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AXIALLY SYMMETRIC MAGNETOSTATICS IN SU_2 GAUGE THEORY

R.W. Tucker *)

CERN -- Geneva

ABSTRACT

A study is made of axially symmetric solutions of the Bogomolny equation having finite energy. The theory is reformulated in the language of quaternion valued differential forms and the techniques of the exterior calculus used throughout. With boundary conditions appropriate to a configuration of definite magnetic charge the forms studied lead unambiguously to the 't Hooft-Polyakov monopole. The implications of this result for the conjectured gauge field-monopole duality are discussed.

*) On sabbatical leave from the Department of Physics, University of Lancaster.

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1. INTRODUCTION

Since the discovery[1] of finite energy configurations in SO(3) gauge models with scalar fields there has been considerable activity in the study of classical solutions to non-linear field theories. The spherically symmetric 't Hooft-Polyakov monopole is topologically stable and may be regarded as a bound state of SO(3) gluons and isovector scalars. It has been conjectured by Montonen and Olive[2] that the gauge theory in a certain limit can be completely re-expressed in terms of local monopole fields playing the role of heavy gauge particles. The spherically symmetric monopole would acquire the status of an elementary field in this dual description. It has also been shown[3] that if monopoles of higher magnetic charge exist they must of necessity be non-spherically symmetric. In the dual description one expects such states, if they exist, to be excitations of the fundamental monopole. However, if such states with magnetic charge $g_n = ng_0$ exist as topologically stable classical solutions of the Bogomolny equations then the duality hypothesis would require either infinite component monopole fields or the existence of gluon composites of mass n times the fundamental gluon mass and electric charge n to maintain the symmetry.

One piece of evidence to support the duality conjecture is Manton's calculation[4] of the classical inter-monopole force for widely separated monopoles. In the limit of a vanishing Higgs potential this result follows from the dual description since a Higgs scalar meson becomes massless and can compensate appropriately the long-range photon exchange. Although this result neglects higher-order exchanges it has led some people[5] to expect classically stable configurations of several monopoles with like magnetic charge. If the inter-monopole forces between two 't Hooft-Polyakov monopoles really vanished to all orders, one would indeed expect an axially symmetric stable monopole configuration of magnetic charge twice that of a single monopole.

In this paper a search for such configurations with finite energy will be made. Since it is intended to use the techniques of quaternionic valued differential forms[6] the first section develops the theory ab initio. Aside from its intrinsic nature, the advantage of this approach resides mainly in the computational ease with which one can manipulate complicated partial differential equations. The quaternionic analysis leads naturally to the use of V and S operators that streamline the purely algebraic aspects of the calculation. After reviewing a derivation of the magnetic Bogomolny[7] equation, the next section defines axially symmetric q vector p forms in terms of V and the Lie derivative. The important question of appropriate boundary conditions is discussed[8] and the basic axially symmetric ansatz for the investigation presented. The equations resulting from this ansatz are developed in a particular gauge and an argument against the existence of finite energy configurations with axial symmetry satisfying the Bogomolny equations is made in the penultimate section. The last section indicates that it is difficult to escape from this conclusion within the framework of the ansatz under investigation.

2. QUATERNIONIC FORMS

Since the algebra of SU_2 is isomorphic to the algebra of the quaternions H I reformulate the theory in this representation. This offers more than formal advantages as will be demonstrated in the following sections. Furthermore, it is not necessary to commit oneself to a particular co-ordinate system at the outset so it is natural to manipulate the intrinsic geometric forms in the theory which will take values in the field of quaternions rather than the real numbers.

A basis of H will be denoted by $\{1,i,j,k\} \equiv \{1,\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ with the multiplication rules

$$\hat{e}_a \hat{e}_b = -\delta_{ab} + \varepsilon_{abc}\hat{e}_c \tag{1}$$

For a typical quaternion

$$q = a_0 + a_i \hat{e}_i \tag{2}$$

where a_0, a_i are, in general, complex numbers. H conjugation is defined by

$$\bar{q} = a_0 - a_i \hat{e}_i$$

and

$$\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1 \tag{3}$$

The useful operations of S and V are defined by

$$2S(q) = q + \bar{q}$$
$$2V(q) = q - \bar{q} \tag{4}$$

Any quaternion q with $S(q) = 0$ $[V(q) = 0]$ will be termed a q vector [q scalar]. The unit quaternions Q form an important class satisfying $Q\bar{Q} = \bar{Q}Q = 1$. They generate q vector rotations. Specifically, for any q vector r we can choose to write $Q = \exp(n\chi/2)$ where n is a unit q vector and χ a real angle. Then

$$r' = Q r \bar{Q} \tag{5}$$

has components which may be identified with those of the vector (r_1,r_2,r_3) rotated by χ about the axis (n_1,n_2,n_3) in R^3. Clearly the unit quaternions are isomorphic to SU_2 elements.

A quaternionic p form is simply a p form with quaternionic components. In a p form basis e^J (where J is a multi-index) such an entity may be expressed as

$$\omega = \sum_J \omega_J e^J$$

where

$$\omega_J = \omega_{J0} + i\omega_{Ji} + j\omega_{Jj} + k\omega_{Jk} \tag{6}$$

SU_2 gauge theory may be generated in terms of a q vector 1 form A in Minkowski space-time. It is defined to be a connection such that under local gauge transformation generated by the element Q

$$A \to QA\bar{Q} + Q\,d\bar{Q} \tag{7}$$

In this expression the exterior derivative d commutes with the generators of H. This transformation ensures that the q vector curvature 2 form F defined by

$$F = dA + A \wedge A \tag{8}$$

transforms as

$$F \to QF\bar{Q}$$

Since $Q\bar{Q} = 1$ it is clear that any q scalar constructed from entities transforming in this way will be locally gauge invariant.

A scalar field Φ in the adjoint representation of SU_2 may be represented by a q vector 0 form transforming under Q as

$$\Phi \to Q\Phi\bar{Q} \tag{9}$$

Its covariant exterior derivative with respect to the connection A is defined as

$$D\Phi = d\Phi + 2V(A\Phi) \tag{10}$$

since this transforms like Φ. As $\bar{A} = -A$ and $\bar{\Phi} = -\Phi$ then $D\Phi$ is also a q vector 1 form.

In order to construct gauge invariant 4 forms on Minkowski space I employ the Hodge operator $\underset{(4)}{*}$ defined with respect to a metric $\underset{(4)}{g}$ and volume element ε:

$$\omega \wedge \underset{(4)}{*} \omega = \underset{(4)}{g_p}(\omega, \omega)\, \varepsilon \tag{11}$$

where $\underset{(4)}{g_p}$ is the metric induced on p forms ω by the tensor

$$\underset{(4)}{g} = -dt \otimes dt + \sum_{a=1}^{3} e^a \otimes e^a \tag{12}$$

It is convenient to use a co-ordinate time t and an orthonormal triad of 1 forms (e^1, e^2, e^3) in 3 space. One readily verifies that $F \wedge \underset{(4)}{*} F$ and $D\Phi \wedge \underset{(4)}{*} D\Phi$ are q scalar 4 forms. These are just the usual kinetic contributions to the action of SU_2 gauge theory with isovector scalar fields.

In the following I study the field equation generated from the action

$$S = -\frac{1}{e^2} \int \left(F \wedge \underset{(4)}{*} F + D\Phi \wedge \underset{(4)}{*} D\Phi \right) \tag{13}$$

adopting the usual procedure of identifying Φ as a Higgs field that satisfies the R^3 asymptotic condition

$$\Phi^2 \longrightarrow \text{constant}. \tag{14}$$

The field equations obtained by varying the forms A and Φ may be written

$$D \underset{(4)}{*} F = V(\underset{(4)}{*} D\Phi \,.\, \Phi) \tag{15}$$

$$D \underset{(4)}{*} (D\Phi) = 0 \tag{16}$$

or in terms of exterior derivatives

$$d \underset{(4)}{*} F + 2V(A \wedge \underset{(4)}{*} F) = V(\underset{(4)}{*} D\Phi \,.\, \Phi) \tag{17}$$

$$d(\underset{(4)}{*} D\Phi) + 2V(A \wedge \underset{(4)}{*} D\Phi) = 0 \tag{18}$$

From the definitions (8) and (10) there follow the identities

$$\mathcal{D} F \equiv dF + 2V(A_\wedge F) = 0 \tag{19}$$

$$\mathcal{D}^2 \Phi \equiv d(\mathcal{D}\Phi) + 2V(A_\wedge \mathcal{D}\Phi) = 2V(F\Phi) \tag{20}$$

As a first step in recovering the static R^3 field equations I define q vector electric and magnetic forms with the 3+1 decomposition

$$F = B + dt_\wedge E \tag{21}$$

$$A = \alpha + \alpha_0\, dt \tag{22}$$

$$\mathcal{D}\Phi = h + h_0\, dt \tag{23}$$

The "field forms" follow from (8) and (10) in terms of the potentials

$$\begin{aligned} B &= \underset{(3)}{d}\alpha + \alpha_\wedge\alpha \\ E &= \dot{\alpha} - \underset{(3)}{d}\alpha_0 + \alpha_0\alpha - \alpha\alpha_0 \\ h &= \underset{(3)}{d}\Phi + 2V(\alpha\Phi) \\ h_0 &= \dot{\Phi} + 2V(\alpha_0\Phi) \end{aligned} \tag{24}$$

The dot differentiates all components of a form with respect to t and the symbol $\underset{(3)}{d}$ means exterior differentiation in the subspace of forms generated by e^1, e^2 and e^3. This decomposition enables one to identify a three-dimensional dual operator $*$ with respect to the metric g where

$$\underset{(4)}{g} = -dt \otimes dt + g \tag{25}$$

Since any Minkowski space form can be written $\alpha + dt_\wedge\beta$ where α and β are independent of dt (although they may have time dependent components) the $\underset{(4)}{*}$ duals can be replaced by $*$ duals according to

$$\underset{(4)}{*}\left[\alpha + dt_\wedge\beta\right] = (*\alpha)_\wedge dt + *\beta \tag{26}$$

With these definitions the field equations (15) and (16) decompose into

$$\left\{ \dot{\tilde{E}} + \underset{(3)}{d}\tilde{B} \right\} + 2V(\alpha_\wedge \tilde{B} + \alpha_0 \tilde{E}) = V(\tilde{h}\,\Phi) \tag{27}$$

$$\left\{ \underset{(3)}{d}\tilde{E} \right\} + 2V(\alpha_\wedge \tilde{E}) = V(\tilde{h}_0 \Phi) \tag{28}$$

$$\underset{(3)}{d}\tilde{h} - \dot{\tilde{h}}_0 + 2V(\alpha_\wedge \tilde{h} - \alpha_0 \tilde{h}_0) = 0 \tag{29}$$

$$\underset{(3)}{d}\tilde{h}_0 + 2V(\alpha_\wedge \tilde{h}_0) = 0 \tag{30}$$

where

$$\begin{aligned} \tilde{h}_0 &= *h_0 \\ \tilde{h} &= *h \\ \tilde{E} &= *E \\ \tilde{B} &= *B \end{aligned} \tag{31}$$

The two identities (19) and (20) become

$$\left\{ \dot{B} - \underset{(3)}{d}E \right\} + 2V(\alpha_0 B - \alpha_\wedge E) = 0 \tag{32}$$

$$\left\{ \underset{(3)}{d}B \right\} + 2V(\alpha_\wedge B) = 0 \tag{33}$$

$$\dot{h} - \underset{(3)}{d}h_0 = 2V(E\Phi - \alpha_0 h + \alpha_\wedge h_0) \tag{34}$$

$$\underset{(3)}{d}h = 2V(B\Phi - \alpha_\wedge h) \tag{35}$$

It is interesting to examine these equations in the case where Φ and A are proportional to a single unit quaternion. If the component of Φ in such a basis is constant only the terms in curly parentheses above survive. These equations are recognized as the 3+1 form of Maxwell's equations.

At this point I restrict the equations to static configurations in which $\alpha_0 = 0$. Thus $h_0 = 0$, $E = 0$, $F = B$ and only two of the field equations survive:

$$d\tilde{h} + 2V(\alpha_\wedge \tilde{h}) = 0 \tag{36}$$

$$d\tilde{B} + 2V(\alpha_\wedge \tilde{B}) = V(\tilde{h}\Phi) \tag{37}$$

The operator $\underset{(3)}{d}$ can now be written d without ambiguity. Similarly, there are only two non-trivial identities

$$dB + 2V(\alpha_\wedge B) = 0 \tag{38}$$

$$dh + 2V(\alpha_\wedge h) = 2V(B\Phi) \tag{39}$$

One now observes in this magnetostatic situation that if

$$B = \pm \frac{1}{\sqrt{2}} \tilde{h} \tag{40}$$

then the two field equations (36) and (37) are satisfied by virtue of the equations (38) and (39). This is the Bogomolny equation which was originally derived as a consequence of minimizing the static field energy of the system.

If the coupling constant is suitably normalized, the field energy within a ball b of R^3 may be written in this case as

$$M = -\int_b S(B_\wedge h) \tag{41}$$

Since [with the aid of (19)]

$$d(B\Phi) = B_\wedge h + 2V(B\Phi_\wedge \alpha) \tag{42}$$

the above integral may be written

$$M = -\int_b S(d(B\Phi)) = -\int_{\partial b} S(B\Phi) \tag{43}$$

where ∂b bounds b containing a non-singular 3 form. If at large distances from some origin $D\Phi$ tends to zero we may interpret $B\Phi$ as a Maxwell magnetic field (see below). For ∂b a large sphere we see that the mass is proportional to the magnetic charge.

3. SYMMETRIC FORMS

I now turn my attention to the study of field configurations that obey the basic equation (40). The imposition of spherical symmetry on gauge invariant quantities was a powerful tool in the discovery of the singly charged magnetic monopole. It has been subsequently shown[3] that monopoles of higher magnetic charge must of necessity be non-spherically symmetric configurations. The next simplest situation occurs for axially symmetric configuration. Suppose the metric is chosen with an azimuthal angle co-ordinate ϕ. A gauge invariant (i.e., q scalar) p form ω_p will be said to be axially symmetric if

$$\mathcal{L}_X \, \omega_p = 0 \tag{44}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field $X = \partial/\partial\phi$. A q vector p form λ_p will be said to be axially symmetric if the action of $\mathcal{L}_X$ can be compensated by an infinitesimal gauge transformation with angle χ, say

$$\delta_I \lambda_p \equiv Q \lambda_p \bar{Q} - \lambda_p = \chi \, V(n\lambda_p) \tag{45}$$

Thus λ_p is axially symmetric about the $n = (n_1, n_2, n_3)$ axis if

$$\mathcal{L}_X \lambda_p - V(n\lambda_p) = 0 \tag{46}$$

It may be recalled that $\mathcal{L}_X$ is a derivation and acting on a 1 form

$$A = \sum_a A_a \, dx^a$$

$$\mathcal{L}_X A = \sum_a (XA_a)\, dx^a + \sum_{a,b} A_a \frac{\partial X_a}{\partial x^b} dx^b \tag{47}$$

where, in general,

$$X = \sum_a X_a \frac{\partial}{\partial x^a}$$

For λ_p to be rotationally symmetric

$$\mathcal{L}_{L_i} \lambda_p - V(\hat{e}_i \lambda_p) = 0 \tag{48}$$

where L_i are the orbital angular momentum generators.

The solutions of definite magnetic charge establish a map between R^3 space (co-ordinated by x^1, x^2, ϕ, say) and a unit sphere in SU_2 space. I co-ordinate the latter by spherical polar angles γ and ψ and choose k to define an axis of symmetry. Then k and $T \equiv i\, e^{-K\psi}$ enable one to construct the axially symmetric unit q vector 0 form:

$$M = K\cos\gamma(x^1,x^2) + T\sin\gamma(x^1,x^2) = K e^{-KT\gamma} \tag{49}$$

where $\psi = m\phi$ for some constant m. The q vector M will cover the SU_2 sphere (possibly several times) in accordance with the function $\gamma(x^1,x^2)$ and m. If one uses spherical polar co-ordinates $(x^1,x^2) = (r,\theta)$ then $\gamma = \theta$ defines the rotationally invariant unit q vector

$$N = K\cos\theta + T\sin\theta \tag{50}$$

In this latter case the q vector 1 form NdN is rotationally symmetric so that an ansatz with this symmetry is

$$A = \frac{1}{2}(K(r) - 1)\, N dN \tag{51}$$

$$\Phi = \frac{1}{\sqrt{2}} \frac{H(r)}{r} N \tag{52}$$

A straightforward calculation verifies that for

$$K = \frac{ar}{\operatorname{sh} ar} \tag{53}$$

$$H = \pm(ar \coth ar - 1) \tag{54}$$

Equation (40) is satisfied and one identifies the Prasad-Sommerfeld solution for the the 't Hooft-Polyakov monopole.

4. BOUNDARY CONDITIONS

The question of appropriate boundary conditions for an axially symmetric field configuration will now be considered. The solution that is sought should produce a finite value for the integral (41). If the equation (40) is satisfied we are then guaranteed a definite magnetic charge. For a specific ansatz this will entail certain regularity conditions on the solutions both asymptotically and in the neighbourhood of monopole centres. For the Higgs field the isotropic asymptotic condition (14) will be adopted throughout since this is connected to the Higgs mechanism that we have tacitly assumed. If a static configuration of several monopoles exists, yielding a total finite energy, then one expects that the Maxwell magnetic field that they produce should coincide asymptotically with that produced from a series of singular Dirac monopoles. Alternatively, one might search for a single magnetic multipole. Only the monopole component of the magnetic field would be detected asymptotically.

The calculations by Manton[4] have indicated that widely separated monopoles of like magnetic charge are non-interacting to a first approximation. The existence of residual forces cannot, however, be ruled out in which case it may be that such monopoles eventually migrate to infinity removing the system from the class of static configurations considered here.

In order to be precise about a possible set of boundary conditions consistent with (14), the field equations and the existence of magnetic charge I first present a solution to the second-order equations (17) and (18) that describes an arbitrary axially symmetric distribution of singular magnetic sources. I then prescribe that any first-order solution to (40) should asymptotically approach this solution in order to describe the long-range magnetic field component. This implies that any multiple magnetic charges should be finitely separated.

The forms

$$A = -\frac{1}{2} M \, dM \tag{55}$$

$$\Phi = \frac{Z}{\sqrt{2}} M \tag{56}$$

for some constant Z will solve the second-order equations (17) and (18) provided $\cos\gamma$ is such that

$$A_M = \cos\gamma \, d\phi \tag{57}$$

generates a Maxwell field F_M that obeys Maxwell's equations

$$d \underset{(4)}{*} F_M = 0 \tag{58}$$

almost everywhere. The argument follows with the aid of some relations in the Appendix. Away from singularities $AM = \frac{1}{2}\, dM$

$$DM = dM + 2V(AM) = 0 \tag{59}$$

Thus

$$D\Phi = 0 \tag{60}$$

and the only remaining field equation to solve is

$$D \underset{(4)}{*} F = 0 \tag{61}$$

Now

$$F = dA + A \wedge A = -\frac{1}{4} dM \wedge dM = \frac{1}{2} m \, d(\cos\gamma) \wedge d\phi M \tag{62}$$

But the Maxwell field generated by A_M is

$$F_M = dA_M = d(\cos\gamma) \wedge d\phi \tag{63}$$

So

$$D \underset{(4)}{*} F = \frac{1}{2} m M D \underset{(4)}{*} F_M = \frac{1}{2} m M \, d \underset{(4)}{*} F_M = 0 \tag{64}$$

A particular solution to (58) which in spherical polar co-ordinates reads

$$r^2 \partial_r^2 (\cos\gamma) + \sin\theta \, \partial_\theta \left(\frac{1}{\sin\theta} \partial_\theta (\cos\gamma) \right) = 0$$

is

$$\cos\gamma(r,\theta) = \sum_{n=1}^{\infty} \frac{c_n}{r^{n-1}} C_n^{-1/2}(\cos\theta) \tag{65}$$

where c_n are arbitrary constants and

$$C_n^{-1/2}(z) = \frac{1}{(2n-1)} \left\{ P_{n-2}(z) - P_n(z) \right\} \tag{66}$$

in terms of Legendre polynomials. For $n = 1$ one recognizes a Dirac monopole with a string along the axis of symmetry. (Half of this string can be removed by the Abelian gauge transformation $A_M \rightarrow A_M - c_1 d\phi$.) The $n = 2$ term corresponds to the magnetic dipole

$$A_M = \frac{2c_2}{r} \sin^2\theta \, d\phi \tag{67}$$

Since the Maxwell equations are linear in $\cos\gamma$ general solutions can be obtained by superposition. Thus a series of Dirac monopoles along the symmetry axis gives the solution

$$\cos\gamma = \sum_i c_{1i} \cos\theta_i \tag{68}$$

where θ_i is the polar angle with respect to the i^{th} monopole origin. Having identified the Maxwell content of (55) and (56) I adopt as boundary conditions for any solution to (40)

$$
\begin{aligned}
A_\infty &= -\tfrac{1}{2} M\, dM \Big|_\infty \\
\Phi_\infty &= \frac{z}{\sqrt{2}} M_\infty
\end{aligned}
\tag{69}
$$

5. THE BASIC ANSATZ

The usual procedure for generating partial differential equations from a quaternionic equation such as (40) is to formulate an ansatz for the forms that suitably couples the internal algebra and tensor structure so that the non-commutative q elements are removed from the problems. In this procedure one must keep in mind that the structure of the ansatz should not contain more degrees of freedom than can be computed from the resulting differential equations[9]. In this respect I examine the axially symmetric forms

$$A = kT\beta + f_1 k\, dT + (f_2 - \tfrac{1}{2})T\, dT \tag{70}$$

$$\Phi = (\lambda_1 k + \lambda_2 T)\sqrt{2} \tag{71}$$

where in a general co-ordinate system (x^1, x^2, ϕ) the real functions $f_1, f_2, \lambda_1, \lambda_2$ are independent of ϕ and β is the 1 form

$$\beta = \beta_1(x^1, x^2)\, dx^1 + \beta_2(x^1, x^2)\, dx^2 \tag{72}$$

With the aid of results in the Appendix it is straightforward to generate from (40) the coupled partial differential equations (choosing polarity $\varepsilon = 1$ for definiteness)

$$* d\beta = 2\, \mathrm{Re}(F \bar{\Lambda})\, d\psi \tag{73}$$

$$(dF + 2i\beta F) \wedge d\psi = i * (d\Lambda + 2i\beta\Lambda) \tag{74}$$

and it is convenient to introduce

$$\Lambda = \lambda_1 + \hat{i}\lambda_2 \tag{75}$$

$$\mathcal{F} = f_1 + \hat{i} f_2 \tag{76}$$

($\hat{i}$ is the complex unit). These equations describe a two-dimensional Abelian gauge field coupled minimally to the doublets Λ and $\mathcal{F}$. They are invariant under the local Abelian gauge transformations

$$\begin{aligned} \mathcal{F} &\to \mathcal{F} e^{\hat{i}\mu} \\ \Lambda &\to \Lambda e^{\hat{i}\mu} \\ \beta &\to \beta - \tfrac{1}{2} d\mu \end{aligned} \tag{77}$$

where μ is an arbitrary real function of x^1 and x^2. The asymptotic boundary conditions (69) require

$$\begin{aligned} \beta_\infty &= -\tfrac{1}{2} d\gamma_\infty \\ \mathcal{F}_\infty &= \frac{\hat{i}}{2}\cos\gamma_\infty\, e^{\hat{i}\gamma_\infty} \\ \Lambda_\infty &= \frac{Z}{2} e^{\hat{i}\gamma_\infty} \end{aligned} \tag{78}$$

Consequently, the equations (73) and (74) may be simplified by writing

$$\Lambda = p\, e^{\hat{i}\gamma} \tag{79}$$

$$\mathcal{F} = (h_1 + \hat{i} h_2)\, e^{\hat{i}\gamma} \tag{80}$$

$$\beta = -\tfrac{1}{2} d\gamma + \sigma \tag{81}$$

and exploiting the gauge invariance (77) to take p, h_1, h_2 and σ real. Taking real and imaginary parts one has

$$d\sigma = 2 p h_1 * d\psi \tag{82}$$

$$2\sigma p = 2 * (\sigma \wedge d\psi)\, h_2 - * (dh_1 \wedge d\psi) \tag{83}$$

$$dp = * (dh_2 \wedge d\psi) + 2 * (\sigma \wedge d\psi)\, h_1 \tag{84}$$

The asymptotic boundary conditions are now simply

$$\sigma \to \infty \tag{85}$$

$$h_1 \to 0 \tag{86}$$

$$h_2 \to \frac{1}{2}\cos\gamma_\infty \tag{87}$$

$$p \to \frac{z}{2} \tag{88}$$

Using the fact that $** = 1$ we can eliminate σ from the set (82)-(84) obtaining the three equations linking p, h_1 and h_2

$$d*dp = 4h_1^2\, p\, d\psi \wedge *d\psi + \frac{dh_1}{h_1} \wedge (*dp - dh_2 \wedge d\psi) \tag{89}$$

$$*dp^2 = \{2d(ph_2) - *(d(h_1^2+h_2^2) \wedge d\psi)\} \wedge d\psi \tag{90}$$

This observation implies that once p and h_1 are found as functions of x^1 and x^2 then in an axially symmetric co-ordinate system (82) takes the form

$$d\sigma = f(x^1, x^2)\, dx^1 \wedge dx^2 \tag{91}$$

for some known function f. Consequently one can always write locally

$$\sigma = \Sigma(x^1, x^2)\, dx^2 \tag{92}$$

say and introduce the single component Σ. The equations (89) and (90) are coupled non-linear partial differential equations in two variables. There are two integrability conditions that follow from (90) by applying $*d$ and $d*$, respectively

$$d*dp^2 + d\{*(d(h_1^2+h_2^2) \wedge d\psi) \wedge d\psi\} = 0 \tag{93}$$

$$d\{*(*[d(h_1^2+h_2^2) \wedge d\psi])\} - 2d*(d(ph_2) \wedge d\psi) = 0 \tag{94}$$

At this point one can make contact with the equations developed by Manton[5)] in his search for axially symmetric monopole configurations. Adopting circular cylindrical co-ordinates $(x^1,x^2) = (\rho,z)$ one recognizes that Eqs. (89)-(94) reproduce his second order equations.

6. ABSENCE OF AXIALLY SYMMETRIC MONOPOLES

In this section I shall analyze (82)-(84) in spherical polar co-ordinates $(x^1,x^2) = (r,\theta)$. The dual operator gives

$$
\begin{aligned}
*\,(dr \wedge d\theta) &= \sin\theta \, d\phi \\
*\,(dr \wedge d\phi) &= -\frac{d\theta}{\sin\theta} \\
*\,(d\theta \wedge d\phi) &= \frac{dr}{r^2 \sin\theta}
\end{aligned}
\tag{95}
$$

and (82)-(84) correspond to the five component equations

$$\dot{\Sigma} = 2 p h_1 \frac{m}{\sin\theta} \tag{96}$$

$$2p\Sigma = \dot{h}_1 \frac{m}{\sin\theta} \tag{97}$$

$$2 h_2 \Sigma = h_1' \tag{98}$$

$$\dot{p}\, r^2 \sin\theta = m h_2' + 2 h_1 m \Sigma \tag{99}$$

$$p' \sin\theta = -m \dot{h}_2 \tag{100}$$

Dot (or prime) indicates partial differentiation with respect to $r(\theta)$.

Multiplying (96) and (97) together and integrating with respect to r one finds

$$\Sigma^2 - m^2 \frac{h_1^2}{\sin^2\theta} = g(\theta) \tag{101}$$

where g is some function only of θ. However, with the choice of boundary conditions (85) and (86) g must clearly be zero and one has

$$\Sigma = \varepsilon \frac{m h_1}{\sin\theta} \qquad (\varepsilon = \pm 1) \tag{102}$$

Equations (96) and (97) are now equivalent to

$$\dot{Y} = 4 p \varepsilon \tag{103}$$

with $\Sigma^2 = e^Y$. The remaining three equations become

$$\varepsilon \sin\theta \, Y' = 4m\, h_2 - 2\varepsilon \cos\theta \qquad (104)$$

$$\dot{p} r^2 = \frac{m h_2'}{\sin\theta} + 2\varepsilon e^{Y} \qquad (105)$$

$$p' = - \frac{m \dot{h}_2}{\sin\theta} \qquad (106)$$

Differentiating (104) with respect to r gives

$$\varepsilon \sin\theta \, \dot{Y}' = 4m\, \dot{h}_2 \qquad (107)$$

However, using (103) and (106) gives

$$\varepsilon \sin\theta \, \dot{Y}' = -4m\, \dot{h}_2 \qquad (108)$$

Since m must not vanish these can only be consistent if h_2 is a function solely of θ. In which case, from (106) p is a function of r. Now, integrating (103) with respect to r gives

$$Y = Y_1(r) + Y_2(\theta) \qquad (109)$$

where $\dot{Y} = 4p\varepsilon$, $Y' = Y_2'$ and Y_2 is to be determined. From (104)

$$h_2 = \frac{\varepsilon}{4m} \left\{ Y' \sin\theta + 2\cos\theta \right\} \qquad (110)$$

Inserting this in (105) and using $4\varepsilon p = \dot{Y}$ we finally obtain

$$r^2 \ddot{Y}_1(r) - \frac{1}{\sin\theta} \partial_\theta \left(Y_2'(\theta) \sin\theta \right) = 8 e^{Y_1(r)} e^{Y_2(\theta)} - 2 \qquad (111)$$

This equation is only consistent if one (or both) of the two functions Y_1 and Y_2 is constant. Since a constant Y_1 is not interesting one must have $Y_2 = \ln c$ say and

$$r^2 \ddot{Y}_1 = 8 c\, e^{Y_1} - 2 \qquad (112)$$

with the solution

$$Y_1 = \ln\left(a^2 r^2 / 4c \sinh^2 ar\right) \qquad (113)$$

for a constant $a = Z^2$. The complete solution consistent with the asymptotic boundary condition is now constructed as

$$\begin{aligned} \Sigma &= \frac{mar}{2\,\mathrm{sh}\,ar} \\ p &= \tfrac{1}{2}\left(\frac{1}{r} - a\coth ar\right) \\ mh_2 &= \frac{\varepsilon}{2}\cos\theta \\ mh_1 &= \varepsilon\,\Sigma\sin\theta \end{aligned} \tag{114}$$

Thus one is led unambiguously back to the spherically symmetric monopole (51), (52) with unit magnetic charge.

7. DISCUSSION

In the last section it was shown that the only consistent monopole solution to the first-order equation (40) with the asymptotic boundary conditions (69) was the known spherically symmetric solution. [It will be recalled that the singular solutions (55), (56) obeyed second-order field equations.] One is naturally led to reconsider the boundary conditions. One possibility is to contemplate modifications to (85) and (86) while keeping the isotropic condition $\Phi^2 = 2\Lambda\bar{\Lambda} = 2p^2 \to Z^2/2$. If Σ and/or h_1 tended to a non-vanishing function of θ on the sphere at infinity one need not have $g(\theta)$ vanishing and the arguments of the last section would not apply. However, examining the field energy (41) for the ansatz (70) one finds

$$M = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\{\dot{p}^2 r^2 + p'^2 + 4p^2\Sigma^2 + 4p^2\frac{h_1^2 m^2}{\sin^2\theta}\right\}\sin\theta\, dr\, d\theta\, d\phi$$

If p^2 is essentially constant outside a large radius, the only way for Σ^2 in the third term to prevent a divergence is for it to vanish asymptotically. The same argument applies to $m^2h_1^2/\sin^2\theta$ in the last term. Since all the terms in the integrand are positive there is no possibility of cancellations between terms.

The conclusion is that any finite energy axially symmetric configuration obeying (40) does not belong to the class encompassed by the ansatz (70), (71) with the boundary conditions (69). Such solutions with magnetic charge $g_n = ng_0$, if they exist, must arise from a more general ansatz than that considered here or from solutions[9] to the second-order equations (36),(37). If this latter case turns out to be the only possibility then (barring degeneracies) they may be safely described as excitations of the fundamental monopole since their energy would exceed n monopole masses due to the Bogomolny bound.

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APPENDIX

In this Appendix I collect some results that are used in the derivation of (53), (54), (73) and (74)

$$e^{-k\psi} j = j e^{k\psi} \tag{A.1}$$

$$e^{-k\psi} i = i e^{k\psi} \tag{A.2}$$

$$M^2 = -1 \tag{A.3}$$

Applying d to (A.3)

$$M\, dM = -dM\, M \tag{A.4}$$

Applying d to the definition of M

$$dM = M j e^{-k\psi} \left[-d\gamma + M \sin\gamma\, d\psi \right] \tag{A.5}$$

Hence

$$V\left(dM\, j e^{-k\psi} \right) = M\, d\gamma \tag{A.6}$$

$$V\left(M\, dM\, j e^{-k\psi} \right) = -M \sin\gamma\, d\psi \tag{A.7}$$

Under exterior multiplication of dM with itself

$$dM \wedge dM = -2M\, d(\cos\gamma) \wedge d\psi \tag{A.8}$$

Other simple but useful results are:

$$j e^{-k\psi} M e^{k\psi} j = M \tag{A.9}$$

$$M \left(e^{k\psi} j \right) = -\left(e^{k\psi} j \right) M \tag{A.10}$$

$$dM \wedge dM \wedge dM = 0 \tag{A.11}$$

The quaternion $T = i\, e^{-k\psi}$ obeys the following relations

$$T^2 = -1 \tag{A.12}$$

$$T\,dT = -dT\,T \tag{A.13}$$

$$dT \wedge dT = 0 \tag{A.14}$$

$$dT = kT\,d\psi \tag{A.15}$$

$$Tk = -kT \tag{A.16}$$

$$(kT)M = -M(kT) \tag{A.17}$$

From the ansatz (70),(71) the computation of B gives

$$\begin{aligned} B &= dA + A \wedge A \\ &= T\{-df_1 + 2f_2\beta\} \wedge d\psi + k\{df_2 + 2f_1\beta\} \wedge d\psi + kT\,d\beta \end{aligned} \tag{A.18}$$

Similarly

$$\begin{aligned} h &= d\Phi + 2v(A\Phi) \\ &= T\sqrt{2}\{d\lambda_2 + 2\lambda_1\beta\} + k\sqrt{2}\{d\lambda_1 - 2\lambda_2\beta\} \\ &\quad + 2\sqrt{2}\,kT\{f_1\lambda_1 + f_2\lambda_2\}\,d\psi \end{aligned} \tag{A.19}$$

Equations (73),(74) now follow by applying * to (A.19) and taking the k, T and kT components of (40).

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