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THE REGGEON FIELD THEORY BEYOND THE CRITICAL POINT

AN INSTANTON VACUUM !

Alan R. White

CERN - Geneva

A B S T R A C T

It is shown by analytic continuation - at finite rapidity - of the S matrix path integral that the Reggeon field theory with bare intercept greater than the critical value contains a renormalised Pomeron pole with intercept less than one as the leading singularity. The multi-Pomeron cuts satisfy Reggeon unitarity and so the complete theory is t channel unitary. A Rayleigh-Schrödinger perturbation expansion, valid beyond the critical point, is obtained by expanding the path integral around a rapidity ("Euclidean time") dependent instanton vacuum. The transition to the new vacuum at the critical point takes place smoothly and without any symmetry breaking. The theory with a point triple Pomeron interaction below the critical point contains all higher-order interactions above this point. Further, all interactions are non-local in impact parameter space.

The results imply that total cross-sections rise asymptotically only at the critical point and so imply a deeper significance for the experimental discovery of rising cross-sections.

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1. INTRODUCTION

Recently several different solutions have been proposed[1-7] to the problem of the Reggeon Field Theory[8-11] (RFT) with the bare Pomeron intercept α_0 greater than the critical value α_{0c}. However, all of these solutions lack the usual virtue of RFT; that is manifest t-channel unitarity[9,10]. In this paper we shall propose a further solution which is manifestly unitary. We shall also argue that our solution is the unique answer to the problem since it is derived by analytic continuation in α_0, at finite rapidity, of the theory defined with $\alpha_0 < \alpha_{0c}$. Before discussing the technical details of our solution and how it relates to the other proposed solutions we would like to briefly discuss the general theoretical and experimental significance of the problem.

It is well known that RFT is a field-theoretic technique for calculating Regge cut corrections to Regge pole exchange at high energy and that it is essential for studying the Pomeron. The most general basis for the RFT is the Regge cut discontinuity formulae[10,12,13] (Reggeon unitarity) which follow from the combination of basic analyticity properties of production amplitudes with multiparticle t-channel unitarity. The Reggeon unitarity equations apply directly to physical, or "renormalized", poles and cuts. From this point of view the bare Pomeron intercept is simply a parameter in an effective Lagrangian[9,10] used to solve the unitarity equations.

Let us suppose first that the renormalized Pomeron pole has trajectory $\alpha(t)$ with $\alpha(0) = 1 + \varepsilon$, $\varepsilon > 0$. The trajectory of the N-Pomeron cut is

$$\alpha_N(t) = N\alpha\left(t/N^2\right) - N + 1$$

and so $\alpha_N(0) = 1 + N\varepsilon$. Consequently, at $t = 0$ all the cuts, as well as the initial pole, violate the Froissart bound. If this situation is relevant to physics then it must be that none of these singularities is on the physical sheet in the angular momentum plane (at least at $t = 0$). They must be on unphysical sheet(s) of some additional singularity or singularities. It also follows from the trajectory formula that for any positive t there must be an infinite number of branch points to the right of any finite point in the angular momentum plane. Assuming that the amplitude is polynomially bounded at fixed t, there must again be at least one further singularity present besides the original pole and cuts, to keep this infinite set of cuts off the physical sheet. (The discontinuity formulae will not allow this role to be played by the Pomeron cuts themselves.) Hence we cannot demonstrate exact multiparticle t-channel unitarity simply by checking the discontinuity formulae for the Pomeron cuts -- in contrast to the situation when $\alpha(0) \leq 1$. In other words, Reggeon unitarity implies multiparticle t-channel unitarity only when $\alpha(0) \leq 1$.

It is already known[14] that we obtain the complete range of renormalized intercepts $-\infty \leq \alpha(0) \leq 1$ by considering $-\infty \leq \alpha_0 \leq \alpha_{0c}$. Therefore from the Reggeon unitarity point of view it appears that we have already obtained the complete range of relevant solutions of RFT by studying $\alpha_0 \leq \alpha_{0c}$.

An alternative view of the RFT, however, is that it is based on the formal "bare" perturbation expansion which can be derived directly from Gribov's original hybrid Feynman graph analysis, or more satisfactorily, from the topological expansion[15] of the underlying strong interaction theory. From this point of view α_0 is related to the parameters of the underlying theory [in particular the coupling constant (s)] and so RFT, viewed as the formal sum of the perturbation expansion, should have a physically meaningful solution for all values of α_0.

For $\alpha_0 \leq \alpha_{0c}$, the two points of view are certainly equivalent. However, the results of Refs. 1, 2, 4-7 suggest that when $\alpha_0 > \alpha_{0c}$, RFT gives total cross-sections which rise asymptotically like $[\ln s]^2$ and so saturate the Froissart bound. In this case the leading singularity in the angular momentum plane is (at least for some positive range of t) not a pole. This could perhaps suggest that the situation with $\alpha(0) > 1$, described above, has occurred. In any case, it appears that both Reggeon unitarity and t-channel unitarity have been lost.

In this paper we shall argue that this unsatisfactory situation does not in fact occur, but rather the correct solution to the $\alpha_0 > \alpha_{0c}$ problem has again a renormalized Pomeron pole, with $\alpha(0) < 1$, as the leading singularity. This property of our solution is shared by that proposed by Abarbanel, Bronzan, Schwimmer and Sugar (ABSS) in Ref. 3. However, in our solution the Pomeron cuts also satisfy Reggeon unitarity and so t-channel unitarity is maintained. In the ABSS solution this is not the case. Our theory with $\alpha_0 > \alpha_{0c}$ is not directly equivalent to a conventional theory with $\alpha_0 < \alpha_{0c}$ for the following reason. If we start with a theory containing just a point triple Pomeron interaction below the critical point ($\alpha_0 < \alpha_{0c}$), then above this point we obtain all one to many Pomeron interactions. The unconventional feature of these interactions is that they are all non-local in impact parameter space. Nevertheless, it may be that if we allowed such interactions below the critical point we could obtain a theory which is completely symmetric about the critical point.

If our solution is correct then it has important physical significance. First, let us note that the evidence for Pomeron factorization over a significant range of momentum transfer is now very good[16]. This strongly indicates that the exchange of one Pomeron pole (and not more), together with its associated cuts, is the physically relevant problem. Our results then imply that if total cross-sections do rise asymptotically we must be exactly at the critical point and the very beautiful critical theory[17,18] can be applied [even if not all phenomenological analyses of experimental data have supported this possibility[19]]. This presumably implies that the parameters of the underlying strong interaction theory satisfy an important constraint. Why this should be so cannot be explained within RFT [note that since RFT can be regarded as an extension of eikonal-type models to incorporate t-channel unitarity our results imply that field-theoretic calculations leading to such models[20] also do not "explain" rising cross-sections]. The only arguments (within RFT) that the intercept should be at the critical point are aesthetic. It is the strongest allowed interaction at high energy and the result is universal scaling diffraction peaks whose shape can be calculated[21-23].

The arguments in this paper are based on the path-integral giving the elastic S-matrix in RFT. We study this integral in detail in Section 2. We argue that, at large but finite rapidity, the integral is analytic in α_0 and so the theory for $\alpha_0 > \alpha_{0c}$ is well-defined by analytic continuation. This analyticity property seems to be an innate property of the RFT S-matrix. However, it is almost certainly required for consistency of RFT with the analyticity properties of the full S-matrix. The boundary conditions in the path-integral play an important role in our analysis and these are also discussed at length in Section 2. In particular, we note that it is essential to use boundary conditions at finite rapidity points in order to exploit the analyticity in α_0.

In Section 3 we begin our analysis of the theory above the critical point by discussing the zero structure of the classical potential. We note that in addition to the stationary points, which are the origin, that is the original "vacuum" and the two fixed-points chosen by ABSS as new vacua (both of which are unsymmetric under the $\psi \leftrightarrow \bar{\psi}$ symmetry of the theory), there exist lines of zeros joining these points which correspond to rapidity-dependent zero energy classical solutions of the equations of motion. Since rapidity is the analogue of Euclidean time in our problem these are *bona fide* instantons[24,25], at least in zero space dimensions.

Because of the non-Hermiticity of the interacting RFT fields, initial and final boundary conditions, or alternatively initial and final states, need not in general be Hermitian conjugate. Consequently, if a new "vacuum" does appear at the critical, or "phase transition" point, which is some mixture of positive energy states below this point, it is likely to have a different specification at initial and final rapidities. In the path-integral such a "vacuum" would naturally manifest itself as a rapidity-dependent classical solution. To study this possibility we consider the analytic continuation of source-dependent classical solutions which define the original vacuum in the infinite rapidity or zero source limit below the critical point. This suggests that the "analytic continuation" of the original vacuum is the symmetric instanton joining the two asymmetric ABSS fixed-points.

In Section 4 we carry out a change of variables in the path-integral. We make a shift of the fields by a source-dependent classical solution of the type discussed in Section 3. In this way we are able to show that the analytic continuation of the integral beyond the critical point can be expressed as an integral around the symmetric instanton.

In Section 5 we derive the perturbation expansion of the S-matrix around the instanton. This is not straightforward since the lack of translation invariance (in rapidity) of the instanton introduces translation dependence into the expansion which we have to eliminate. We find that this is most simply done in the Rayleigh-Schrödinger form of the expansion. In this expansion only certain rapidity orderings of the vacuum interactions (introduced by the shift) and the triple Pomeron interaction can contribute to the translation invariant S-matrix. The final result is an expansion which for $\alpha_0 \to \infty$ contains a leading Pomeron pole with intercept $(2 - \alpha_0)$ and manifestly satisfies Reggeon unitarity. It is most simply described as a unitary mixture of the two asymmetric ABSS expansions. The $\psi \leftrightarrow \bar{\psi}$ symmetry of the theory is preserved because of the symmetry of the instanton. The loss of this symmetry was regarded by ABSS as the major fault of their solution. Actually the symmetry is a necessary requirement for Reggeon unitarity and in our view it is the lack of unitarity which is the more fundamental criticism of the solution. The close relation of our solution to that of ABSS should ensure that ours shares with it the property that the Green's functions scale with the same exponents when the critical point is approached from above as when it is approached from below (at least in the ε-expansion). Since the transition to the new vacuum takes place smoothly in our theory, the phase transition can also be regarded as a second-order transition. (Although it is important to note that the transition only takes place at infinite rapidity.)

Finally in Section 6 we make a few general comments on our theory and then discuss its relation to other solutions. In particular, it is most interesting to compare with the results obtained by Amati, Ciafaloni, Le Bellac and Marchesini[5)] (ACLM) and confirmed by Cardy[7)] in what now seems to be the most attractive lattice spin model[26)] for the RFT. This model is directly analogous to that which has been successfully used to describe the phase transition in real $\lambda\phi^4$ field theory[27)]. Therefore it would be unfortunate if our theory appeared to be in contradiction with the spin model. Actually we argue that this is not the case and that our results simply show that the S-matrix should be defined in terms of a new vacuum rather than the original vacuum as is done by ACLM and Cardy. This new vacuum is clearly identifiable as our instanton. In the model it appears as a zero norm state which is degenerate with the original vacuum beyond the critical point. We argue that the choice of the new vacuum, and hence our results, is actually suggested by the model if the right interpretation is put on the results obtained.

2. GENERAL PATH-INTEGRAL FORMALISM

We consider the Lagrangian

$$\mathcal{L}(\bar{\psi},\psi,\alpha_0',\Delta_0,r_0,\cdots) = \mathcal{L}_0 + \mathcal{L}_3 + \mathcal{L}_4 + \cdots \tag{1}$$

where

$$\mathcal{L}_0(\bar{\psi},\psi,\alpha_0',\Delta_0) = \frac{1}{2}\bar{\psi}\overleftrightarrow{\frac{\partial}{\partial y}}\psi + \alpha_0'\nabla\bar{\psi}.\nabla\psi + \Delta_0\bar{\psi}\psi \tag{2}$$

$$\mathcal{L}_3(\bar{\psi},\psi,r_0) = i\frac{r_0}{2}\bar{\psi}(\bar{\psi}+\psi)\psi \tag{3}$$

$$\mathcal{L}_4(\bar{\psi},\psi,\lambda_{01},\lambda_{02}) = \lambda_{01}(\bar{\psi}^3\psi + \bar{\psi}\psi^3) + \lambda_{02}\bar{\psi}^2\psi^2 \tag{4}$$

where $\Delta_0 = 1 - \alpha_0$ and the terms not specified in (1) contain five and more Pomeron interactions, derivative interactions, etc. For most of our discussion we shall set λ_{01}, λ_{02} and all higher coupling constants to zero so that $\mathcal{L}$ is given by $\mathcal{L}_0 + \mathcal{L}_3$. However, we intend our analysis to apply to the general situation and will occasionally take λ_{01} and λ_{02} non-zero to illustrate this point. There will also be an upper cut-off in all momentum integrations which we shall not always make explicit.

We shall, for simplicity, consider eikonal-type (or "independent emission") couplings of Pomerons to external particles. In this case the elastic S-matrix (two particles → two particles) can be written in rapidity (y) and impact parameter ($\underline{x}$) space as

$$S_{g\bar{g}}(y_2-y_1,\underline{x}_2-\underline{x}_1,\alpha_0',\Delta_0,r_0,\cdots) = \int d\psi\, d\bar{\psi}\; e^{-\int dy\, d^2\underline{x}\, \mathcal{L}_{g\bar{g}}} \tag{5}$$

The functional integral $\int d\psi d\bar{\psi} = \int d(\mathrm{Re}\,\psi)d(\mathrm{Im}\,\psi)/\pi$ is over the complex ψ-plane at each point ($\underline{x}$,y) and

$$\mathcal{L}_{g\bar{g}}(\bar{\psi},\psi,\alpha_0',\cdots) = \mathcal{L} + ig\,\delta(y-y_1)\delta^2(\underline{x}-\underline{x}_1)\bar{\psi} + i\bar{g}\,\delta(y-y_2)\delta^2(\underline{x}-\underline{x}_2)\psi \tag{6}$$

g and $\bar{g}$ must be real and have the same sign to ensure the negative sign of the two Pomeron cut in lowest order perturbation theory.

We shall evaluate all path-integrals we consider by (complex) Gaussian integration[28,29,30] and so to evaluate $S_{g\bar{g}}$ we define a "potential" $V_{g\bar{g}}(\bar{\psi},\psi)$ by writing

$$\mathcal{L}_{g\bar{g}} = \mathcal{L}_0 + V_{g\bar{g}} \tag{7}$$

Next we note that we can formally rewrite (5) as

$$S_{g\bar{g}} = \exp\left[-\int dy\, d^2\underline{x}\; V_{g\bar{g}}\left(\frac{\delta}{\delta J(\underline{x},y)}, \frac{\delta}{\delta \bar{J}(\underline{x},y)}\right)\right]\Bigg|_{J=\bar{J}=0}$$

$$\times \int d\psi\, d\bar{\psi}\, \exp\left[-\int dy\, d^2\underline{x}\,\left(\mathcal{L}_0 + J\bar{\psi} + \bar{J}\psi\right)\right] \tag{8}$$

This expression can be evaluated if we can evaluate

$$\int d\psi\, d\bar{\psi}\; e^{-\int dy\, d^2\underline{x}\,\left[\mathcal{L}_0 + J\bar{\psi} + \bar{J}\psi\right]}$$

$$\equiv \int d\psi\, d\bar{\psi}\, \exp\Big[-\int dy\, dy'\, d^2\underline{x}\, d^2\underline{x}'\,\Big\{\bar{\psi}(y,\underline{x})\, A(y-y',\underline{x}-\underline{x}')\,\psi(y',\underline{x}')$$

$$+ J(y',\underline{x}')\,\bar{\psi}(y,\underline{x}) + \bar{J}(y,\underline{x})\,\psi(y',\underline{x}')\Big\}\Big] \tag{9}$$

So far this is simply a definition of $A(y - y', \underline{x} - \underline{x}')$, but if we now use the usual matrix complex Gaussian integration formula (generalized to a continuous matrix) we obtain for (9)

$$\exp\left[\int dy\, dy'\, d^2\underline{x}\, d^2\underline{x}'\; \bar{J}(y,\underline{x})\, \Delta(y-y', \underline{x}-\underline{x}')\, J(y',\underline{x}')\right] \tag{10}$$

apart from a normalization factor which we take to be one to give $S_{00} = 1$. $\Delta(y,\underline{x})$ is the usual Pomeron propagator, defined in this formalism by

$$\int dy''\, d^2\underline{x}''\;\Delta(y-y'', \underline{x}-\underline{x}'')\, A(y''-y', \underline{x}''-\underline{x}') = \delta(y-y')\,\delta^2(\underline{x}-\underline{x}') \tag{11}$$

which implies that Δ has the Fourier representation

$$\Delta(y,\underline{x}) = \frac{-i}{(2\pi)^3}\int_{-i\infty}^{+i\infty} dE \int d^2\underline{k}\; e^{-Ey+i\underline{k}.\underline{x}}\; \frac{1}{(E-\alpha_0'\underline{k}^2-\Delta_0)} \qquad (12)$$

$$= \frac{1}{4\pi}\frac{e^{-\Delta_0 y - \underline{x}^2/4\alpha_0' y}}{\alpha_0' y} \qquad y>0 \qquad (13)$$

$$= 0 \qquad y<0 \qquad (14)$$

Writing $J_1 = J(y_1,\underline{x}_1)$, etc. we also obtain from (10)

$$\frac{\delta}{\delta J_1}\frac{\delta}{\delta \bar{J}_2}\bigg|_{J=\bar{J}=0} \equiv \frac{\delta}{\delta J_1}\frac{\delta}{\delta \bar{J}_2}\bigg|_{J=\bar{J}=0} \exp\left[-\int dy\, d^2\underline{x}\,(\mathcal{L}_0 + J\bar{\psi} + \bar{J}\psi)\right] \qquad (15)$$

$$= \Delta(y_2-y_1,\ \underline{x}_2-\underline{x}_1) \qquad (16)$$

and in the same notation

$$\frac{\delta}{\delta J_1}\bigg|_{J=\bar{J}=0} = \frac{\delta}{\delta \bar{J}_1}\bigg|_{J=\bar{J}=0} = 0 \qquad (17)$$

and

$$\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\bigg|_{J=\bar{J}=0} = \frac{\delta}{\delta \bar{J}_1}\frac{\delta}{\delta \bar{J}_2}\bigg|_{J=\bar{J}=0} = 0 \qquad (18)$$

Finally, we can write in general,

$$\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_m}\frac{\delta}{\delta \bar{J}_1}\cdots\frac{\delta}{\delta \bar{J}_n}\bigg|_{J=\bar{J}=0} = \delta_{nm}\sum_C \prod_{i=1}^{m} \Delta_{i\,c(i)} \qquad (19)$$

where the sum $\sum_C$ is over all contractions C in which each J_i, i = 1, ..., m, is paired with $\bar{J}_{C(i)}$ and

$$\Delta_{i\,c(i)} = \Delta\left(y_i - y_{c(i)},\ \underline{x}_i - \underline{x}_{c(i)}\right) \qquad (20)$$

Clearly we can evaluate (5) in the form (8) by expanding exp $\left[- \int dyd^2x\, V_{g\bar{g}}\right]$ in a power series and evaluating each term in the series using (19). As anticipated, the result will be the familiar Reggeon calculus perturbation expansion in rapidity and impact parameter space. This procedure has the virtue that it is a self-contained method for relating directly the perturbation expansion of RFT and the path-integral (5). No intermediate quantum mechanical concepts, treating the fields as operators in a Hilbert space of states, need be introduced. To some extent this is fortunate because, as yet, there does not exist a complete formalism of this kind for the interacting non-Hermitian RFT fields. However, we have made no mention of boundary conditions on the fields integrated over in the above and it will be essential in the following to understand what boundary conditions are implicit in this procedure. We shall not be able to avoid quantum mechanics altogether.

First note that because of the special form of $V_{g\bar{g}}(\bar{\psi},\psi)$ (no terms of the form $\psi^2,\psi^3,\bar{\psi}^2,\bar{\psi}^3$, etc., which would represent vacuum production or absorption of Pomerons) and the "causal" property of $\Delta(y,\underline{x})$ given by (16), all y-integrations in the perturbation expansion will be restricted to the range

$$y_1 \leq y \leq y_2 \tag{21}$$

y_1 and y_2 being the points at which the sources g_1 and g_2 act. In fact, we could take the initial range of integration in (5) to be over any interval $Y_1 \leq y \leq Y_2$ and obtain the same result provided only that $Y_1 < y_1$ and $Y_2 > y_2$. Note, however, that if we take $Y_1 = y_1$ and $Y_2 = y_2$, then when we carry out the differentiation (15) at an end-point we will obtain an extra factor of 1/2 in the source differentiation if (as is correct) we use a localized source rather than a point source. We shall refer to this point again.

There is an immediate implication of the finiteness of the y-integrations. Since the propagator $\Delta(y,\underline{x})$ falls off exponentially at large $\underline{x}$ for fixed y, Δ_0 and α_0' (it is only at $y = \infty$ that this property fails), $S_{g\bar{g}}$ should be analytic as a function of Δ_0 for fixed values of $(y_1 - y_2)$, $(\underline{x}_1 - \underline{x}_2)$ and sufficiently small values of the other parameters g, $\bar{g}$, r_0, etc. This would be closely analagous to the analyticity, as a function of temperature, of the partition function of a finite volume thermodynamic system. We shall not attempt to prove this analyticity property of $S_{g\bar{g}}$ here, but shall simply assume it. For $(y_1 - y_2)$ sufficiently large it is in any case very likely to be a necessary requirement for the consistency of the RFT with the analyticity properties of the underlying strong interaction S-matrix, for the following reasons.

Suppose the RFT is an asymptotic approximation to the full S-matrix for some range of the parameters of the theory in which the RFT S-matrix is analytic. If a singularity in Δ_0 is then generated in RFT for finite but arbitrarily large y, the occurrence of this singularity in the full S-matrix can only be avoided if the validity of the RFT as an asymptotic approximation fails. Since we expect the dependence of RFT on Δ_0 to be related to the dependence of the underlying theory on the coupling constant(s) of the theory, analyticity in Δ_0 should directly reflect analyticity in the coupling constant(s). More generally we can argue that if the underlying theory satisfies both s- and t-channel unitarity in the initial range of the parameters, a singularity in Δ_0 would give a further imaginary part to the S-matrix which would violate unitarity.

The above discussion applies also in the (realistic) case that we consider in which there is an upper cut-off in the $\underline{k}^2$ integration in (12). This is equivalent to introducing a lattice in impact parameter space which only simplifies the task of proving analyticity properties. However, we know that in this case there is a "phase-transition" at infinite rapidity $y_2 - y_1$, for some finite value Δ_{0c} of Δ_0. In fact what we mean by this statement is that the asymptotic limit of $S_{g\bar{g}}$ as $y_2 - y_1 \to \infty$ is singular at $\Delta_0 = \Delta_{0c}$ while, as we have argued above, $S_{g\bar{g}}$ itself is not. Consequently the theory for $\Delta_0 < \Delta_{0c}$ is <u>defined</u> by the path-integral (5) if we are careful to analytically continue in Δ_0, with <u>$y_2 - y_1$ finite</u> (but sufficiently large).

In order to discuss boundary conditions in (5) we first review some standard results[30] for a one-dimensional (non-relativistic) quantum mechanical system, in which we have two Hermitian conjugate operators ψ and $\bar{\psi}$ satisfying the canonical commutation relation

$$[\psi, \bar{\psi}] = 1 \tag{22}$$

We can work in the holomorphic representation in which $\bar{\psi}$ and ψ have the representation

$$\bar{\psi} = z \quad , \quad \psi = \frac{d}{dz} \tag{23}$$

and act on the space of entire functions of type 1/2 with scalar product

$$(f_1, f_2) = \int f_1^*(z) f_2(z) \; e^{-|z|^2} \frac{dz\, dz^*}{2\pi i} \tag{24}$$

The matrix element of an operator O, say, is given by

$$(f_1, O f_2) = \int f_1^*(\bar{z}) \; O(\bar{z}, z) f_2(z) \; e^{-|\bar{z}|^2 - |z|^2} \frac{d\bar{z} d\bar{z}^*}{2\pi i} \frac{dz\, dz^*}{2\pi i} \tag{25}$$

where $O(\bar{z},z)$, a function of the two complex variables z and $\bar{z}$, is defined to be the kernel of O. $O(\bar{z},z)$ can always be found by writing

$$O(\bar{z},z) = \sum_{n,m} O_{nm} \frac{z^n}{\sqrt{n!}} \frac{\bar{z}^m}{\sqrt{m!}} \tag{26}$$

and noting that the functions $\psi_n(z) = z^n/\sqrt{n!}$ are the eigenfunctions of the "number operator" $\bar{\psi}\psi$. It can then be shown that an operator $A(\bar{\psi},\psi)$ which is a normal-ordered operator function of $\bar{\psi}$ and ψ has the kernel

$$e^{\bar{z}z} A(\bar{z},z) \tag{27}$$

Consequently, the kernel of an operator is closely analagous to a "matrix element between eigenstates of ψ".

Using this formalism, making the usual decomposition of a time interval $(t'' - t')$ into small intervals of length ε, making the appropriate convolutions of kernels and letting $\varepsilon \to 0$, it can be shown that the kernel of an evolution operator

$$U(t'', t') = e^{-iH(t''-t')} \tag{28}$$

where H (the Hamiltonian) is a normal-ordered function of ψ and $\bar{\psi}$, is given by the path-integral

$$U(\bar{z}, z; t''-t') = \int d\bar{\psi}\, d\psi \, \exp \tfrac{1}{2}\left[\bar{z}\,\psi(t'') + z\,\bar{\psi}(t')\right] \times \exp\left[-i\int_{t'}^{t''} dt \left\{ \frac{1}{2i}\, \bar{\psi} \overleftrightarrow{\frac{\partial}{\partial t}} \psi + H(\bar{\psi},\psi)\right\}\right] \tag{29}$$

The above discussion is brief, but it should be sufficient to show that we can regard (29) as a quantum-mechanical path-integral over trajectories $\psi(t)$ and $\bar{\psi}(t)$ with z the initial value of $\psi(t')$ and $\bar{z}$ the final value of $\bar{\psi}(t'')$. This can also be seen by noting that the "boundary term"

$$\exp \tfrac{1}{2}\left[\bar{z}\,\psi(t'') + z\,\bar{\psi}(t')\right] \tag{30}$$

is just what is required to give the classical solution of the equations of motion for ψ and $\bar{\psi}$ [with the boundary conditions $\psi(t') = z$, $\bar{\psi}(t'') = \bar{z}$] as a stationary point of the functional integrand in (29). Consequently, the path-integral can be evaluated perturbatively by expanding the integrand around this point. This will be an important idea for us, although we shall not go into any details since we shall not need them. However, for a complete discussion of this point we refer the reader to Faddeev's lecture notes[30].

We now wish to compare (29) with (5) in order to have a quantum mechanical interpretation of (5) which goes beyond our Gaussian integration procedure and which makes the question of boundary conditions clearer. Let us first regard $S_{g\bar{g}}$ as the analytic continuation to imaginary times ($y = it$) and imaginary coupling constant (ir_0) of the same quantity defined within a Hermitian RFT. We can also temporarily ignore the $\underline{x}$-dependence for simplicity. In the Hermitian theory we can safely apply the holomorphic representation of the fields regarded as operators. Also we can take at least the four-Pomeron interaction in (1) to be non-zero to ensure the boundedness of the potential and the Hamiltonian spectrum, since this is implicit in the derivation of (29).

There are now several different ways of comparing (5) with (29), which we can list as follows:

i) The most straightforward identification is to write

$$y_1 = it' \,, \quad y_2 = it'' \,, \quad \bar{z} = 2i\bar{g} \,, \quad z = 2ig \tag{31}$$

Apart from the $\underline{x}$-dependence, this immediately identifies (5) and (29) if we also write

$$H = \Delta_0 \bar{\psi}\psi + \mathcal{L}_3(\bar{\psi},\psi) + \mathcal{L}_4(\bar{\psi},\psi) + \ldots \tag{32}$$

The integration region is taken to be $y_1 \leq y \leq y_2$ and $2ig$ and $2i\bar{g}$ are respectively regarded as boundary values for ψ and $\bar{\psi}$. The factor of 2 here is because of the factor of 1/2 which we noted earlier, occurs if we integrate only up to the end-points at $y = y_1, y_2$ and not beyond. This point becomes more complicated when we take the impact parameter dependence of the sources into account. It is essentially the reason that we shall not use these boundary conditions.

ii) A second possibility is to use the freedom of choice of the range of the y-integration to write

$$Y_1 = it' \,, \quad Y_2 = it'' \qquad (Y_2 > y_2 \,, \ Y_1 < y_1) \tag{33}$$

$$\bar{z} = z = 0 \tag{34}$$

Now we must write

$$H_{g\bar{g}} = \Delta_0 \bar{\psi}\psi + V_{g\bar{g}}(\bar{\psi},\psi) \tag{35}$$

and integrate over $Y_1 < y < Y_2$ with the boundary condition

$$\psi(Y_1) = \bar{\psi}(Y_2) = 0 \tag{36}$$

ig and $i\bar{g}$ are interpreted as (rapidity dependent) sources of $\bar{\psi}$ and ψ giving a rapidity dependent "Hamiltonian".

iii) A third more complicated possibility is to write $Y_1 = it'$, $Y_2 = it''$ as in (ii), but instead of (34) we write

$$z\ ,\ \bar{z} \qquad \text{fixed but undetermined} \tag{37}$$

We take $H_{g\bar{g}}$ to be defined as in (35) and then take the limit

$$Y_1 \to -\infty\ ,\ Y_2 \to +\infty \tag{38}$$

To show that this limit gives $S_{g\bar{g}}$ we must evaluate (29) (with H given by $H_{g\bar{g}}$) before the limit is taken. As we have noted, the most direct way to do this is to develop a perturbation expansion around the classical solution satisfying the boundary conditions $\psi(t') = z$, $\bar{\psi}(t'') = \bar{z}$. This solution is shown qualitatively (for $\Delta_0 > 0$; we shall discuss the one-dimensional classical problem more in the next section) in Fig. 2.1 and, as is shown in the figure, does go over to the classical solution to the boundary conditions of (ii) when the limit (38) is taken. Hence it is plausible that the complete perturbation expansion around one classical solution goes to the expansion around the other in this limit.

An alternative derivation of this last result, which involves ideas that are important for us, is obtained by using (19) to evaluate (29) by Gaussian integration as we did for $S_{g\bar{g}}$. In this case we expand both $\exp\left[-\int V_{g\bar{g}}\right]$ and $\exp 1/2\left[\bar{z}\psi + z\bar{\psi}\right]$ as power series and replace the powers of the fields by derivatives with respect to sources as in (8). In addition to the usual Reggeon graphs propagating between the internal sources g and $\bar{g}$, we obtain further graphs from contractions of powers of the "boundary fields" $\psi(Y_2)$, $\bar{\psi}(Y_1)$ with the internal fields and sources. As is implied in (1), placing non-zero boundary conditions on the fields is equivalent to adding sources of ψ and $\bar{\psi}$ at the boundary (of strengths $\bar{z}/2i$ and $z/2i$, respectively) in addition to the internal sources. The complete graphical expansion for $U(\bar{z},z,\bar{g},g)$ is illustrated in Fig. 2.2 as a sum over integrals of the usual Green's functions of RFT defined by the Lagrangian (1).

Now suppose that all the Green's functions go to zero when any of the rapidities involved goes to infinity (as is the case when the leading singularity in the E-plane is a pole with renormalized intercept $\Delta > 0$). In this case all terms in the expansion of $U(z,\bar{z},g,\bar{g})$ go to zero in the limit (38) except those involving only the internal sources. That is

$$U(z,\bar{z},g,\bar{g}) \underset{Y_2,-Y_1 \to \infty}{\longrightarrow} S_{g\bar{g}} \tag{39}$$

This is in fact the conventional field theory method for defining vacuum to vacuum amplitudes in the path-integral formalism[28]. It appears to be the most general approach to the problem of boundary conditions since it is independent of the boundary values z and $\bar{z}$ specified. It can also be used to justify our neglect of boundary values in the first part of this section. That is, (39) is the implicit statement about boundary conditions to which we referred earlier. However, this approach is only clearly justified as long as the Green's functions of the field used to evaluate the path-integral go to zero at large rapidity. In particular this condition breaks down if there is a phase transition at infinite rapidity, as in our case. In general, a path-integral is non-analytic at a phase transition just because of this point. In our case we want to avoid using (39) because it hides the fact that due to the special form of $V_{g\bar{g}}$, we can use finite boundary conditions as in (ii) which ensure the analyticity of $S_{g\bar{g}}$. Unfortunately, as we discuss in the next sections, (ii) is also not suitable for our purpose. We shall use yet another possiblity for relating (5) and (29) which we now discuss.

iv) We again write $Y_1 = it'$, $Y_2 = it''$. We take H to be $H_{g\bar{g}}$ defined as in (35) and fix z and $\bar{z}$ at some finite undetermined values. We then note from the perturbation expansion shown in Fig. 2.2 that the only part of $U(z,\bar{z},g,\bar{g},Y_1,Y_2,y_1,y_2)$ which is translation invariant with respect to displacements of y_1 and y_2 (with Y_1 and Y_2 kept fixed) is the sum of disconnected graphs involving interactions propagating separately between y_1 and y_2, and Y_1 and Y_2. That is all graphs of the form shown in Fig. 2.3. Hence we can write

$$S_{g\bar{g}} = \mathrm{Tr}\left[\frac{U(z,\bar{z},g,\bar{g})}{U(z,\bar{z},0,0)}\right] \tag{40}$$

where Tr implies that we take the translation invariant part. Alternatively, since the only translation invariant part of $U(z,\bar{z},g,0)$ or $U(z,\bar{z},0,\bar{g})$ is $U(z,\bar{z},0,0)$, we can rewrite (40) in the form

$$S_{g\bar{g}} = \mathrm{Tr}\left[\frac{U(z,\bar{z},g,\bar{g})\;U(z,\bar{z},0\,0)}{U(z,\bar{z},g,0)\;U(z,\bar{z},0,\bar{g})}\right] \tag{41}$$

This second form is more convenient for the field shift we discuss in Section 4.

Before leaving this general discussion of the path-integral formalism we briefly discuss the inclusion of the $\underline{x}$-dependence of $S_{g\bar{g}}$ in the formalism. Clearly it is straightforward to write

$$H_{g\bar{g}} = \int d^2\underline{x}\;\; \alpha_0 \nabla\bar{\psi}(\underline{x},y).\nabla\psi(\underline{x},y) + \Delta_0\bar{\psi}\psi + V_{g\bar{g}}(\psi,\bar{\psi}) \tag{42}$$

and

$$[z\bar{\psi} + \bar{z}\psi] = \left[\int d^2\underline{x}\;\; Z(\underline{x})\,\bar{\psi}(\underline{x},y_1) + \bar{Z}(\underline{x})\,\psi(\underline{x},y_2)\right] \tag{43}$$

As we mentioned earlier we have a cut-off in our $\underline{k}^2$-integrations which is equivalent to introducing a lattice in $\underline{x}$-space. This implies that the $\underline{x}$-integrations in (42) and (43) can be replaced by lattice summations.

We can also introduce a boundary in $\underline{x}$-space, but because the equations of motion contain only second-order derivatives in $\underline{x}$, there will be no boundary terms provided only that we specify the fields to have zero derivative at the boundary. Alternatively, we can eliminate boundary terms by the "periodic" boundary condtion

$$\psi(\underline{B},y) = \psi(-\underline{B},y)\;,\quad \bar{\psi}(\underline{B},y) = \bar{\psi}(-\underline{B},y) \tag{44}$$

where both $(\underline{B},y)$ and $(-\underline{B},y)$ lie on the boundary. Note that either form of boundary condition is compatible with the field shift that we discuss in Section 4 which is rapidity-dependent but constant as a function of impact parameter.

It is important to note that we only obtain the complete Reggeon graphs (even for finite external $\underline{x}_1 - \underline{x}_2$) once we let the boundary go to infinity (in contrast to the rapidity integrations). The presence of a boundary implies a cut-off in all the internal impact parameter integrations. Of course, it is an essential prerequisite for letting the boundary go to infinity, that the propagator goes to zero sufficiently fast at large impact parameter. As we have noted, this is the case for all values of Δ_0.

3. THE CLASSICAL POTENTIAL AND THE INSTANTON SOLUTIONS

We define the "classical" potential V_c to be the one-dimensional Hamiltonian of (32)

$$V_c(\psi, \bar\psi) = \Delta_0 \bar\psi \psi + \tfrac{1}{2} i r_0 \bar\psi (\bar\psi + \psi) \psi \tag{45}$$

We note that V_c has three planes of zeros in the four-dimensional complex $(\psi, \bar\psi)$ space

$$\text{(i)} \quad \bar\psi = 0 \quad , \quad \text{(ii)} \quad \psi = 0 \qquad \text{(iii)} \quad \bar\psi + \psi = \frac{2i\Delta_0}{r_0} \tag{46}$$

The zeros intersect at stationary points of V_c, that is, at

$$\text{(a)} \quad \psi = \bar\psi = 0 \; , \quad \text{(b)} \quad \bar\psi = \frac{2i\Delta_0}{r_0}, \; \psi = 0 \quad , \quad \text{(c)} \quad \bar\psi = 0, \; \psi = \frac{2i\Delta_0}{r_0} \tag{47}$$

There is also a further stationary point (d), $\psi = \bar\psi = i\Delta_0/3r_0$ where $V_c \neq 0$.

Because the sources ig and $i\bar g$, of $\bar\psi$ and ψ respectively, are pure imaginary, it will be useful to consider the (Im ψ, Im $\bar\psi$) plane. This plane is shown in Fig. 3.1 for $r_0 > 0$. The integration contour of (5), that is $\bar\psi = \psi^*$, appears as a line, shown dotted in Fig. 3.1, passing through the origin. Note that when $\Delta_0 = 0$, the line of zeros (iii) coincides with the integration contour. It is this effect which is essentially responsible for the phase transition, although, of course, renormalization effects[14] shift the transition to $\Delta_0 = \Delta_{0c}$ (< 0) instead of $\Delta_0 = 0$.

We also note that lines on which V_c is constant correspond to rapidity-dependent solutions of the classical equations of motion. Such solutions have been studied in detail by Amati, Caneschi and Jengo[31], and we shall essentially just quote their results. In particular, of course, the lines (47) correspond to zero energy solutions which, since they asymptote to the stationary points of the potential when we take the infinite "imaginary time" limit (that is, the infinite rapidity limit in our language), can be regarded as "instanton" solutions[24,25]. In Fig. 3.2 we have drawn some contours of fixed V_c indicating the direction of the classical solutions. The analytic form of the solutions can be found by integrating

$$y = \int^{\psi} \frac{dy}{d\psi}\, d\psi = \int^{\psi} d\psi \Big/ \frac{\partial H}{\partial \bar\psi} \tag{48}$$

$$= \frac{1}{r_0} \int^{\psi} dx \Big/ x \left[\left(\Delta_0/r_0 + x \right)^2 + \frac{4E}{r_0 x} \right]^{\frac{1}{2}} \tag{49}$$

where E is the "energy", that is, the value of V_c.

A particular way of specifying a solution is to give ψ at an initial rapidity Y_1 -- this fixes the lower integration point in (49), and $\bar{\psi}$ at a final rapidity Y_2 -- E then becomes a function of $\psi(Y_2)$ which can be solved for. We shall therefore denote a general (one-dimensional) classical solution by

$$\chi_{z\bar{z}} = \left(\psi_{z\bar{z}}(y), \bar{\psi}_{z\bar{z}}(y) \right) \tag{50}$$

where $\psi(Y_1) = z$, $\bar{\psi}(Y_2) = \bar{z}$. The Y_1, Y_2 dependence of $\chi_{z\bar{z}}$ will be implicit in the definition of z and $\bar{z}$. Because of the translational invariance of the equations of motion

$$\chi_{z\bar{z}y_0} = \left(\psi_{z\bar{z}}(y-y_0), \bar{\psi}_{z\bar{z}}(y-y_0) \right) \tag{51}$$

is also a solution [which can, of course, be specified by $\psi(Y_1 + y_0) = z$, $\bar{\psi}(Y_2 + y_0) = \bar{z}$].

From Fig. 3.2 it can be seen that for $\Delta_0 > 0$ our specification of $\chi_{z\bar{z}}$ is unique if we disallow infinite values of the fields, except if Im z and Im $\bar{z}$ are both positive and $Y = Y_2 - Y_1$ is sufficiently small. This is illustrated in Fig. 3.3 where we have shown two solutions with the same z and $\bar{z}$. For Y sufficiently large ($Y > Y_c$, say) only the solution nearest the origin is possible. This can be seen from the general property of all solutions, which follows from (49), that they must approach one (or more) of the zero energy solutions when $Y \to \infty$. That is, they must spend an infinite rapidity interval approaching one of the stationary points. It is important that Y_c depends on Δ_0, and that $Y_c \to \infty$ when $\Delta_0 \to 0$. This is related to the convergence of the expansion of $S_{g\bar{g}}$, in a power series in g and $\bar{g}$. We shall discuss this point further in Section 6. For the moment we note that if we take Im z, Im z < 0, $\Delta_0 > 0$, then $\chi_{z\bar{z}}$ is unique for all finite Y. Also, when $Y \to 0$, $\chi_{z\bar{z}}$ collapses into the origin as shown in Fig. 3.4.

We begin our consideration of analytic continuation in Δ_0 at finite rapidity by tracing the analytic continuation of $\chi_{z\bar{z}}$ in the case Im z, Im $\bar{z}$ < 0. A finite Y path cannot cross the stationary point (d) nor, since the orbits around (d) have finite period, can such a path shrink to a point in the (Im ψ, Im $\bar{\psi}$) plane if Y is sufficiently large. Therefore if $z = \bar{z}$, then as we continue through $\Delta_0 = 0$ to Δ_0 very large and negative, the result must be as shown in Fig. 3.5. The most important feature of this continuation is that if we take either $Y \to \infty$ (or z, $\bar{z} \to 0$) <u>after the continuation</u>, we obtain the result shown in Fig. 3.6. That is $\chi_{z\bar{z}}$ asymptotes to the triangle formed by lines (i), (ii), and (iii) or the three instanton solutions which in $\Delta_0 > 0$ can be denoted by (note that $Y = \infty$)

$$\text{(i) } \chi_{\mu,0} \quad , \text{(ii) } \chi_{0,0} \quad , \text{(iii) } \chi_{0,\mu} \qquad \left(\mu = 2i\Delta_0/r_0\right) \tag{52}$$

In fact since each instanton requires an infinite rapidity interval to traverse, it follows that if we take the $Y \to \infty$ limit symmetrically, that is $Y_2 = -Y_1 \to +\infty$, then $\chi_{\mu 0}$ must be traversed at $y = -\infty$, $\chi_{0\mu}$ is traversed at $y = +\infty$, while χ_{00} is traversed in the interval $y \in (-\infty, +\infty)$. Note that if we take $Y \to \infty$ at any of the intermediate stages shown in Fig. 3.5 we also obtain χ_{00}. If $z \neq \bar{z}$ the continuation of Fig. 3.5 is again correct provided that Y is sufficiently large.

We now observe firstly that $\chi_{z\bar{z}}$ represents the classical contribution to a path-integral of the form (5) (ignoring the $\underline{x}$-dependence for the moment) in the presence of sources of the type $ig, i\bar{g}$. Secondly, $\chi_{z\bar{z}}$ reduces to the "vacuum", i.e. the origin, when either $z, \bar{z} \to 0$ or $Y \to \infty$, provided that $\Delta_0 > 0$. The above continuation strongly suggests that beyond the phase transition (that is $\Delta_0 < 0$ for the classical potential) it is χ_{00} which we should take as the "vacuum" of the theory . We shall make this idea specific in the next section by expanding the path-integral around $\chi_{z\bar{z}}$.

It is also interesting to plot the analytic continuation of the classical trajectory in the presence of the internal sources which is shown in Fig. 2.1. This is directly related to our specification of $S_{g\bar{g}}$ using (40). The analytic continuation is shown in Fig. 3.7. It is, of course, closely related to the continuation of $\chi_{z\bar{z}}$, but it emphasizes that in $\Delta_0 < 0$ we can expect the sources g and $\bar{g}$ to give rise to excitations around χ_{00}. Note that if we attempt to plot the continuation of the classical solution with the boundary conditions (ii), we encounter an ambiguity as shown in Fig. 3.8. Graphically we cannot see which of the two alternatives shown gives the analytic continuation of the action. This is one reason why we do not use these boundary conditions.

It is important to remark that while in a conventional field theory it would be nonsense to consider an instanton or "time dependent" vacuum, in the RFT this is not the case. Since the Hamiltonian is non-Hermitian the initial and final states, defined as left and right eigenstates in some appropriate space of states (which has still to be formulated), need not be identical. As will be clear by the end of Section 5, it is the change of "vacuum" at $y = \pm\infty$ which is the essential ingredient in our analysis.

Finally we note that for the zero energy solutions it is trivial to integrate (49) to obtain

$$\chi_{\mu,0} = \frac{i\Delta_0}{r_0}\left(1 - \tanh\frac{\Delta_0}{2}y\,,\ 0\right) \tag{53}$$

$$\chi_{0,0} = \frac{i\Delta_0}{r_0}\left(1 + \tanh\frac{\Delta_0}{2}y\,,\ 1 - \tanh\frac{\Delta_0}{2}y\right) \tag{54}$$

$$\chi_{0,\mu} = \frac{i\Delta_0}{r_0}\left(0\,,\ 1 + \tanh\frac{\Delta_0}{2}y\right) \tag{55}$$

We have, of course, included only $\mathcal{L}_3(\psi,\bar\psi)$ in our study of V_c, but it should be clear that it is the zero structure of V_c near the origin that is central in our analysis. This will only be changed in an inessential way if we add (sufficiently small) higher couplings. For example, if we take $\lambda_{02} = 2\lambda_{01} = 2\lambda$ so that

$$\mathcal{L}_4 = \lambda \bar\psi \left(\bar\psi + \psi\right)^2 \psi \tag{56}$$

then χ_{00} still has the form

$$\chi_{00} = a\left(1 + \tanh by,\ 1 - \tanh by\right) \tag{57}$$

but with

$$a = \frac{i r_0 - \left[-r_0^2 - 16\Delta_0\lambda\right]^{\frac{1}{2}}}{8\lambda} \underset{\lambda\to 0}{\longrightarrow} \frac{i\Delta_0}{r_0} \tag{58}$$

$$b = -\frac{\left[i r_0 - \left(-r_0^2 - 16\Delta_0\lambda\right)^{\frac{1}{2}}\right]\left[-16\Delta_0\lambda - r_0^2\right]^{\frac{1}{2}}}{16\lambda} \underset{\lambda\to 0}{\longrightarrow} \frac{\Delta_0}{2} \tag{59}$$

4. THE FIELD SHIFT

We take $S_{g\bar g}$ to be given by (41) with $H_{g\bar g}$ given by (42) and we take $z(\underline{x})$ and $\bar z(x)$ in (43) to be independent of $\underline{x}$. $U(z,\bar z,g,\bar g)$ will be well defined by the perturbation expansion of Fig. 2.2, since we know that the usual RFT Green's functions go to zero at large impact parameter. This ensures the convergence of the $\underline{x}$-integration in (43).

We now make a change of variables in U by writing

$$\left(\psi(y,\underline{x}),\ \bar\psi(y,\underline{x})\right) = \left(\phi(y,\underline{x}),\ \bar\phi(y,\underline{x})\right) + \chi_{z\bar z} \tag{60}$$

First we remark that since $\bar\psi_{z\bar z}(y)$ is not the complex conjugate of $\psi_{z\bar z}(y)$, this shift is not a simple change of variables in (41). Instead (60) represents a contour deformation in the integrations over both Re ψ and Im ψ for each y. The integrand will be an analytic function of both Re ψ and Im ψ regarded as independent complex variables and Cauchy's theorem can be used to move the integration contours, as specified by (60), provided that the asymptotic behaviour of the integrand allows this. This is certainly the case if we take a small four-Pomeron interaction of the form (56) to be present.

The effect of the shift in $U(z,\bar{z},g,\bar{g})$ is firstly to produce an over-all factor which is the action of the $\chi_{z\bar{z}}$ trajectory and which will cancel in (41). Since $\chi_{z\bar{z}}$ satisfies the equations of motion, the terms in $V_c(\psi,\bar{\psi})$ linear in the shifted fields ϕ and $\bar{\phi}$ will cancel against the remaining rapidity derivative terms if we make an integration by parts. The resulting boundary term is

$$\exp \tfrac{1}{2}\Big[\int d^2\underline{x}\ \bar{\psi}_{z\bar{z}}(Y_2)\,\phi(Y_2,\underline{x}) - \bar{\psi}_{z\bar{z}}(Y_1)\,\phi(Y_1,\underline{x}) - \bar{\phi}(Y_2,\underline{x})\,\psi_{z\bar{z}}(Y_2) + \bar{\phi}(Y_1,\underline{x})\,\psi_{z\bar{z}}(Y_1) \Big] \tag{61}$$

But since

$$\psi(Y_1,\underline{x}) = \psi_{z\bar{z}}(Y_1) = z \ , \quad \bar{\psi}(Y_2,\underline{x}) = \bar{\psi}_{z\bar{z}}(Y_2) = \bar{z} \tag{62}$$

it follows that

$$\phi(Y_1,\underline{x}) = \bar{\phi}(Y_2,\underline{x}) = 0 \tag{63}$$

and so (61) reduces to

$$\exp \tfrac{1}{2}\Big[\int d^2\underline{x}\ \ \bar{z}\,\phi(Y_2,\underline{x}) + z\,\bar{\phi}(Y_1,\underline{x}) \Big] \tag{64}$$

which exactly cancels the original boundary term (43), apart from a factor

$$\exp\Big[\int d^2\underline{x}\ \ \bar{z}z \Big] \tag{65}$$

which will cancel in (41). [Note that we could keep the volume in $\underline{x}$-space finite during these manipulations in order to keep factors such as (65) finite.] The $g\bar{\psi}_{z\bar{z}}$ and $\bar{g}\psi_{z\bar{z}}$ terms in U will also cancel in (41) and so finally we can write

$$S_{g\bar{g}} = \mathrm{Tr}\left[\frac{U_s(g,\bar{g})\ U_s(0,0)}{U_s(g,0)\ \ U_s(0,\bar{g})} \right] \tag{66}$$

where U_s is defined analagously to (29) but with no boundary term. That is

$$U_s(g,\bar{g}) = \int d\bar{\phi}\,d\phi\ \exp -\Big[\int_{Y_1}^{Y_2} dy \int d^2\underline{x}\ \ \mathcal{L}_{sg\bar{g}}(\bar{\phi},\phi) \Big] \tag{67}$$

where $\mathcal{L}_{sg\bar{g}}$ is defined as in (6), but with $\mathcal{L}\to\mathcal{L}_s$, where

$$\mathcal{L}_s = \mathcal{L}_0(\bar{\phi},\phi,\alpha_0',-\Delta_0) + \mathcal{L}_2(\bar{\phi},\phi,\Delta_0) + \mathcal{L}_3(\bar{\phi},\phi,r_0) \tag{68}$$

Clearly $\mathscr{L}_2$ is the new ingredient in $\mathscr{L}_s$, its precise form is

$$\mathscr{L}_2(\bar{\phi},\phi,\Delta_0,y) = \frac{i r_0}{2}\left[\bar{\psi}_{z\bar{z}}\phi^2 + \psi_{z\bar{z}}\bar{\phi}^2 + 2\left(\frac{2\Delta_0}{i r_0} + \bar{\psi}_{z\bar{z}} + \psi_{z\bar{z}}\right)\bar{\phi}\phi\right] \tag{69}$$

Since $\bar{\psi}_{z\bar{z}}$ and $\psi_{z\bar{z}}$ are y-dependent, $\mathscr{L}_2$ is also y-dependent.

We now analytically continue $S_{g\bar{g}}$ as a function of Δ_0 in the form (66) with Y_1 and Y_2 kept finite but sufficiently large. When $\Delta_0 << 0$ we can take the limit $-Y_1$, $Y_2 \to \infty$ so that $\chi_{z\bar{z}}$ asymptotes to the three instanton trajectories (52), given by (53)-(55). As we noted in the last section, only the trajectory χ_{00} is covered in the finite y-region $y \in (-\infty,+\infty)$ and so the sources g and $\bar{g}$ which act at the finite points y_1 and y_2, respectively, will act during this interval.

Although the trajectories $\chi_{\mu 0}$ and $\chi_{0\mu}$ have retreated to $-\infty$ and $+\infty$, respectively, we must check that they do not give a finite contribution to $S_{g\bar{g}}$. We can make a perturbation expansion on each of these trajectories using the Gaussian integration method of Section 2 if we treat both $\mathscr{L}_2$ and $\mathscr{L}_3$ as part of the interaction and hence take them to be part of $V_{g\bar{g}}(\bar{\phi},\phi)$. On $\chi_{0\mu}$, $\bar{\psi}_{z\bar{z}}$ is zero whilst $\psi_{z\bar{z}}$ is given by (53). Since only the $\bar{\phi}^2$ and $\bar{\phi}\phi$ terms in (69) are non-zero there will be no vacuum processes. Also there is neither an initial boundary term nor any internal source. Consequently we can only obtain a non-zero contribution from $\chi_{0\mu}$ if interactions produced by the $\bar{\phi}^2$ vacuum production term can communicate with the interactions that take place along the χ_{00} trajectory. For this to be possible such interactions must be able to survive for the infinite rapidity interval needed to pass through the stationary point (c).

At (c) we have

$$\bar{\psi}_{z\bar{z}} = 0 \quad , \quad \psi_{z\bar{z}} = -\frac{2i\Delta_0}{r_0} \tag{70}$$

When substituted into (69) this, of course, gives just the unsymmetric theory obtained by a constant shift, studied in detail by ABSS. However, we know from the ABSS results that the Green's functions of this theory go to zero at infinite rapidity when $\Delta_0 << 0$. Consequently, the interactions produced along the $\chi_{0\mu}$ trajectory cannot communicate with those along the χ_{00} trajectory. Therefore the $\chi_{0\mu}$ trajectory and similarly the $\chi_{\mu 0}$ trajectory give zero contribution to $S_{g\bar{g}}$.

Finally we are left with evaluating the χ_{00} contribution to $S_{g\bar{g}}$. First we note that if we replace Y_1 and Y_2 by $Y_1 - y_0$ and $Y_2 - y_0$, respectively, throughout all of our previous discussion then we will arrive at (69) with $\bar{\psi}_{z\bar{z}}$ replaced by $\bar{\psi}_{z\bar{z}y_0}$ and $\psi_{z\bar{z}}$ replaced by $\psi_{z\bar{z}y_0}$. Hence $S_{g\bar{g}}$ must be given by (66) if U_s is evaluated from (67)-(69) using any $\psi_{z\bar{z}y_0}$. That is, $S_{g\bar{g}}$ must be independent of y_0.

This is important because if we have introduced any new translational invariant part into U_s by taking the limit $-Y_1, Y_2 \to +\infty$, it will depend on y_0. Therefore we can eliminate this possibility by calculating (66) with an arbitrary χ_{00y_0}.

$$\bar{\Psi}_{0,0,y_0} = \frac{-i\Delta_0}{r_0}\left(1 - \tanh\frac{\Delta_0}{2}(y-y_0)\right) \tag{71}$$

$$\Psi_{0,0,y_0} = -\frac{i\Delta_0}{r_0}\left(1 + \tanh\frac{\Delta_0}{2}(y-y_0)\right) \tag{72}$$

That is, we use the perturbation expansion obtained from the general method of Section 2 and pick out that part of the expansion which is both translation invariant in y_1 and y_2, and independent of y_0.

Finally we note that since there is no finite rapidity classical solution to the boundary conditions (ii) of Section 2, a shift of the kind discussed above would necessarily introduce new boundary terms. This is a second reason why we have not used these conditions.

5. THE PERTURBATION EXPANSION

It is straightforward to derive a perturbation expansion for $U_s(g,\bar{g})$ using the method of Section 2. Apart from the shift of $\Delta_0 \to -\Delta_0$ in $\mathscr{L}_0$, which ensures that our bare propagator is well defined in $\Delta_0 < 0$, rather than $\Delta_0 > 0$, the only difference from the usual RFT expansion is the presence of $\mathscr{L}_2$. Since

$$\Psi_{0,0,y_0} + \bar{\Psi}_{0,0,y_0} = -\frac{2i\Delta_0}{r_0} \tag{73}$$

we have

$$\mathcal{L}_2(\bar{\phi},\phi,\Delta_0) = \frac{\Delta_0}{2}\left[\delta^+_{y_0}\phi^2 + \delta^-_{y_0}\bar{\phi}^2\right] \tag{74}$$

where

$$\delta^{\pm}_{y_0}(y) = 1 \pm \tanh -\frac{\Delta_0}{2}(y-y_0) \tag{75}$$

To discuss both y_0-independence and translation invariance we go to Fourier transform space. The Fourier transforms of $\delta^{\pm}_{y_0}$ are

$$\delta^{\pm}_{y_0}(E,\underline{k}) = \frac{2\pi}{\Delta_0}\frac{e^{Ey_0}}{\sin\frac{\pi}{\Delta_0}(E \mp \varepsilon)}\,\delta^2(\underline{k}) \tag{76}$$

the only singularities of which are the poles at $E = 0, \pm\Delta_0, \pm 2\Delta_0, \ldots$. The $\mp\varepsilon$ simply indicates to which side of the imaginary axis the pole at $E = 0$ should be taken when the Fourier integral is along this axis. Note that only this pole has a residue independent of y_0, and so we can write

$$\delta^{\pm}_{y_0}(E,\underline{k}) = \frac{2\,\delta^2(\underline{k})}{E \mp \varepsilon} + R^{\pm}_{y_0}(E,\underline{k}) \equiv \delta^{\pm} + R^{\pm}_{y_0} \tag{77}$$

In the following the y_0-independent part of a graph will be obtained by neglecting R'_{y_0}. This corresponds to taking θ-functions in y space, i.e. keeping only the leading asymptotic behaviour of $\delta^{\pm}y_0$ as $y \to \pm\infty$. It is perhaps obvious from the plot of $\delta^{\pm}y_0$ in Fig. 5.1 that this is the only y_0-independent feature of these functions.

We could be more sophisticated at this point. For example, we could argue that since $S_{g\bar{g}}$ is independent of y_0 we will calculate

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dy_0\, S_{g\bar{g}}\, e^{\alpha y_0} \Big/ \delta(\alpha) \tag{78}$$

If we close all E contours in the left half-plane, the pole at $E = 0$ will give $S_{g\bar{g}}$ for (78) while for the other poles the y_0-contour can be closed in the right half-plane to give zero because of the e^{Ey_0} factor in (76). Hence we would re-obtain the result that we should keep only $\delta^{\pm}$ in (77). Another alternative would be to exploit the analogy between y_0 and a collective coordinate[32], although we shall not pursue this interesting possibility here.

The Feynman rules we need are identical to the usual RFT rules in $(E,\underline{k})$ space[9,18] except for the graphs involving the ϕ^2 and $\bar{\phi}^2$ vertices. These vertices have "coupling constant" Δ_0 and a form factor $\delta^{\pm}$, as appropriate, for the energy and (zero) momentum entering the vertices. We shall now formulate a simple general prescription for the translation invariant part of the perturbation expansion by studying some of the simplest diagrams in detail. We represent the ϕ^2 and $\bar{\phi}^2$ vertices as shown in Fig. 5.2. Remembering that we are to divide out $U_s(g,0)$ and $U_s(0,\bar{g})$ from $U_s(g,\bar{g})$, the simplest graph involving the new vertices is shown in Fig. 5.3. Labelling the energies and momenta as shown, we obtain for the complete diagram (apart from $i\pi$ factors)

$$\Delta_0^2 \int_{-i\infty}^{+i\infty} dE_\alpha\, dE_\beta\, e^{E_\alpha y_1 - E_\beta y_2} \int_{-i\infty}^{+i\infty} dE \int d^2\underline{k}\, e^{i\underline{k}\cdot(\underline{x}_1-\underline{x}_2)}$$

$$\times\Big[(E_\alpha - \alpha_0'\underline{k}^2 + \Delta_0)(E_\beta - \alpha_0'\underline{k}^2 + \Delta_0)(E - E_\alpha - E_\beta - \alpha_0'\underline{k}^2 + \Delta_0)(E - E_\alpha + \varepsilon)(E - E_\beta + \varepsilon)\Big]^{-1} \tag{79}$$

If there is a translation invariant part to this diagram, then the E integration should produce a δ-function in $(E_\alpha - E_\beta)$.

The singularities and integration contour in the E-plane are shown in Fig. 5.4. It is clear that in the form (78) there is no singularity at $E_\alpha = E_\beta$, since there is no pinching of the E-contour by the poles shown. However, we can rewrite (78) in another form, which, although it will not generate a δ-function in this diagram, will in more complicated diagrams.

For $-y_1$, $y_2 > 0$ we can close the E_α and E_β contours in the right half-planes so that they enclose the real axis as shown in Fig. 5.5. If we also close the E-contour in the right half-plane we arrive at the situation shown in Fig. 5.6. We have plotted the movement of the various poles in the E-plane during the E_α and E_β integrations. There is still no pinching of the E-contour when $E_\alpha = E_\beta$ since the pole at $E_\alpha + E_\beta + \alpha'\underline{k}^2 - \Delta_0$ is always to the right of the poles at E_α and E_β (remembering $\Delta_0 < 0$). We conclude that this diagram has no translation invariant part. The same analysis generalizes immediately to all diagrams of the form shown in Fig. 5.7.

The above discussion would have been quite different if there had been at least one singularity in the right half E-plane which was independent of both E_α and E_β. We shall see this as we discuss further diagrams. Consider next the diagrams of Figs. 5.8 and 5.9. The complete expression for Fig. 5.8 is

$$\Delta_0 r_0^2 \int_{-i\infty}^{+i\infty} dE_\alpha\, dE_\beta \int d^2\underline{k}\, \frac{e^{E_\alpha y_1 - E_\beta y_2}\, e^{i\underline{k}\cdot(\underline{x}_1 - \underline{x}_2)}}{(E_\beta - \alpha_0'\underline{k}^2 + \Delta_0)(E_\alpha - \alpha_0'\underline{k}^2 + \Delta_0)(E_\beta - E_\alpha + \varepsilon)}$$

$$\times \int dE'\, d^2\underline{k}' \left[(E' - \alpha_0'\underline{k}'^2 + \Delta_0)(E_\alpha + E' - \alpha_0'(\underline{k}+\underline{k}')^2 + \Delta_0)(E_\beta - E_\alpha - E' - \alpha'\underline{k}'^2 + \Delta_0)\right]^{-1} \qquad (80)$$

The singularity structure in the E_β-plane, after the loop integration, is shown in Fig. 5.10. The two Reggeon cuts in the E_β and $E_\beta - E_\alpha$ channels appear at $\alpha'(\underline{k}^2/2) - 2\Delta_0$ and $E_\alpha - 2\Delta_0$, respectively. If we again close both the E_α and E_β contours in the right half-plane, we obtain Fig. 5.11.

There is now a pinch of the E_β contour by the pole, at $E_\beta = E_\alpha$, on the two parts of the E_α contour. This gives a contribution to I_2 which can be written in the form

$$\int dE_\alpha\, dE_\beta\, \delta(E_\alpha - E_\beta)\, e^{-E_\beta(y_2 - y_1)}\, F(E_\beta) \qquad (81)$$

where the integration is along the real axis and $F(E_\beta)$ is obtained by taking the discontinuity of the integrand due to singularities in the E_β-channel, i.e. singularities whose location depends only on E_β and the momentum $\underline{k}$ in this channel (apart from α_0', Δ_0, etc.).

Expression (81) is in general the same result as is obtained by simply replacing $[E_\beta - E_\alpha + \varepsilon]^{-1}$ by $\delta(E_\beta - E_\alpha)$ provided that the singularities in the E_β-channel are the only ones in the E_β-plane which are reflected into the E_α-plane by the replacement. For the integral (80) this is the case, since it is clear from Fig. 5.10 that all the singularities in the E_β-plane are either in the E_β-channel or the $(E_\beta - E_\alpha)$ channel.

The analysis of the diagram of Fig. 5.9 is almost identical to that of Fig. 5.8. The only difference is that in the E_β-plane the branch-point at $\alpha'(k^2/2) - 2\Delta_0$ does not appear but instead there is an extra pole at $E_\beta = E_\alpha - \Delta_0$. In this case only the pole at $E_\beta = \alpha_0'\underline{k}^2 - \Delta_0$ contributes to (81).

We consider next the diagram of Fig. 5.12. We shall not give the complete expression for this diagram but on the basis of the previous examples simply look at the singularity structure in the E-plane after the internal loop integration. From Fig. 5.13 it is clear that closing the E_α and E_β contours in the right half-plane will again produce a δ-function which picks out only the discontinuity in the E-channel. That is, only the branch-point at $\alpha_0'(k'/2) - 2\Delta_0$ is involved in the pinch generating the δ-function. The coefficient of the δ-function is again the same as obtained by replacing the poles at $E - E_\alpha + \varepsilon = 0$ and at $E - E_\beta + \varepsilon = 0$ by δ-functions since the only singularities in the E-plane besides that in the E channel are in the $E - E_\alpha$ and $E - E_\beta$ channels.

As final simple examples we consider the diagrams of Fig. 5.14 whose E-plane is shown in Fig. 5.15, and the diagram of Fig. 5.16 whose E-plane is shown in Fig. 5.17. There is no branch-point in Fig. 5.14 which is independent of E_α and E_β, while in Fig. 5.17 there is one. The diagram of Fig. 5.14 has no discontinuity in the E-channel. Hence no $\delta(E_\alpha - E_\beta)$ is generated and there is no translation invariant part to the diagram. That of Fig. 5.16 does have such a translation invariant part, but it is no longer given by replacing the poles at $E - E_\alpha + \varepsilon = 0$ and $E - E_\beta + \varepsilon = 0$ by δ-functions. It is first necessary to eliminate the singularity at

$$E - E_\alpha - E_\beta = \alpha_0' \underline{k}^2/2 - 2\Delta_0 \tag{82}$$

which would be reflected in the E_α and E_β planes by this replacement.

From our simple examples we can formulate the following rule -- a diagram containing a $\bar{\phi}^2$ (ϕ^2) vertex has a translation invariant part if, and only if, there is a discontinuity in the channel which represents the total energy present before (after) the insertion. Further in a diagram which has such a discontinuity the translation invariant part is given by keeping only this discontinuity in the associated energy plane. To make these statements precise in a way which will apply to an arbitrary diagram, we must go to a rapidity-ordered or Rayleigh-Schrödinger form of the perturbation expansion.

The Rayleigh-Schrödinger expansion[14] is obtained by performing all energy integrations in the diagrams of the present expansion by picking up propagator poles. In general one diagram in the original expansion gives several in this new expansion each of which corresponds to a possible rapidity ordering of the vertices. The rules for writing down the new diagrams are that we write the same factors for the vertices as in the old expansion (except that $r_0 \to \sqrt{2\pi}\, r_0$), but we write propagators for each multi-Pomeron intermediate state. For an n-Pomeron state we write

$$i \left[E - n(-\Delta_0) - \alpha_0' \sum_{i=1}^{n} \underline{k}_i^2 \right]^{-1} \tag{83}$$

where E is the energy associated with the state. We impose momentum conservation at all vertices and integrate over all loop momenta. We also integrate over the energies entering the ϕ^2 and $\bar{\phi}^2$ vertices with the $\delta^{\pm}$ form factors as before.

There is a further manipulation of the Rayleigh-Schrödinger expansion which we also need. We sum over all graphs that differ only by the rapidity-ordering of interactions that are in parts of the graphs that are disconnected either from the earliest interaction to $y = +\infty$ or from $y = -\infty$ to the latest interaction. The sum of such graphs is then given by a single graph in which the product of propagators for each of the disconnected states is replaced by a product of propagators of the form (81) for each intermediate state in the connected sub-graphs. This process is illustrated in Fig. 5.18. The product of propagators represented by dotted lines in the four graphs summed over is replaced by the product of propagators for the dotted lines in the final graph. The energy that appears in the new propagators is the energy flowing through the connected sub-graphs. We call this form of the Rayleigh-Schrödinger expansion "the rapidity-ordered expansion", and refer to the graphs as rapidity-ordered graphs.

It is now straightforward to generalize the above analysis of the simplest graphs. Every rapidity-ordered graph has a discontinuity in a certain channel if, and only if, it contains a propagator in that channel. In a graph containing many ϕ^2 and $\bar{\phi}^2$ vertices an over-all $\delta(E_\alpha - E_\beta)$ will be generated if there is a multiple pinch of the integration over all energies entering and leaving through the vertices. This requires singularities in all corresponding total energies. This is equivalent to saying there must be a discontinuity, and hence a propagator, in every channel representing the energy present after (before) the production (absorption) at a $\bar{\phi}^2$ (ϕ^2) vertex. Alternatively, there must be at least one triple Pomeron interaction which is connected to the main body of the graph, between each ϕ^2 and $\bar{\phi}^2$ vertex and the next.

Consequently all rapidity-ordered graphs which do not have this property have no translation invariant parts. Those graphs which do will necessarily have propagators and hence singularities in only the total energy channel and the energies incoming at the ϕ^2 and $\bar{\phi}^2$ vertices. In these graphs the replacement of all poles in the vertex form factors by δ-functions is allowable since this simply eliminates all singularities in vertex energy channels and leaves only the total energy channels. Hence we arrive at the final prescription for calculating the translation invariant, y_0-independent perturbation expansion.

$S_{g\bar{g}}$ is given by the sum of all rapidity-ordered graphs satisfying the requirement that there be a propagator in the total energy after (before) a $\bar{\phi}^2$ (ϕ^2) vertex. The graphs are evaluated with zero energy and momentum entering at the $\bar{\phi}^2$ and ϕ^2 vertices (<u>no</u> form factors) and with a coupling constant Δ_0 for these vertices.

The sum of graphs giving the propagator (i.e. the $g\bar{g}$ term in $S_{g\bar{g}}$) is shown in Fig. 5.19. The sum is over all number and orderings of the ϕ^2 and $\bar{\phi}^2$ vertices. Each circle represents a Green's function calculated without the ϕ^2 and $\bar{\phi}^2$ vertices and each set of hatched lines represents a sum over propagators of the form (83) (including no propagator).

An equivalent description of the set of graphs giving $S_{g\bar{g}}$ is to say draw all the usual RFT Rayleigh-Schrödinger graphs for a theory with bare intercept $-\Delta_0$ and with all bare Pomeron interactions λ_{1N}, N = 1, 2, ... producing N Pomerons from one. λ_{11} will, of course, simply modify the bare intercept to $\Delta_0' = -\Delta_0 + \lambda_{11}(0)$. The λ_{1N} are momentum dependent vertices given by the sum over graphs shown in Fig. 5.20. Each λ_{1N} therefore contains a complete Green's function evaluated with only triple Pomeron interactions. Again the $\bar{\phi}^2$ vertex has zero energy and momentum entering. This implies that the energy is set to zero in all the propagators appearing in λ_{1N}.

It is interesting to note now that the graph of Fig. 5.16 discussed above gives the two rapidity-ordered graphs shown in Fig. 5.21. Only the graph (i) has a total energy propagator and so contributes to $S_{g\bar{g}}$. The second graph (ii) is not translation invariant. It was because it was necessary to subtract this graph from the complete graph of Fig. 5.16 that we were unable to simply set the vertex form factor poles equal to δ-functions in our earlier discussion.

Finally we discuss the convergence of the expansion for $S_{g\bar{g}}$. This parallels closely the ABSS discussion of their expansion. Consider first graphs not involving the ϕ^2 or $\bar{\phi}^2$ vertices. Each successive term in the perturbation series will have an extra factor of r_0^2, two extra propagators and one extra momentum loop integration. When $-\Delta_0$ is very large, the propagators are of order Δ_0, while the loop integration is of order Λ -- the cut-off in the momentum integrals. Thus each term will be down from the preceding one by the dimensionless factor δ_0 where

$$\delta_0 = \frac{r_0^2 \Lambda}{\Delta_0^2} \tag{84}$$

Now consider the graphs involving the $\bar{\phi}^2$ and ϕ^2 vertices. The addition of such a vertex to a diagram always adds one extra loop integration, two r_0 vertices, three propagators and a factor Δ_0 for the vertex. Hence these graphs are also an expansion in the parameter δ_0. The expansion should therefore be convergent (or rather Borel summable as we discuss in the next section) when $\delta_0 \sim 0$ and in particular when $-\Delta_0 \to \infty$. This region is, of course, far from the phase transition region where $\delta_0 \sim 0(1)$. However, it is sufficient for us to conclude that beyond the critical point the leading singularity in the E_α-plane is a simple pole since we can write for the new propagator $\tilde{\Gamma}(E,\underline{k}^2)$

$$\tilde{\Gamma}(E,\underline{k}^2) = i\left[E - \alpha_0' \underline{k}^2 + \Delta_0 + O\left(\frac{r_0^2 \Lambda}{\Delta_0^2}\right)\right]^{-1} \tag{85}$$

Also the graphs of Fig. 5.19 satisfy perturbative Reggeon unitarity (as is clear from the fact that we can write them in the form of the usual RFT expansion with the vertices λ_{1N}). Therefore the expansion for $S_{g\bar{g}}$ that we have obtained from (41) satisfies Reggeon unitarity and also t-channel unitarity.

Note that the non-translation invariant graphs that we have dropped cannot be classified into powers of δ_0 since such graphs contain arbitrary numbers of $\bar{\phi}^2$ and ϕ^2 vertices without any accompanying triple Pomeron vertices. Hence we cannot begin to prove the convergence of this part of the complete expansion. We must simply assume that we have extracted the complete invariant part of $S_{g\bar{g}}$ so that the non-invariant terms in the expansion do not sum up (in some sense) to something which has an invariant part.

6. COMMENTS AND DISCUSSION

In this section we shall make some brief general comments on the theory we have developed in the previous sections and then discuss the relation of our results to the other proposed solutions to the problem.

First we note that our new Green's functions have no fixed $\underline{k}^2$ singularities on the physical sheet. They have only the usual multi-Pomeron cuts. As a result, the new Green's functions go to zero at large impact parameter as did the old Green's functions. This might seem strange since we have made a field shift which is constant in the impact parameter. The reason is that the only part of the shift that was important in extracting the translation invariant part of the perturbation expansion was the asymptotic part at $y = \pm\infty$. Hence if we keep y finite, but go to large impact parameter, we see no effect from the shift.

It is interesting that by relating the S-matrix in rapidity space above and below the critical point we have in effect related the integrals over the imaginary part in the angular momentum or energy planes. This is fortunate since we expect that in E-space the Green's functions will develop asymptotically divergent parts which lead to the divergence of the normal Fourier or Sommerfeld-Watson integral.

In Section 3 we noted that the one-dimensional classical solution for fixed g/r_0, $\bar{g}/r_0$, $\Delta_0 > 0$, is singular at some $Y = Y_c$, since for $Y < Y_c$ the solution is not unique. Also as $\Delta_0 \to 0$ this singularity approaches $g, \bar{g} = 0$, $Y = \infty$. Since our understanding of the phase-transition in higher dimensions is so closely related to the one-dimensional problem near $\Delta_0 = 0$, it seems likely that a similar singularity (or singularities -- with or without impact parameter dependence) will be present when $\Delta_0 \sim \Delta_{0c}$ in the complete S-matrix in higher dimensions. If this is the case then there are many important implications. Firstly the expansion of $S_{g\bar{g}}$ in powers of g and $\bar{g}$, with the coefficients given by the Green's functions, will not be convergent in the neighbourhood of $\Delta_0 = \Delta_{0c}$, $Y = \infty$. Hopefully it will still be an asymptotic expansion, valid for g/r_0, $\bar{g}/r_0 \lesssim 0$. Nevertheless if we are to apply the large Y asymptotic expansion for $S_{g\bar{g}}$ obtained from that for the Green's functions, the sign of g/r_0 and $\bar{g}/r_0$ must be correct. Similarly if the finite Y analyticity in Δ_0, on which we have relied, is to hold, we must also fix g/r_0, $\bar{g}/r_0 < 0$. Fortunately this constraint is physically sensible since this sign is also fixed in the same way if s-channel unitarity in the form of the AGK cutting rules is imposed on the theory[15,33]) (when $\Delta_0 > 0$).

Since the shifted perturbation expansion (including the expansion in Green's functions) converges around $Y = \infty$ for $\Delta_0 \ll \Delta_{0c}$ and the original expansion converges around $Y = \infty$ for $\Delta_0 \gg \Delta_{0c}$, the two expansions converge on different sides of the supposed singularity. Hence Green's functions of the two theories will not be related, although the S-matrices are. Therefore all the multi-Pomeron couplings to the external particles (that is all powers of g and $\bar{g}$) are essential for relating the theories above and below the critical point. We chose the eikonal couplings since they are the simplest couplings known to be consistent with s-channel unitarity. It seems to us therefore that the inclusion of all multi-Pomeron couplings and the constraint on the signs of g and $\bar{g}$ can be regarded as constraints of s-channel unitarity which, although not required by the t-channel unitarity formulation of the RFT, are essential to ensure that the $\alpha_0 > \alpha_{0c}$ problem has a sensible solution. It would hardly be surprising if s-channel unitarity was essential in a situation where the Froissart bound might, in principle, be violated.

It will be important to eventually check whether the theory we have derived is completely consistent with s-channel unitarity. In particular we can ask whether the AGK rules can be applied for discontinuities. Since we believe that these rules are a general consequence of multi-Regge theory and the singularity structure of multiparticle amplitudes[10], this should be the case. In any case it will obviously be necessary to extend our theory to multiparticle amplitudes and to inclusive cross-sections, as has been done for the theory below the critical point.

It is perhaps worth noting that we have derived a Rayleigh-Schrödinger expansion for the shifted theory rather than a Lagrangian perturbation expansion. Of course, we can expand all vertices as power series in the momenta, introduce energy integrations, and write a corresponding Lagrangian involving all derivative interactions. Nevertheless it seems to be consistently the case that the Rayleigh-Schrödinger expansion is best suited to new problems in the RFT. Since it is this expansion which was originally derived from the hybrid Feynman graph expansion and which is also most directly related to Reggeon unitarity, we might consider whether the Lagrangian formalism has <u>any</u> advantages in the RFT.

We begin our comparison with other work on the subject by discussing the dimensional dependence of the convergence of our perturbation expansion. It is now generally understood that the convergence of perturbation expansions in conventional quantum field theories is controlled by instanton solutions of the corresponding classical equations of motion[24,25,34,35]. An expansion around a vacuum which is a minimum of some classical potential will be Borel summable and so define a theory only if there does not exist a real instanton having finite action relative to the minimum. The non-existence of spontaneous symmetry breaking of a discrete symmetry in one dimension (due to the tunnelling effect) and of a continuous symmetry in two dimensions can both be understood as due to the existence of instantons[25].

In the RFT it seems that due to the non-Hermiticity of the interacting fields, we must consider both instantons and stationary points of the classical potential on the same basis as potential "vacua" around which the path-integral may be expanded. [Perhaps we should remark that it is because we consider an instanton vacuum that we avoid the "no-go theorem" of Ellis and Savit[36] that says a field shift to a new vacuum is not possible in the RFT.] Presumably the convergence of the perturbation expansion around any vacuum is likely to not be Borel summable whenever there exist other potential vacua with a finite action relative to the one considered.

As we discussed in Section 3, in one dimension (zero impact parameter dimension) there exist several instantons and stationary points, all of which in fact have finite action relative to each other. Consequently the expansion around any one vacuum and in particular our instanton is likely to not be Borel summable. We conclude then that it is unlikely that we can draw any conclusions from our perturbation expansion in one dimension and so there is not any potential conflict of our results with the known results for this case[37,38]. This is directly analogous to the failure of the shifted perturbation expansion in one-dimensional $\lambda\phi^4$ due to an instanton, or equivalently the tunnelling effect[25,35].

In higher dimensions we can regard all the one-dimensional solutions as impact parameter independent solutions. In this case, however, all the other solutions have infinite action relative to our instanton and so our perturbation expansion should be Borel summable. Again this would be analagous to the behaviour of the shifted perturbation expansion in higher dimensions in $\lambda\phi^4$. The difference, of course, is that the path-integral in $\lambda\phi^4$ is not analytic at the phase-transition point because of the "infinite volume" of the theory. That is, finite distance Green's functions always involve infinite volume interactions because of the presence of vacuum processes.

The analyticity of the path-integral defining $S_{g\bar{g}}$ is the vital point that has enabled us to avoid the procedure, suggested by the analogy with $\lambda\phi^4$, which ABSS used to continue to $\Delta_0 < \Delta_{0c}$. This is to make the continuation in the presence of an external source. Such a continuation is, in principle, not unique since an arbitrary extra variable has been introduced into the theory. If the source has no physical interpretation it is impossible to be sure the correct continuation has been made. In fact it should be clear from Sections 2 and 3 that the ABSS solution is not the correct analytic continuation of the theory.

The comparison of our results with those of ACLM and Cardy in the spin model is more complicated in that these authors study the space of states suggested by the model and this is difficult to compare with the path-integral formalism. Nevertheless the following remarks suggest a comparison. We use Cardy's notation but the results are the same as obtained by ACLM. For $\alpha_0 > \alpha_{0c}$ there exists a second state $|1\rangle$ which is degenerate with the vacuum $|0\rangle$ and both are zero eigenstates of the Hamiltonian. The phase transition is characterized by the occurrence of matrix elements

$$\langle 0|\psi|1\rangle = \langle 1|\bar{\psi}|0\rangle = \frac{2|\Delta_0|}{r_0}\sigma \tag{86}$$

$$\langle 1|\psi|1\rangle = \langle 1|\bar{\psi}|1\rangle = i\frac{2|\Delta_0|}{r_0}\sigma \tag{87}$$

where $\sigma = 1$ at $\alpha_0 = \infty$ and is zero for $\alpha_0 < \alpha_{0c}$. All other matrix elements are zero for all α_0. Because of (86) and (87), if we define

$$|\emptyset\rangle = |0\rangle + i\,|1\rangle \tag{88}$$

then within the two-dimensional space of $|0\rangle$ and $|1\rangle$, we have

$$\psi\,|\emptyset\rangle = \frac{2|\Delta_0|\sigma}{r_0}|\emptyset\rangle \quad , \quad \bar{\psi}\,|\emptyset\rangle = 0 \tag{89}$$

and equivalently

$$\langle\emptyset|\bar{\psi} = \frac{2|\Delta_0|\sigma}{r_0}\langle\emptyset| \quad , \quad \langle\emptyset|\psi = 0 \tag{90}$$

$\langle\emptyset|$ is the left-hand eigenstate of the Hamiltonian corresponding to $|\emptyset\rangle$ and is not the Hermitian conjugate. In fact

$$\langle\emptyset| = \langle 0| + i\langle 1| \tag{91}$$

where $\langle 0|$ and $\langle 1|$ are defined so that

$$\langle 0|0\rangle = \langle 1|1\rangle = 1 \quad , \quad \langle 0|1\rangle = \langle 1|0\rangle = 0 \tag{92}$$

Consequently for a spectral decomposition of the Hamiltonian, $|\emptyset\rangle$ has norm

$$\langle\emptyset|\emptyset\rangle = \langle 0|0\rangle - \langle 1|1\rangle = 0 \tag{93}$$

A consequence of (89) and (90) is that if we consider

$$\langle\emptyset|\,O\,|\emptyset\rangle \tag{94}$$

where O is some normal-ordered operator function of ψ and $\bar{\psi}$, then no zero energy eigenstate appears as an intermediate state in this matrix element.

The S-matrix is defined by ACLM to be

$$\langle 0|\exp\left[-\int dy\, d^2\underline{x}\; H_{g\bar{g}}(\psi,\bar{\psi})\right]|0\rangle \tag{95}$$

and from (86) and (87) we see that $|1\rangle$ will appear as an intermediate state in this formula. This effect is responsible for the $(\ln s)^2$ rise of the total cross-section obtained by ACLM. However, if instead we define the S-matrix as

$$\langle\emptyset|\exp\left[-\int dy\, d^2\underline{x}\; H_{g\bar{g}}(\psi,\bar{\psi})\right]|\emptyset\rangle \tag{96}$$

then there is no zero energy intermediate state. Consequently the total cross-section will go to zero asymptotically as in our theory. Note also from (89) and (90) that $|\emptyset\rangle$ is a right-hand eigenstate of ψ and $\bar{\psi}$ whose eigenvalues (at $\Delta_0 = -\infty$) are identical with the initial values of ψ and $\bar{\psi}$ in our instanton χ_{00}. Similarly, the left-hand eigenvalues of $\langle\emptyset|$ correspond to the final values of ψ and $\bar{\psi}$ in χ_{00}. As we discussed in Section 2, in the holomorphic representation, our path-integral can be regarded as the matrix element of the evolution operator between eigenstates of ψ and $\bar{\psi}$ with the eigenvalues given by the boundary-values specified for these fields. For $\Delta_0 \to -\infty$, we have shown that the boundary-values in the path-integral are those of χ_{00}.

From these observations it seems clear that the analogue of our theory in the spin model would be to define the S-matrix with the boundary conditions of (96) rather than (95). In fact if we demand, within the spin model, that we have a space of states consistent with Reggeon unitarity, that is a unique vacuum with only finite energy Reggeon excitations, then the only possibility is to define $|\emptyset\rangle$ as a new vacuum. In this sense the spin model derivation of the matrix elements (86) and (87) can be regarded as supporting our results. We note that this is basically the point of view one adopts when deducing from the spin model for $\lambda\phi^4$ that there is a degenerate ground state and consequent spontaneous symmetry breaking.

It seems natural also to relate the existence of the matrix elements of (86) and (87) to the existence of the instanton structure of Section 3 with the off-diagonal elements $\langle 0|\psi|1\rangle = \langle 0|\psi|\emptyset\rangle$ and $\langle 1|\bar{\psi}|0\rangle = \langle\emptyset|\bar{\psi}|0\rangle$ identified with the instantons connecting the origin to our instanton. Both the matrix elements and the instantons can be viewed as providing the smooth transition to the new vacuum. (Although it is only through our analytic continuation arguments that the transition is seen to actually take place.) It would be consistent with the argument that the infinite action of these instantons relative to ours makes our perturbation expansion convergent, if the correct space of states in the spin model was the zero norm states. The old vacuum, having infinite relative norm, would not be in this space of states and so clearly would not contribute to the completeness relation for the eigenstates of the Hamiltonian.

Finally, we note that in the remainder of Refs. 1-7 exact solutions to the problem are not claimed. Cardy's approach[2)] of summing the RFT diagrams by making an eikonal sum between every pair of interaction points in rapidity space is developed in Ref. 6 to obtain an exact integral equation for the "Froissaron" propagator in E and $\underline{k}$ space. However, the equation has not been solved exactly and there is no discussion of whether the approximate solution obtained is related to the theory in $\alpha_0 < \alpha_{0c}$ by analytic continuation.

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REFERENCES

1) J.B. Bronzan, Phys. Rev. D 9, 2347 (1974).

2) J.L. Cardy, Nuclear Phys. B75, 413 (1974).

3) H.D.I. Abarbanel, J.B. Bronzan, R.L. Sugar and A. Schwimmer, to be published in Phys. Rev. D (1976).

4) A. Czechowski, Nuclear Phys. B113, 323 (1976), CERN preprint TH 2217 (1976).

5) D. Amati, M. Ciafaloni, M. Le Bellac and G. Marchesini, Nuclear Phys. B112, 107 (1976).
D. Amati, G. Marchesini, M. Ciafaloni and G. Parisi, CERN preprint TH 2185 (1976).

6) M.S. Dubovikov, B.Z. Kopeliovich, L.I. Lapidus and K.A. Ter-Martirosyan, Dubna preprint D2-9789 (1976).

7) J.L. Cardy, SLAC-PUB-1784 (1976).

8) The original RFT papers are V.N. Gribov, Soviet Phys. JETP 26, 414 (1968), V.N. Gribov and A.A. Migdal, Soviet J. Nuclear Phys. 8, 583, 783 (1967). An introductory review of the subject is given in Ref. 9, while reviews covering most recent work can be found in Refs. 10 and 11.

9) H.D.I. Abarbanel, J.B. Bronzan, R.L. Sugar and A.R. White, Phys. Rep. C21, 119 (1975).

10) A.R. White, Lectures at the Les Houches Institute of Theoretical Physics (1975) (North-Holland Publ. Co., Amsterdam, 1975).

11) M. Moshe, Lectures at the Canadian Institute of Particle Physics Summer School, San Diego preprint UCSD-10P10-166 (1976).

12) V.N. Gribov, I. Ya Pomeranchuk and K.A. Ter-Martirosyan, Phys. Rev. B139, 184 (1965).

13) A.R. White, Nuclear Phys. B50, 93, 130 (1972); Phys. Rev. D 10, 1236 (1974).

14) R.L. Sugar and A.R. White, Phys. Rev. D 10, 4074 (1974).

15) M. Ciafaloni, G. Marchesini and G. Veneziano, Nuclear Phys. B98, 472, 493 (1976).

16) W. Lockman et al., contribution to the 18th Internat. Conf. on High-Energy Physics, Tbilisi, USSR (1976).

17) A.A. Migdal, A.M. Polyakov and K.A. Ter-Martirosyan, Phys. Letters 48B, 239 (1974); Zh. Eksper. Teor. Fiz. 67, 84 (1974).

18) H.D.I. Abarbanel and J.B. Bronzan, Phys. Rev. D 9, 2397 (1974).

19) A recent analysis taking energy conservation in particle production into account has concluded that it is possible that we are approaching the critical theory, A. Capella and A.B. Kaidalov, Nuclear Phys. B111, 477 (1976).

20) H. Cheng and T.T. Wu, Phys. Rev. Letters 24, 1456 (1970).

21) H.D.I. Abarbanel, J. Bartels, J.B. Bronzan and D. Sidhu, Phys. Rev. D 12, 2459, 2799 (1975).

22) W.R. Frazer and M. Moshe, Phys. Rev. D 12, 2370, 2386 (1975).

23) W.R. Frazer, H. Hoffman, H. Fulco and R.L. Suger, UCSB/UCSD preprint (1976).

24) A.M. Polyakov, Phys. Letters 59B, 82, 85 (1975).

25) J. Zinn-Justin, Lectures given at the Basko Polje Summer School, Saclay preprint D.Ph-T/76/99.

26) In addition to Refs. 5 and 7, the spin model has also been derived by R.C. Brower, M.A. Furman and K. Subbarao, UCSC preprint UCSC 76/112 (1976).

27) B. Stoeckly and D.J. Scalapino, Phys. Rev. B11, 205 (1975).

28) E.S. Abers and B.W. Lee, Phys. Rep. 9C, 1 (1973).

29) J. Zinn-Justin, Lectures at the Bonn Summer Institute (1974), Saclay preprint D.Ph-T/74/88.

30) L.D. Faddeev, Lectures at the Les Houches Summer School (1975) (North-Holland Publ. Co., Amsterdam, 1975).

31) D. Amati, L. Caneschi and R. Jengo, Nuclear Phys. B101, 397 (1975).

32) L.D. Faddeev and V.E. Korepin, CERN preprint TH 2187 (1976).

33) J.L. Cardy and P. Suryani, Phys. Rev. D 13, 1064 (1976).

34) L.N. Lipatov, JETP Letters 24, 179 (1976), and Leningrad preprints.

35) E. Brezin, J.C. Le Guillou and J. Zinn-Justin, Saclay preprints D.Ph-T/76/102, 119.

36) J. Ellis and R. Savit, Nuclear Phys. B104, 435 (1976).

37) V. Alessandrini, D. Amati and R. Jengo, Nuclear Phys. B108, 425 (1976); R. Jengo, Nuclear Phys. B108, 447 (1976).

38) J.B. Bronzan, J.A. Shapiro and R.L. Sugar, Phys. Rev. D 14, 618 (1976).

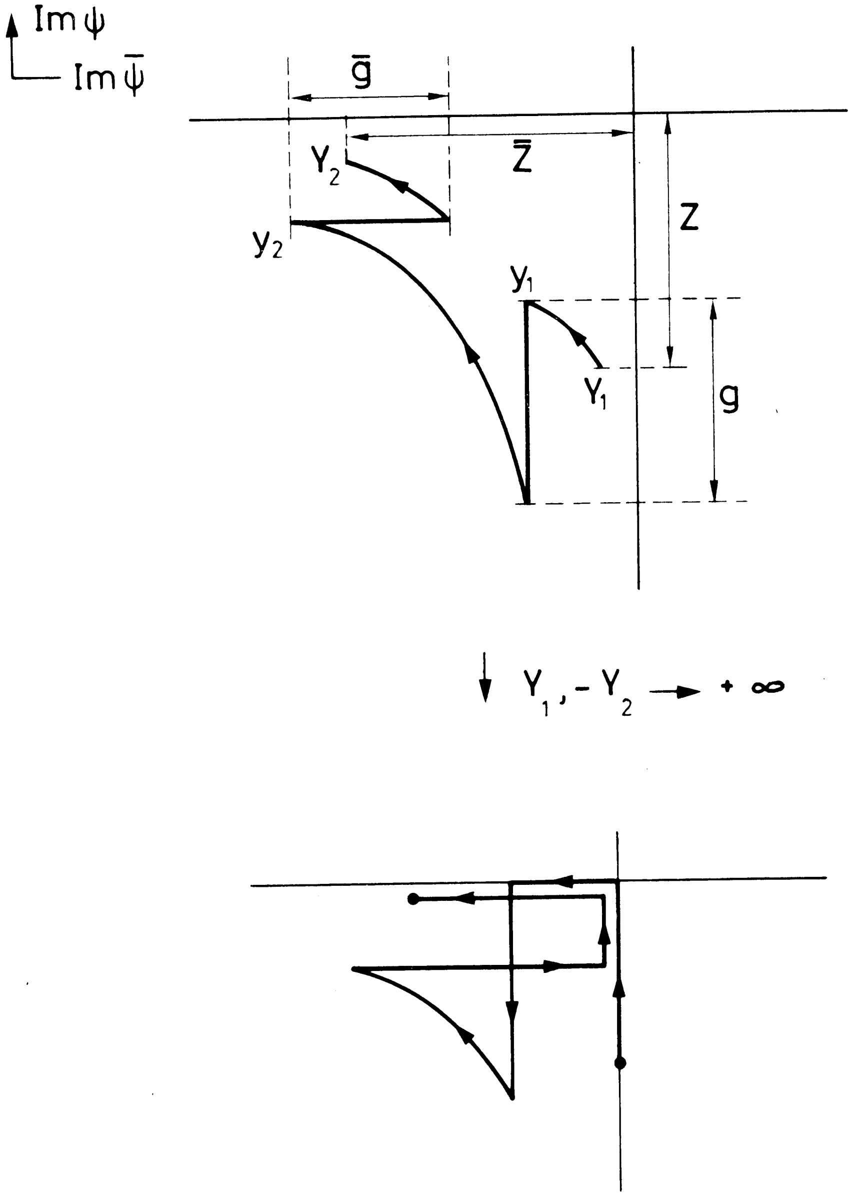
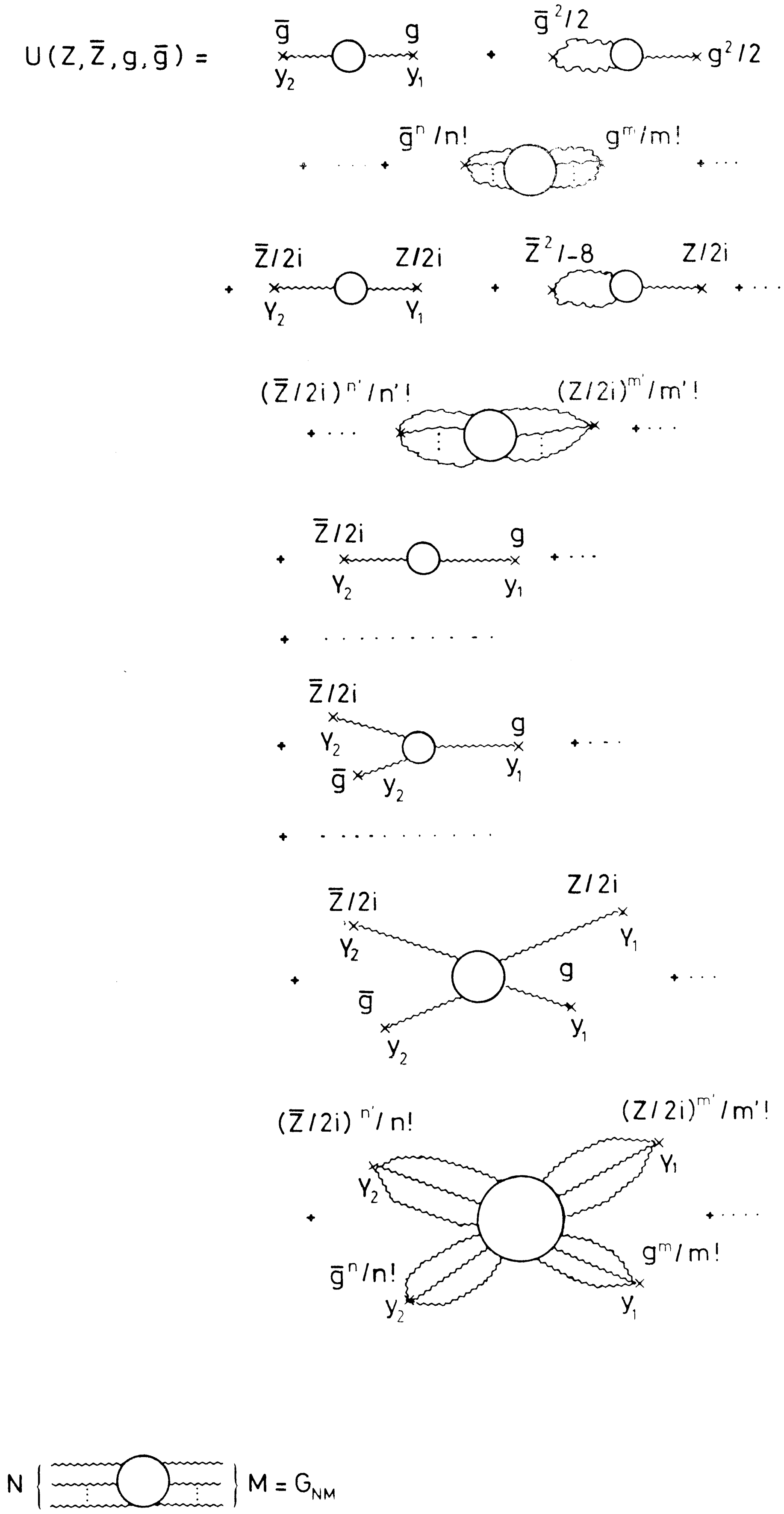


FIG. 2.1



U(Z, Z̄, g, ḡ) =
ḡ
g
y2
y1
ḡ²/2
g²/2
ḡⁿ/n!
gᵐ/m!
Z̄/2i
Z/2i
Y2
Y1
Z̄²/-8
(Z̄/2i)ⁿ'/n'!
(Z/2i)ᵐ'/m'!
(Z̄/2i)ⁿ'/n!
gᵐ/m!
ḡⁿ/n!
N
M = G_NM

FIG. 2.2

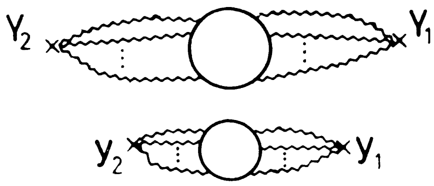


FIG. 2.3

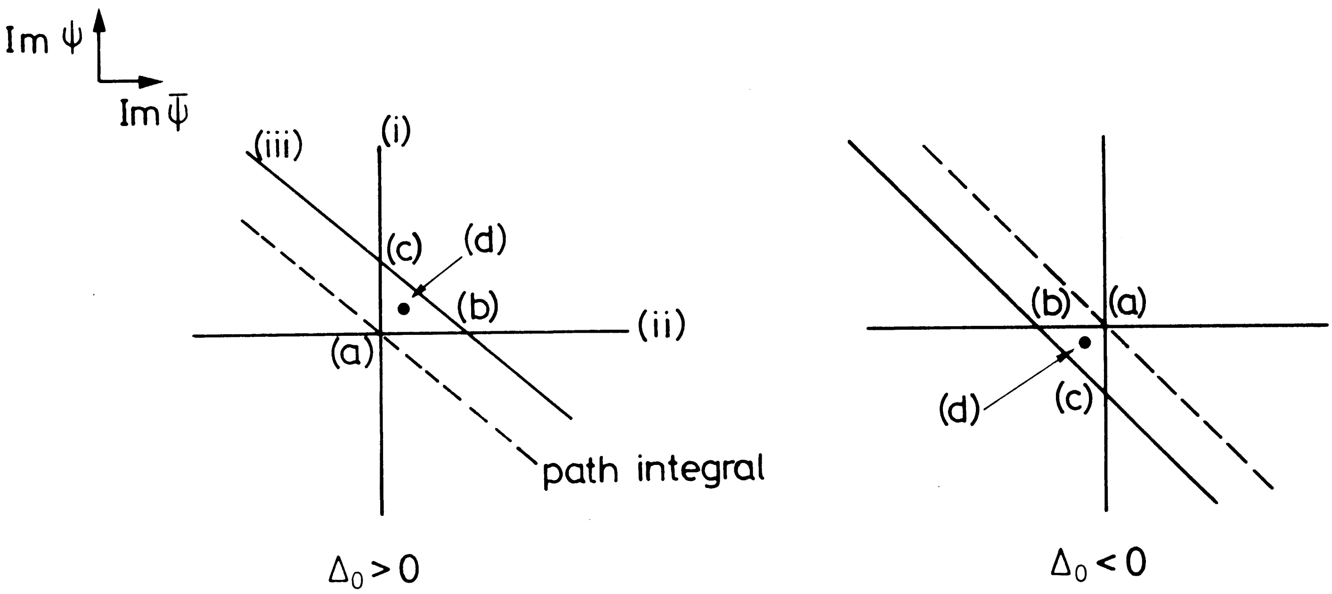


FIG. 3.1

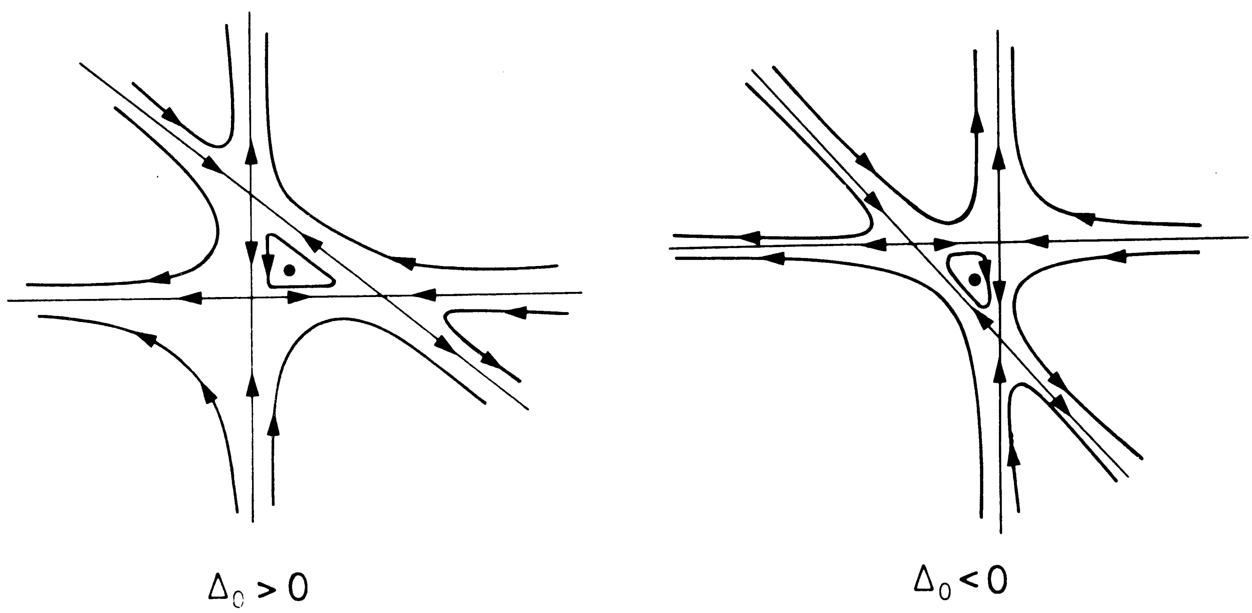


FIG. 3.2

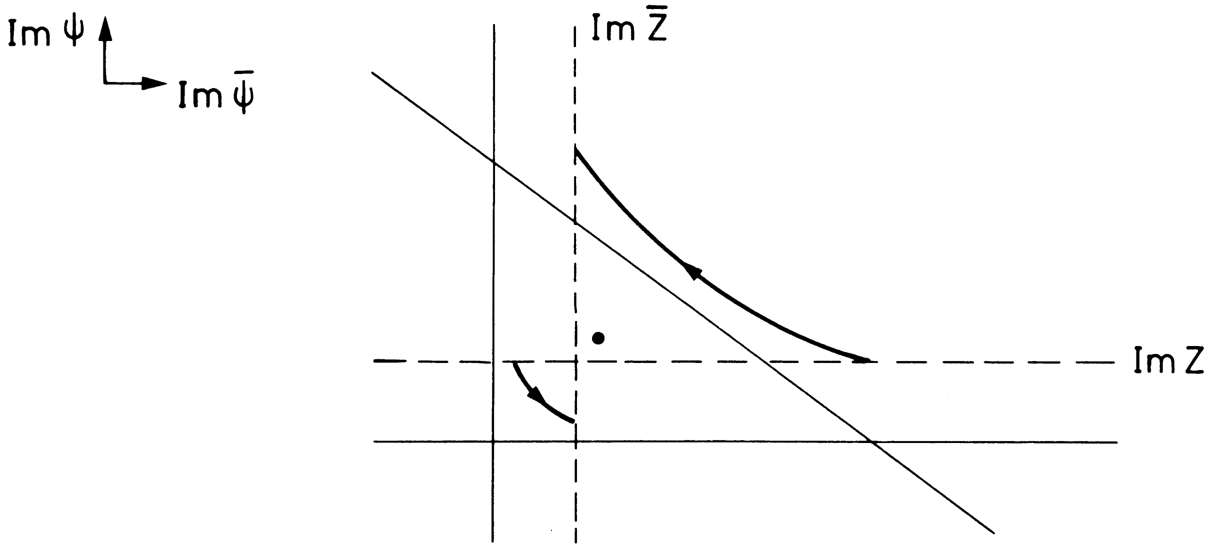


FIG. 3.3

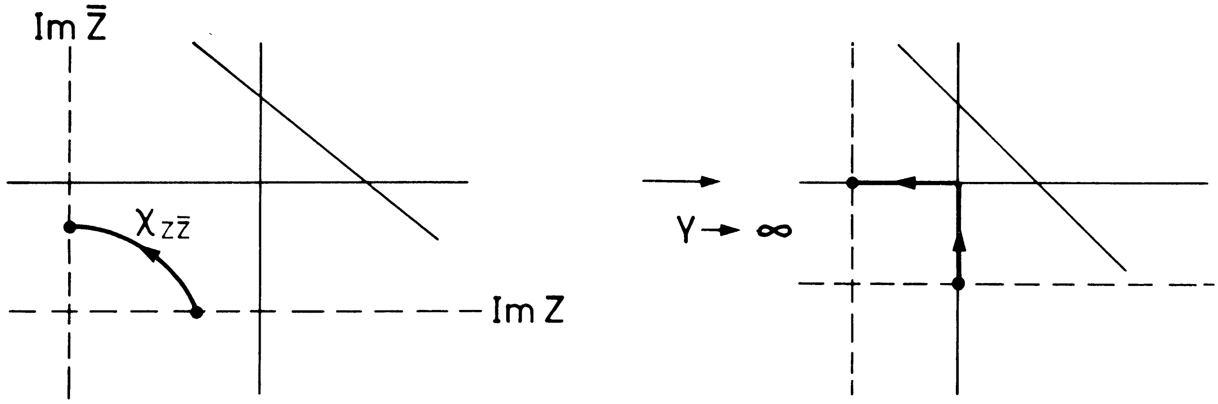


FIG. 3.4

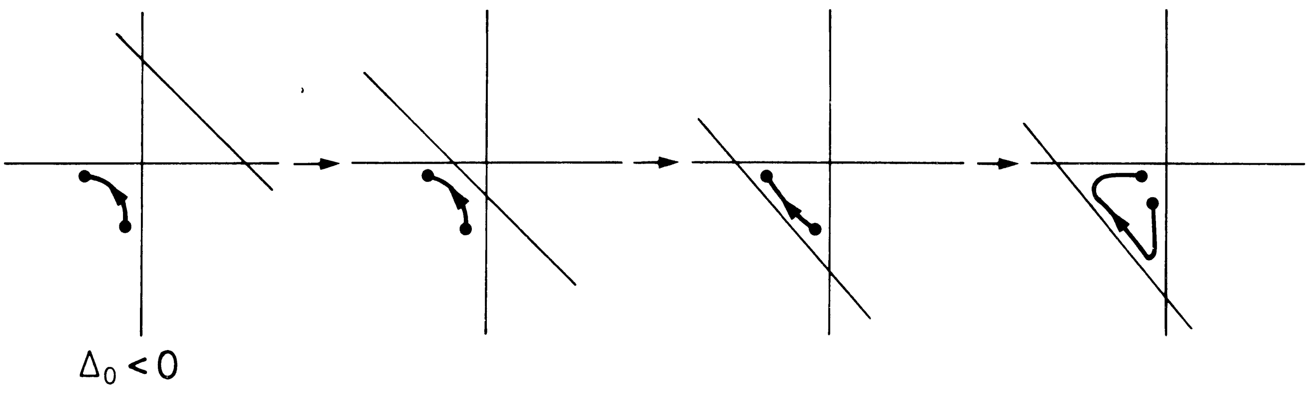


FIG. 3.5

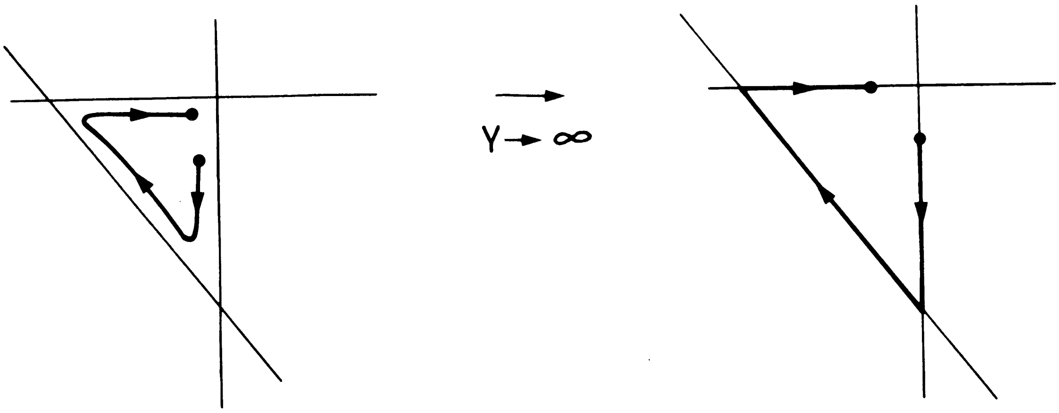


FIG. 3.6

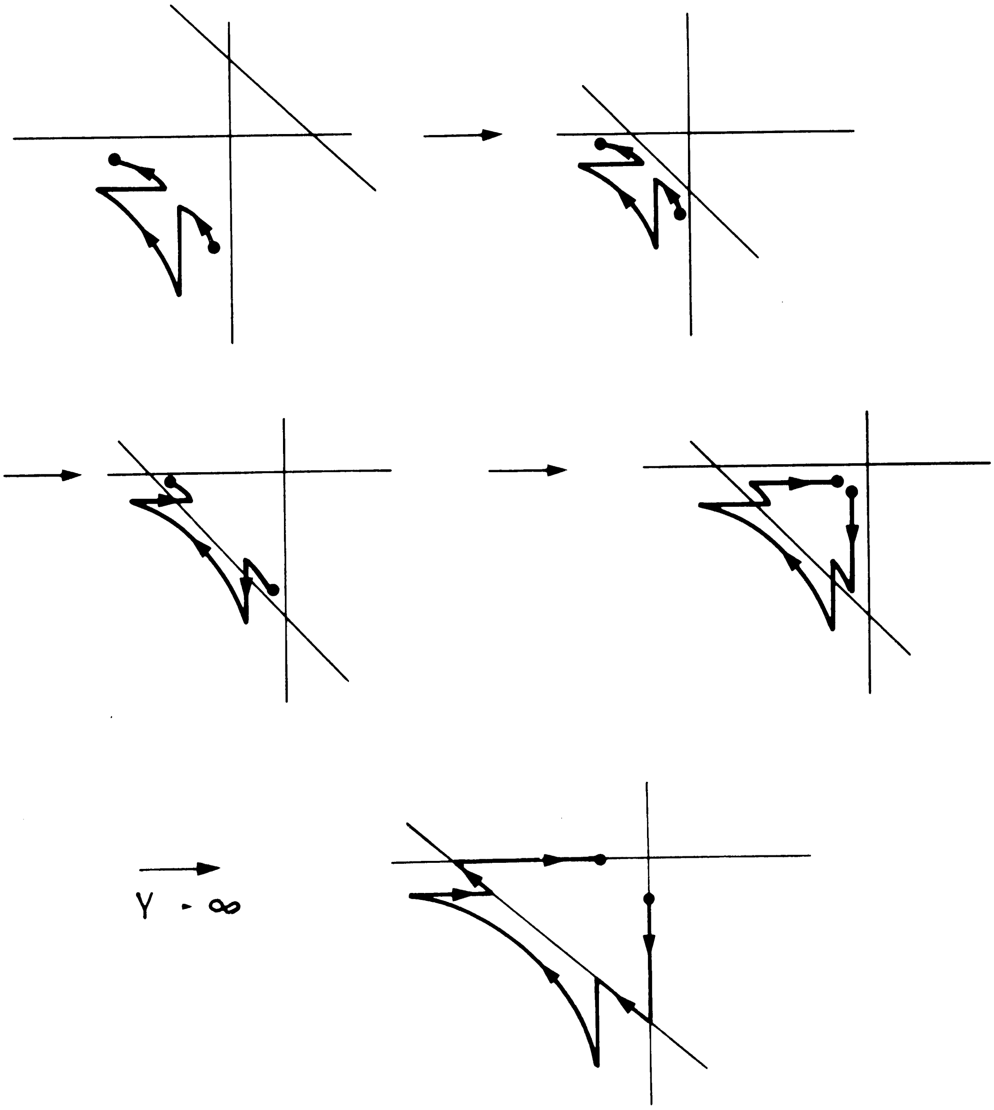


FIG. 3.7

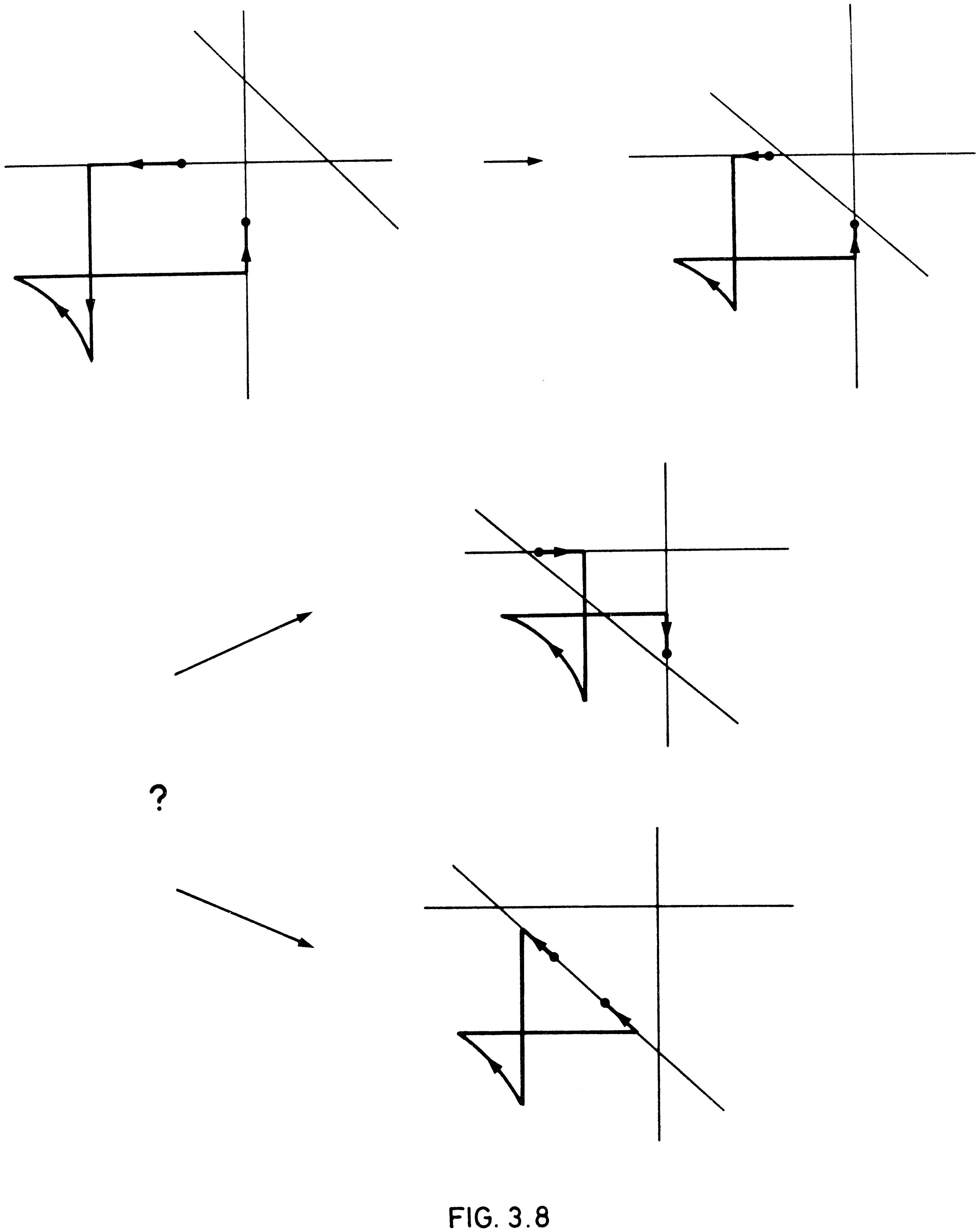


FIG. 3.8

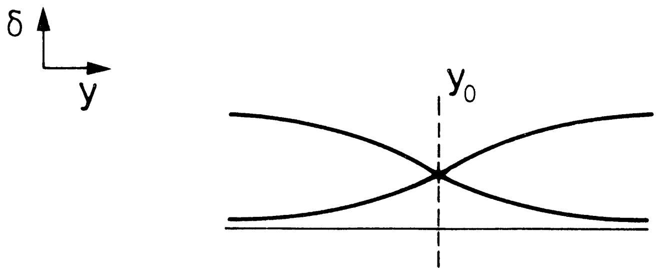
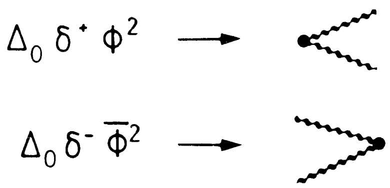


FIG. 5.1



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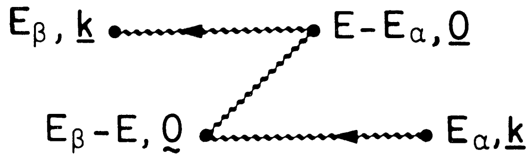


FIG. 5.3

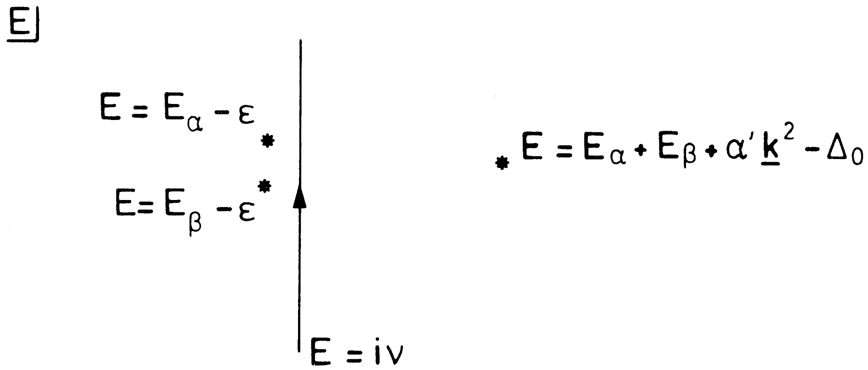


FIG. 5.4

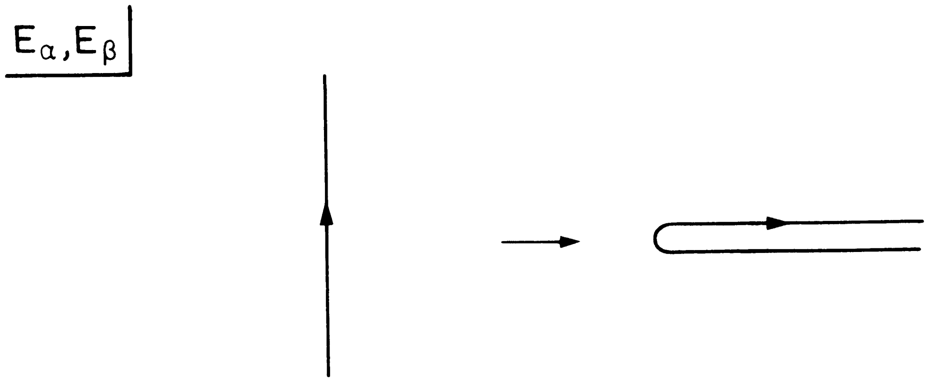


FIG. 5.5

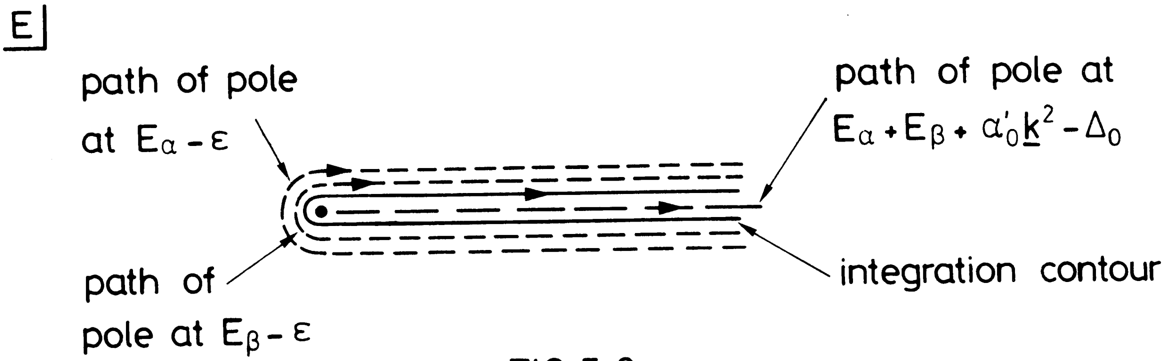


FIG. 5.6

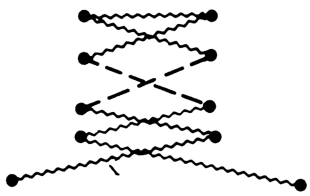


FIG. 5.7

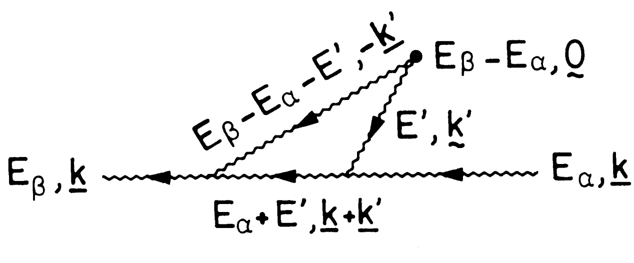


FIG. 5.8

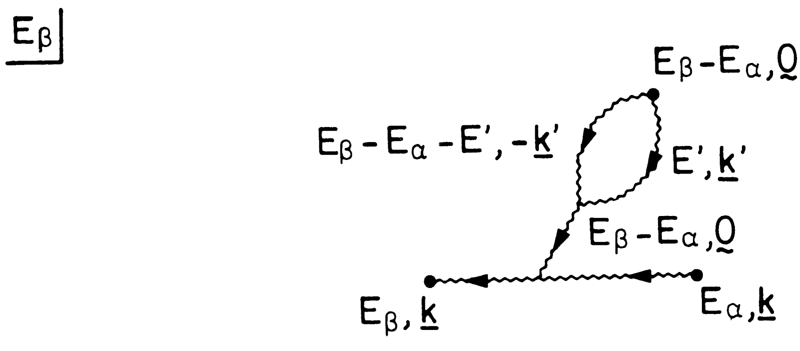


FIG. 5.9

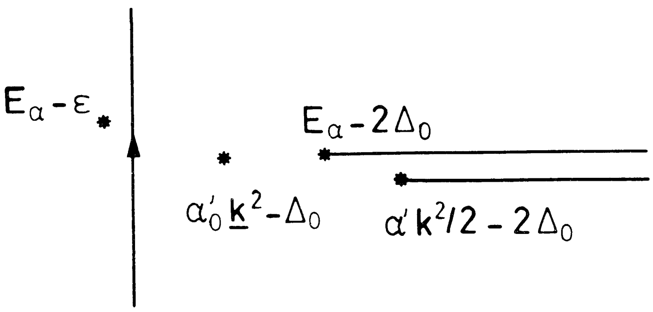


FIG. 5.10

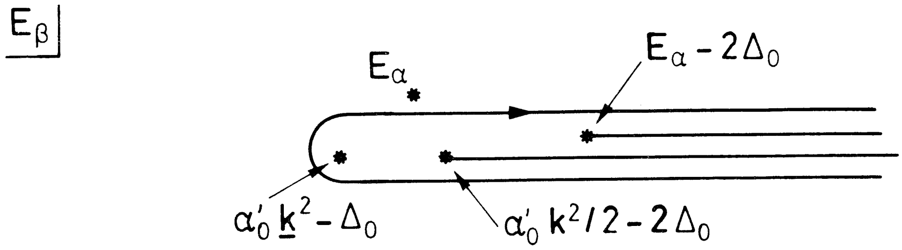


FIG. 5.11

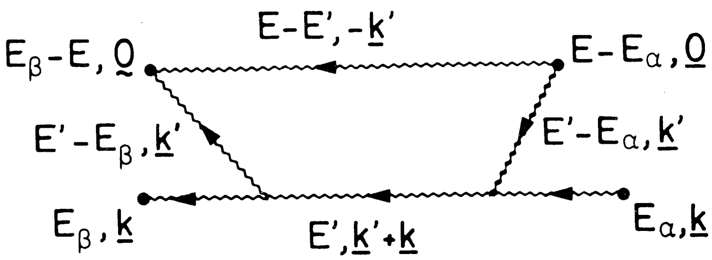


FIG. 5.12

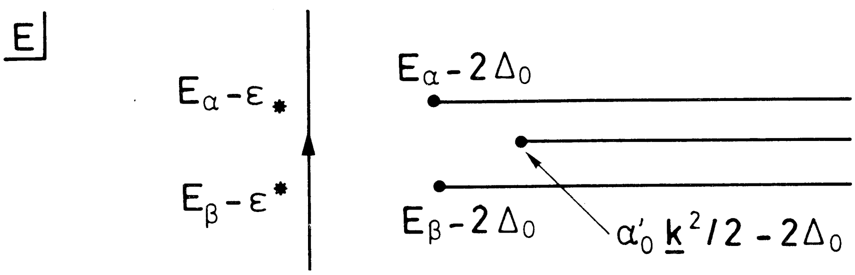


FIG. 5.13

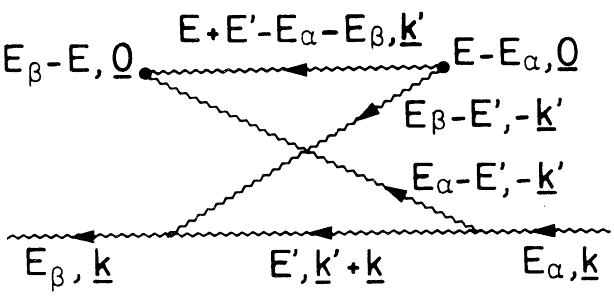


FIG. 5.14

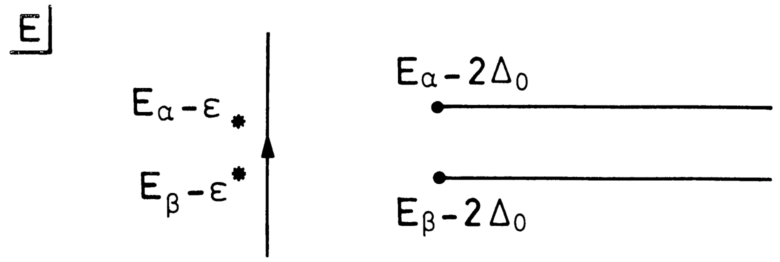


FIG. 5.15

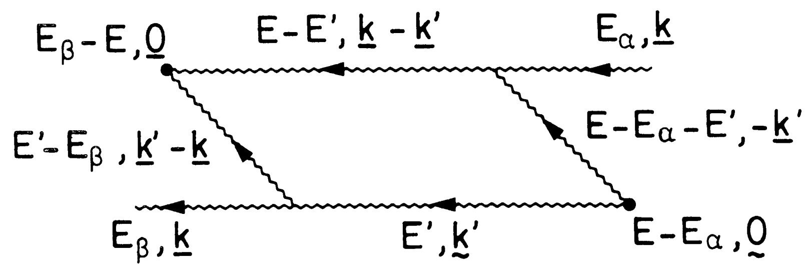


FIG. 5.16

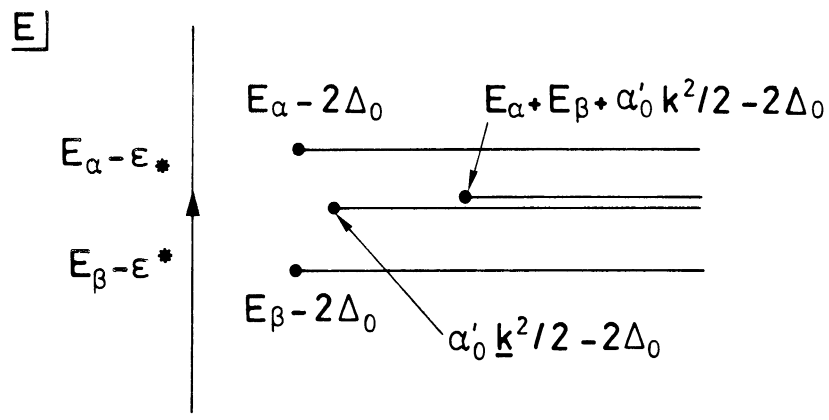
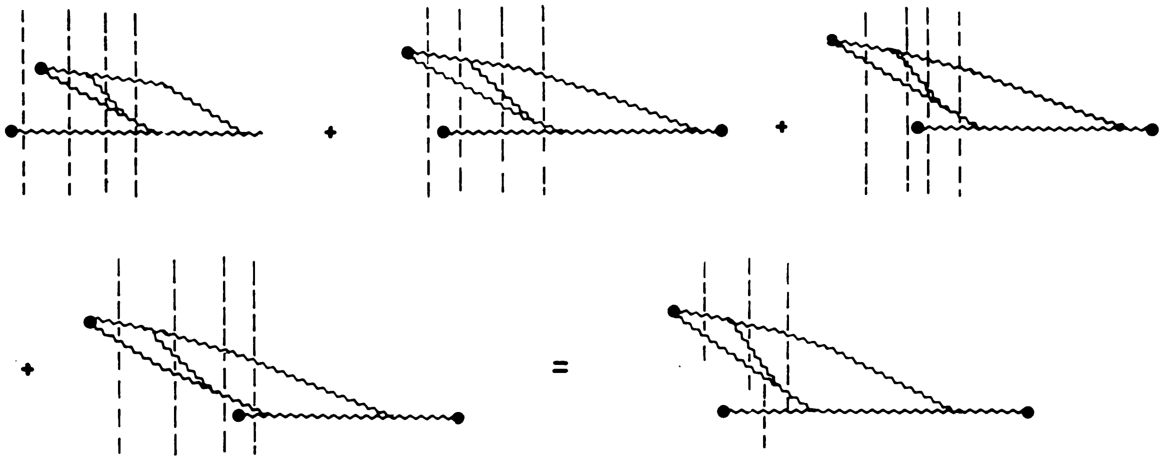


FIG. 5.17



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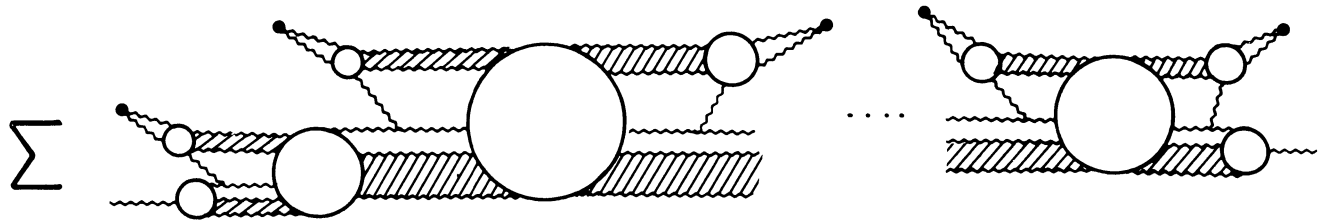

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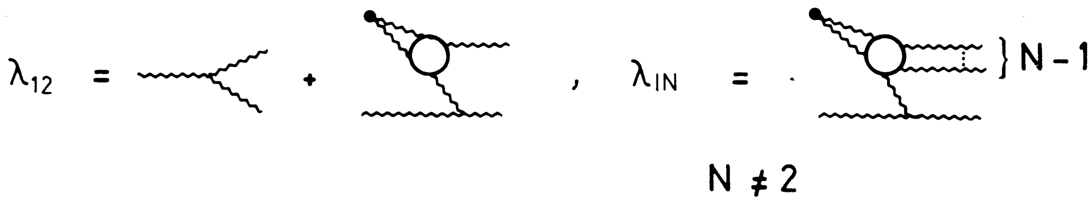

FIG. 5.20

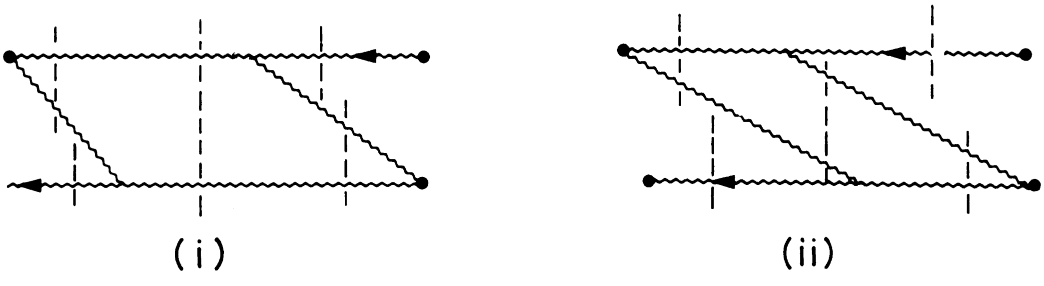

FIG. 5.21