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A MODEL FOR THE DEEP INELASTIC SCALING FUNCTIONS SATISFYING UNITARITY, CROSSING AND ANALYTICITY

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A B S T R A C T

The unitarity relation for virtual Compton scattering in the deep inelastic region derived in a recent paper is solved. The solution exhibits scaling behaviour in deep inelastic scattering and deep annihilation under rather general conditions on the spectrum, the coupling of the final hadron states and the driving force. The non-forward scaling function is explicitly constructed and discussed.

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1. - INTRODUCTION

In a previous paper [1] we have derived a unitarity relation for deep inelastic scattering which is valid under the following assumptions :

(i) the spectrum of the final hadron states in deep inelastic electron-proton scattering is approximated by a continuous set of two-particle states consisting of a nucleon and a vector meson of increasing mass ;

(ii) the electromagnetic current is dominated by the same continuous set of vector mesons (generalized vector meson dominance).

For the sake of simplicity we studied in Ref. 1) a model of scalar particles and scalar currents only. In this model we obtained in the Bjorken limit for the leading term $\bar{G}_B(q^2,Q^2,s,t) = G_B(q^2,Q^2,s,\tau)$ of the double mass discontinuity *) of the retarded Compton amplitude T_B^{ret} the following <u>unitarity relation</u> [see Eq. (26) of Ref. 1)] :

$$\mathrm{disc}_s\, G_B(q^2,Q^2,s,\tau) = \int_{m_0^2}^{\infty} \frac{dm^2}{s}\, \rho_B(m^2,s)\, \Theta(s-m^2)$$

$$\cdot \int_{-s}^{0} d\tau' \int_{-s}^{0} d\tau'' \frac{\Theta(K_B(\tau,\tau',\tau''))}{\sqrt{K_B(\tau,\tau',\tau'')}}\, G_B(q^2,m^2,s+i\varepsilon,\tau')\, G_B(m^2,Q^2,s-i\varepsilon,\tau'') \tag{1}$$

where ρ_B and K_B are given by

$$\rho_B(m^2,s) = \frac{1}{(2\pi)^4}\, \frac{1}{\alpha'(m^2)\,\gamma^2(m^2)}\, \frac{s-m^2}{s} \tag{2}$$

$$K_B(\tau,\tau',\tau'') = 2\left(\tau\tau' + \tau\tau'' + \tau'\tau''\right) - \tau^2 - \tau'^2 - \tau''^2 \tag{3}$$

and the relation between t and τ is

$$\tau = \frac{1}{(1-x)(1-y)}\left[t + M^2 \frac{(x-y)^2}{(1-x)(1-y)}\right],\quad x = \frac{q^2}{s},\quad y = \frac{Q^2}{s}\,. \tag{4}$$

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*) $\bar{G}_B(q^2,Q^2,s,t) = \mathrm{disc}_{q^2}\,\mathrm{disc}_{Q^2}\, T_B^{ret}(q^2,Q^2,s,t)$.

The quantity $\alpha'(m^2)$ stands for the spectral density of the scalar mesons and $\gamma(m^2)$ is the scalar meson-current coupling constant [see Eq. (5) of Ref. 1] *).

It is the purpose of this paper to construct a solution of the unitarity relation (1), which in turn determines the behaviour of the structure function in the deep inelastic region.

The paper is organized as follows. In Section 2 we present a solution of the unitarity relation (1) diagonal in the masses q^2 and Q^2 of the currents which fulfils analyticity and s-u crossing symmetry. In Section 3 we show that the solution has scaling behaviour under rather general conditions on the spectrum and coupling of the final hadron states and on the driving force. The non-forward scaling function emerging is calculated for a simple Born term ansatz, its main features are discussed. In Section 4 we outline briefly a method how to construct a non-diagonal solution of our unitarity relation. Section 5 lists the conclusions.

2. - CONSTRUCTION OF A DIAGONAL SOLUTION OF THE UNITARITY RELATION

We start from an impact parameter representation for G_B :

$$G_B(q^2,Q^2,s,\tau) = \int_0^\infty db\, b\, J_0(b\sqrt{-\tau})\, \Gamma(q^2,Q^2,s,b) . \tag{5}$$

Making use of the relation [2]

$$\int_{-s}^{0} d\tau' \int_{-s}^{0} d\tau'' \frac{\Theta(K_B(\tau,\tau',\tau''))}{\sqrt{K_B(\tau,\tau',\tau'')}}\; J_0(b'\sqrt{-\tau'})\, J_0(b''\sqrt{-\tau''})$$
$$= \frac{\pi}{8b'}\; J_0(b'\sqrt{-\tau})\, \delta(b'-b'') \tag{6}$$

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*) For details of the kinematical notation, we refer the reader to Ref. 1).

we obtain from Eq. (1) the following <u>unitarity relation for $\Gamma(q^2,Q^2,s,b)$</u> :

$$\mathrm{disc}_s\, \Gamma(q^2,Q^2,s,b) =$$

$$\frac{\pi}{8}\int_{m_0^2}^{\infty}\frac{dm^2}{s}\,\rho_B(m^2,s)\,\Theta(s-m^2)\,\Gamma(q^2,m^2,s+i\varepsilon,b)\,\Gamma(m^2,Q^2,s-i\varepsilon,b)\,. \qquad (7)$$

With the help of the inverse Γ^{-1} of Γ, defined by

$$\int_{m_0^2}^{\infty} dm^2\, \Gamma(q^2,m^2,s,b)\,\Gamma^{-1}(m^2,Q^2,s,b) = \delta(q^2-Q^2)\quad , \qquad (8)$$

Eq. (7) leads to

$$\mathrm{disc}_s\, \Gamma^{-1}(q^2,Q^2,s,b) = -\frac{\pi}{8}\,\frac{\rho_B(q^2,s)}{s}\,\Theta(s-q^2)\,\delta(q^2-Q^2)\,. \qquad (9)$$

The solution of this equation can be written in terms of a dispersion relation. An important point is the preservation of s–u crossing symmetry. We recall that $\bar{G}_B(q^2,Q^2,s,t,u)$ was defined as the leading term of an asymptotic expansion in the Bjorken limit, i.e., for fixed t, therefore

$$u = q^2+Q^2+2M^2-s-t \xrightarrow[\substack{q^2,Q^2,s\to\infty\\ t\ \text{fixed}}]{} q^2+Q^2-s \equiv u_B\,. \qquad (10)$$

From the crossing relation

$$\bar{G}_B(q^2,Q^2,s,t,u_B) = \bar{G}_B(q^2,Q^2,u_B,t,s) \qquad (11)$$

it follows

$$G_B(q^2,Q^2,s,\tau,u_B) = G_B\left(q^2,Q^2,u_B,\frac{u_B^2}{s^2}\tau,s\right) \qquad (12)$$

and thus for the impact parameter amplitude Γ

$$\Gamma(q^2,Q^2,s,b,u_B) = \frac{s^2}{u_B^2}\,\Gamma\left(q^2,Q^2,u_B,\frac{s}{u_B}b,s\right)\,. \qquad (13)$$

The crossing relation for the inverse Γ^{-1} is rather involved. Since, however, Eq. (9) exhibits the $\delta(q^2-Q^2)$ function, it is sufficient to give the relation for $q^2 = Q^2$ only

$$\Gamma^{-1}(q^2,q^2,s,b,u_B) = \frac{u_B^2}{s^2}\,\Gamma^{-1}\left(q^2,q^2,u_B,\frac{s}{u_B}b,s\right). \tag{14}$$

Assuming a dispersion relation for Γ, the inverse Γ^{-1} satisfies the following dispersion relation in s and u_B keeping q^2, Q^2 and b fixed

$$\begin{aligned}\Gamma^{-1}(q^2,Q^2,s,b,u_B) &= B^{-1}(q^2,Q^2,s,b,u_B)\\ &-\delta(q^2-Q^2)\,\frac{u_B^2}{8}\int_{q^2}^{\infty}ds'\,\frac{\rho_B(q^2,s')}{s'}\left(\frac{P}{u_B'^2}\right)\left[\frac{1}{s'-s}+\frac{1}{s'-u_B}\right].\end{aligned} \tag{15}$$

The subtraction-like form of the dispersion integral is needed in order to fulfil the crossing relation (14). The term B^{-1} plays the rôle of the inverse of a t channel Born term, i.e., it is a subtraction and it is therefore a polynomial in s and u_B. We ensure the correct crossing property of B^{-1} by introducing it as the inverse of the impact parameter transform of an explicitly crossing symmetric function.

The simplest assumption about the Born term

$$B^{-1}(q^2,Q^2,s,b,u_B) = \frac{\delta(q^2-Q^2)}{V(q^2,s,b,u_B)} \tag{16}$$

leads to

$$\Gamma(q^2,Q^2,s,b,u_B) = \delta(q^2-Q^2)\,\frac{V(q^2,s,b,u_B)}{1-u_B^2\left[L(q^2,s)+L(q^2,u_B)\right]V(q^2,s,b,u_B)} \tag{17}$$

where the function $L(q^2,s)$ is given by

$$\begin{aligned}L(q^2,s) &= \frac{1}{8(2\pi)^4\alpha'(q^2)\gamma^2(q^2)}\int_{q^2}^{\infty}ds'\,\frac{s'-q^2}{s'^2(s'-s)}\,\frac{P}{(2q^2-s')^2}\\ &= \frac{1}{8(2\pi)^4(q^2)^3\alpha'(q^2)\gamma^2(q^2)}\,I(\omega),\end{aligned} \tag{18}$$

with

$$I(\omega)=\frac{1}{4(1-\omega)}+\frac{\mathcal{P}}{4(1+\omega)}+\mathcal{P}\frac{\omega}{(1-\omega)^2(1+\omega)^2}\ln\omega\ ,\quad \omega=1-\frac{s}{q^2}. \tag{19}$$

The function $L(q^2,u_B)$ is obtained from Eq. (18) through the substitution $\omega \to (-\omega)$.

The expression (17) for $\Gamma(q^2,Q^2,s,b,u_B)$ is the solution of the unitarity relation (7) in the deep inelastic region. By construction it is s-u crossing symmetric and has the correct analyticity properties.

By the impact parameter transformation (5) we find $\bar{G}_B(q^2,Q^2,s,t,u_B)$

$$\bar{G}_B(q^2,Q^2,s,t,u_B)=\delta(q^2-Q^2)\frac{\omega^2}{(1-\omega)^2}\int_0^\infty db\, b\, J_0(b\sqrt{-t})$$

$$\cdot\frac{V\left(q^2,(1-\omega)q^2,\frac{\omega}{1-\omega}b,(1+\omega)q^2\right)}{1-\frac{(1+\omega)^2}{8(2\pi)^4q^2\alpha'(q^2)\gamma^2(q^2)}\left[I(\omega)+I(-\omega)\right]V\left(q^2,(1-\omega)q^2,\frac{\omega}{1-\omega}b,(1+\omega)q^2\right)} \tag{20}$$

which is a diagonal solution of the unitarity relation, Eq. (1).

3. - SCALING LIMIT AND SCALING FUNCTION

In Ref. 3), we have shown that the deep inelastic scaling function $F(\omega)$ satisfies a double dispersion relation [Eq. (13), Ref. 3)]. A straightforward generalization of the arguments given there leads to a representation for the non-forward scaling function $F(\omega, \Omega, t)$, which, for the case where $\bar{G}_B$ contains a $\delta(q^2-Q^2)$ function, can be brought into the form *)

$$F(\omega,\Omega,t) = \frac{\frac{1}{2}(\omega+\Omega)-1}{\pi^3}\int_{-\infty}^{1} d\omega' \frac{\varphi(\omega',t)}{(\omega'-\omega)(\omega'-\Omega)} \quad , \Omega = 1-\frac{s}{Q^2} \, . \tag{21}$$

Here the spectral function $\varphi(\omega, t)$ is defined as the scaling limit **)

$$\left(\frac{x-1}{x}=\omega \, , \, \frac{y-1}{y}=\Omega\right)$$

$$\delta(x-y)\varphi\left(\frac{x-1}{x}, t\right) = \lim_{\substack{q^2\to\infty \\ x,y,t \text{ fixed}}} s\, g(q^2,Q^2,s,t,u_B) \tag{22}$$

with g being the triple discontinuity of the retarded Compton amplitude, i.e.,

$$\begin{aligned} g(q^2,Q^2,s,t,u_B) &= \mathrm{disc}_s\, \bar{G}_B(q^2,Q^2,s,t,u_B) \\ &= \mathrm{disc}_s\, \mathrm{disc}_{q^2}\, \mathrm{disc}_{Q^2}\, T_B^{ret}(q^2,Q^2,s,t,u_B) \, . \end{aligned} \tag{23}$$

From the solution (20) of the unitarity relation (1), one obtains

$$g(q^2,Q^2,s,t,u_B) = \frac{1}{s}\,\delta(x-y)\,\pi\, C(q^2)\,\frac{\omega^3}{(1-\omega)^4}\,\Theta(-\omega)$$

$$\cdot\int_0^\infty db\, b\, J_0(b\sqrt{-t}) \frac{V^2\left(q^2,(1-\omega)q^2,\frac{\omega}{1-\omega}b,(1+\omega)q^2\right)}{\left[1-C(q^2)\frac{1+\omega}{1-\omega}V\right]^2+\left[\pi C(q^2)\frac{\omega}{(1-\omega)^2}V\right]^2} \, , \tag{24}$$

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*) Here, the non-forward scaling function F is obtained from the structure function by extracting $1/q\cdot Q$ instead of $1/q^2$ as in Ref. 3), where the forward direction was treated only.

**) In a scalar model where the currents have the canonical dimension 2, Eq. (22) states the correct scaling behaviour.

where we used the notation

$$C(q^2) = \frac{1}{8(2\pi)^4 q^2 \alpha'(q^2)\, \gamma^2(q^2)} \quad . \tag{25}$$

With no further restrictions on the Born term V, Eq. (24) has the correct 1/s scaling behaviour required by Eq. (22) only if

$$V\left(q^2,(1-\omega)q^2,\frac{\omega}{1-\omega}b,(1+\omega)q^2\right) \xrightarrow[\substack{q^2\to\infty \\ \omega,\, b \text{ fixed}}]{} \left(\frac{q^2}{M^2}\right)^n V_B^{(n)}(\omega,b) \tag{26}$$

$$C(q^2) \xrightarrow[q^2\to\infty]{} \left(\frac{M^2}{q^2}\right)^{2n} C_B^{(n)} \quad , \quad n \geq 0 \ . \tag{27}$$

Assuming this kind of behaviour, we obtain for the spectral function $\varphi^{(n)}(\omega,t)$ <u>for n = 0</u> :

$$\varphi^{(0)}(\omega,t) = \pi\,\Theta(-\omega)\, C_B^{(0)} \frac{\omega^3}{(1-\omega)^4} \cdot \int_0^\infty db\, b\, J_0(b\sqrt{-t}) \frac{\left[V_B^{(0)}(\omega,b)\right]^2}{\left[1-C_B^{(0)}\frac{1+\omega}{1-\omega}V_B^{(0)}(\omega,b)\right]^2+\left[\pi C_B^{(0)}\frac{\omega}{(1-\omega)^2}V_B^{(0)}(\omega,b)\right]^2} \tag{28}$$

and <u>for n > 0</u> :

$$\varphi^{(n)}(\omega,t) = \pi\,\Theta(-\omega)\, C_B^{(n)} \frac{\omega^3}{(1-\omega)^4} \int_0^\infty db\, b\, J_0(b\sqrt{-t}) \left[V_B^{(n)}(\omega,b)\right]^2 . \tag{29}$$

We should like to recall at this point that scaling has been achieved under two assumptions :

(i) the condition (27), which implies a restriction on the spectrum $\alpha'(q^2)$ of the mesons and the "photon" coupling to the mesons $\gamma(q^2)$; for the spectrum of the kind found in dual models, i.e., $\alpha'(q^2) = \alpha' =$ = const., Eq. (27) requires

$$\gamma(q^2) \xrightarrow[q^2\to\infty]{} \left(\frac{q^2}{M^2}\right)^{n-1/2} \cdot \gamma_B^{(n)} \quad , \quad n \geq 0 \ . \tag{30}$$

The non-forward scaling function, Eq. (37), obtained from the unitarity relation, Eq. (1), in conjunction with the driving force, Eq. (32), and the scaling conditions on the spectral density, photon-meson coupling and the Born term, has the following properties :

(i) the t dependence as expressed by the function $\phi(t)$, Eq. (38), gives the correct lowest t channel threshold at $t = 4\mu^2$; the branch point at $t = 0$ showing up in Eq. (38) is fictitious ; the fact that the t dependence factorizes in Eq. (37) is a consequence of the rough approximation of the K_o Bessel function in the denominator of Eq. (36) ;

(ii) the scaling function is analytic in the entire complex ω and Ω planes, apart from the cuts

$$-\infty < \omega \leq 0 \quad , \quad -\infty < \Omega \leq 0 \, ;$$

the poles seemingly apparent in the function $H(\omega)$ appear in unphysical sheets only ;

(iii) specializing $F(\omega,\Omega,t)$ to forward direction $F(\omega,\omega,0) = F(\omega)$ one has

$$F(\omega) = \frac{2}{\pi^4 C_B^{(0)}} \frac{\varkappa^2}{(1+\varkappa)^2} (\omega-1) \frac{dH}{d\omega}(\omega) \; ; \tag{39}$$

the threshold behaviour at $\omega = 1$ is linear ; obviously the cut in ω extends over the interval $-\infty < \omega \leq 0$; the behaviour for large ω is proportional to $\ln\omega/\omega$;

(iv) as regards the scaling function $\bar{F}(\omega)$ of the annihilation channel $(-1 \leq \omega < 0)$ one simply has [see Ref. 3]

$$\bar{F}(\omega) = -F(-\omega) \quad , \tag{40}$$

since $F(\omega)$, Eq. (39), has no cut in the interval $0 < \omega < 1$; as a consequence of the simple crossing relation, Eq. (40), $\bar{F}(\omega)$ has the same threshold behaviour at $\omega = -1$ as $F(\omega)$ at $\omega = 1$; for $\omega \to 0$, $\bar{F}(\omega)$ goes to a constant.

Now we turn to the case $n > 0$, for which the spectral function $\varphi^{(n)}(\omega, t)$ is given by Eq. (29). For the Born term given by Eq. (32) resp. (35) the scaling function does not exist, since in this case $\varphi^{(n)}$ turns out to be given by const./ω for $t = 0$. The simplest modification in accordance with crossing symmetry is

$$\overline{U}(q^2, s, t, u_B) = \frac{\mu^2 f(q^2)}{\mu^2 - t} (s - u_B)^2 . \tag{41}$$

This leads in forward direction to

$$\varphi^{(n)}(\omega, 0) = \text{const.} \times \omega^3 . \tag{42}$$

Obviously, in this case, the representation (21) needs three subtractions and the high ω behaviour of the scaling function $F(\omega)$ is proportional to $\omega^3 \ln \omega$. Of course, the three subtraction constants are not determined in this approach. Nevertheless this example illustrates that arbitrarily high ω behaviour can be accommodated.

4. - CONSTRUCTION OF A NON-DIAGONAL SOLUTION OF THE UNITARITY RELATION

Up to now we have dealt with solutions which were strictly diagonal in q^2 and Q^2. This property of the solution was a consequence of the diagonal ansatz of the Born term, Eq. (16). In the following we shall exhibit an example of an explicitly non-diagonal solution of the unitarity equation (7). Obviously, we have to start from a non-diagonal Born term which we choose to be

$$\begin{aligned} B(q^2, Q^2, s, b, u_B) = {} & \delta(q^2 - Q^2) V(q^2, s, b, u_B) \\ & + v(q^2, s, b, u_B) v(Q^2, s, b, u_B) \end{aligned} \tag{43}$$

which corresponds to a strong diagonal driving force and a factorizable non-diagonal contribution. One easily checks that $B^{-1}(q^2, Q^2, s, b, u_B)$ defined in the sense of Eq. (8) is given by

$$B^{-1}(q^2,Q^2,s,b,u_B) = \frac{\delta(q^2-Q^2)}{V(q^2,s,b,u_B)} + \mathcal{R}(q^2,s,b,u_B)\mathcal{R}(Q^2,s,b,u_B) \tag{44}$$

with

$$\mathcal{R}(q^2,s,b,u_B) = \frac{v(q^2,s,b,u_B)}{\chi_1(s,b,u_B)\,V(q^2,s,b,u_B)} \tag{45}$$

and

$$\chi_1^2(s,b,u_B) = -1 - \int_{m_0^2}^{\infty} dm^2 \frac{v^2(m^2,s,b,u_B)}{V(m^2,s,b,u_B)} \, . \tag{46}$$

Using for B^{-1} in the dispersion relation (15) the expression (44), one obtains for Γ

$$\Gamma(q^2,Q^2,s,b,u_B) = \delta(q^2-Q^2)\,\mathcal{D}(q^2,s,b,u_B) + \mathcal{D}(q^2,s,b,u_B)\frac{\mathcal{R}(q^2,s,b,u_B)\mathcal{R}(Q^2,s,b,u_B)}{\chi_2^2(s,b,u_B)}\mathcal{D}(Q^2,s,b,u_B) \tag{47}$$

with

$$\mathcal{D}(q^2,s,b,u_B) = \frac{V(q^2,s,b,u_B)}{1-u_B^2\left[L(q^2,s)+L(q^2,u_B)\right]V(q^2,s,b,u_B)} \tag{48}$$

and

$$\chi_2^2(s,b,u_B) = -1 - \int_{m_0^2}^{\infty} dm^2\, \mathcal{R}^2(m^2,s,b,u_B)\,\mathcal{D}(m^2,s,b,u_B) \, . \tag{49}$$

We should like to point out that the diagonal part of Eq. (47) is identical to the diagonal solution, Eq. (17). In order to construct from the solution (47) for Γ the scaling function $F(\omega,\Omega,t)$ one has to repeat all the steps after Eq. (19). The scaling function $F(\omega,\Omega,t)$ obtained this way consists of two terms. The first one, arising from the diagonal part of the driving force is the same as above. The second term has its origin in the non-diagonal part of the Born term, Eq. (43).

For a determination of the scaling function one could supplement the diagonal Born term used in Section 3 by a non-diagonal, factorizable term. From the complexity of the solution, Eq. (47), it is obvious that the calculation of the scaling function has to be done by numerical methods.

5. - CONCLUSION

From a deep inelastic unitarity equation [1] for the leading term in the Bjorken limit of the virtual Compton amplitude in a scalar model we have shown under rather general conditions on the spectrum of the intermediate states and the driving force that scaling behaviour prevails. In addition we have constructed explicit solutions to this unitarity relation which fulfil the analyticity requirements in all three channels s, t, u and s-u crossing symmetry. This then leads to a non-forward scaling function which satisfies the constraints of locality and spectral conditions [3], which originate from the fact that the non-forward scaling function is the Bjorken limit of a matrix element of the local commutator of currents. It is therefore a model for the one-particle matrix element of a bilocal operator.

In particular, starting from a simple Born term of the form of a single particle exchange in the t channel and diagonal in the masses q^2 and Q^2 of the currents, we have presented a non-forward scaling function with the following properties in forward direction :

(i) linear threshold behaviour at $\omega = 1$,

(ii) high ω behaviour of the type $\ln\omega/\omega$,

(iii) identical threshold behaviour for scattering and annihilation.

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