Dănuţ Marcu
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NOTE ON TURÁN'S GRAPH

DĂNUŢ MARCU, Bucharest

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Graphs considered in the paper are finite, undirected and simple (without loops or multiple edges), [1, 2] being followed for terminology and notation. We denote by $S(p,q)$ the Stirling number of the second kind, that is, the number of partitions of a p-set into q classes.

A *k-partite complete graph* is a graph consisting of k independent sets, such that two vertices are adjacent if and only if they belong to different independent sets.

Turán's graph, denoted by $T(n,k)$, is a k-partite complete graph with n vertices, for which m parts contain $t+1$ vertices and $k-m$ parts contain t vertices, where $n = kt + m$ and $0 \leqslant m \leqslant k-1$. According to [3], $T(n,k)$ is the unique (up to an isomorphism) graph with n vertices which does not contain $(k+1)$-cliques and has the chromatic number equal to k, its number of edges being maximal in the class of graphs with these properties.

A $(k+r)$*-colouring* of a graph with n vertices and the chromatic number equal to k is a partition of its vertex set into $k+r$ classes $(0 \leqslant r \leqslant n-k)$ such that two vertices belonging to the same class are not adjacent, the order of class being indifferent.

Theorem 1. *The number $C(n,k,r)$ of $(k+r)$-colourings of $T(n,k)$ is given by*

$$C(n,k,r) = \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\dots+n_k = k+r}} \left(\prod_{i=1}^{m} S(t+1,n_i)\right) \cdot \left(\prod_{i=m+1}^{k} S(t,n_i)\right).$$

Proof. By n_i for $i = 1, \dots, k$ let us denote the number of classes of the partition of the i-th part of $T(n,k)$ induced by a $(k+r)$-colouring of $T(n,k)$. Then

$$n_1 + \dots + n_k = k + r$$

and

$$n_i \geqslant 1 \quad \text{for } i = 1, \ldots, k.$$

One can observe that all colourings with $k+r$ classes of $T(n,k)$ are obtained without repetitions from the divisions of $k+r$ into k parts, two divisions being considered different if they differ only by the order of terms.

Obviously, $C(n,k,r) = 1$ for $r = 0$ and $r = n-k$, and $C(n,k,r) = 0$ for $r > n-k$. □

Theorem 2. *If we denote $[\lambda]_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$, then the chromatic polynomial of $T(n,k)$ is equal to*

$$P\big(T(n,k);\lambda\big) = \sum_{\substack{p_1+\ldots+p_{t+1}=m\\ q_1+\ldots+q_t=k-m}} \binom{m}{p_1,\ldots,p_{t+1}}\binom{k-m}{q_1,\ldots,q_t} \times \prod_{i=2}^{t}\big(S(t+1,i)\big)^{p_i}\cdot\prod_{j=2}^{t-1}\big(S(t,j)\big)^{q_j}[\lambda]_{p\oplus q},$$

where

$$p\oplus q = p_1 + 2p_2 + \ldots + (t+1)p_{t+1} + q_1 + 2q_2 + \ldots + tq_t.$$

Proof. Obviously, the chromatic polynomial of a graph consisting of p isolated vertices is equal to

$$\lambda^p = \sum_{k=1}^{p} S(p,k)[\lambda]_k.$$

Thus, having in view the method of Read [4], we obtain

$$P\big(T(n,k);\lambda\big) = \Big(\sum_{p=1}^{t+1} S(t+1,p)[\lambda]_p\Big)^m \Big(\sum_{q=1}^{t} S(t,q)[\lambda]_q\Big)^{k-m},$$

where, by definition,

$$[\lambda]_p[\lambda]_q = [\lambda]_{p+q} \quad \text{for all } p \text{ and } q.$$

Using the multinomial formula we obtain the result. □

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Author's address: Str. Pasului 3, Sect. 2, 70241 Bucharest, Romania.