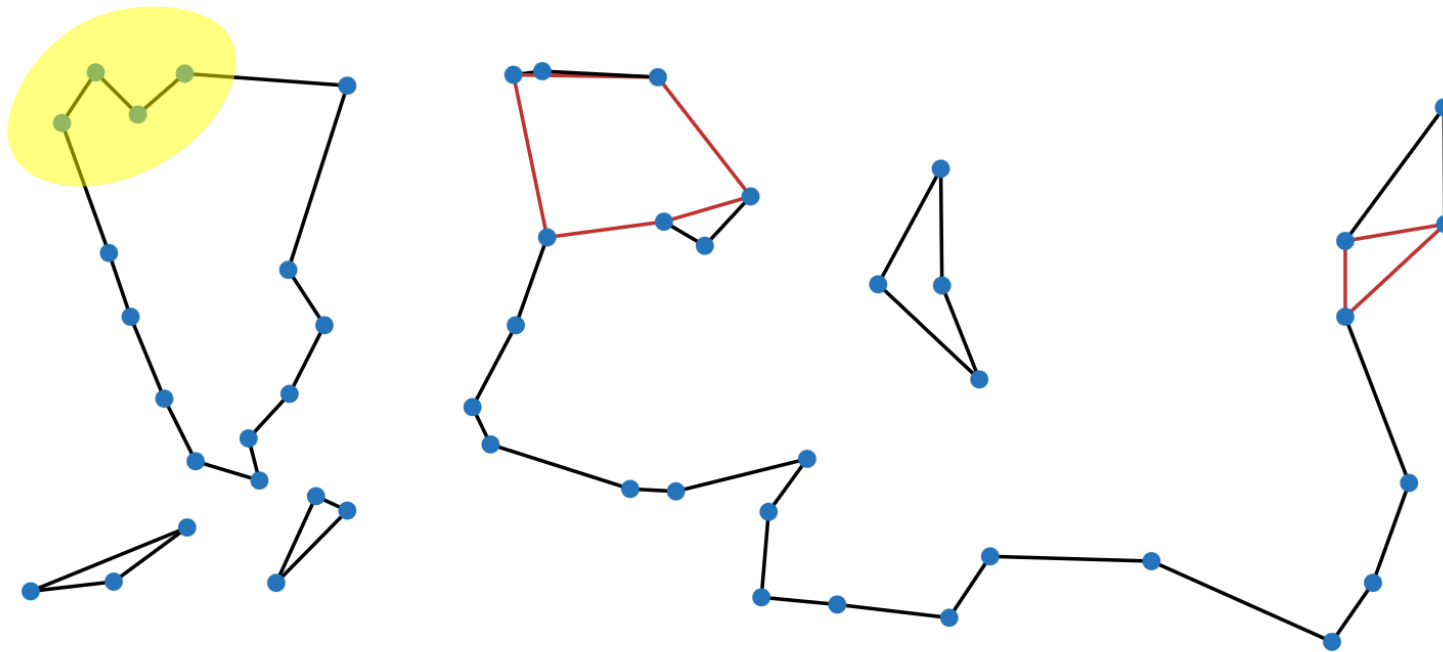
random seed = 3.14, # points = 50



TSP App **DIY** Go Tour Bound LP Cuts Branch-and-Bound

Open Controls

Build a Cut



Added a subtour cut. Bound = 41517 (+31.5).

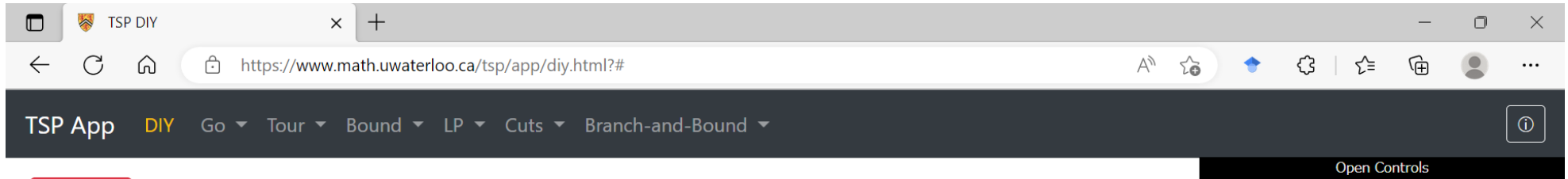
部分巡回路除去制約の追加：例 (7)

18/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 2$

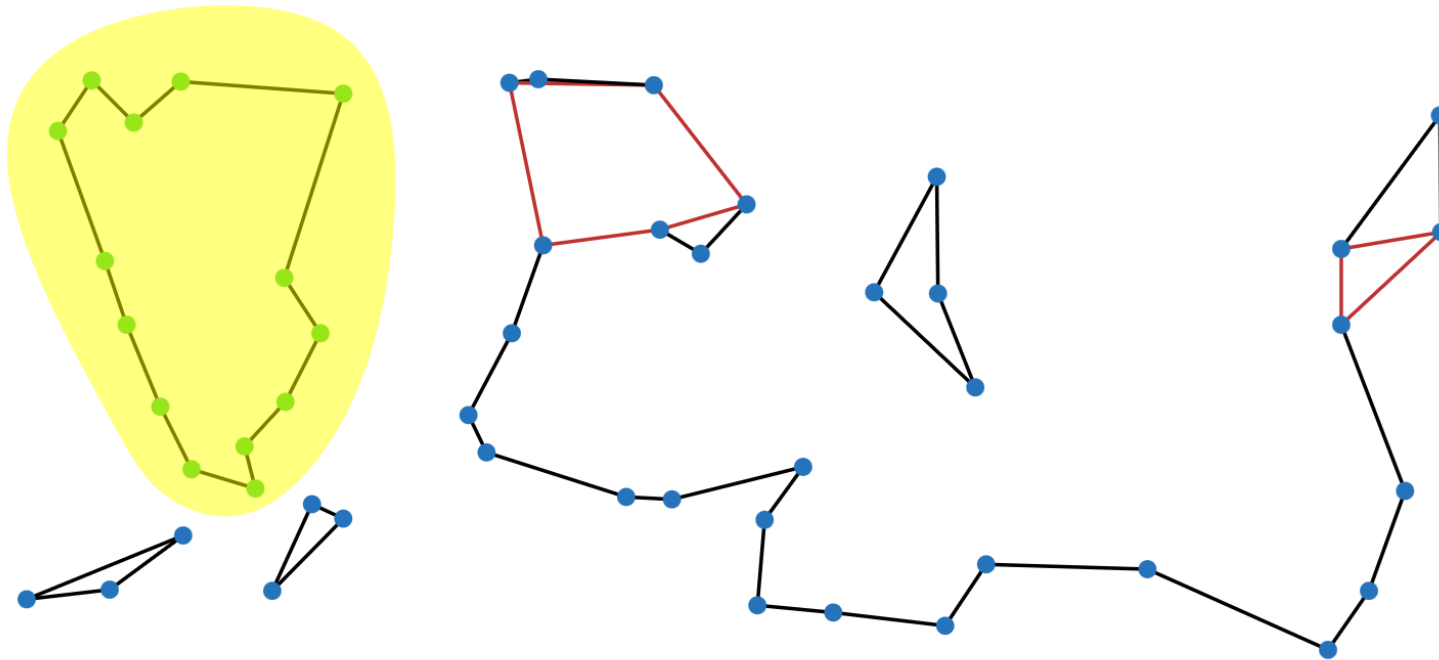
random seed = 3.14, # points = 50



TSP App **DIY** Go Tour Bound LP Cuts Branch-and-Bound

Open Controls

Add Cut



Selected set has cut value 0. Violated!

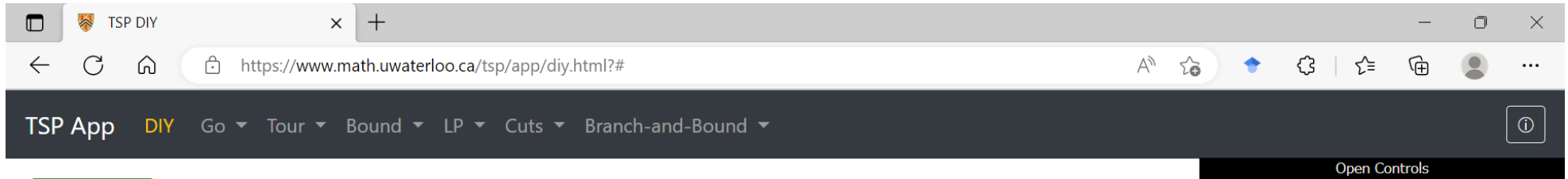
部分巡回路除去制約の追加：例 (8)

19/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 3$

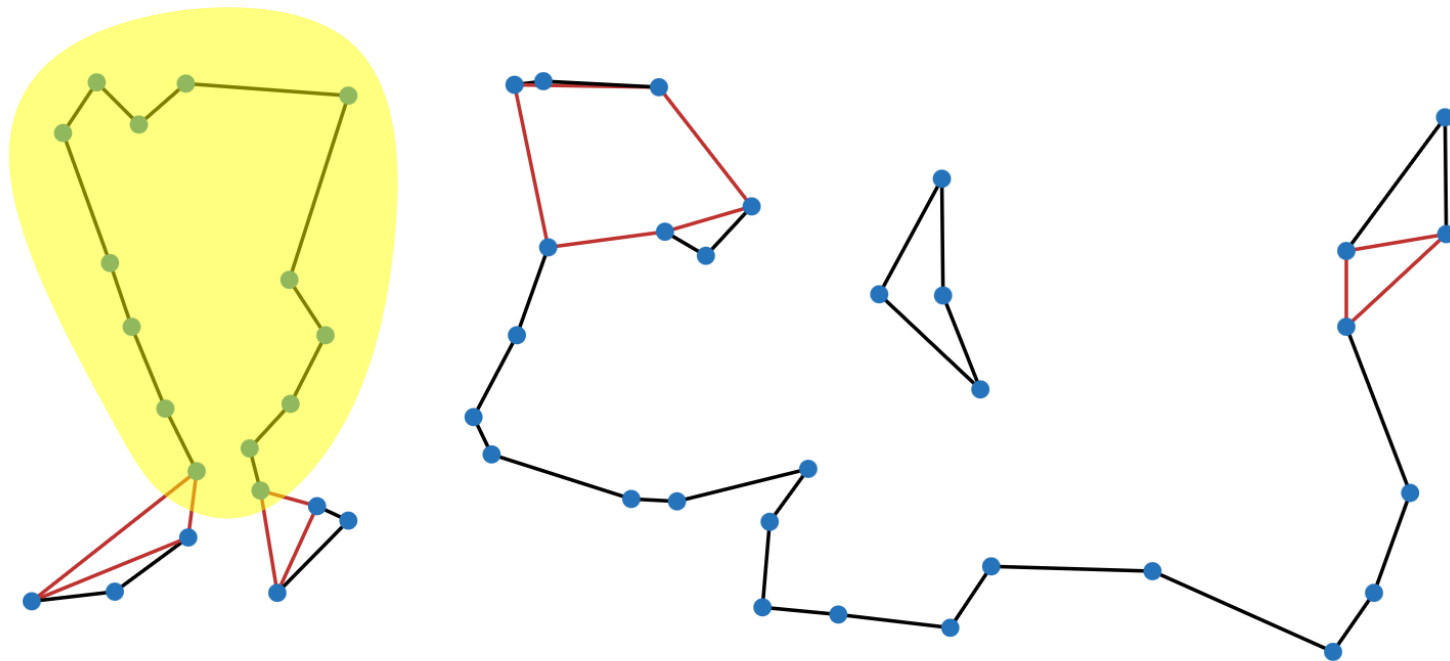
random seed = 3.14, # points = 50



TSP App **DIY** Go Tour Bound LP Cuts Branch-and-Bound

Open Controls

Build a Cut



Added a subtour cut. Bound = 41689 (+51).

部分巡回路除去制約の追加：例 (9)

20/77

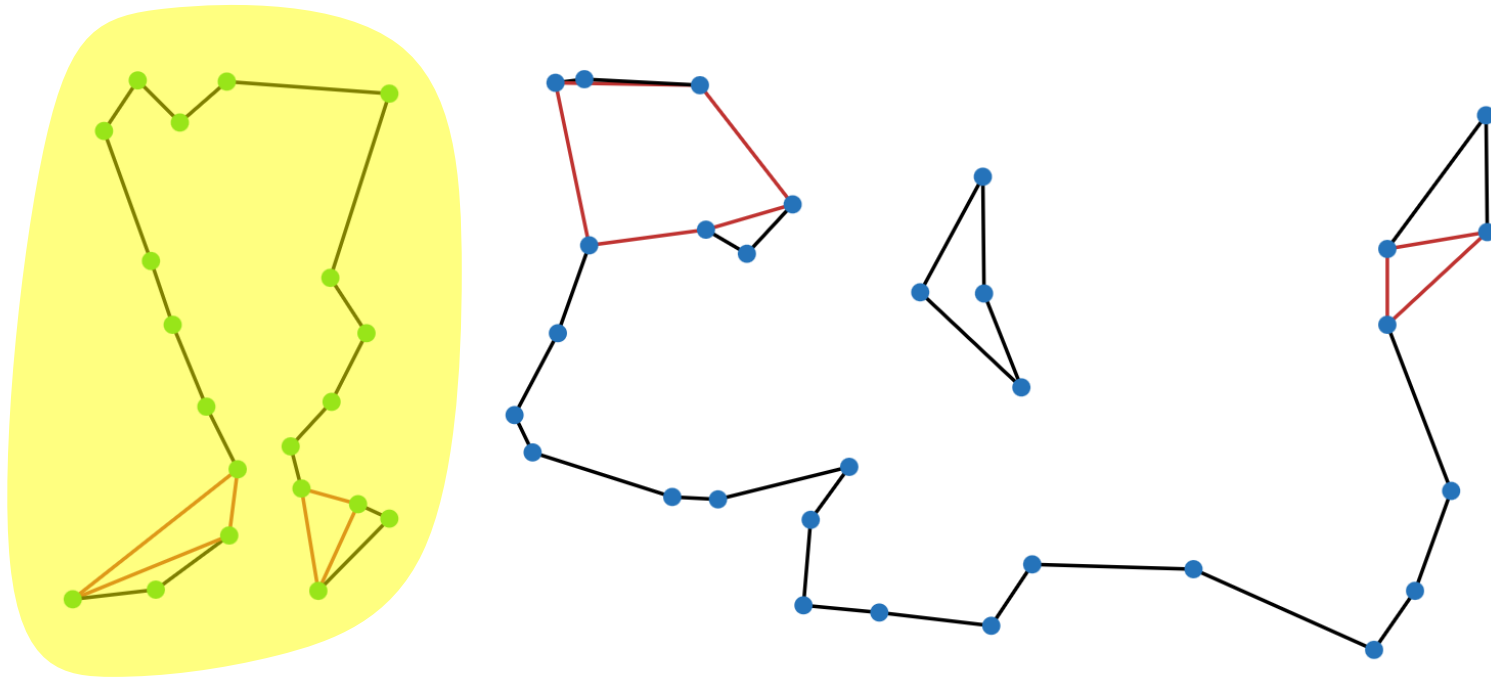
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 3$

random seed = 3.14, # points = 50

The screenshot shows the top part of a web browser window. The address bar contains the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. Below the address bar is a dark navigation bar with several tabs: "TSP App", "DIY" (highlighted in yellow), "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". On the right side of this bar is an information icon. Below the navigation bar is a black button labeled "Open Controls".

Add Cut



Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (10)

21/77

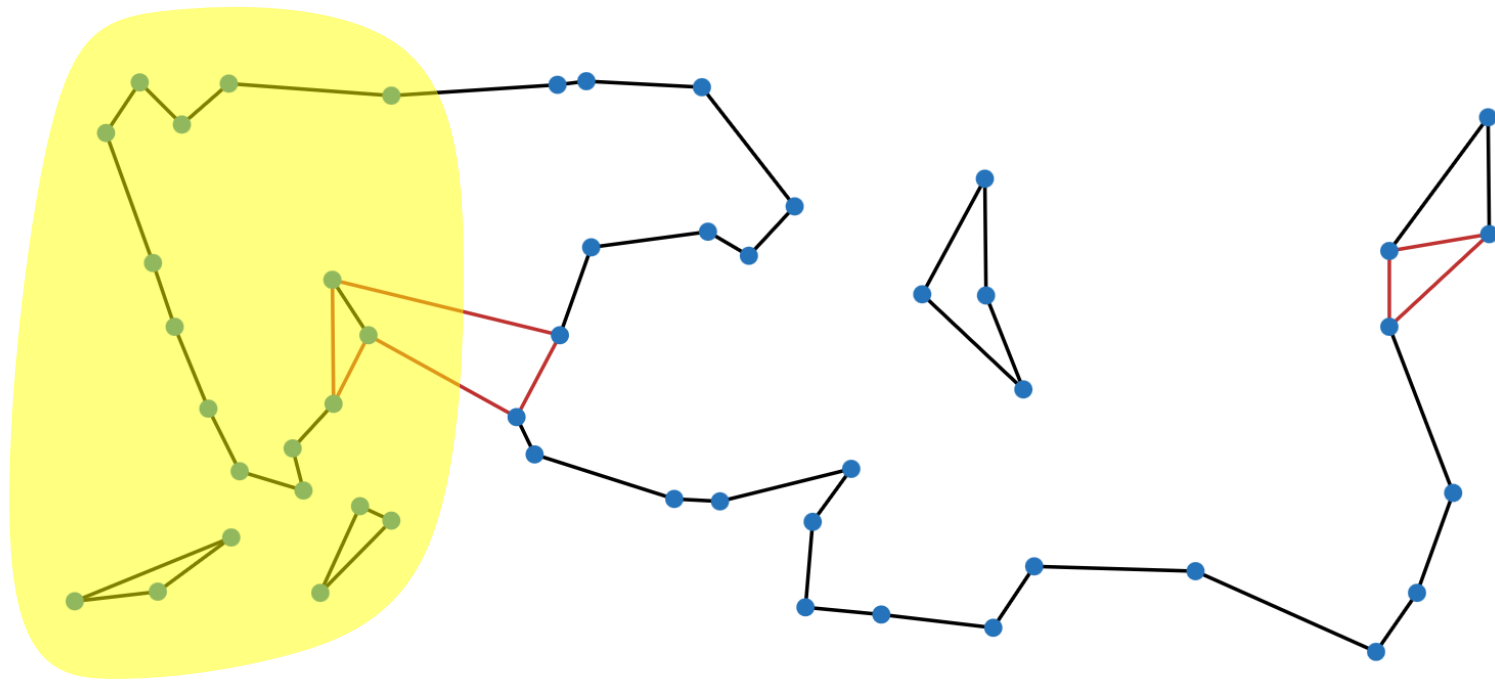
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 4$

random seed = 3.14, # points = 50

The screenshot shows the TSP App interface. The browser address bar displays the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The navigation menu includes "TSP App", "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible. The main area shows a tour of 50 points, with a subtour cut highlighted in yellow. The cut is a cycle of points that is not part of the current tour. The tour is shown as a black line connecting the points, and the cut is shown as a red line connecting the points in the highlighted area.

Build a Cut



Added a subtour cut. Bound = 42230.5 (+541.5).

部分巡回路除去制約の追加：例 (11)

22/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 4$

random seed = 3.14, # points = 50

The screenshot shows the TSP App interface. At the top, there is a navigation bar with "TSP App" and "DIY" highlighted. Below the navigation bar, there is a red "Add Cut" button. The main area displays a tour of 50 points (blue dots) connected by black lines. A subset of points is highlighted in green and enclosed in a yellow oval. The bottom of the interface shows a status bar with the text "Selected set has cut value 0. Violated!".

Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (12)

23/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 5$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the "DIY" mode is selected. The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible. The main area displays a tour of 50 points (blue dots) connected by black lines. A subtour cut is highlighted in red, consisting of a triangle of three points. This triangle is further highlighted with a yellow oval. To the right of the main tour, there are three separate subtours, each consisting of three points connected by black lines.

Build a Cut

Open Controls

Added a subtour cut. Bound = 42710 (+479.5).

部分巡回路除去制約の追加：例 (13)

24/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

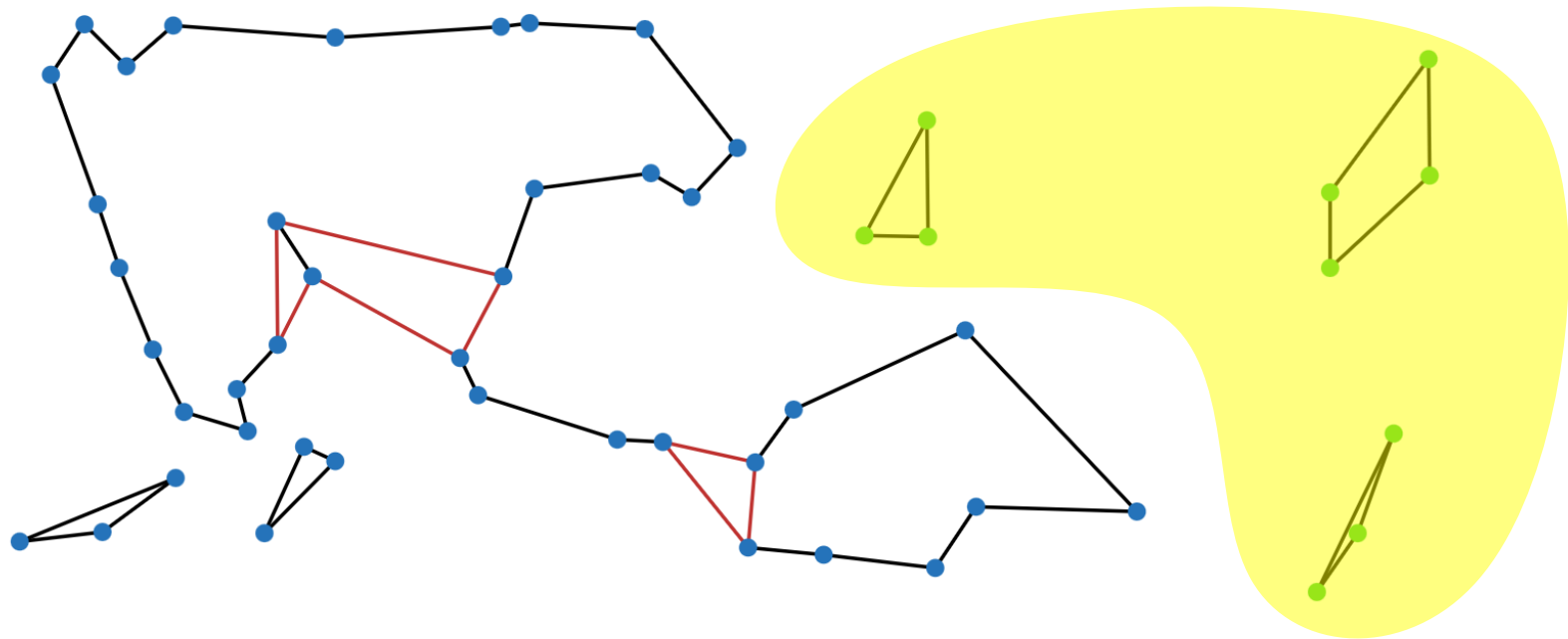
$|\mathcal{F}| = 5$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser, there is a dark navigation bar with the text "TSP App" and several menu items: "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A red button labeled "Add Cut" is positioned on the left. On the right side of the navigation bar, there is an "Open Controls" button. The main area of the browser displays a Traveling Salesman Problem (TSP) tour with 50 blue points. A yellow shaded region highlights a specific set of points and their connections, which is the focus of the current step in the algorithm.

Add Cut

Open Controls



Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (14)

25/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 6$

random seed = 3.14, # points = 50

The screenshot shows the TSP App interface. The browser address bar displays <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The application header includes "TSP App" and a navigation menu with "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green "Build a Cut" button is visible on the left. The main area shows a tour of 50 points (blue dots) connected by black lines. A yellow shaded region highlights a subtour cut consisting of several green dots and lines. At the bottom, a status bar indicates "Added a subtour cut. Bound = 42770 (+60)." and an "Open Controls" button is on the right.

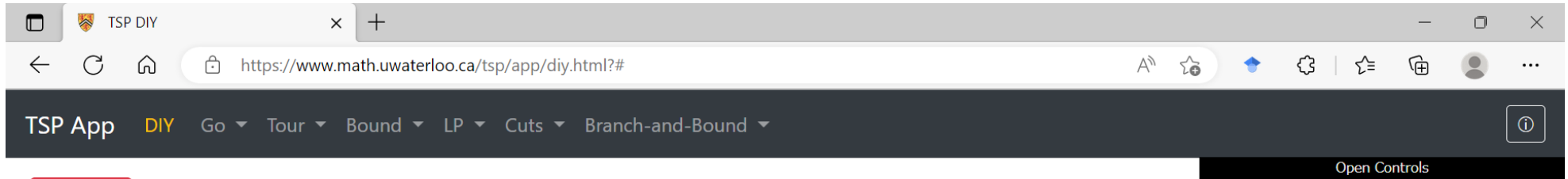
部分巡回路除去制約の追加：例 (15)

26/77

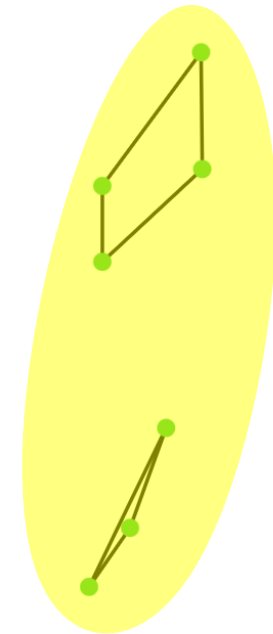
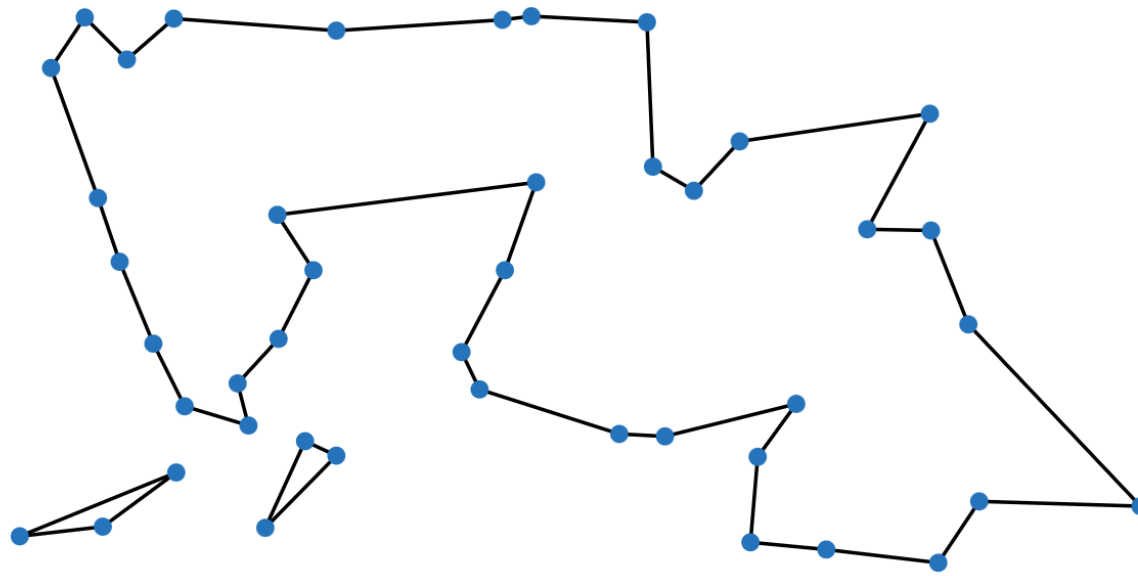
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 6$

random seed = 3.14, # points = 50



Add Cut



Selected set has cut value 0. Violated!

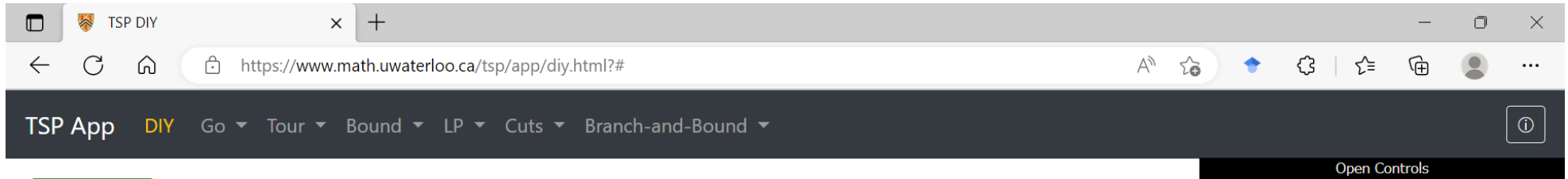
部分巡回路除去制約の追加：例 (16)

27/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$$|\mathcal{F}| = 7$$

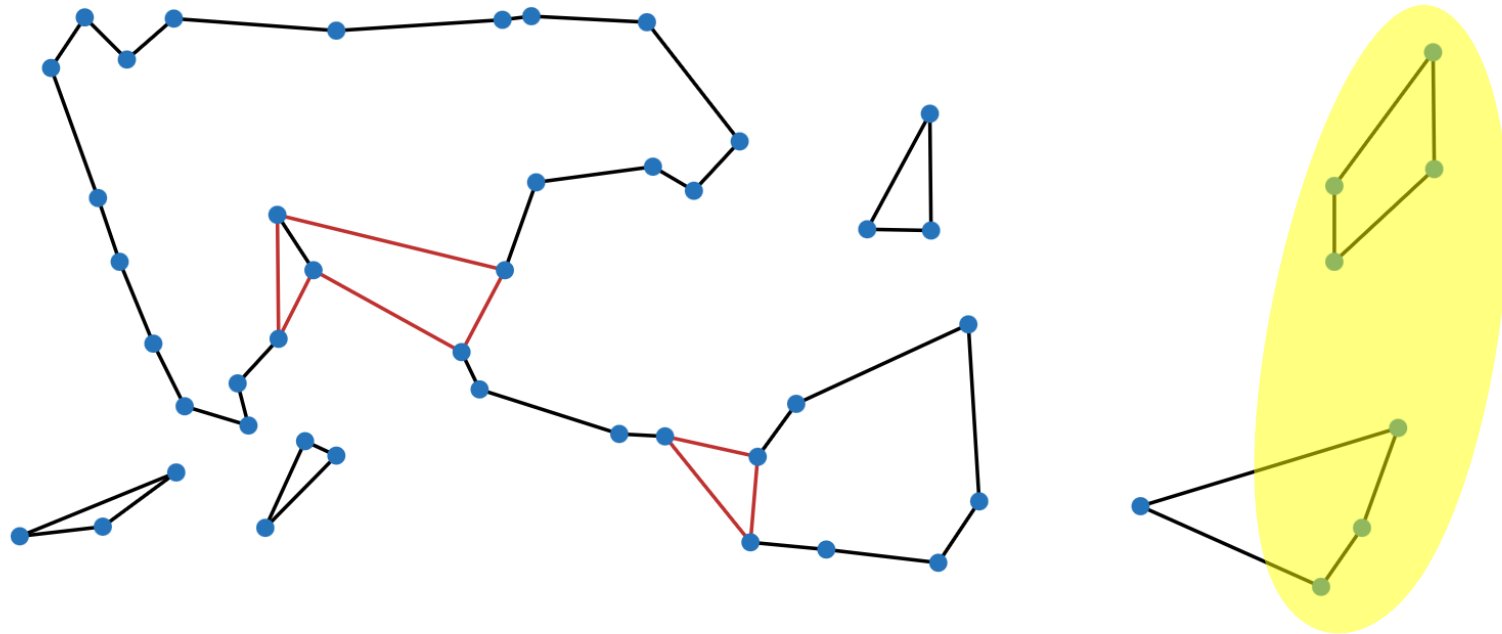
random seed = 3.14, # points = 50



TSP App **DIY** Go Tour Bound LP Cuts Branch-and-Bound

Open Controls

Build a Cut



Added a subtour cut. Bound = 43175 (+405).

部分巡回路除去制約の追加：例 (17)

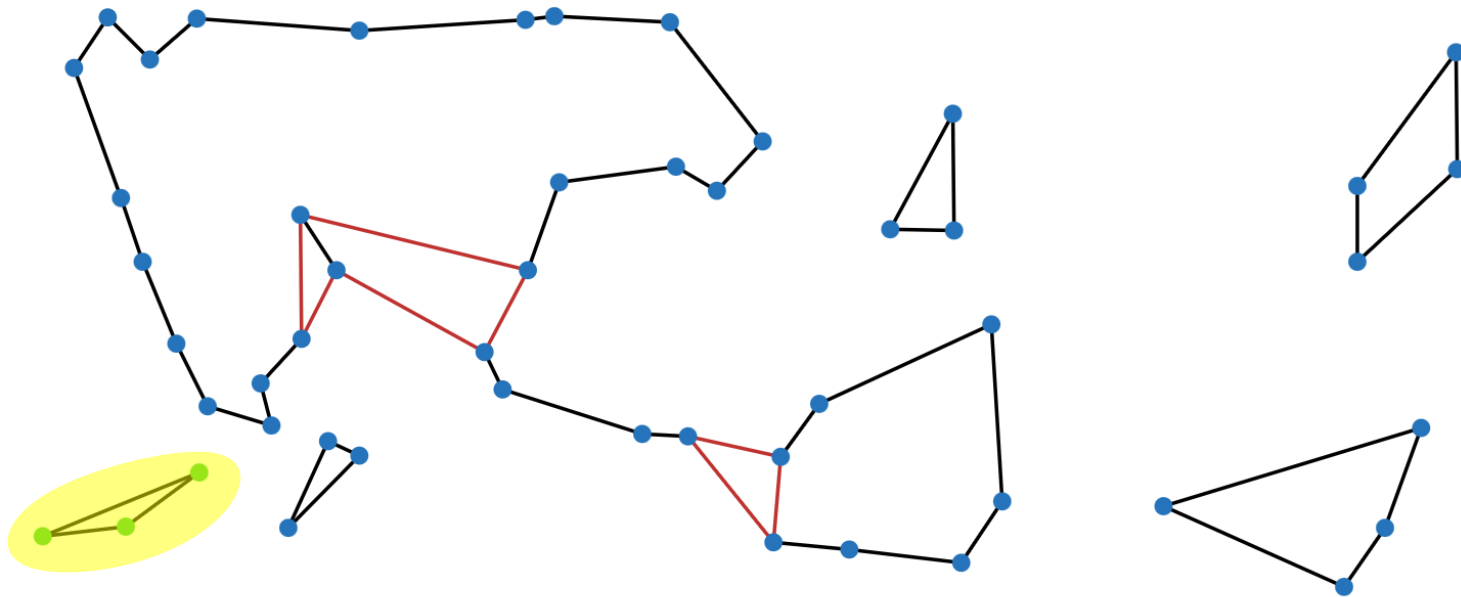
28/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 7$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser, the application interface includes a navigation menu with options: TSP App, DIY, Go, Tour, Bound, LP, Cuts, and Branch-and-Bound. A red button labeled "Add Cut" is positioned on the left. On the right, there is a button labeled "Open Controls". The main area displays a tour of 50 points, with a subset of points highlighted in green and enclosed in a yellow oval.



Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (18)

29/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

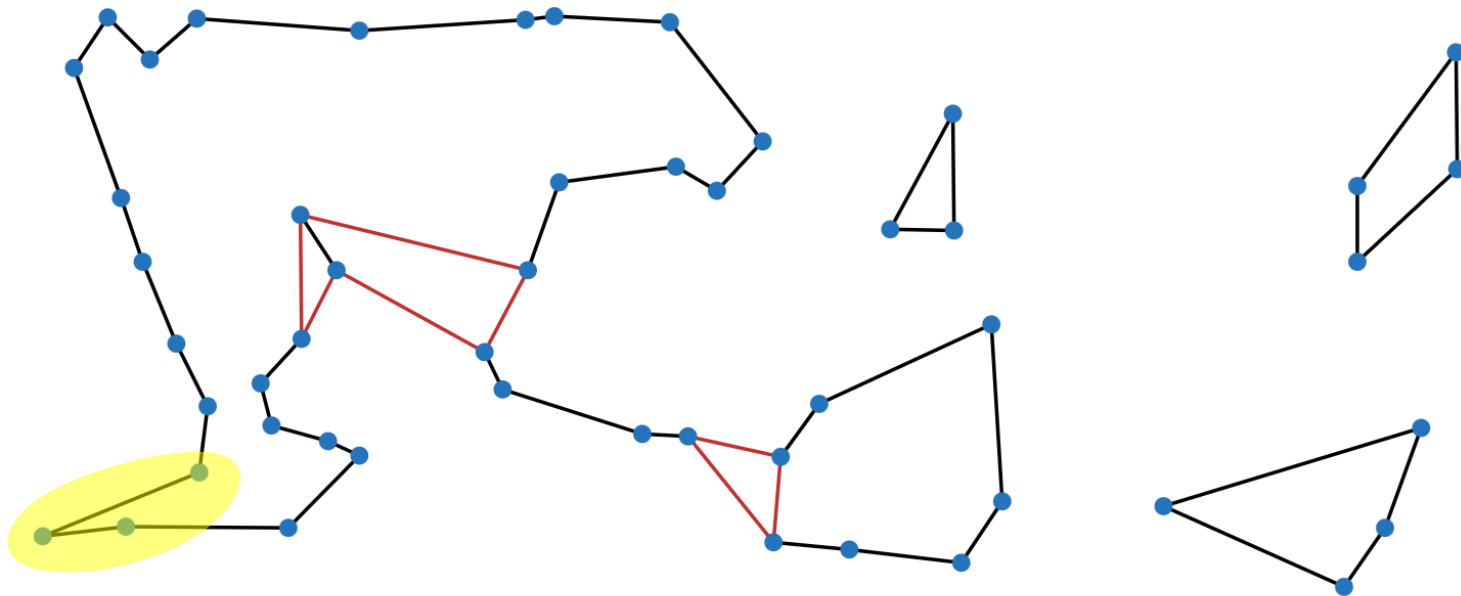
$|\mathcal{F}| = 8$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser, the TSP App interface is shown. It includes a navigation menu with options: TSP App, DIY, Go, Tour, Bound, LP, Cuts, and Branch-and-Bound. A green button labeled "Build a Cut" is on the left. On the right, there is an "Open Controls" button. The main area displays a tour of 50 points. A subtour cut is highlighted in red, consisting of a cycle of points. A yellow oval highlights a specific area of the tour.

Build a Cut

Open Controls



Added a subtour cut. Bound = 43463 (+288).

部分巡回路除去制約の追加：例 (19)

30/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

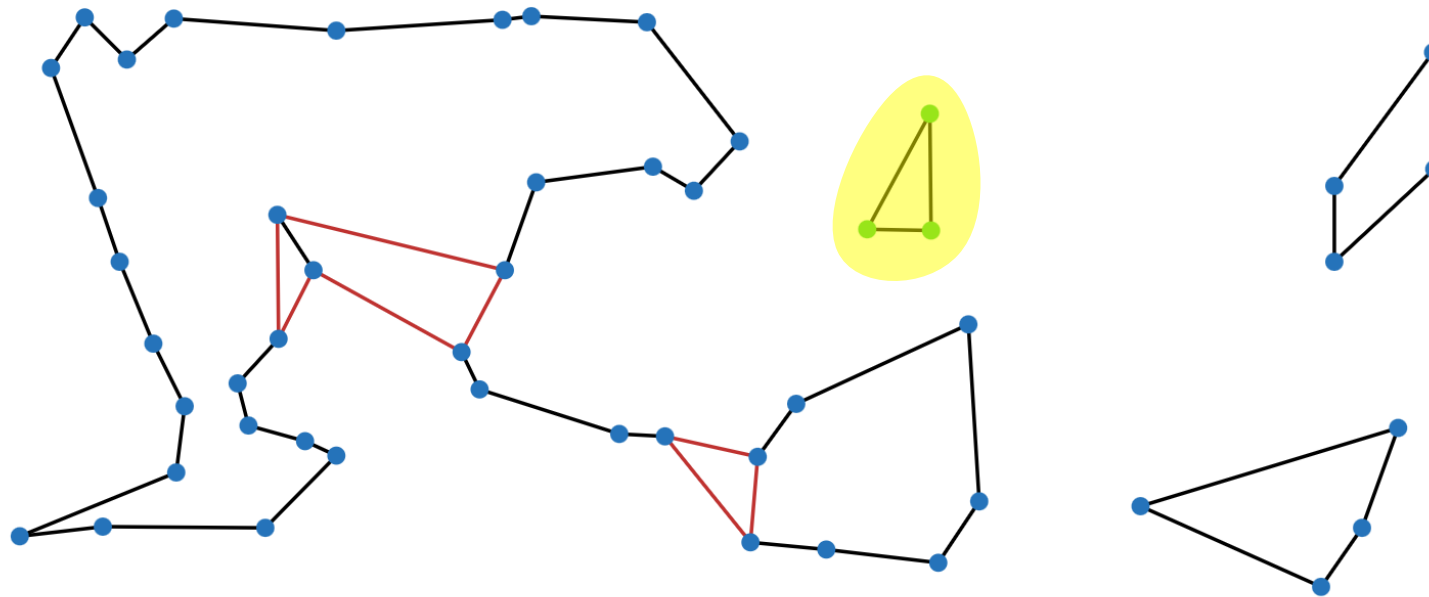
$|\mathcal{F}| = 8$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the "DIY" tab is selected. The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A red "Add Cut" button is visible in the top left. The main area displays a tour of 50 points (blue dots) connected by black lines. A specific triangle formed by three points is highlighted in yellow, indicating it is the selected set for a cut.

Add Cut

Open Controls



Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (20)

31/77

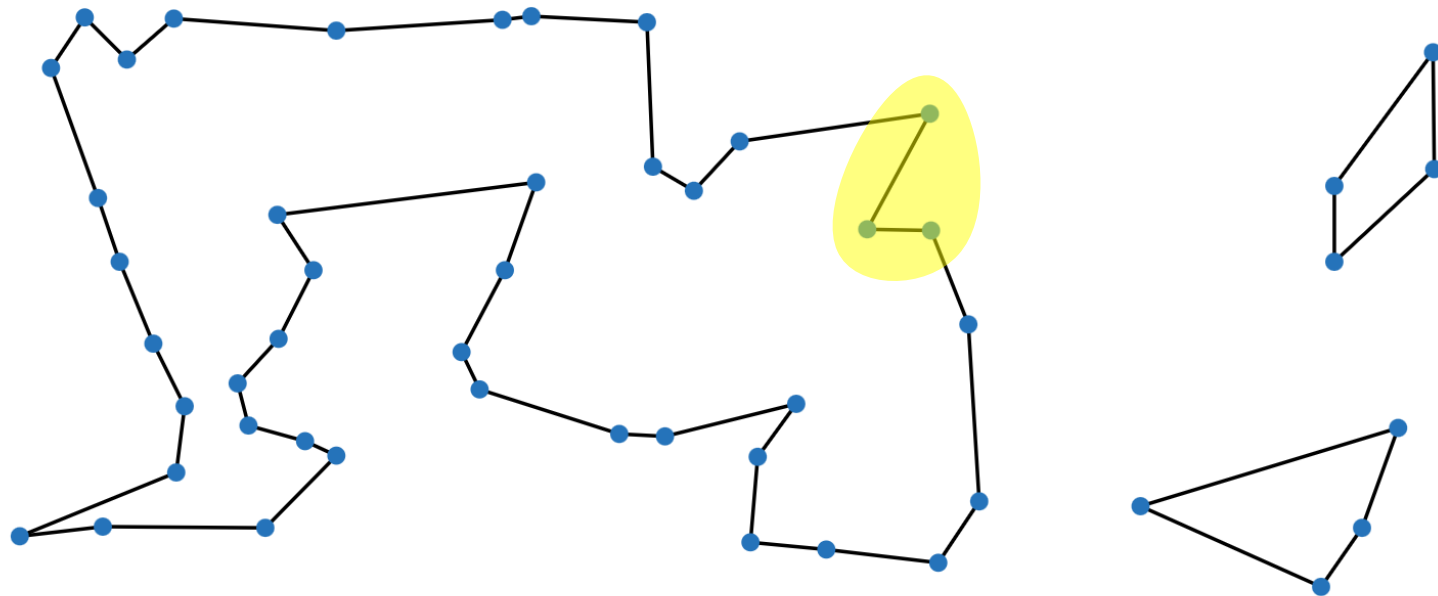
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 9$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the mode is "DIY". The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible. The main area displays a tour of 50 points (blue dots) connected by black lines. A subtour cut is highlighted in yellow, consisting of three points and their connecting edges. To the right, two separate subtours are shown, each consisting of three points and their connecting edges. A black bar at the bottom right contains the text "Open Controls".

Build a Cut



Added a subtour cut. Bound = 43523 (+60).

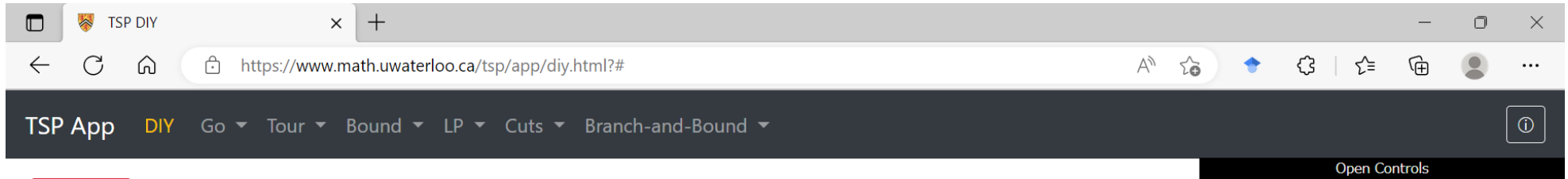
部分巡回路除去制約の追加：例 (21)

32/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$$|\mathcal{F}| = 9$$

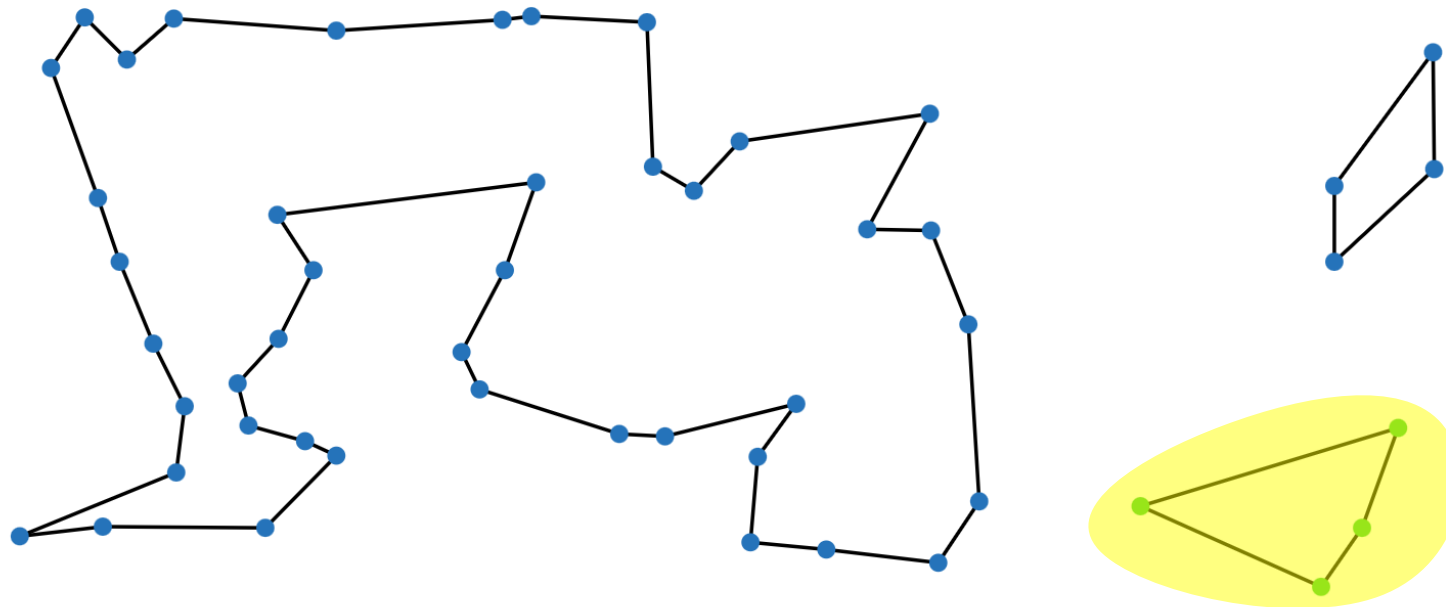
random seed = 3.14, # points = 50



TSP App **DIY** Go Tour Bound LP Cuts Branch-and-Bound

Open Controls

Add Cut



Selected set has cut value 0. Violated!

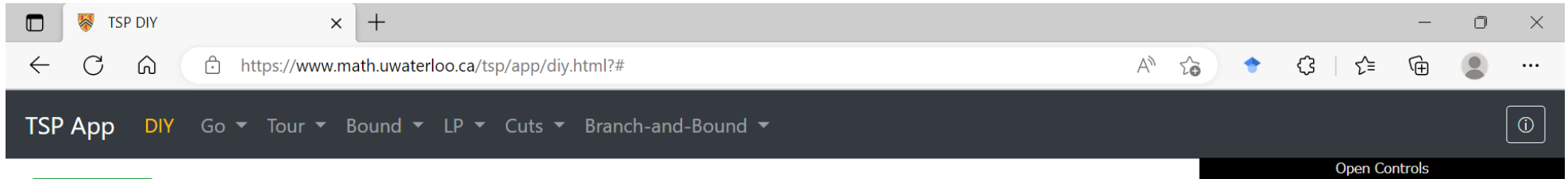
部分巡回路除去制約の追加：例 (22)

33/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 10$

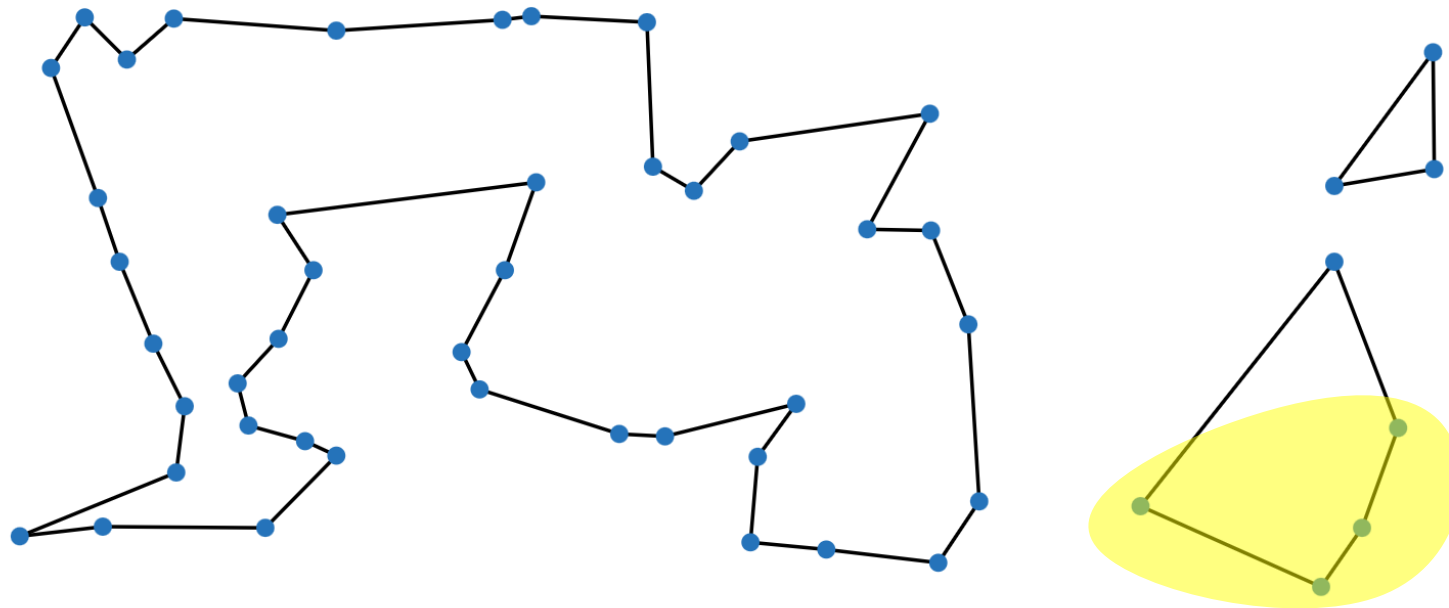
random seed = 3.14, # points = 50



TSP App **DIY** Go Tour Bound LP Cuts Branch-and-Bound

Open Controls

Build a Cut



Added a subtour cut. Bound = 44435 (+912).

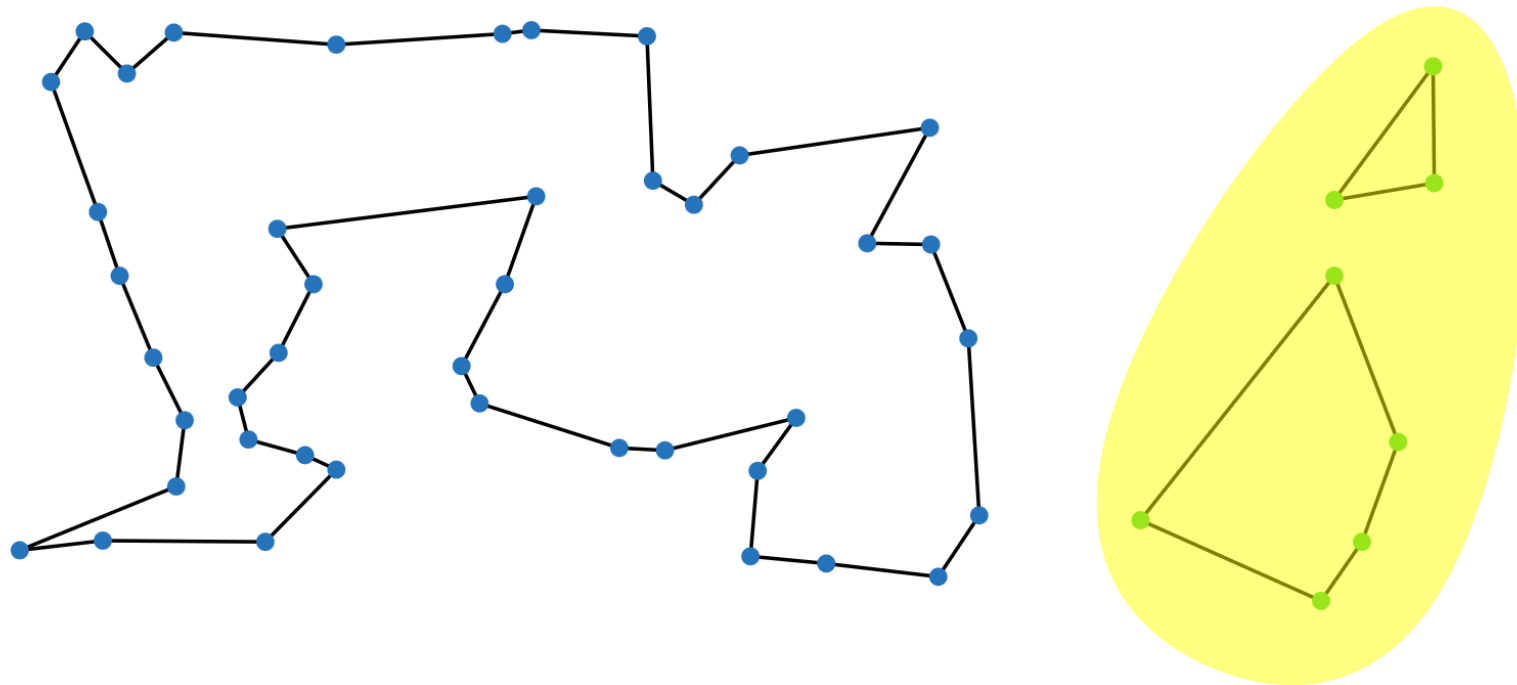
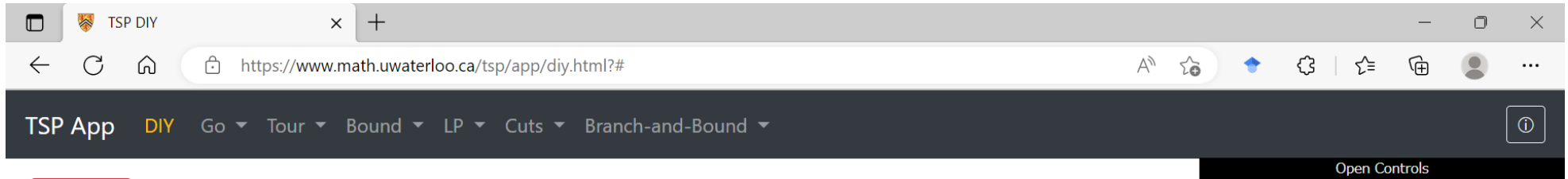
部分巡回路除去制約の追加：例 (23)

34/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 10$

random seed = 3.14, # points = 50



Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (24)

35/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 11$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the "DIY" mode is selected. The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible. The main area displays a tour of 50 points (blue dots) connected by black lines. A yellow oval highlights a subtour cut consisting of 11 points (green dots) connected by green lines. An "Open Controls" button is located in the bottom right corner of the interface.

Build a Cut

Open Controls

Added a subtour cut. Bound = 44796 (+361).

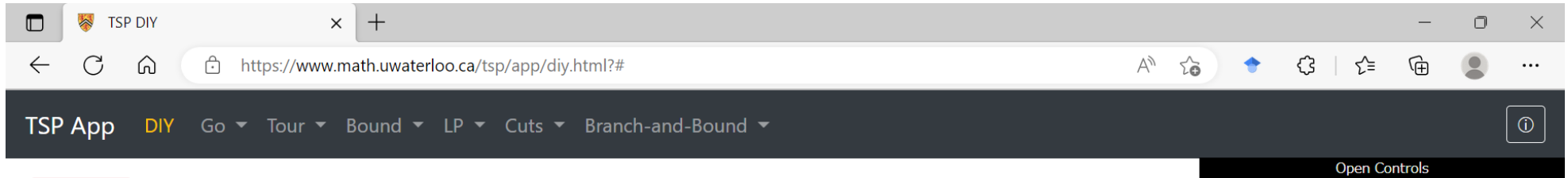
部分巡回路除去制約の追加：例 (25)

36/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 11$

random seed = 3.14, # points = 50



The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the sub-page is "DIY". The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A red "Add Cut" button is visible on the left. The main area displays a tour of 50 points (blue dots) connected by black lines. A yellow oval highlights a triangle formed by three points on the right side of the tour.

Add Cut

Open Controls

Selected set has cut value 0. Violated!

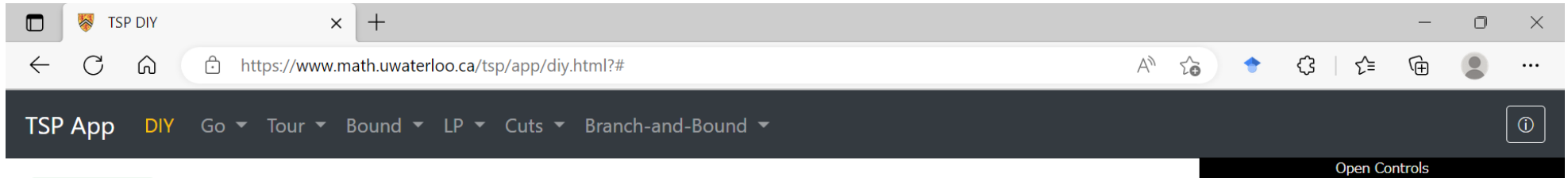
部分巡回路除去制約の追加：例 (26)

37/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 12$

random seed = 3.14, # points = 50



The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the "DIY" mode is selected. The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible. The main area displays a tour of 50 points (blue dots) connected by black lines. A subtour cut is highlighted in yellow, consisting of three points and their connecting edges.

Build a Cut

Open Controls

Added a subtour cut. Bound = 44826 (+30).

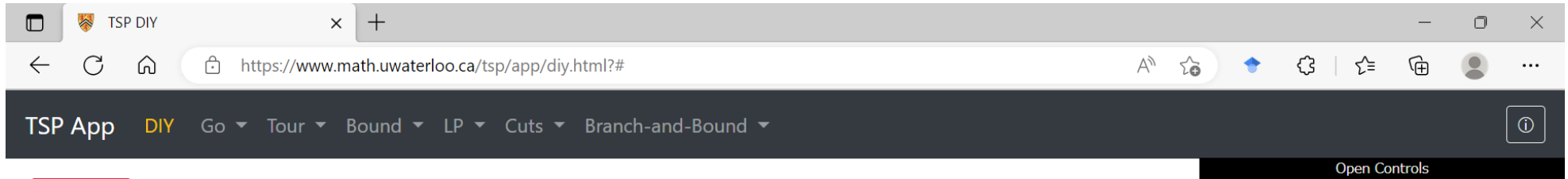
部分巡回路除去制約の追加：例 (27)

38/77

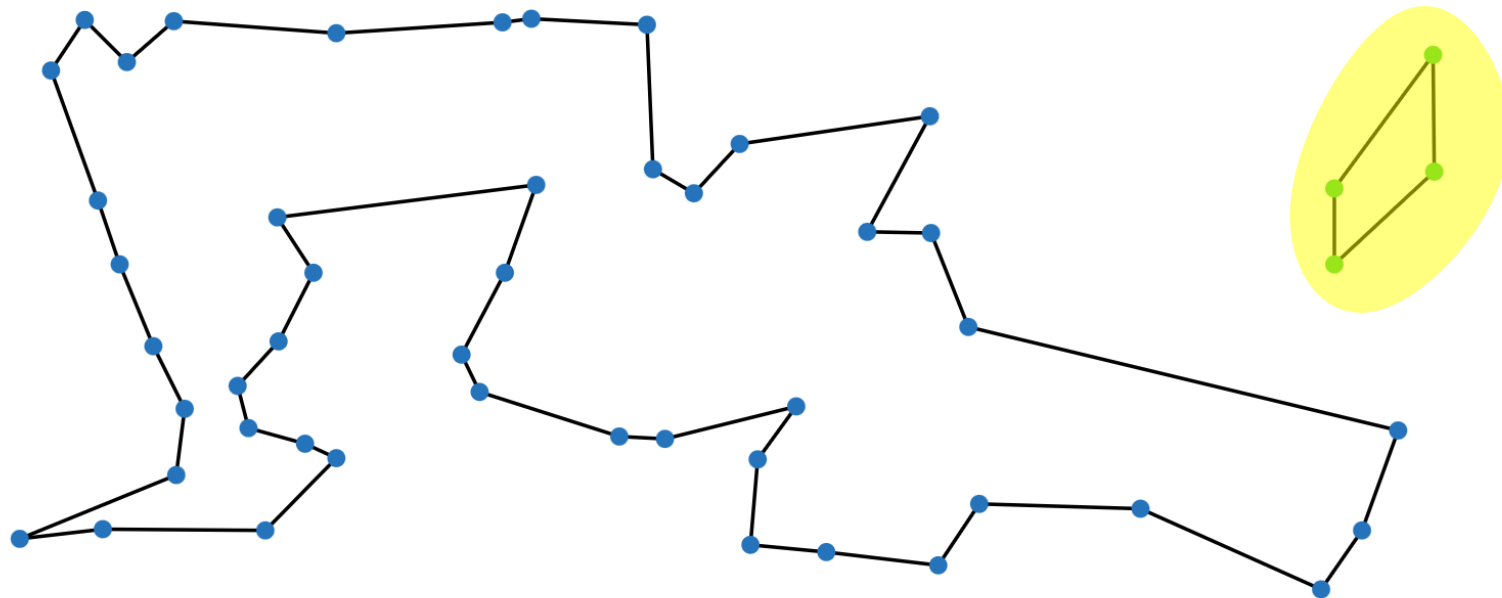
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 12$

random seed = 3.14, # points = 50



The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser, the application interface includes a dark navigation bar with the text "TSP App" and several menu items: "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A red button labeled "Add Cut" is positioned on the left. On the right side of the interface, there is a button labeled "Open Controls".



Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (28)

39/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 13$

random seed = 3.14, # points = 50

The screenshot shows the TSP App interface. The browser address bar displays the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The application header includes "TSP App" and a navigation menu with "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible on the left. The main area shows a tour of 50 points (blue dots) connected by black lines. A subtour cut is highlighted in red, consisting of a cycle of points. A yellow oval highlights a specific part of the tour on the right side.

Build a Cut

Open Controls

Added a subtour cut. Bound = 44982 (+156).

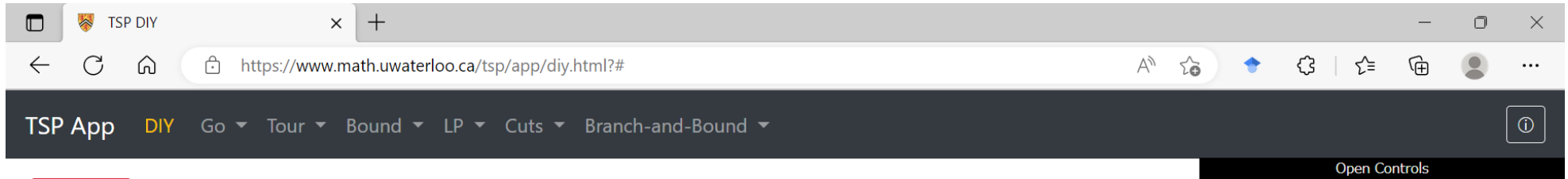
部分巡回路除去制約の追加：例 (29)

40/77

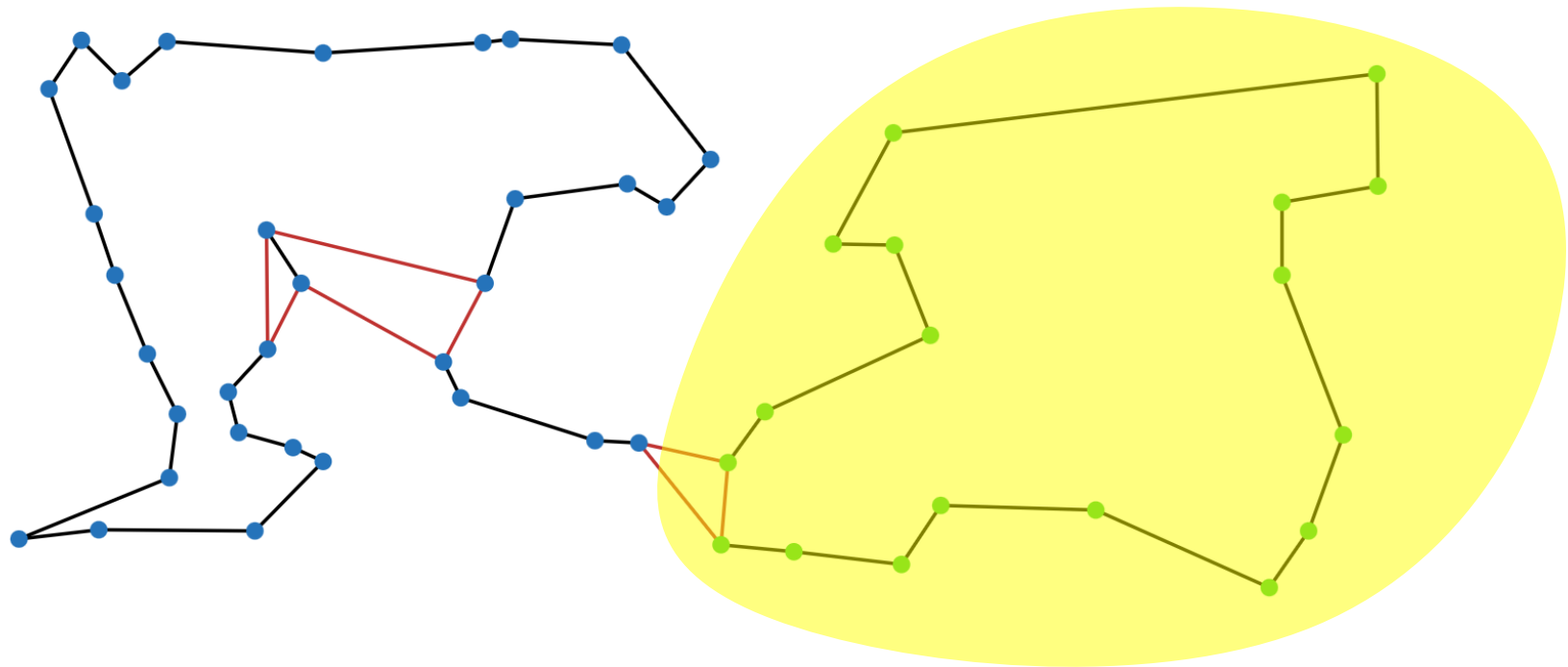
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 13$

random seed = 3.14, # points = 50



The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser, the application interface includes a dark navigation bar with the text "TSP App" and a menu with options: "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A red button labeled "Add Cut" is positioned on the left. On the right side of the interface, there is a button labeled "Open Controls".



Selected set has cut value 1. Violated!

部分巡回路除去制約の追加：例 (30)

41/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 14$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the sub-page is "DIY". The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible. The main area displays a tour of 50 points. A subtour cut is highlighted in yellow, consisting of a closed loop of points on the right side of the tour.

Build a Cut

Open Controls

Added a subtour cut. Bound = 45203 (+221).

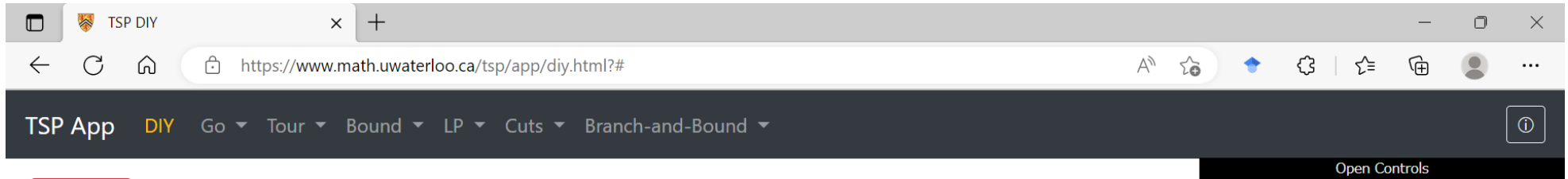
部分巡回路除去制約の追加：例 (31)

42/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 14$

random seed = 3.14, # points = 50



Add Cut

Open Controls

Selected set has cut value 0. Violated!

部分巡回路除去制約の追加：例 (32)

43/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 15$

random seed = 3.14, # points = 50

The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the "DIY" mode is selected. The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green button labeled "Build a Cut" is visible. The main area displays a tour of 50 points, with a subtour cut highlighted in yellow. The subtour cut is a closed loop of points that is not part of the main tour. The text "Open Controls" is visible in the bottom right corner.

Build a Cut

Open Controls

Added a subtour cut. Bound = 45234 (+31).

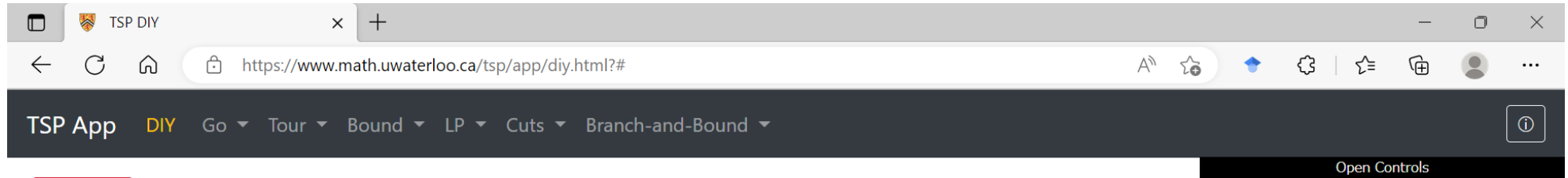
部分巡回路除去制約の追加：例 (33)

44/77

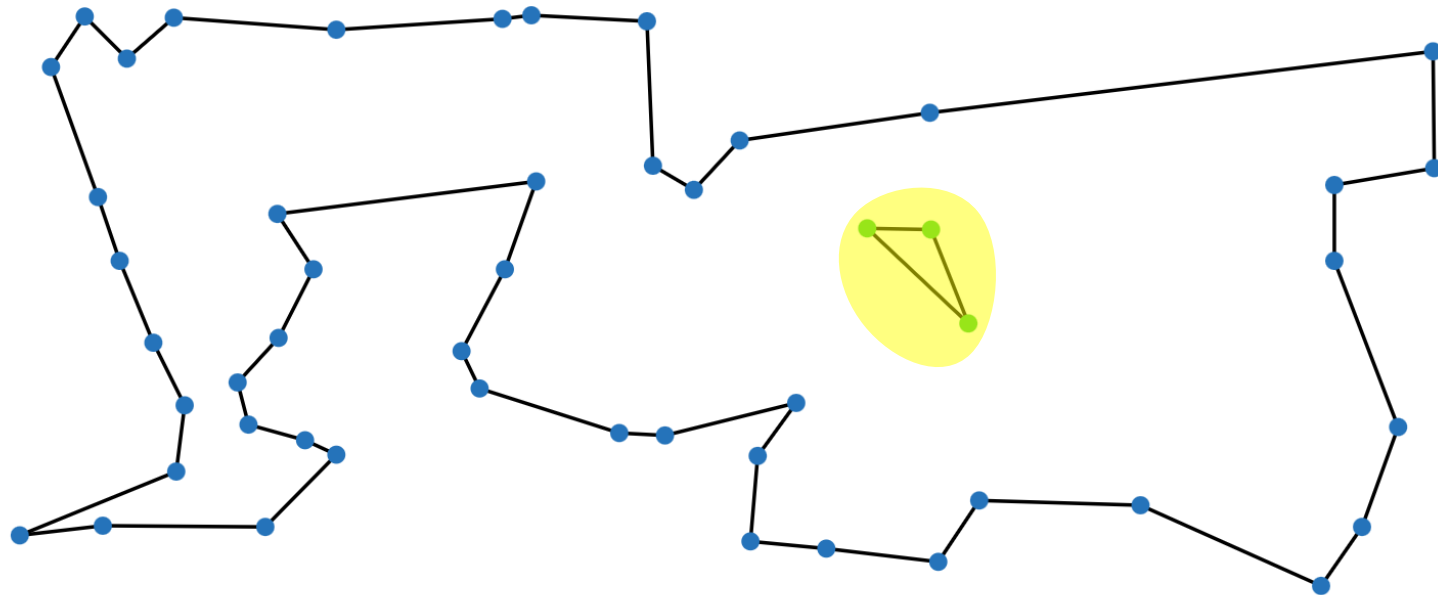
<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 15$

random seed = 3.14, # points = 50



The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The browser's address bar and navigation icons are visible. Below the browser, the TSP App interface is shown with a dark header containing the text "TSP App" and a menu with options: "DIY", "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A red button labeled "Add Cut" is on the left, and a black button labeled "Open Controls" is on the right.



Selected set has cut value 0. Violated!

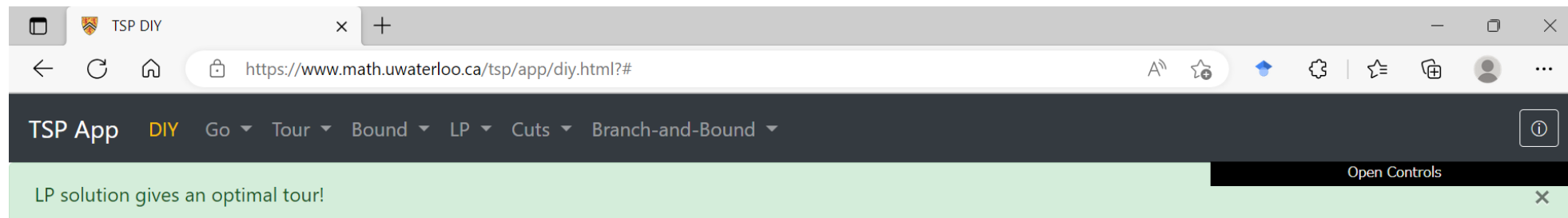
部分巡回路除去制約の追加：例 (34)

45/77

<https://www.math.uwaterloo.ca/tsp/app/diy.html>

$|\mathcal{F}| = 16$

random seed = 3.14, # points = 50

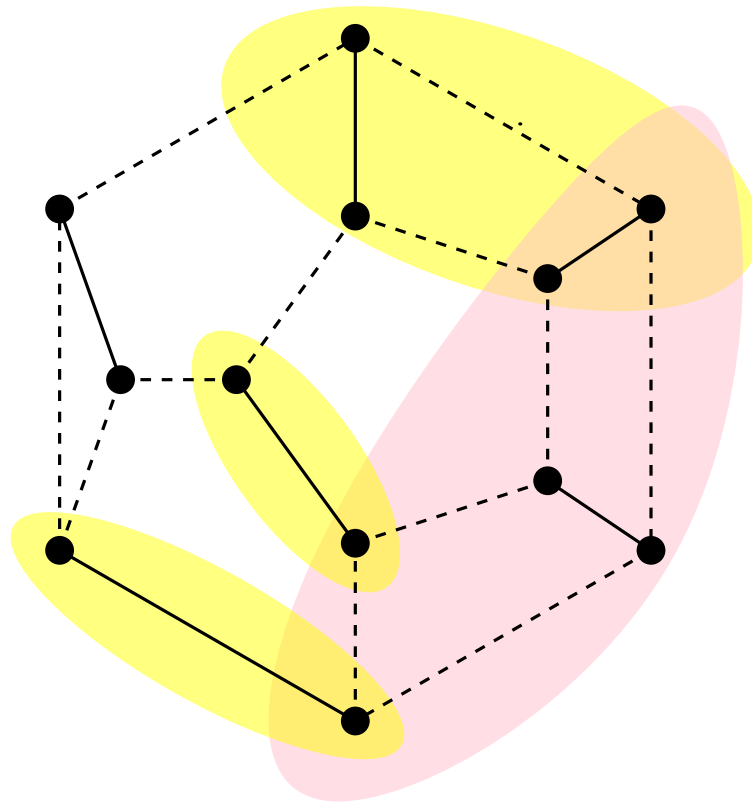


The screenshot shows a web browser window with the URL <https://www.math.uwaterloo.ca/tsp/app/diy.html?#>. The page title is "TSP App" and the sub-page is "DIY". The navigation menu includes "Go", "Tour", "Bound", "LP", "Cuts", and "Branch-and-Bound". A green notification bar at the top says "LP solution gives an optimal tour!". Below the notification is a tour of 50 points connected by black lines. A sub-tour of three points is highlighted with a yellow oval.

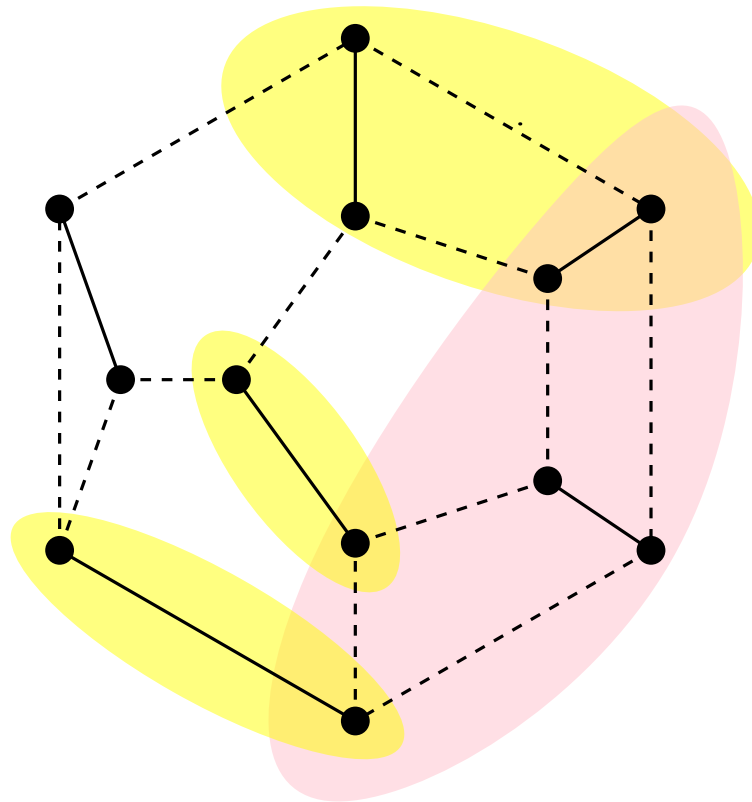
P の最適解が見つかった！

Optimal tour. It's length is 45252.

- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



線形計画緩和をどう解くか？ (3) 再掲

47/77

線形計画緩和 R

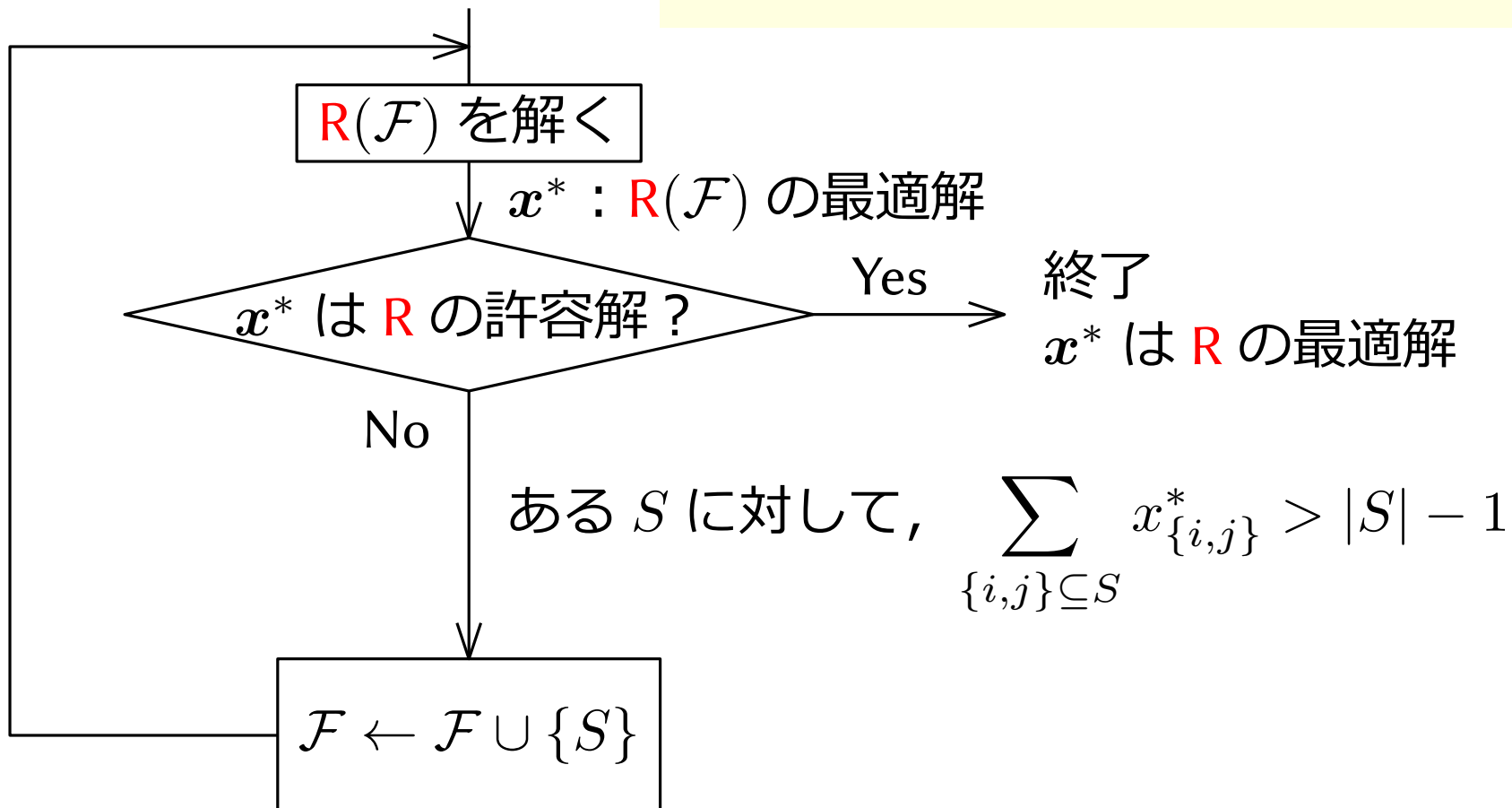


$$\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}}$$

線形計画緩和 $R(\mathcal{F})$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}}$$



線形計画緩和をどう解くか？ (3) 再掲

47/77

線形計画緩和 R

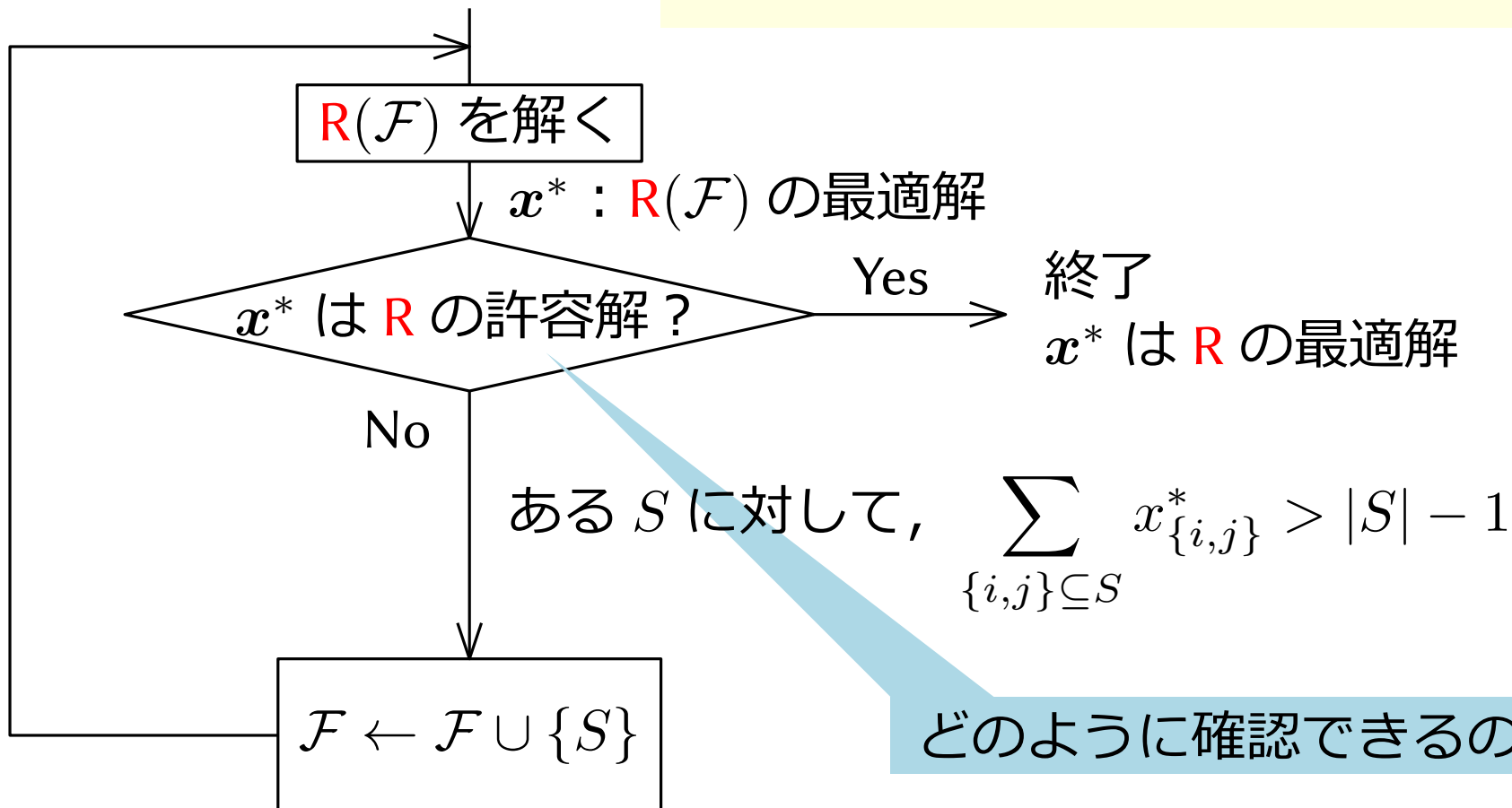
$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}}$$



$$\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$$

線形計画緩和 $R(\mathcal{F})$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}}$$



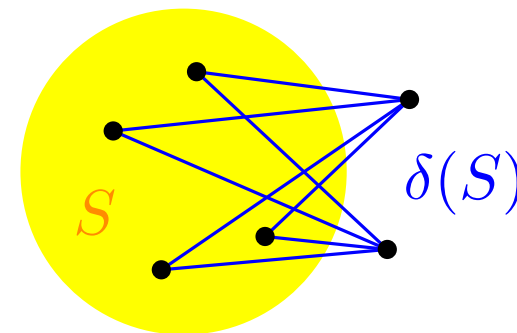
どのように確認できるのか？

線形計画緩和 R

$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

線形計画緩和 R'

$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \geq 2 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$



線形計画緩和 R

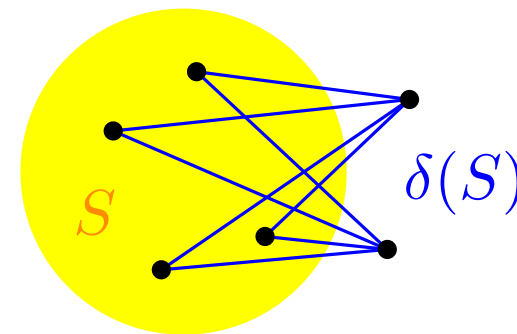
$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

線形計画緩和 R'

$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \geq 2 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

性質

R の許容領域 = R' の許容領域



線形計画緩和 R

$$\begin{aligned}
 \min. \quad & \sum_{e \in E} d(e)x_e \\
 \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\
 & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\
 & 0 \leq x_e \leq 1 \quad \forall e
 \end{aligned}$$

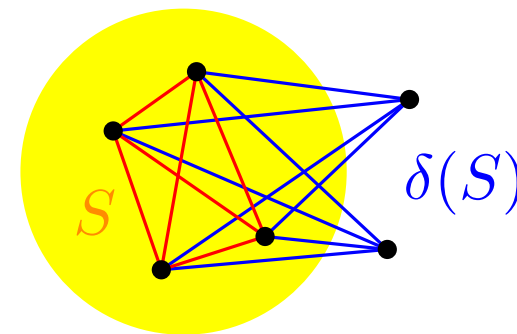
線形計画緩和 R'

$$\begin{aligned}
 \min. \quad & \sum_{e \in E} d(e)x_e \\
 \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\
 & \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \geq 2 \quad \forall S \\
 & 0 \leq x_e \leq 1 \quad \forall e
 \end{aligned}$$

性質

R の許容領域 = R' の許容領域

$$\begin{aligned}
 & 2 \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} + \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \\
 &= \sum_{i \in S} \sum_{j \in V - \{i\}} x_{\{i,j\}} = \sum_{i \in S} 2 = 2|S|
 \end{aligned}$$

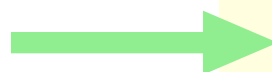


線形計画緩和 R

$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

線形計画緩和 R'

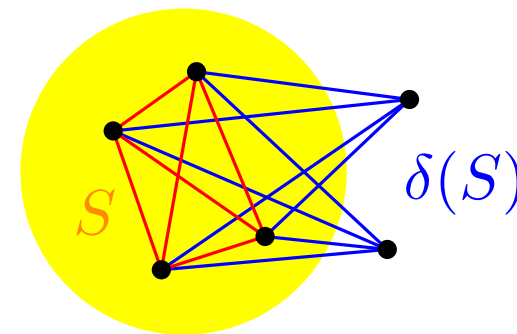
$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \geq 2 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$



性質

R の許容領域 = R' の許容領域

$$\begin{aligned} \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} &= 2|S| - 2 \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \\ &\geq 2|S| - 2(|S| - 1) = 2 \end{aligned}$$

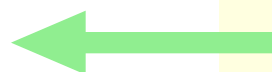


線形計画緩和 R

$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

線形計画緩和 R'

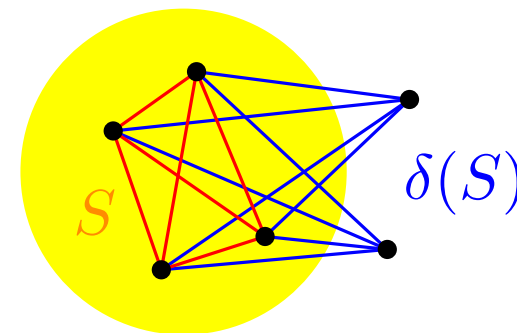
$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \geq 2 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$



性質

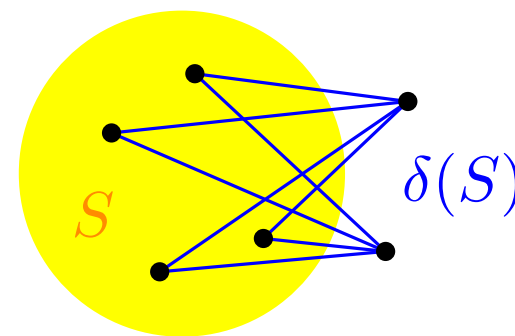
R の許容領域 = R' の許容領域

$$\begin{aligned} \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} &= |S| - \frac{1}{2} \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \\ &\leq |S| - \frac{1}{2} \cdot 2 = |S| - 1 \quad \square \end{aligned}$$



線形計画緩和 $R'(\mathcal{F}) \longrightarrow$ 最適解 x^*

$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \in V \\ & \sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}} \geq 2 \quad \forall S \in \mathcal{F} \\ & 0 \leq x_e \leq 1 \quad \forall e \in E \end{aligned}$$



解くべき問題 (分離問題)

次を満たす $S \in 2^V - \{\emptyset, V\}$ があるか？

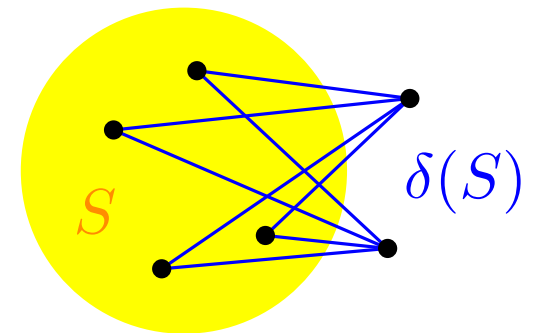
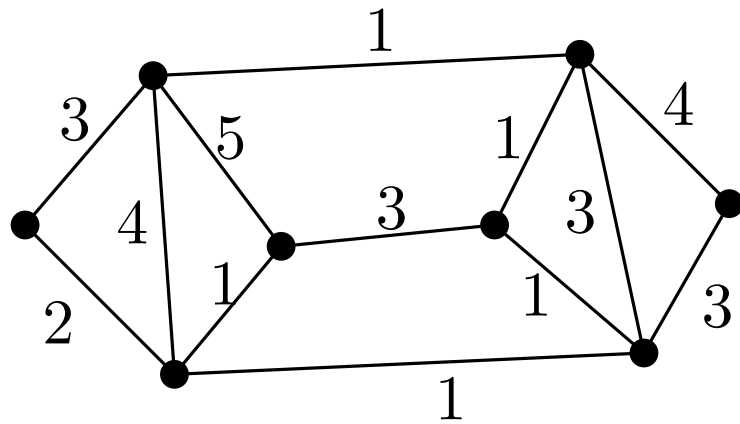
$$\sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}}^* < 2$$

〜 最小カット問題 に帰着できる

定義：最小カット問題 (minimum cut problem)

- 入力：無向グラフ $G = (V, E)$, 非負辺重み $w \in \mathbb{R}^E$
- 出力：次の量を最小にする $S \in 2^V - \{\emptyset, V\}$

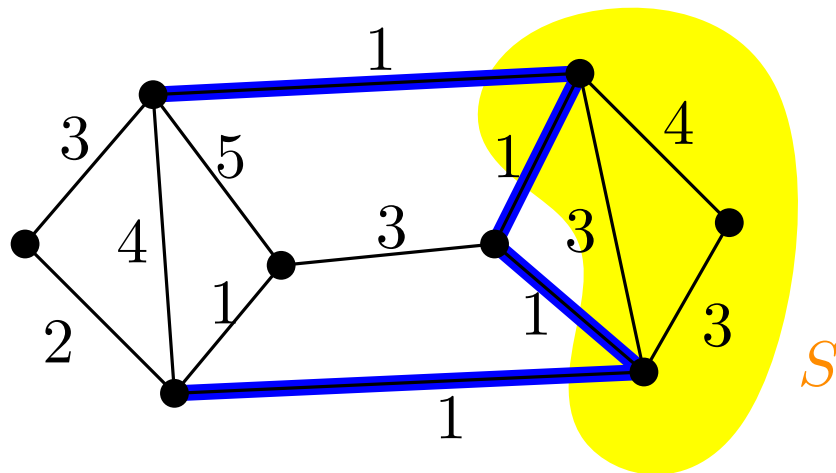
$$\sum_{e \in \delta(S)} w_e$$



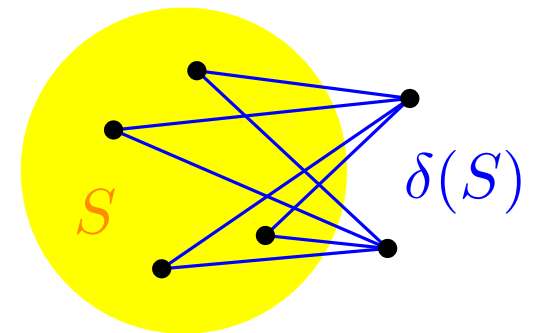
定義：最小カット問題 (minimum cut problem)

- 入力：無向グラフ $G = (V, E)$, 非負辺重み $w \in \mathbb{R}^E$
- 出力：次の量を最小にする $S \in 2^V - \{\emptyset, V\}$

$$\sum_{e \in \delta(S)} w_e$$



$$\sum_{e \in \delta(S)} w_e = 4$$



線形計画緩和 $R'(\mathcal{F}) \longrightarrow$ 最適解 x^*

解くべき問題 (分離問題)

次を満たす $S \in 2^V - \{\emptyset, V\}$ があるか？

$$\sum_{\{i,j\} \in \delta(S)} x_{\{i,j\}}^* < 2$$

アルゴリズム :

1. x^* を辺重みとして, **最小カット問題** を解く
(S^* を最小カット問題の最適解とする)
2. $\sum_{\{i,j\} \in \delta(S^*)} x_{\{i,j\}}^* < 2$ ならば, Yes (S^* を出力)
3. $\sum_{\{i,j\} \in \delta(S^*)} x_{\{i,j\}}^* \geq 2$ ならば, No

事実

最小カット問題は多項式時間で解ける (高速に解ける)

- 最大流最小カット定理の応用
($O(|V|^2)$ 回だけ最小 s, t カットを計算すればよい)
- Nagamochi-Ibaraki のアルゴリズム ('92)
 $\leadsto O(|E||V| + |V| \log |V|)$
- Karger の乱択アルゴリズム ('00)
 $\leadsto O(|E| \log^3 |V|)$

現在, 理論上最速の決定性 (乱数不使用) アルゴリズム

- Li のアルゴリズム ('21)
 $\leadsto O(|E|^{1+o(1)})$

線形計画緩和 R

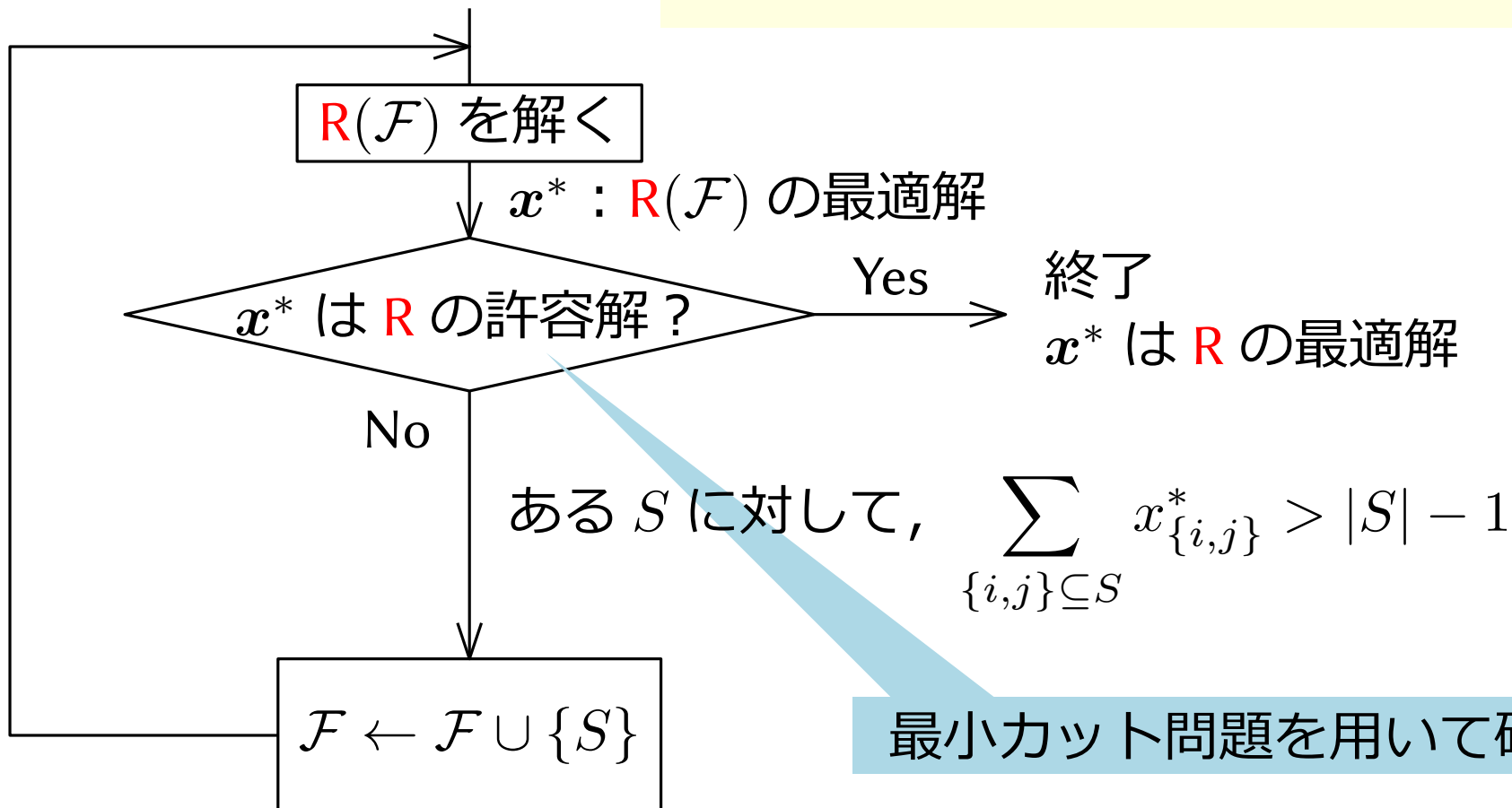
$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{2^V - \{\emptyset, V\}}$$



$$\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$$

線形計画緩和 $R(\mathcal{F})$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in \underline{\mathcal{F}}$$



解きたい問題 **P** $\xrightarrow{\text{緩和}}$ 線形計画緩和 **R**

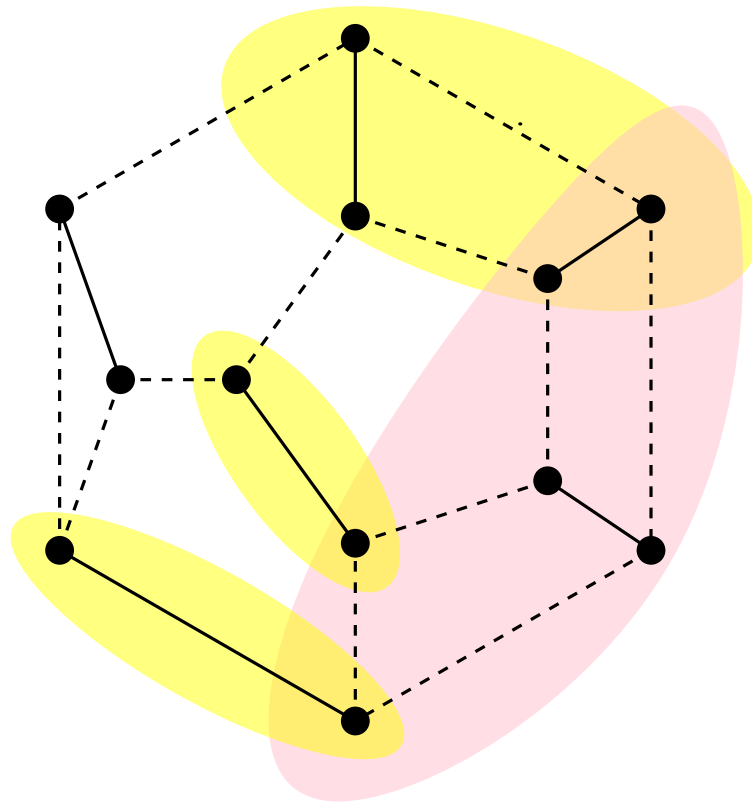
$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & x_e \in \{0, 1\} \quad \forall e \end{aligned}$$

$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \\ & \sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{aligned}$$

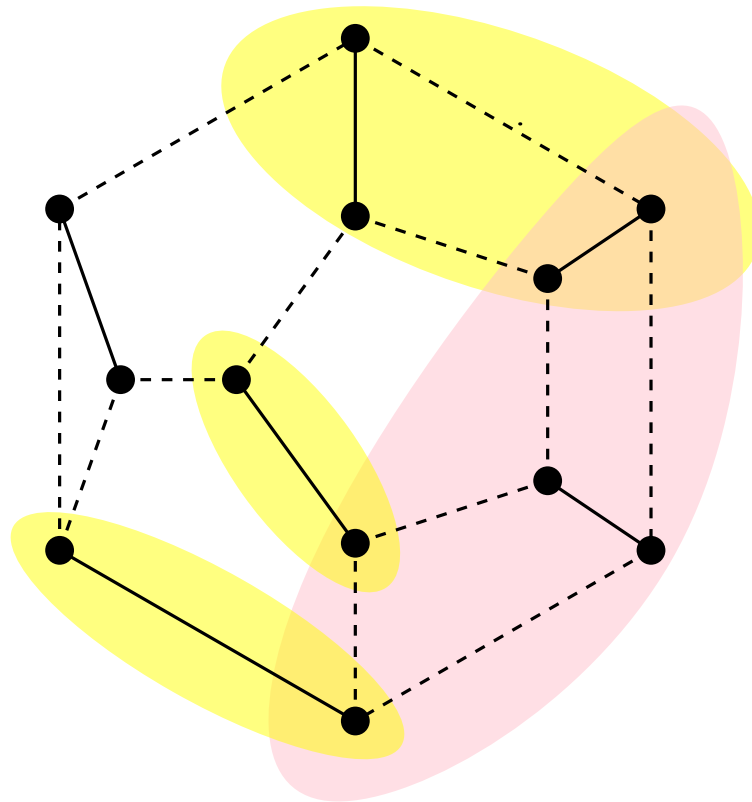
線形計画緩和 **R** の最適解を使って、元の問題 **P** を解く

- 分枝限定法
- 分枝切除法
- …

- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



解きたい問題 P

$$\begin{aligned} \min. \quad & \sum_{e \in E} d(e)x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = 2 \quad \forall i \\ & \sum_{e \subseteq S} x_e \leq |S| - 1 \quad \forall S \\ & x_e \in \{0, 1\} \quad \forall e \end{aligned}$$

目標

よい切除平面を与えること

→ 今から紹介するもの：^{くし} 櫛不等式

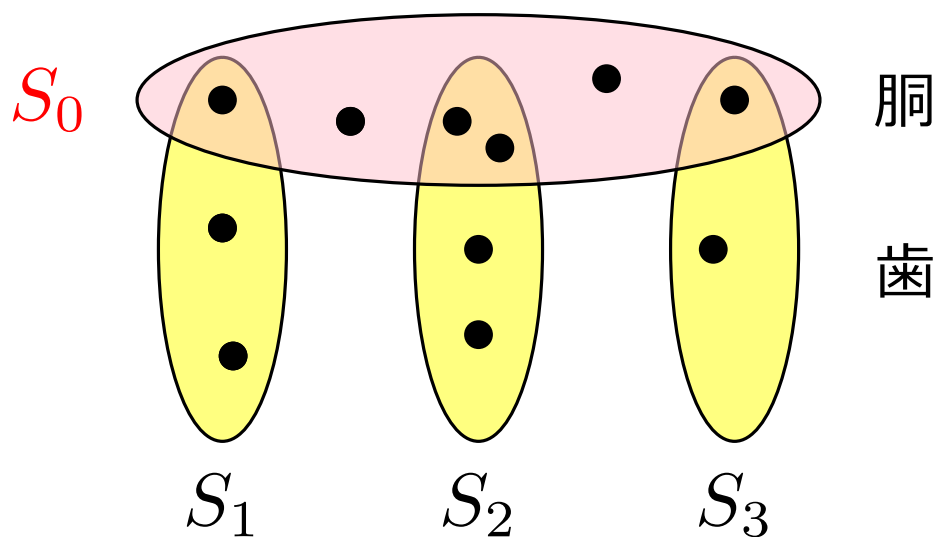
有限集合 V

定義：櫛 (comb)

(Grötschel, Padberg '79)

櫛 とは, 次を満たす $k + 1$ 個の集合 S_0, S_1, \dots, S_k

- $k \geq 3$ は奇数
- S_1, S_2, \dots, S_k は互いに素
- $S_0 \cap S_i \neq \emptyset, S_i - S_0 \neq \emptyset (\forall i \in \{1, 2, \dots, k\})$

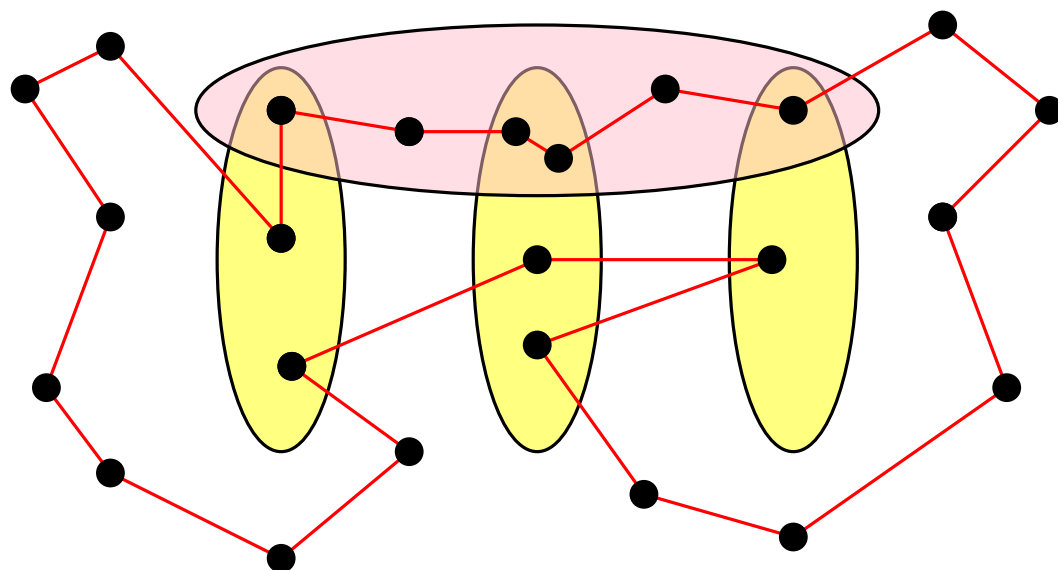


定義：櫛不等式 (comb inequality) (Grötschel, Padberg '79)

任意の櫛 S_0, S_1, \dots, S_k に対して,

$$\sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq \sum_{i=0}^k |S_i| - \frac{3k+1}{2}$$

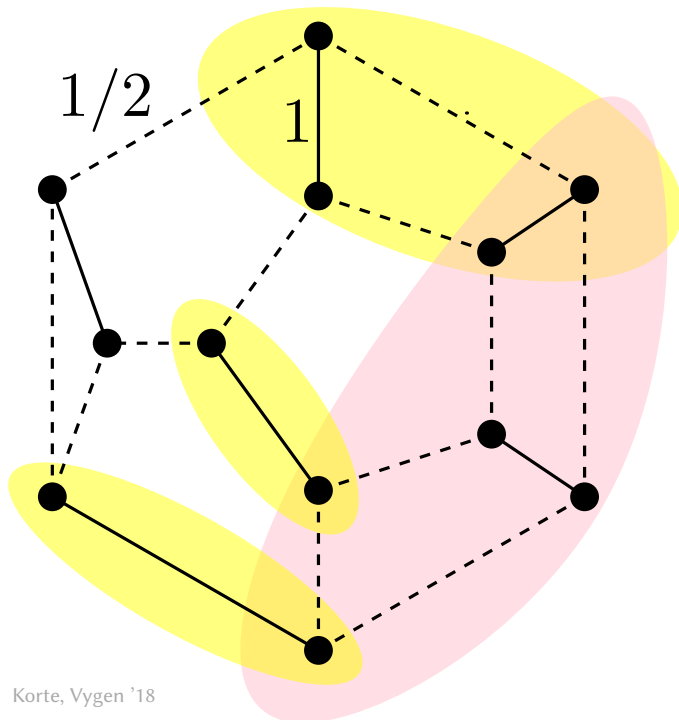
$$5 + 1 + 1 + 0 \quad 6 + 3 + 4 + 2 - \frac{3 \cdot 3 + 1}{2}$$



定義：橢不等式 (comb inequality) (Grötschel, Padberg '79)

任意の橢 S_0, S_1, \dots, S_k に対して,

$$\sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq \sum_{i=0}^k |S_i| - \frac{3k+1}{2}$$



これは線形計画緩和 R の最適解 (適当な距離のもとで)

$$\text{左辺} = \frac{9}{2} + 3 + 1 + 1 = \frac{19}{2} = 9.5$$

$$\text{右辺} = 5 + 4 + 2 + 2 - \frac{3 \cdot 3 + 1}{2} = 8$$

→ 橢不等式でこの解を切り落とせる

定義：櫛不等式 (comb inequality) (Grötschel, Padberg '79)

任意の櫛 S_0, S_1, \dots, S_k に対して,

$$\sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq \sum_{i=0}^k |S_i| - \frac{3k+1}{2}$$

解きたい問題 P

$$\begin{aligned} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{e \in \delta(i)} x_e = 2 & \forall i \\ & \sum_{e \subseteq S} x_e \leq |S| - 1 & \forall S \\ & x_e \in \{0, 1\} & \forall e \end{aligned}$$

性質：

P の任意の許容解は
櫛不等式を満たす

証明： x を P の許容解とする

$$\sum_{e \in \delta(i)} x_e = 2 \quad i \in S_0$$

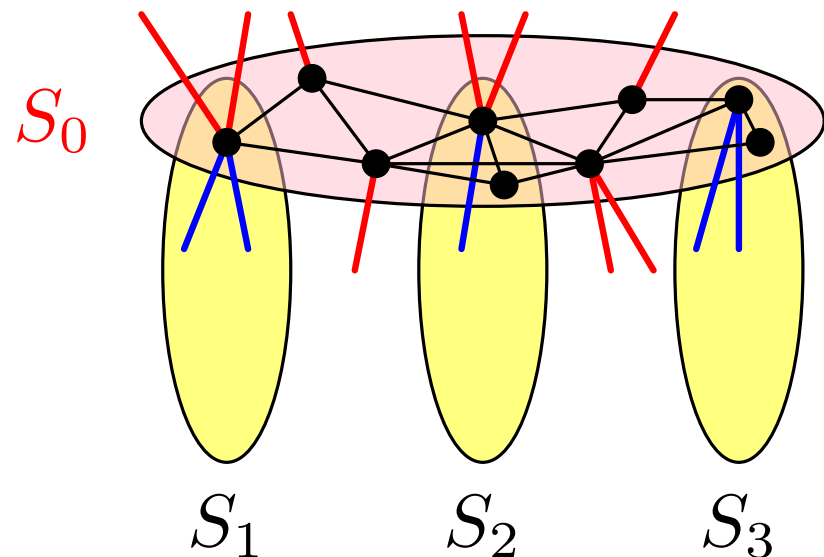
$$-x_e \leq 0 \quad e \in \delta(S_0), e \notin S_1, \dots, S_k$$

解きたい問題 P

$$\text{s.t.} \quad \sum_{e \in \delta(i)} x_e = 2 \quad \forall i$$

$$\sum_{e \subseteq S} x_e \leq |S| - 1 \quad \forall S$$

$$x_e \in \{0, 1\} \quad \forall e$$



証明： x を P の許容解とする

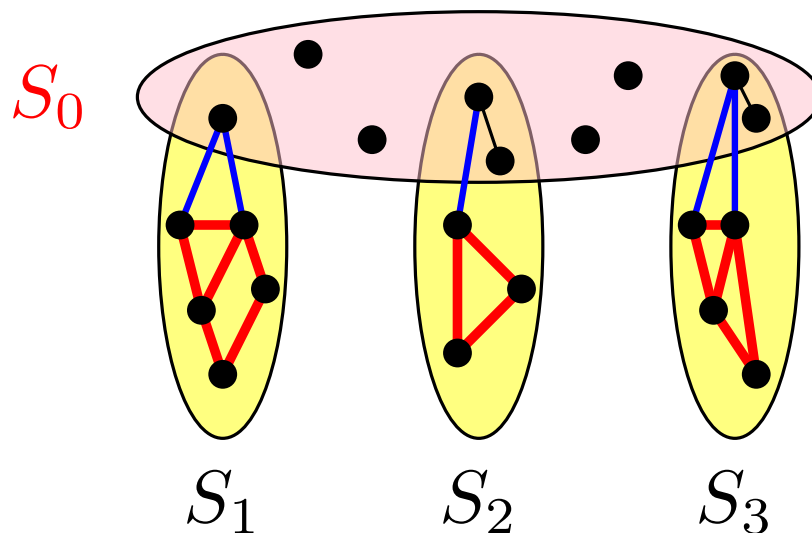
$$\sum_{e \subseteq S_i} x_e \leq |S_i| - 1 \quad i \in \{1, \dots, k\}$$

$$\sum_{e \subseteq S_i - S_0} x_e \leq |S_i - S_0| - 1 \quad i \in \{1, \dots, k\}$$

$$\sum_{e \subseteq S_i \cap S_0} x_e \leq |S_i \cap S_0| - 1 \quad i \in \{1, \dots, k\}$$

解きたい問題 P

$$\begin{aligned} \text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = 2 && \forall i \\ & \sum_{e \subseteq S} x_e \leq |S| - 1 && \forall S \\ & x_e \in \{0, 1\} && \forall e \end{aligned}$$



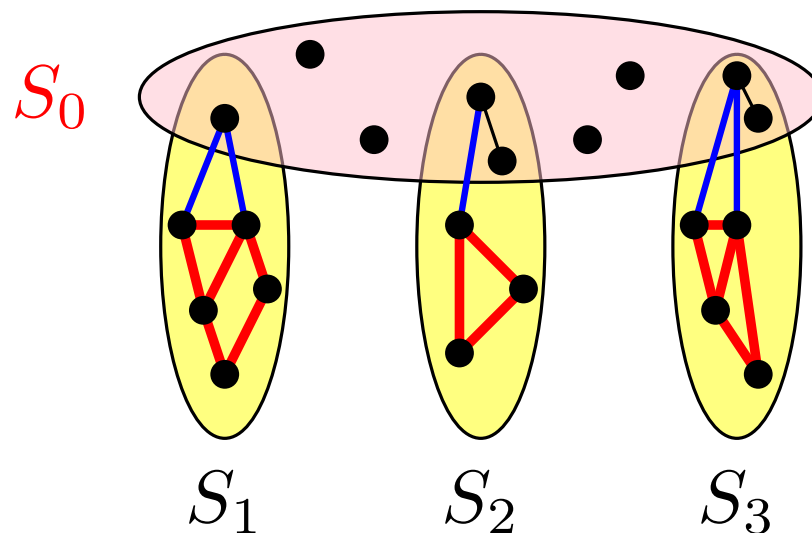
証明： x を P の許容解とする

出てきた式をすべて足すと次が得られる

$$2 \sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq 2 \sum_{i=0}^k |S_i| - 3k$$

解きたい問題 P

$$\begin{array}{ll} \text{s.t.} & \sum_{e \in \delta(i)} x_e = 2 & \forall i \\ & \sum_{e \subseteq S} x_e \leq |S| - 1 & \forall S \\ & x_e \in \{0, 1\} & \forall e \end{array}$$



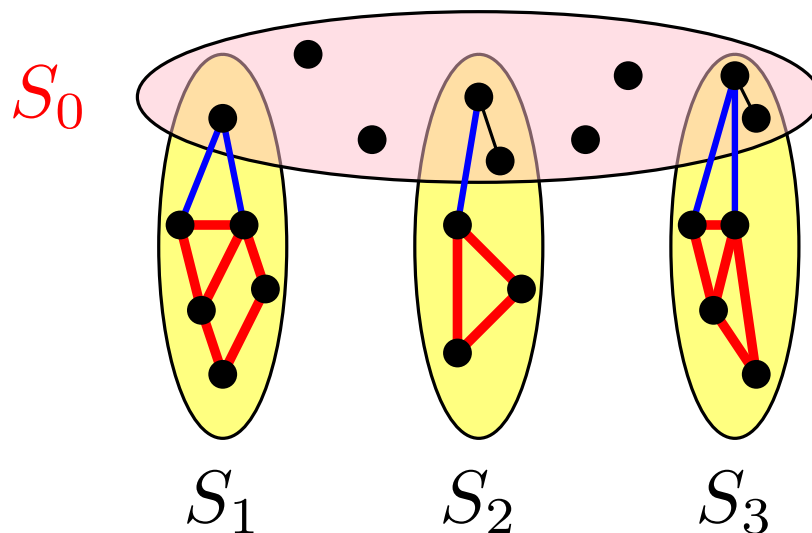
証明： x を P の許容解とする

出てきた式をすべて足すと次が得られる

$$\underbrace{2 \sum_{i=0}^k \sum_{e \subseteq S_i} x_e}_{\text{偶}} \leq \underbrace{2 \sum_{i=0}^k |S_i|}_{\text{偶}} - \underbrace{3k}_{\text{奇}}$$

解きたい問題 P

$$\begin{array}{ll}
 \text{s.t.} & \sum_{e \in \delta(i)} x_e = 2 & \forall i \\
 & \sum_{e \subseteq S} x_e \leq |S| - 1 & \forall S \\
 & x_e \in \{0, 1\} & \forall e
 \end{array}$$



証明： x を P の許容解とする

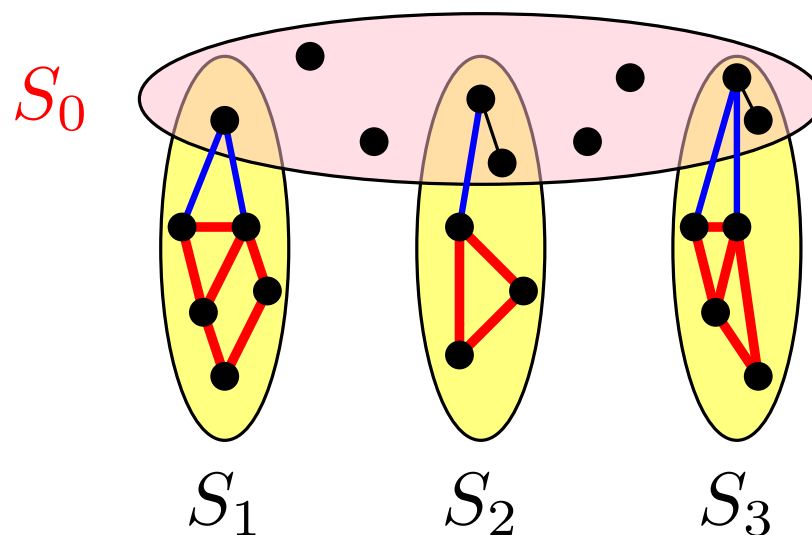
出てきた式をすべて足すと次が得られる

$$2 \sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq 2 \sum_{i=0}^k |S_i| - 3k$$

$$\therefore 2 \sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq 2 \sum_{i=0}^k |S_i| - 3k - 1 \quad \square$$

解きたい問題 P

$$\begin{array}{ll} \text{s.t.} & \sum_{e \in \delta(i)} x_e = 2 & \forall i \\ & \sum_{e \subseteq S} x_e \leq |S| - 1 & \forall S \\ & x_e \in \{0, 1\} & \forall e \end{array}$$



証明： x を P の許容解とする

出てきた式をすべて足すと次が得られる

$$2 \sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq 2 \sum_{i=0}^k |S_i| - 3k$$

$$\therefore 2 \sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq 2 \sum_{i=0}^k |S_i| - 3k - 1 \quad \square$$

定義：櫛不等式 (comb inequality) (Grötschel, Padberg '79)

任意の櫛 S_0, S_1, \dots, S_k に対して,

$$\sum_{i=0}^k \sum_{e \subseteq S_i} x_e \leq \sum_{i=0}^k |S_i| - \frac{3k + 1}{2}$$

目標 : 櫛不等式も切除平面として使いたい

$$\begin{array}{ll} \min. & \sum_{e \in E} d(e)x_e \\ \text{s.t.} & \sum_{e \in \delta(i)} x_e = 2 \quad \forall i \\ & \sum_{e \subseteq S} x_e \leq |S| - 1 \quad \forall S \\ & 0 \leq x_e \leq 1 \quad \forall e \end{array} \longrightarrow \text{最適解 } x^*$$

解くべき問題 (分離問題)

次を満たす櫛 S_0, S_1, \dots, S_k があるか？

$$\sum_{i=0}^k \sum_{e \subseteq S_i} x_e^* > \sum_{i=0}^k |S_i| - \frac{3k+1}{2}$$

解くべき問題 (分離問題)

次を満たす櫛 S_0, S_1, \dots, S_k があるか？

$$\sum_{i=0}^k \sum_{e \subseteq S_i} x_e^* > \sum_{i=0}^k |S_i| - \frac{3k+1}{2}$$

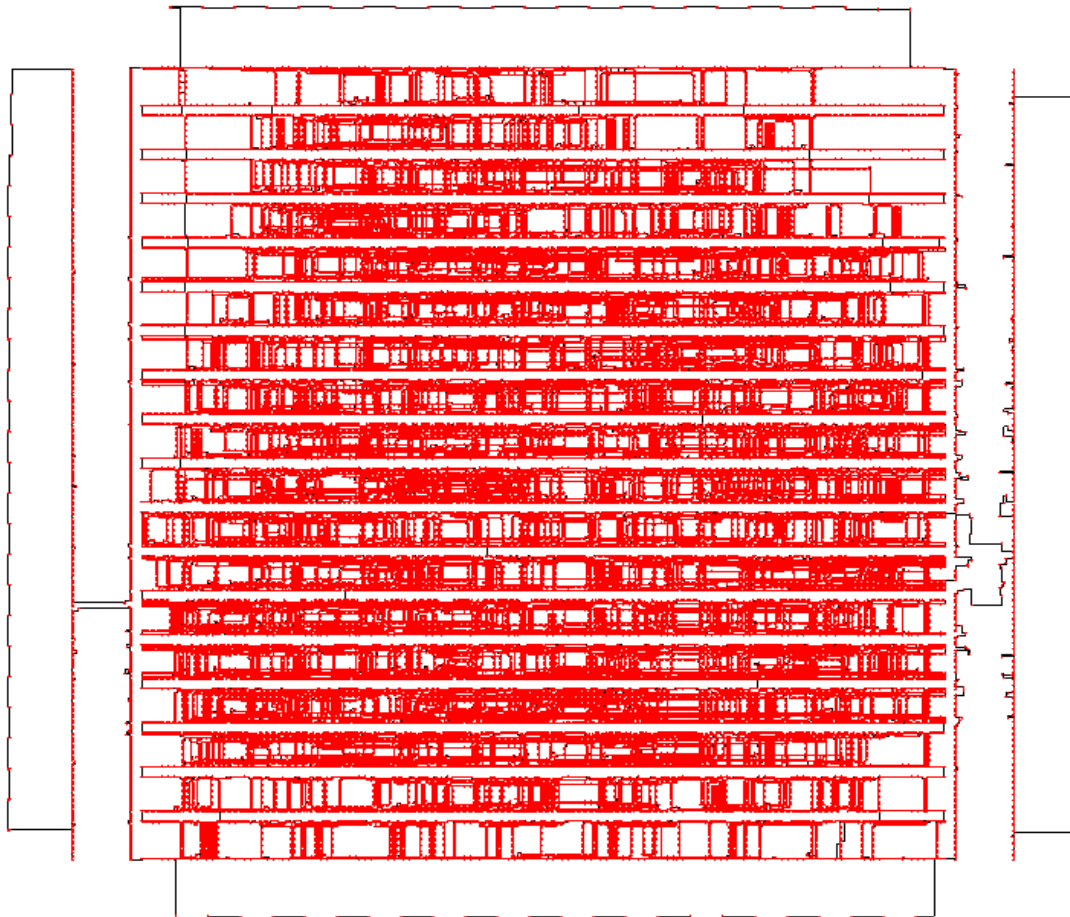
未解決問題

櫛不等式に対する分離問題が多項式時間で解けるか？

多項式時間で解ける場合の例

- $|S_i| = 2$ ($\forall i = 1, \dots, k$) のとき (Padberg, Rao '82)
- k が定数のとき (Carr '97)

実践ではヒューリスティクスが使われる (例えば, Applegate et al. '03)



VLSI インスタンス pla85900
(Applegate et al. 2006)

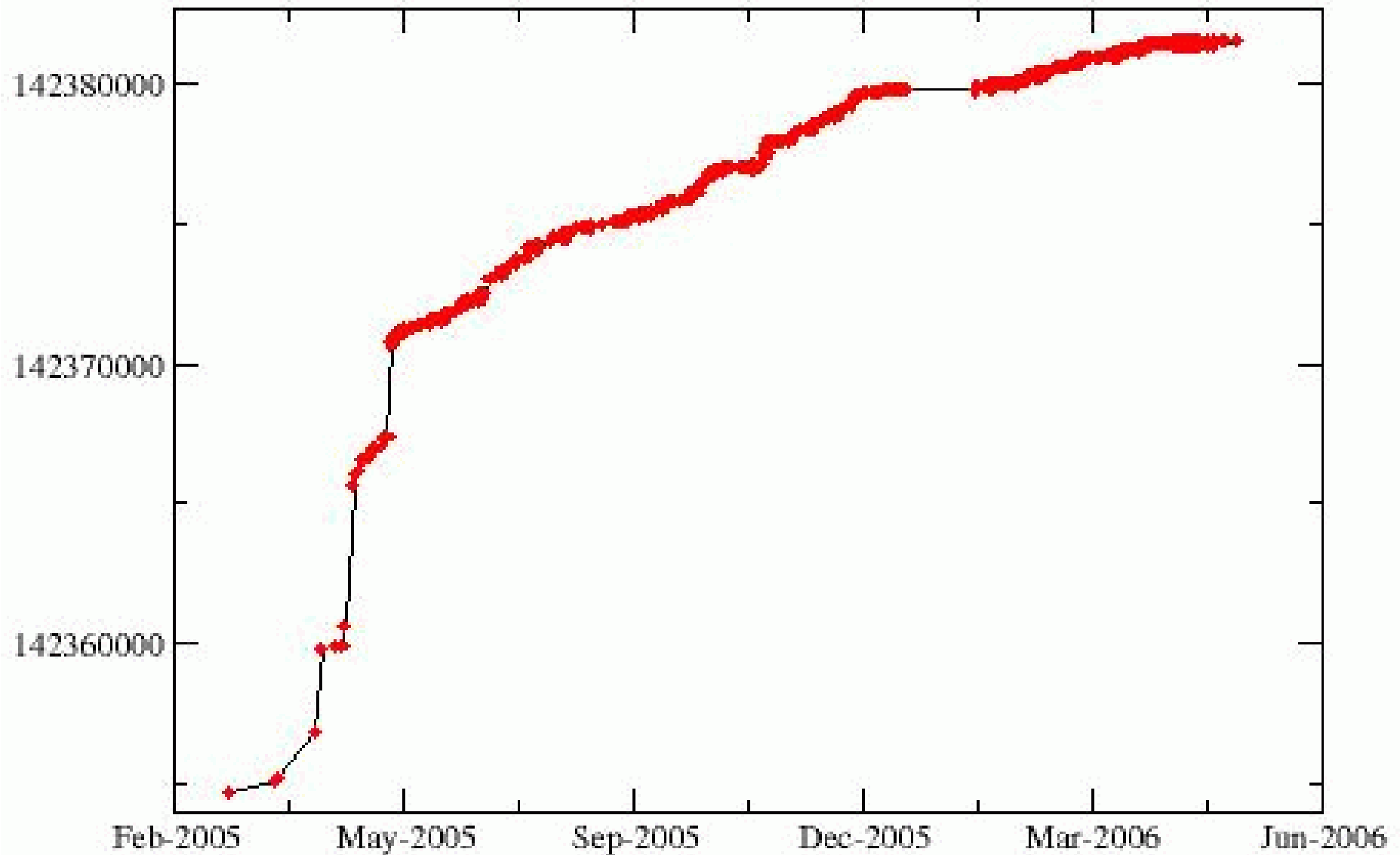
TSPLIB の最大インスタンス

<https://www.math.uwaterloo.ca/tsp/pla85900/index.html>

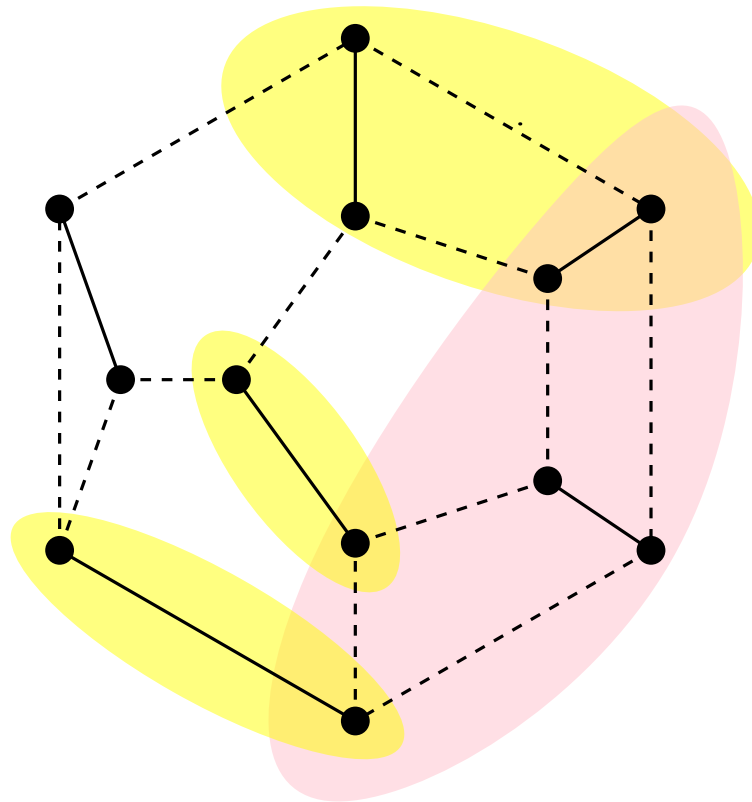
- よい許容解を見つけるヒューリスティクス
- よい下界を見つける切除平面 (今日の講義の内容)
- 並列計算 (ハイパフォーマンスコンピューティング)
- その他

<https://www.math.uwaterloo.ca/tsp/pla85900/index.html>

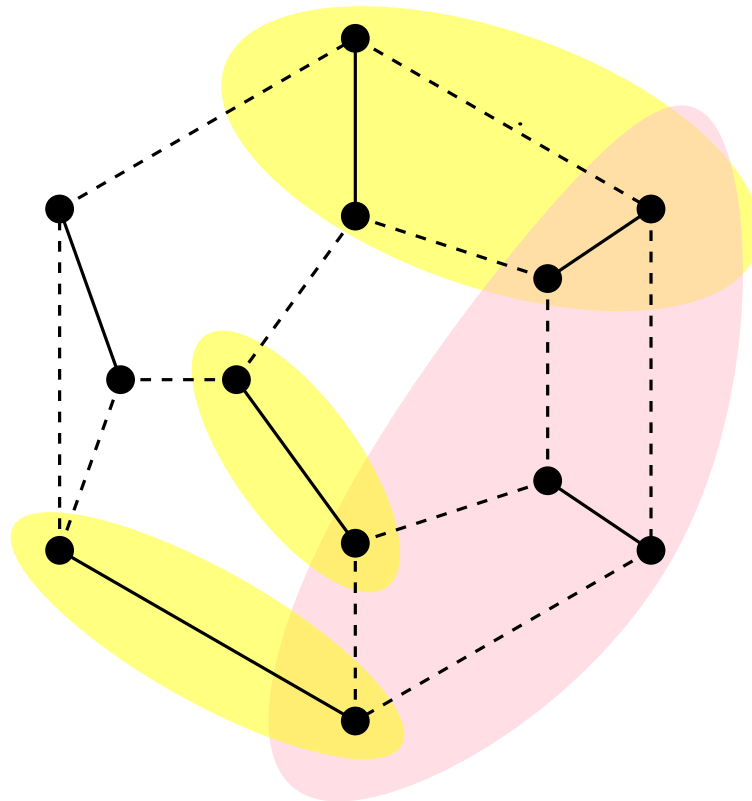
下界が更新される様子



- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



- 復習：巡回セールスマン問題の整数計画モデリング
- 巡回セールスマン問題に対する分枝切除法
- 楕円不等式
- ファセット定義不等式



性質 (証明はしない)

(Grötschel, Padberg '79)

n を都市数とする

- $n \geq 4$ で, $2 \leq |S| \leq n - 2$ のとき,
部分巡回路除去制約は
ハミルトン閉路多面体のファセットを定義する
- $n \geq 6$ のとき,
櫛不等式は
ハミルトン閉路多面体のファセットを定義する

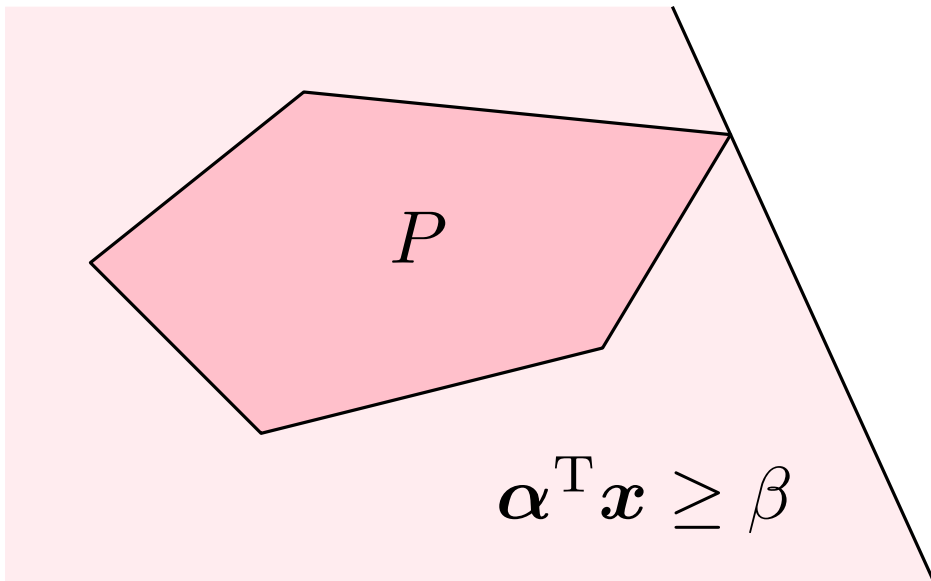
$$A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

$$\text{凸多面集合 } P = \{\mathbf{z} \in \mathbb{R}^n \mid A\mathbf{z} \geq \mathbf{b}\}$$

定義：面 (face)

P の **面** とは, P に対する妥当不等式 $\alpha^T \mathbf{x} \geq \beta$ を用いて次のように書ける集合のこと

$$P \cap \{\mathbf{x} \mid \alpha^T \mathbf{x} = \beta\}$$



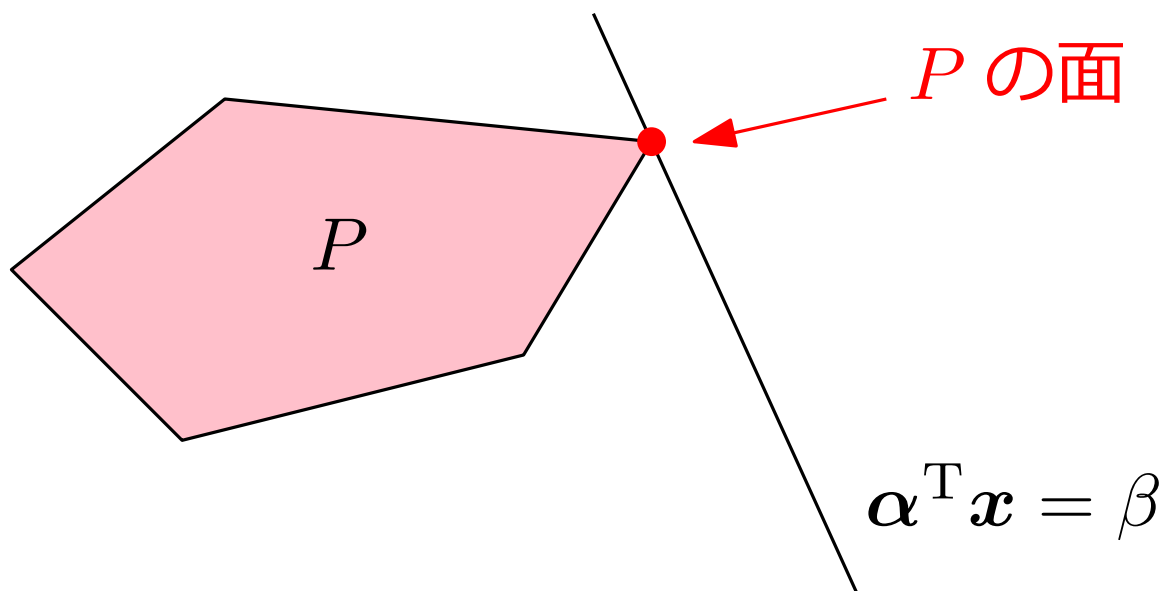
$$A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

$$\text{凸多面集合 } P = \{\mathbf{z} \in \mathbb{R}^n \mid A\mathbf{z} \geq \mathbf{b}\}$$

定義：面 (face)

P の **面** とは, P に対する妥当不等式 $\alpha^T \mathbf{x} \geq \beta$ を用いて次のように書ける集合のこと

$$P \cap \{\mathbf{x} \mid \alpha^T \mathbf{x} = \beta\}$$

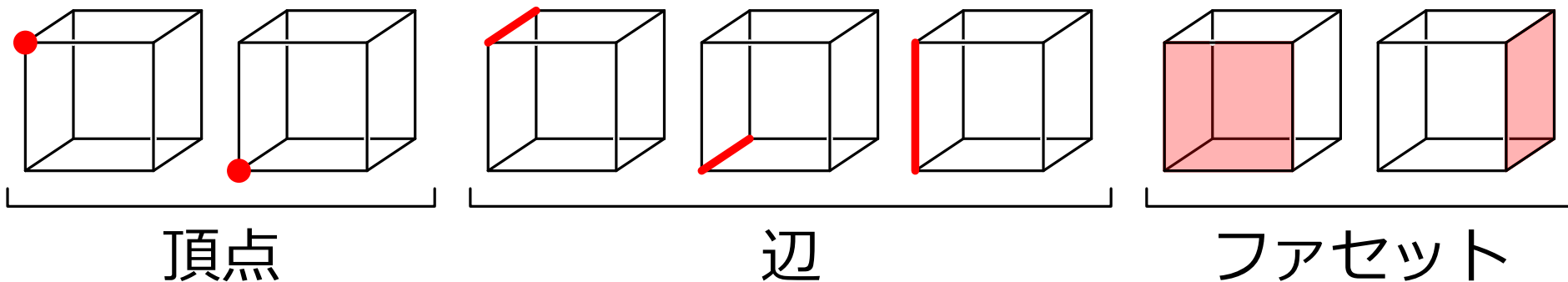


凸多面集合 $P \subseteq \mathbb{R}^n$

定義：頂点 (vertex), 辺 (edge), ファセット (facet)

P の面 F が, P の

- **頂点** であるとは, $\dim F = 0$ のこと
- **辺** であるとは, $\dim F = 1$ のこと
- **ファセット** であるとは, $\dim F = \dim P - 1$ のこと



定義：ハミルトン閉路多面体 (Hamilton cycle polytope)

ハミルトン閉路多面体 とは
問題 P の許容領域の凸包のこと

問題 P

Dantzig, Fulkerson, Johnson ('54)

$$\text{minimize } \sum_{e \in E} d(e)x_e$$

$$\text{subject to } \sum_{j \in V - \{i\}} x_{\{i,j\}} = 2 \quad \forall i \in V$$

$$\sum_{\{i,j\} \subseteq S} x_{\{i,j\}} \leq |S| - 1 \quad \forall S \in 2^V - \{\emptyset, V\}$$

$$x_e \in \{0, 1\} \quad \forall e \in E$$

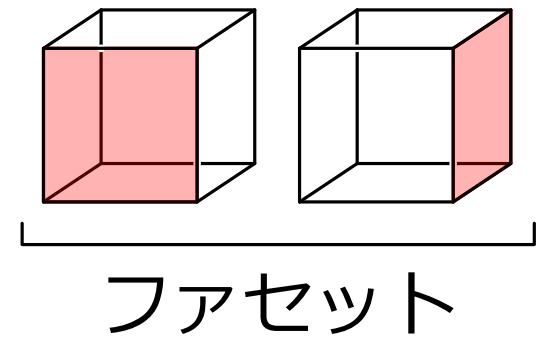
性質 (証明はしない)

(Grötschel, Padberg '79)

n を都市数とする

- $n \geq 4$ で, $2 \leq |S| \leq n - 2$ のとき,
部分巡回路除去制約は
ハミルトン閉路多面体のファセットを定義する
- $n \geq 6$ のとき,
櫛不等式は
ハミルトン閉路多面体のファセットを定義する

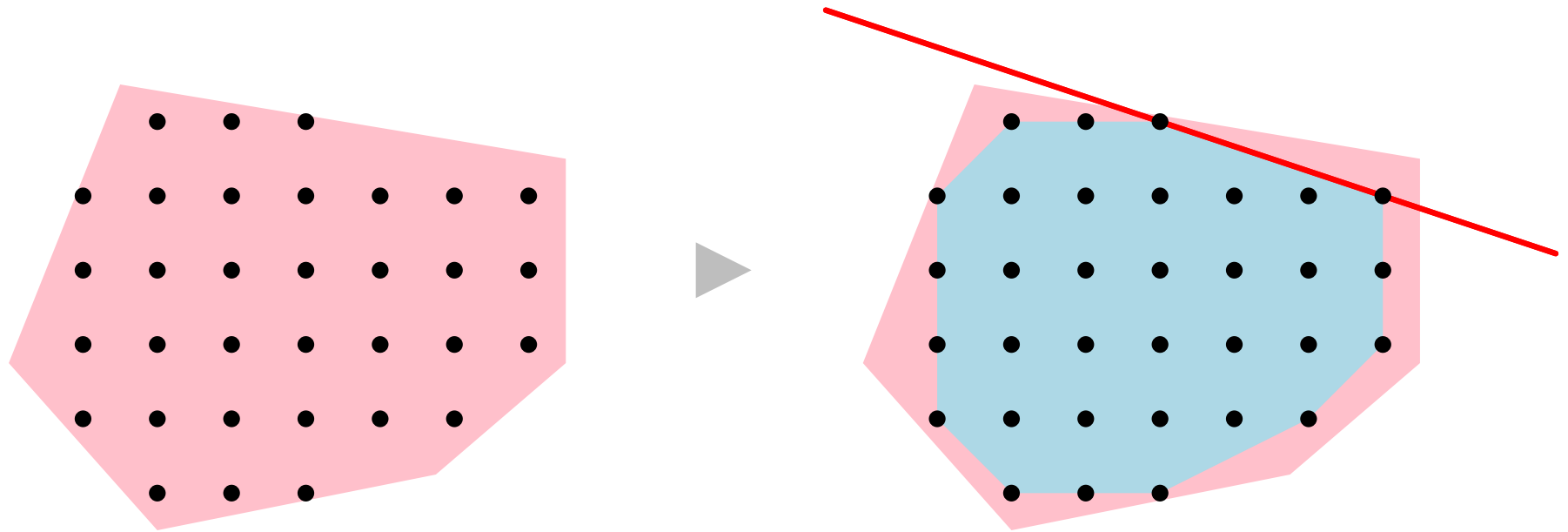
- ファセットを定義する不等式は
凸多面体を記述するために **必要** なもの
- \therefore 追加する不等式としてよい



上記 2 種類以外にもファセットを定義する不等式が研究されている

まとめ：追加する不等式に求められる性質^{76/77}

1. 解きたい問題 P の許容解がすべて満たす
2. 分離問題が簡単に解ける
3. 問題 P の許容領域の凸包のファセットを定義する



巡回セールスマン問題以外の問題でも、
追加する不等式が考えられている

次回の内容

列生成法

- 変数の数が多い問題を解く手法

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline \mathbf{a}_1 & \mathbf{a}_2 & & & & \mathbf{a}_n \\ \hline \end{array} \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \\ \hline \\ \hline \\ \hline x_n \\ \hline \end{array} = x_1 \begin{array}{|c|} \hline \\ \hline \end{array} + x_2 \begin{array}{|c|} \hline \\ \hline \end{array} + \cdots + x_n \begin{array}{|c|} \hline \\ \hline \end{array}$$