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BINDING EFFECTS AND A SUM RULE IN $Q\bar{Q}$ RADIATIVE DECAYS

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A B S T R A C T

The influence of binding is considered for the radiative decay of a non-relativistically bound $^3S_1$ $Q\bar{Q}$ state to a photon and an elementary spin-zero boson by mixing with all relevant $Q\bar{Q}$ states. For a pseudoscalar boson, the sum reduces to a single term and one recovers the result of free quark perturbation theory in the limit $\omega r < 1$. For a scalar boson, binding effects can be important in general, but in the combined limit $(mr)^{-1} \ll \omega r \ll 1 \ll mr$, the free quark perturbation theory result is also approached, thanks to the sum rule

$$R_{nS}(0) = \sum_m R_{mP'}(0) \langle mP|r|nS\rangle$$

among S- and P-wave radial wave functions and dipole matrix elements. Simple examples of the sum rule are discussed.

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The Higgs particles H of unified electroweak interactions can be produced in radiative quarkonium decay [1]. The decay rate of any $^3S_1$ $Q\overline{Q}$ bound state V into $H+\gamma$ is related to that into e^+e^-:

$$\frac{\Gamma(V \to H+\gamma)}{\Gamma(V \to e^+e^-)} = \frac{2g_Y^2}{e^2}\,(1 - \frac{M_H^2}{M_V^2})\ . \tag{1}$$

Here g_Y is the Yukawa coupling of the Higgs boson to the quark Q. Equation (1) is valid for either scalar or pseudoscalar H. Equation (1) is derived in the limit of a loosely bound $Q\overline{Q}$ system. Under what conditions might one expect significant deviations from the result (1) due to hadronic interactions between Q and $\overline{Q}$?

Our result is that the expression (1) for the decay of a $^3S_1$ state into a photon and an elementary spinless particle H (which may or may not be a Higgs boson) is expected to be valid even for soft photon energies ω , as long as these energies satisfy $\Delta E << \omega << <r>^{-1}$, where ΔE is a characteristic level spacing and $<r>$ is the size of the system. (For smaller ω , the characteristic electric or magnetic dipole rates $\Gamma \sim \omega^3$ are recovered [2-4].)

Here

$$\omega = \frac{M_V}{2}\,(1 - \frac{M_H^2}{M_V^2})\ . \tag{2}$$

The result follows from a duality between a free-quark and bound state description that is essentially trivial when H is pseudoscalar, but that is based on an interesting sum rule when H is scalar. This relation is

$$R_{nS}(0) = \sum_m R_{mP}'\ (0)\ <mP|r|nS>\ , \tag{3}$$

where $R_{nS}(r)$, $R_{mP}(r)$ are radial S- and P-wave $Q\overline{Q}$ wave functions, and $<mP|r|nS>$ is a dipole matrix element (radial integral).

The relation (3) is implicit in several related treatments of this problem [2-4]. Nevertheless its derivation and implications are so simple we feel they are worth pointing out.

We calculate the amplitudes for $V \to H+\gamma$ via standard second-order perturbation theory, taking account both of spin-zero [5] and spin-one intermediate states. If A_o is the amplitude for the process $V_n \to H+\gamma$ in the tree approximation of Ref.[1], we find the amplitudes to be

$J^P(H) = 0^-$:

$$A = A_o \frac{\omega}{R_{nS}(0)} \sum_m R_{mS}(0) < mS|j_o(\frac{\omega r}{2})|nS> \times \left[\frac{1}{E_m(^1S_o) - E_n(^3S_1) + \omega} + \frac{1}{E_m(^3S_1) - \omega}\right] \tag{4}$$

$J^P(H) = 0^+$:

$$A = A_o \left\{\frac{M_n}{R_{nS}(0)} \sum_m \frac{R_{mS}(0) < mS|j_o(\frac{\omega r}{2})|nS>}{E_m(^3S_1) - \omega} + \frac{M_n}{m_Q^2} \frac{3}{R_{nS}(0)} \sum_m \frac{R_{mP}'(0) < mP| \frac{2}{\omega r} j_1\left(\frac{\omega r}{2}\right)\left[\frac{\partial}{\partial r} - \frac{\omega^2 r}{4}\right] |nS>}{E_m(^3P_o) - E_n(^3S_1) + \omega}\right\}, \tag{5}$$

where M_n is the mass of V_n and m_Q is the quark mass. Equivalent results have been obtained in Ref.[2].

For $J^P(H) = 0^-$, the result of Ref. 1 is obtained when (a) $\omega r < 1$, $j_o(\omega r/2) \approx 1$, so only the state with m=n contributes to the sum; (b) $\omega << E_m$ (as will hold if $\omega < <r>^{-1}$ for a nonrelativistic system), so only the first term in the square bracket in (4), corresponding to the 0^- intermediate state, is important; and (c) $\omega >> (\Delta E)_{HFS}$ so that the term $\omega/(E_m - E_n + \omega)$ approaches 1. Similar results have been noted in Refs. [2, 3, and 6].

In the limit $\omega << E_n(^3S_1) - E_n(^1S_o)$, the amplitude in (4) will behave as ω, as befits a magnetic dipole transition.

For $J^P(H) = 0^+$, when $\omega r \rightarrow 0$, the expression (5) becomes

$$A = A_o \frac{\omega}{M_n}\left[\frac{M_n}{M_n - \omega} + \frac{M_n}{R_{nS}(0)} \sum_m \frac{R_{mP}'(0) < mP|r|nS>}{E_m(^3P_o) - E_n(^3S_1) + \omega}\right]. \tag{6}$$

Here again, as $\omega \rightarrow 0$, the amplitude is proportional to ω, as it should be for an electric dipole transition.

The ratio A/A_o in Eq. (6) approaches 1 when (a) $\omega << M_n$, (b) $|E_m(^3P_o) - E_n(^3S_1)| << \omega$, in which case terms in the sum in (6) dominate over the first term in the square brackets; and (c) the sum rule (3) holds. This last condition is just a consequence of nonrelativistic quantum mechanics, as we now show.

Consider the expectation value of the commutator

$$\delta^3(\vec{r})[r_i, p_j] = i\delta^3(\vec{r})\delta_{ij} \tag{7}$$

in the state $|nS\rangle$. Insert a complete set of intermediate states:

$$\langle nS|\delta^3(\vec{r})|nS\rangle i\delta_{ij} = \sum_m \{ \langle nS|\delta^3(\vec{r})r_i \mid m \rangle \langle m|p_j|nS\rangle$$

$$- \langle nS|\delta^3(\vec{r})p_j|m\rangle \langle m|r_i|nS\rangle\}. \tag{8}$$

Only P states contribute to the sum. In configuration space we can express the P state wave functions in Cartesian form:

$$\Psi^k_{mP}(\vec{r}) = \sqrt{\frac{3}{4\pi}}\,\frac{r^k}{r}\,R_{mP}(r) \quad (k = 1, 2, 3). \tag{9}$$

Equation (8) then reads

$$\delta_{ij}\,|\Psi_{nS}(0)|^2 = \sum_{m,k} \int d^3r\, \Psi^*_{nS}(\vec{r})\delta^3(\vec{r})\partial_j\, \Psi^k_{mP}(\vec{r})$$

$$\times \int d_3r' \quad \Psi^{k*}_{mP}(r')r_i'\; \Psi_{nS}(r'). \tag{10}$$

Here only the second term in (8) has contributed. Near r = 0 we find

$$\partial_j\, \Psi^k_{mP}(\vec{r}) \sim \sqrt{\frac{3}{4\pi}}\, R_{mP}'\,(0)\delta_{jk}\,. \tag{11}$$

After performing an elementary angular average, we arrive at the result (3).

The sum rule (3) in general is more slowly convergent than those for squares of dipole matrix elements such as the Thomas-Reiche-Kuhn [7,8] or Wigner and Kirkwood [9,8] sum rules, examples of which are

$$TRK\!:\;\; m_Q \sum_{m=2}^{\infty} (E_{mP} - E_{nS}) \mid \langle mP|r|nS\rangle|^2 = 3 \tag{12}$$

$$W\text{-}K\!:\;\; m_Q \sum_{n=1}^{\infty} (E_{nS} - E_{mP})|\langle nS|r|mP\rangle|^2 = -\,1\,. \tag{13}$$

Here $m_Q = 2\mu$, where μ is the reduced mass of the $Q\overline{Q}$ system. We compare the convergence of (3) and (12) in Fig. 1 for a $b\overline{b}$ quarkonium system described by a potential that reproduces the data [10]. The Wigner-Kirkwood sum rules converge extremely rapidly, as the dipole matrix elements $<nS|r|mP>$ turn out to be very small for $n > m$ *.

The most elementary illustration of the sum rules is provided by the harmonic oscillator. The only non-zero dipole matrix elements for the oscillator between S and P waves link adjacent values of $N \equiv E - 3/2$ (we take unit mass and coupling strength):

$$<N+1, P\,|r|N, S> = \sqrt{\frac{3+N}{2}} \tag{14}$$

$$<N-1, P\,|r|N,S> = -\sqrt{\frac{N}{2}} \tag{15}$$

The sum rules (12), (13), and (3) are all satisfied with only two terms.

In contrast to the harmonic oscillator, the sum rules for the hydrogen atom, with both discrete and continuous spectra, receive contributions from very many states. Our sum rule (3) for the 1S state of hydrogen has only 19% of its total from the bound states. The bulk (59%) comes from the nearby continuum $(0 < E < \alpha^2 m_e/2)$, with 22% from higher energies. For the corresponding Thomas-Reiche-Kuhn sum rule (12), the contributions are 57% from the bound states and 43% from the continuum.

Let us sum up. We find that as $\omega \rightarrow 0$, bound-state modifications [2-5] to the free-quark calculation of $^3S_1$ *quarkonium* $\rightarrow \gamma +$ *elementary spin–zero boson* indeed lead to a rate behaving as ω^3 , as expected for a dipole transition. Nonetheless there *are* bound-state limits in which the free-quark results are obtained. These involve photon energies small compared to inverse dimensions but large compared to level spacings. We find this condition, $\Delta E << \omega << <r>^{-1}$, satisfied for a wide range of possibilities for a pseudoscalar boson, but less likely for a scalar boson. Indeed, much more substantial

*Our notation for P states corresponds to that used in atomic spectroscopy, where m is the principal quantum number. Thus the lowest P state is 2P.

deviations from Eq. (1) have been found in the scalar case [4], when considering radiative Υ and Υ' decays, than for the pseudoscalar case. We interpret these results in terms of a substantially larger level spacing $E_m(0^{\pm}) - E_n(1^-)$ in the scalar case than in the pseudoscalar case.

Our main result and the purpose of this note: When $\Delta E << \omega << <r>^{-1}$ holds for scalar boson prodution, it is the sum rule (3), a consequence of nonrelativistic quantum mechanics, that guarantees the result (1) if an explicit perturbation-theoretic sum is performed. We have illustrated this sum rule and find it to be quite slowly convergent in practical cases. It suggests several points for further study. First, its slow convergence emphasizes the importance of Green's function methods [2,3] for evaluation of the perturbative sum. Second, modifications of properties of highly excited bound states (e.g., by coupling to continuum states) could affect evaluation of a perturbation calculation for scalar boson production. Third, one can imagine useful generalizations of (3) arising if we multiply (7) by an arbitrary function such as a plane wave $\exp(i\vec{k}\cdot\vec{r})$. Related generalizations of this sort have indeed been examined in Ref. [4].

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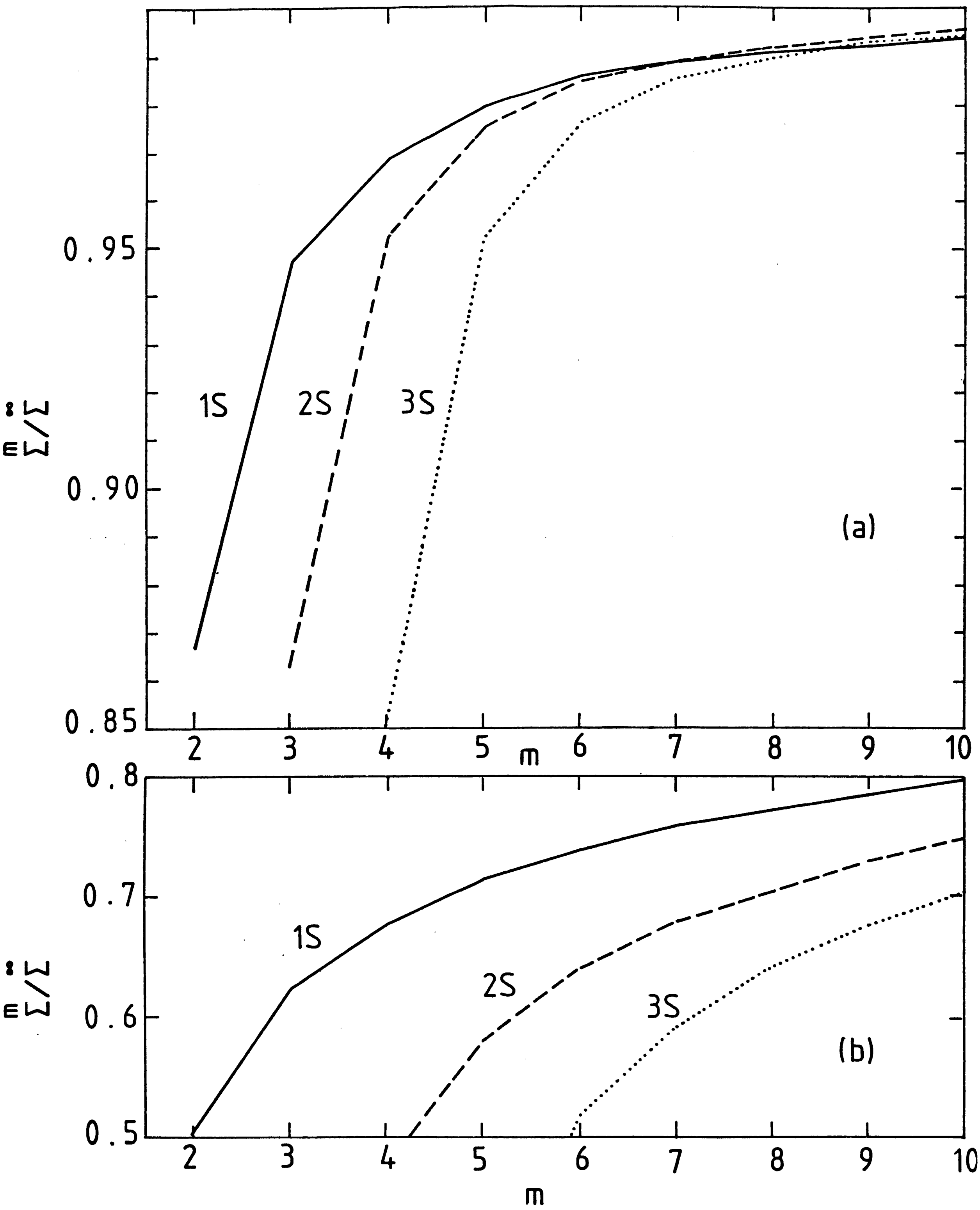
Figure Caption

Comparison of sum rules involving dipole matrix elements for quarkonium systems. We choose the confining potential discussed in Ref. 9, with $m_Q = 4.9\ GeV/c^2$.

(a) Thomas-Reiche-Kuhn sum rule (12);

(b) wave function sum rule (3).



0.95
0.90
0.85
1S
2S
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(a)
2
3
4
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