CONTENTS

PREFACE TO THE FIRST RUSSIAN EDITION	XIII
PREFACE TO THE SECOND RUSSIAN EDITION	xv
Chapter 0. INTRODUCTION	1
Chapter 1. METRIC SPACES	4
1.1. Function Spaces. Order Relations	4
1.11 ZORN's lemma	5
1.12 ZERMELO's theorem	5
1.2. Metric Spaces	6
1.21 Metric space	6
1.22 Limit of a sequence	7
1.23 Closure	8
1.24 Continuous functions	8
1.25 Homeomorphism	8
1.3. Examples of Metric Spaces	9
1.31 The real line R	9
1.32 The Euclidean n -dimensional space E_n	9
1.33 The space $C[0, 1]$ of continuous functions with the CHEBYSHEV metric	9
1.34 The space m of bounded number sequences	9
1.35 The space c of convergent number sequences	10
1.36 The space $M[0, 1]$ of bounded real functions	10
1.37 The space $\widetilde{M}[0, 1]$ of bounded measurable functions	10
1.38 The space s of all sequences of numbers	12
1.39 The space $S[0, 1]$ of convergence in measure	14
1.3(10) The space $L_2[0, 1]$ of p-integrable functions	14
1.3(11) The space $l_p(p \ge 1)$ of real number sequences	14
1.3(12) The space $l_p^{(n)}$	15
1.3(13) The complex spaces	15
1 3(14) Non-metrizable spaces	15

vi Contents

1.4.	Complete Spaces. Some Examples	15
1.41	Definitions	15
	The space E_n	16
	The space $C[0, 1]$	16
1.44	The space m	17
1.45	The space c	17
1.46	The space $\widetilde{M}[0, 1]$	18
1.47	The spaces $L_p[0, 1]$ and l_p .	18
1.5.	The Completion of Metric Spaces	18
1.6.	Theorems on Complete Spaces	23
1.7.	The Contraction Mapping Principle	25
1 71	Solutions of systems of linear algebraic equations by the iteration method	27
1.71	Existence and uniqueness of the solution of an integral equation	29
1.72	Application to partial differential equations	30
1.,,	•	22
1.8.	Separable Spaces	32
	The separability of the <i>n</i> -dimensional Euclidean space E_n	33
	The separability of the n -dimensional Euclidean space $-n$. The separability of the spece $C[0, 1]$	33
	The separability of the space l_p	33
	The separability of the space $L_p[0, 1]$	34
	The separability of the space s	34
1.85	The inseparability of the space m	34
1,00	The inseparation of the space in	
	Chapter 2. NORMED LINEAR SPACE	36
2.1.	Linear Spaces	36
2.11	Definitions	36
	Linear manifolds	38
_	Direct sum	39
2.14	Factor (quotient) spaces	41
2.15	The relation between the real and the complex spaces	42
2.2.	Normed Linear Spaces	43
2,21	Definitions	43
	Series of elements of a BANACH space	49
2.3.	Linear Topological Spaces	50

		Contents	Vii
2.4.	Abstract Hilbert Space		54
	Introduction		54
	Axioms of abstract Hilbert space		54
	Orthogonality		56
	Orthonormal system		58
	Complete orthonormal system (V. A. STEKLOV)		60
2.46	Isomorphism between separable HILBERT spaces	•	61
2.5.	Generalized Derivatives and Spaces of S. L. Sobolev		62
2.51	The equivalence of the two definitions of generalized derivative		65
2.52	SOBOLEV's formula		73
	Inclusion theorem		80
	Chapter 3. LINEAR OPERATORS		83
3.1.	Linear Operators		83
3 11	Definitions		83
	Simplest properties of linear operators		85
	The space of operators		86
3.14	The ring of linear continuous operators		86
	Functions of operators		87
3.2,	Linear Operators in Normed Linear Spaces		90
3.21	The norm of an operator		94
3.3.	Linear Functionals		97
			97
3.3	Basic concepts		99
3.4.	The Space of Bounded Linear Operators		100
3.4	Uniform and pointwise convergence of operators		100
3.4	2 Application to interpolation theory		
3.5	. Inverse Operators		104
3.5	1 Inverse operators and algebraic equations		104 105
3.5	2 Theorems on inverse operator		110
3.5	3 Operators depending on a parameter		110
3.6	6. Banach Spaces with Basis		111
3 6	51 Definition		111
٠.٠	/		

	Chapter 4. LINEAR FUNCTIONALS	118
4.1.	Hahn-Banach Extension Theorem and its Corollaries	118
4.2.	The General Form of Linear Functionals in certain Functional Spaces	123
4.21	Linear functionals on the n -dimensional space E_n	123
4.22	The general form of linear functionals on s	124
4.23	The general form of linear functionals on $C[0, 1]$. The RIESZ theorem	124
4.24	The general form of linear functionais on l_p	12 7
4.25	The general form of linear functionals in the space $L_p[0, 1]$	129
4.2 6	The general form of Linear functionals on HILBERT space	133
4.3.	Conjugate Spaces and Adjoint Operators	134
4.31	Examples of conjugate spaces	134
4.32	Reflexive spaces	135
4.33	The adjoint operators	137
4.34	The matrix form of operators in spaces with basis	139
4.35	The inner product, orthogonal elements, biorthogonal systems	141
4.36	The spaces conjugate to eomplex linear spaces	144
4.4.	Weak Convergence of Sequences of Functionals and Elements	144
4.41	Weak convergence	144
4.42	Application to the theory of quadrature formulae	145
4.43	Weak convergence of sequences of elements of a space. Elementary theorems	146
4.44	Weak convergence in concrete spaces	149
C	hapter 5. COMPACT SETS IN METRIC AND NORMED SPACE	S 151
5.1.	Definitions. General Theorems	151
5.11	The existence theorem for an extremum	152
5.12	Criterion for compactness of sets in metric spaces	154
5,2.	Criteria for Compactness in Some Functional Spaces	159
5.21	The space $C[0, 1]$	159
	Space $L_p[0, 1]$	162
5.23	Space Q . For illustration, proof of an existence theorem in the calculus of	
	variations	165
	Conditions of compactness in space with basis	166
5.25	Space l_p	167

r

	Cont	ents ix
5.3.	Finite Dimension and Compactness	167
5.4 .	The Problem of the Best Approximation	168
5.5.	Weak Compactness	171
5.6.	Universality of the Space $C[0, 1]$	172
	Chapter 6. COMPLETELY CONTINUOUS OPERATORS	176
6,1.	Completely Continuous Operators	176
	Definition	176
6.12	Approximation of completely continuous operators in a BANACH space with a basis of finite-dimensional operators	180
6.2.	Linear Operator Equations with Completely Continuous Operators	181
6. 2 1	Two lemmas	181
6.3.	Schauder Principle and its Applications	192
6.31	Three lemmas	192
6.32	The SCHAUDER fixed point principle	195
6.4.	Completely Continuous Inclusion Operators of Sobolev	197
	hapter 7. ELEMENTS OF SPECTRAL THEORY OF SELF-ADJ	OINT
C	OPERATORS ON HILBERT SPACES	205
7.1 .	Self-adjoint Operators	205
7.11	Adjoint operators on HILBERT spaces	205
	Adjoint operators	205
7.13	Self-adjoint operators	206
7.14	Quadratic forms	207
7.2.	Unitary Operators. Projection Operators	208
7.21	Unitary operators	208
7.22	Projection operators	2 09
7.3 .	Positive Operators. Square Roots of Positive Operators	212
7.31	Positive operators	212
	Square roots of positive operators	214

x Contents

7.4.	Spectrum of Self-adjoint Operators	215
7 / 1	Invariant subspaces	218
	Continuous spectra and point spectra	220
	Operators with pure point spectrum	220
	Spectral Decomposition of a Self-adjoint Operator	223
7 51	Resolution of the identity	223
	Spectral theorem for self-adjoint operators	227
	Operator functions. Resolvent. Spectra	228
	Eigenvalues of a self-adjoint operator	2 32
7.6.	Non-bounded Linear Operators. Basic Concepts and Definitions	233
7 61	Adjoint operators	2 35
	Closed operators. Closure of operators	236
	Graph of an operator	237
	Invariant subspaces. Reducibility	2 39
7.7.	Self-adjoint Operators and Extension of Symmetric Operators Deficiency Indices	240
771	Self-adjoint operators	240
	Extension of a symmetric operator. Deficiency indices	241
7.8.	Spectral Expansion of Non-bounded Self-adjoint Operators. Functions of Self-adjoint Operators	246
		246
	The STIELTIES integral	249
	The two lemmas	252
	Integral representation of operators	254
7.84	Functions of the operator	
7.9.	Examples of Non-bounded Operators	260
7.91	Operator of multiplication by independent variable	260
	Differential operators	262
Ch	apter 8. SOME PROBLEMS OF DIFFERENTIAL AND INTEGRA	L
-11	CALCULUS IN NORMED LINEAR SPACES	272
8.1.	Differentiation and Integration of Abstract Functions of Real Variables	272
8.11	Differentiation	273
	2 Integration	2 78
	Solution of differential equations	284

		Contents	хi
8.2.	The Difference Scheme and the Theorem of P. Lax	2	285
8.3.	Differential of an Abstract Function	2	92
8.31	Strong differential (Frechet differential)		292
8.32	Weak differential (GATEUX differential)	2	293
8.4.	Theorem of Inverse Operator. Newton's Method	2	97
8.41	Newton's method	3	300
8.5.	Homogeneous Forms and Polynomials	3	303
R 51	Multiplication of elements	-	303
	n-ary linear forms		304
8.53	Properties of <i>n</i> -ary homogeneous linear forms		305
	Polynomials	:	3 06
8.6.	Differentials and Derivatives of Higher Order	;	308
			308
	Notations		309
8.62	TAYLOR'S formula		
8.7.	Differentiation of Functions of Two Variables	:	314
8.8.	Theorems on Implicit Functions		315
8.9.	Applications of Implicit Function Theorem		319
8.91	Dependence of the solution of the equation		319
8.92	Application to eigenvectors		320 322
8.93	Equations depending on parameters		322
	Variational equations		323
8.95	Applications to differential equations		
8.10). Tangent Manifolds		324
Q 1(1) The particular case		324
	2) The general ease		325
	3 Locally linear space		328
	I. Extrema		330
3,11	APPENDICES		332
	AIILA		332
Apr	pendix I. Inequalities		332
	The class I n > 1		202

xii Contents

I.12 HÖLDER's inequality	333
I.13 MINKOWSKI's inequality	335
Appendix II. Continuity in the Mean of Functions of the Class $L_p(G)$	337
Appendix III. Bolya-Brouwer Theorem	339
Appendix IV. Two Definitions of the n-th Derivative of a Function of Real Variables	344
REFERENCES	347
BIBLIOGRAPHY	349
INDEX OF SYMBOLS	351
INDEX	353
TABLE OF BASIC PROPERTIES OF IMPORTANT FUNCTIONAL SPACES	360