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From Galilean-invariant to relativistic wave equations

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Through an imaginary change of coordinates in the Galilei algebra in 4 space dimensions and making use of an original idea of Dirac and Lévy-Leblond, we are able to obtain the relativistic equations of Dirac and of Bargmann and Wigner starting with the (Galilean-invariant) Schrödinger equation.

I. INTRODUCTION

In the last decade a series of papers by several people[1-7] have been written with the common purpose of trying to shed some light on the rather intrincate world of both relativistic and Galilean field equations for the different spin particles. Remarkable among them are the systematic studies of the Galilei group carried out by Lévy-Leblond,[1] the selection of good variables in order to make possible the introduction of minimal electromagnetic coupling in a correct way (a program developed by Hagen and Hurley),[2-4] the incorporation in the playground of the light-cone-frame coordinates[5] by Chang, Root, and Yan,[6] and the formulation by Singh and Hagen of general Lagrangian theories for bosons and fermions.[7]

In even more recent papers a procedure has been found[8] which allows for the rederivation of the Galilean-invariant equations of Lévy-Leblond[1] and Hagen[2] starting with the relativistic equations of Dirac, Bargmann and Wigner, Proca, Rarita and Schwinger, and Singh and Hagen for different spin particles. Finally, the discovery that the ordinary Poincaré Lie algebra can be obtained as a subalgebra of the Galilei Lie algebra in 4 space dimensions by means of an imaginary coordinate transformation[9] has opened the way to the inverse procedure, i.e., the derivation of ordinary relativistic field equations starting from Galilean-invariant equations and, in particular, from the Schrödinger equation. These possibilities are explored in the present paper.

The organization of this work is as follows. In Sec. II we provide a review of the mathematical aspects of the procedure: The commutation relations of the Poincaré algebra in Minkowski space are obtained from the Galilei algebra in one additional space dimension. Section III is devoted to the physical applications, namely, the derivation of Dirac's equation starting from Schrödinger's equation trivially generalized to a $(4+1)$-dimensional world. The equations of Bargmann and Wigner are analogously obtained. Finally, Sec. IV is devoted to our conclusions.

II. FROM THE GALILEI TO THE POINCARÉ LIE ALGEBRA

Let $x^a=(x^0,x^1,x^2,x^3,x^4)$ denote a point in $(4+1)$-dimensional space-time. Let us introduce the coordinate transformation

$$\bar{x}^0=ix^4,\quad \bar{x}^i=x^i\quad (i=1,2,3),\quad \bar{x}^4=cx^0\,, \tag{2.1}$$

where c is an arbitrary constant.[9] The Galilei group in $4+1$ dimensions can be put in the form[10,11]

$$G=[\mathrm{SO}(4)\times \mathrm{T}_4^{(v)}]\times[\mathrm{T}_1\otimes \mathrm{T}_4]\,, \tag{2.2}$$

where $\mathrm{T}_4^{(v)}$ is the subgroup of generators of Galilean boosts and T_4 that of translations in the 4-space. If l_i,λ_i $(i=1,2,3)$ are the generators of SO(4), g_r $(r=1,2,3,4)$ those of Galilean boosts, and d_a $(a=0,1,2,3,4)$ the generators of time and space translations, respectively, and if we denote with a bar the corresponding generators in the new coordinate system, we have

$$\bar{l}_i=l_i,\quad \bar{d}_i=d_i,\quad \bar{g}_i=i\lambda_i\quad (i=1,2,3),\quad \bar{d}_0=-id_4\,. \tag{2.3}$$

Following Ref. 8, we take the new generators of boosts to be $k_i=\bar{M}^{0i}=-i\lambda_i=-\bar{g}_i$, where M_{ab} are those of the group $\mathrm{SO}(4)\times\mathrm{T}_4^{(v)}$.

Once the coordinate transformation (2.1) has been carried out, it is easy to see that the commutation relations of the subset of new generators l_i, k_i, d_i $(i=1,2,3)$, and $h\equiv\bar{d}_0$ are given by

$$\begin{aligned}&[l_i,l_j]=i\epsilon_{ijk}l_k,\quad [l_i,k_j]=i\epsilon_{ijh}k_h,\quad [l_i,d_j]=i\epsilon_{ijk}d_k\,,\\ &[k_i,k_j]=-i\epsilon_{ijh}l_h,\quad [d_i,d_j]=[l_i,h]=[d_i,h]=0\,,\\ &[k_i,h]=id_i,\quad [k_i,d_j]=i\delta_{ij}h\quad (i,j,k,h=1,2,3)\,,\end{aligned} \tag{2.4}$$

and constitute the commutation relations of the Lie algebra of the Poincaré group in ordinary Minkowski space. The rest of the transformed generators are

$$\bar{\lambda}_i=-\frac{1}{c}g_i,\quad \bar{g}_4=\frac{i}{c}g_4,\quad \bar{d}_4=\frac{1}{c}d_0\,. \tag{2.5}$$

As c is an arbitrary constant, it can be made to tend to infinity. By doing this, Eqs. (2.4) are not affected and $\bar{d}_4$ is transformed into a central element of the new algebra, but notice that, for any value of c, $\bar{d}_4$ always remains a central element of the Poincaré (sub)algebra (2.4), in which we are primarily interested.

III. FROM THE SCHRÖDINGER TO THE DIRAC AND BARGMANN-WIGNER EQUATIONS

It is now our aim to construct a wave equation invariant under the Galilean $(4+1)$-dimensional kinematical group. Let E denote the energy of the particle, $p_r=(p_1,p_2,p_3,p_4)$ its impulse, and m its mass. The equality

$$2mE-p_rp_r=0 \tag{3.1}$$

must be satisfied. It provides the scalar equation

$$\left(2im\frac{\partial}{\partial x_0}+\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_r}\right)\phi(\vec{x},x_4,x_0)=0\,, \tag{3.2}$$

which we call the $4+1$ Schrödinger equation. After the coordinate change (2.1) this equation becomes

$$\left(\Box-2imc\frac{\partial}{\partial\bar{x}_4}\right)\bar{\phi}(\vec{x},\bar{x}_4,\bar{x}_0)=0\,, \tag{3.3}$$

where $\Box\equiv\partial^2/\partial\bar{x}_0{}^2-(\partial/\partial x_i)\partial/\partial x_i$ and $\bar{\phi}(\vec{x},\bar{x}_4,\bar{x}_0)=\phi(\vec{x},x_4,x_0)$. When the operator $\partial/\partial\bar{x}_4$ acts as a multiplicative constant $-i\eta$ [or, which is the same, when one is restricted to solutions $\bar{\phi}$ whose dependence on $\bar{x}_4$ has the form $\bar{\phi}=\exp(-i\eta\bar{x}_4)\bar{\phi}_1(\vec{x},\bar{x}_0)$], Eq. (3.3) is transformed into

$$(\Box+M^2)\bar{\phi}=0\,, \tag{3.4}$$

where

$$M^2=-2m\eta c\,. \tag{3.5}$$

Equation (3.4) is a Klein-Gordon equation for a particle of mass M given by (3.5). We thus see that the Schrödinger equation gives rise to a relativistic Klein-Gordon equation after properly reducing one space dimension and use of (2.1).

Following Dirac[12] and Lévy-Leblond,[1] let us try to linearize Eq. (3.2). We must find six quantities A, B_r $(r=1,2,3,4)$, and C such that

$$(AE+B_rp_r+C)\psi(\vec{x},x_4,x_0)=0 \tag{3.6}$$

will be a good $(4+1)$-dimensional Galilean equation, i.e., it must include the mass condition (3.1). If we act on the left of (3.6) with the operator $A'E+B_r'p_r+C'$, where A', B_r', and C' is a new set of six quantities, we easily find

$$\begin{aligned}[A'AE^2+\tfrac{1}{2}(B_r'B_s+B_s'B_r)p_rp_s\\ +(A'B_r+B_r'A)Ep_r+(A'C+C'A)E\\ +(C'B_r+B_r'C)p_r+C'C]\psi(\vec{x},x_4,x_0)=0\,.\end{aligned}$$

If we now require that this be identical with the equation $(2mE-p_rp_r)\psi=0$, the following relations must be satisfied:

$$B_a'B_b+B_b'B_a=-2\delta_{ab}\quad(a,b=1,2,3,4,5,6)\,, \tag{3.7}$$

where we have defined convenient new matrices:

$$\begin{aligned}B_5&=i\left(A+\frac{C}{2m}\right),\quad B_5'=i\left(A'+\frac{C'}{2m}\right),\\ B_6&=A-\frac{C}{2m},\quad B_6'=A'-\frac{C'}{2m}.\end{aligned} \tag{3.8}$$

A solution to this system is

$$\begin{aligned}&B_1=R\gamma_1,\ldots,B_5=R\gamma_5,B_6=-iR\,,\\ &B_1'=-\gamma_1R^{-1},\ldots,B_5'=-\gamma_5R^{-1},B_6'=-iR^{-1}\,,\end{aligned} \tag{3.9}$$

where the γ matrices are those given in Ref. 13 and where R is any invertible 4×4 matrix. A particularly clear expression for (3.6) is obtained with the following choice of (high-energy) representation:

$$\begin{aligned}\gamma_i&=\begin{pmatrix}0&-i\sigma_i\\ i\sigma_i&0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ \gamma_5&=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad R=\begin{pmatrix}0&i\\i&0\end{pmatrix},\end{aligned} \tag{3.10}$$

in which case there results

$$\begin{aligned}&E\varphi+(\vec{\sigma}\cdot\vec{p}+ip_4)\chi=0\,,\\ &(\vec{\sigma}\cdot\vec{p}-ip_4)\varphi+2m\chi=0\,.\end{aligned} \tag{3.11}$$

φ and χ are two-component objects and $\psi=(\varphi,\chi)$. This is the linear version of (3.6) and it is invariant under a certain 4×4 linear (projective) representation of the $4+1$ Galilean group.[1]

Let us now perform the change of coordinates (2.1) in (3.11):

$$\begin{aligned}&ic\frac{\partial}{\partial\bar{x}_4}\bar{\varphi}+(\vec{\sigma}\cdot\vec{p}+H)\bar{\chi}=0\,,\\ &(\vec{\sigma}\cdot\vec{p}-H)\bar{\varphi}+2m\bar{\chi}=0\,.\end{aligned} \tag{3.12}$$

Here H is the new energy operator $H=i\partial/\partial\bar{x}_0$ and the bars on functions mean transformed functions in the same sense as before. If, again, $\partial/\partial\bar{x}_4$ acts as a multiplicative constant $-i\eta$, (3.12) becomes

$$\begin{aligned}&\eta c\bar{\varphi}+(\vec{\sigma}\cdot\vec{p}+H)\bar{\chi}=0\,,\\ &(\vec{\sigma}\cdot\vec{p}-H)\bar{\varphi}+2m\bar{\chi}=0\,.\end{aligned} \tag{3.13}$$

Successively adding and substracting both Eqs. (3.13), we get

$$\begin{aligned}&(\vec{\sigma}\cdot\vec{p})(\bar{\chi}+\bar{\varphi})+H(\bar{\chi}-\bar{\varphi})+(2m\bar{\chi}+\eta c\bar{\varphi})=0\,,\\ &(\vec{\sigma}\cdot\vec{p})(\chi-\bar{\varphi})+H(\bar{\chi}+\bar{\varphi})-(2m\bar{\chi}-\eta c\bar{\varphi})=0\,,\end{aligned}$$

which strongly resemble Dirac's equation for a relativistic particle of spin $\frac{1}{2}$. We only need to let $c=-2m$ and define the new field variables

$$\varphi' \equiv \frac{1}{\sqrt{2}}(\bar{\chi}+\bar{\varphi}), \quad \chi' \equiv \frac{1}{\sqrt{2}}(\bar{\chi}-\bar{\varphi}), \tag{3.14}$$

hence finding

$$\begin{aligned} &(H-M)\varphi' + \vec{\sigma}\cdot\vec{p}\chi' = 0, \\ &\vec{\sigma}\cdot\vec{p}\varphi' + (H+M)\chi' = 0, \end{aligned} \tag{3.15}$$

where

$$M = 2m = -\eta c\,. \tag{3.16}$$

Equation (3.15) is exactly that of Dirac for a particle of mass M given by (3.16), in the common low-energy representation of the γ matrices.

Equation (3.16) relating the masses of the Schrödinger and Dirac particles deserves a short comment. Whereas it implies the Klein-Gordon relation (3.5), it is not implied by the latter, i.e., it actually is a stronger restriction on M than (3.5). This is bound to the process of linearization of the Schrödinger equation (3.2) and will also be seen to hold for higher-spin particles.

The generalization of the above procedure to the case of a spin-s particle by means of the Bargmann-Wigner formulation is straightforward. Nevertheless, it must be remarked that the notion of spin in 4 space dimensions is quite unphysical because it is related to the spatial symmetry group SO(4). We therefore should not speak of such things as "Galilean $4+1$ wave equations for a particle of spin s." On the other hand, we would also like to be able to obtain relativistic equations for higher-spin particles. The way out of this problem will be to postulate a family of Galilean $(4+1)$-invariant equations which, after the change of coordinates (2.1), yield the corresponding relativistic equations for the considered value of the spin. In this way we omit any reference to the "spin" of a $4+1$ Galilean particle but find, after all, that true physical spin comes out in a natural way once the above-mentioned procedure is carried out.

It appears to be a natural ansatz to take the following Bargmann-Wigner type of equations:

$$\begin{aligned} &(\Lambda\otimes I\otimes\cdots\otimes I)\phi(\vec{x},x_4,x_0) = 0, \\ &(I\otimes\Lambda\otimes\cdots\otimes I)\phi(\vec{x},x_4,x_0) = 0, \\ &\ldots, \\ &(I\otimes I\otimes\ldots\otimes\Lambda)\phi(\vec{x},x_4,x_0) = 0, \end{aligned} \tag{3.17}$$

where we have a set of $2s$ equations and where Λ is given by (3.6) in the representation (3.10), i.e.,

$$\Lambda = \begin{pmatrix} E & \vec{\sigma}\cdot\vec{p}+ip_4 \\ \vec{\sigma}\cdot\vec{p}-ip_4 & 2m \end{pmatrix}. \tag{3.18}$$

Through the coordinate change (2.1), (3.17) is converted into

$$\begin{aligned} &(\Lambda'\otimes I\otimes\cdots\otimes I)\phi'(\bar{x}_0,\vec{x}) = 0, \\ &(I\otimes\Lambda'\otimes\cdots\otimes I)\phi'(\bar{x}_0,\vec{x}) = 0, \\ &\ldots, \\ &(I\otimes I\otimes\ldots\otimes\Lambda')\phi'(\bar{x}_0,\vec{x}) = 0, \end{aligned} \tag{3.19}$$

where we again have a set of $2s$ equations and where

$$\Lambda' = \begin{pmatrix} H-M & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & H+M \end{pmatrix}. \tag{3.20}$$

As they stand, Eqs. (3.19) and (3.20) are exactly those of Bargmann and Wigner for a relativistic particle of spin s and mass M.

The system (3.17) having been established, we may now use the method described in Ref. 13 to transform a multispinorial wave equation into a tensorial one. It is easy to prove that, using our procedure, the equations of Proca and Rarita and Schwinger in ordinary Minkowski space are obtained starting from Eq. (3.17) with $s=1$ and $s=\frac{3}{2}$, respectively.

IV. CONCLUSIONS

The usual way to relate relativistic with Galilean-invariant field equations has always been to take the nonrelativistic limit of the relativistic expressions, i.e., to write down the constant c where it belongs and let it tend to infinity. In this way one obtains the nonrelativistic equations of Schrödinger and Pauli corresponding to the different spin particles.[7] An alternative way to connect both types of equations was proposed in Ref. 8, based on the known fact that the use of the light-cone frame[5] led to some kind of Galilean symmetries in $2+1$ dimensions. This is so because in the new coordinates a $(2+1)$-dimensional Galilean subgroup appears in the commutation relations of the Poincaré group. Using this light-cone procedure, the nonrelativistic equations of Lévy-Leblond, Hagen and Hurley, and Schrödinger and Pauli for the different spin particles were obtained, all of them, that is clear, in one space dimension less.

The demonstration in a latter work[9] that the $(4+1)$-dimensional Galilei group yields an ordinary Poincaré subgroup through an imaginary coordinate transformation opened the way to the reciprocal procedure, i.e., the obtaining of relativistic equations starting from Galilean-invariant ones.

In the present paper we have taken this way and developed a systematic method to obtain relativistic equations for any spin by linearization of the Schrödinger equation in $4+1$ dimensions and then extension of the Bargmann-Wigner method. This procedure avoids any allusion to the unphysical notion of SO(4) spin yielding, however, meaningful relativistic equations in Minkowski space. The explicit expansion of the multispinorial formulation in terms of tensors—performed with coherent results for the cases $s=1$ and $s=\frac{3}{2}$—leads to relativistic equations in the tensor-spinor representation (Proca, Rarita and Schwinger, etc.). It seems clear to us that the tensorialization of multispinorial equations and the reduction through (2.1) of one space dimension are commutative processes for any value of the spin.

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[1]J.-M. Lévy-Leblond, J. Math. Phys. 4, 776 (1963); Commun. Math. Phys. 4, 157 (1967); 6, 286 (1967); Riv. Nuovo Cimento 4, 99 (1974).
[2]C. R. Hagen, Commun. Math. Phys. 18, 97 (1971).
[3]W. J. Hurley, Phys. Rev. D 4, 3605 (1971).
[4]C. R. Hagen and W. J. Hurley, Phys. Rev. Lett. 24, 1381 (1970); W. J. Hurley, Phys. Rev. D 3, 2339 (1971).
[5]J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970); J. D. Bjorken, J. B. Kogut, and D. E. Soper, *ibid*. 3, 1382 (1971).
[6]S. Chang, R. G. Root, and T. Yan, Phys. Rev. D 7, 1133 (1973); 7, 1147 (1973).
[7]L. P. S. Singh and C. R. Hagen, Phys. Rev. D 9, 898 (1974); 9, 910 (1974).
[8]E. Elizalde and J. Gomis, Nuovo Cimento 35A, 336 (1976); 35A, 347 (1976); 35A, 367 (1976); Nucl. Phys. B122, 535 (1977).
[9]E. Elizalde, J. Math. Phys. 19, 526 (1978); E. Elizalde and J. Gomis, *ibid*. 19, 1790 (1978).
[10]E. G. Sudarshan and N. Múkunda, *Classical Dynamics* (Wiley, New York, 1974).
[11]H. Bacry, *Lecons sur la Thêorie des Groupes et les Symmêtries des Particules Élêmentaires* (Gordon and Breach, Paris, 1967).
[12]P. A. M. Dirac, Proc. R. Soc. London A117, 610 (1928).
[13]D. Lurié, *Particles and Fields* (Wiley, New York, 1968).