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VERTEX BOOTSTRAP FOR THE FINE STRUCTURE CONSTANT

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A B S T R A C T

The possibility of converting the Baker-Johnson-Adler eigenvalue equation $F(\alpha)=0$ for the fine structure constant into a simpler condition is discussed in the framework of the self-consistent (bootstrap) formulation of massless quantum electrodynamics. It is shown that the imposition of a strong convergence condition on the vertex bootstrap equation in the canonical (generalized Landau) gauge leads to an eigenvalue equation for α, $\tilde{Z}_{1c}(\alpha)=0$, which is consistent with the gauge covariance of the theory and implies $F(\alpha)=0$. The function $\tilde{Z}_{1c}(\alpha)$ may be expanded in a power series in α through Feynman graphs.

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1. - INTRODUCTION

In an earlier paper [1], massless finite quantum electrodynamics (QED) [2)-4)] was formulated as a field theoretical "bootstrap" in nearly all gauges. The resulting equations determine, in principle, the coupling constant $\bar{e}$ of the renormalized massless theory ; following Adler's recent conjecture [2] $\bar{e}$ was identified with the physical charge e.

One may envisage this "bootstrap" approach as comprising essentially two equations :

a) a homogeneous non-linear integral equation for the vertex function in which the electron propagator is expressed in terms of the vertex function through the Ward identity ; we shall hereafter refer to this as the vertex bootstrap equation or simply as the bootstrap equation ; it is given in Eqs. (II.3) and (II.5) of Ref. 1) ;

b) an equation expressing the vanishing of the renormalized photon self-energy. This is in fact a rearrangement of Adler's equation $F(\alpha) = 0$ in terms of the renormalized quantities which enter the vertex bootstrap. It is given in Eq. (II.6) of Ref. 1). We shall refer to it as the self-energy condition.

The purpose of the present work is to obtain a limiting condition for the vertex bootstrap, as a consequence of which the self-energy condition will be satisfied trivially. The fine structure constant α might thus be obtainable directly from the vertex bootstrap, via an alternate, maybe simpler, eigenvalue condition.

Our hope for the success of such a program hinges on the following plausibility argument : the vertex bootstrap is a consequence of the vanishing of a certain vertex renormalization constant Z. (We do not specify Z for the moment because we will show subsequently that it is not to be identified with the conventional renormalization constant Z_1 of QED.) On the other hand, the simplest expression for the photon self-energy in terms of the renormalized vertex function Γ_μ and the renormalized electron propagator S is

$$\Pi_{\mu\nu} = Z \int \Gamma_\mu S S \gamma_\nu \qquad (1.1)$$

where the integral is of course defined through the usual Feynman rules corresponding to the graph of Fig. 1. The vanishing of Z in (1.1) would then seem to imply the vanishing of $\Pi_{\mu\nu}$. At this stage the argument is as yet incomplete because of divergences. However, in QED, the gauge arbitrariness may be responsible for spurious divergences which cancel in gauge invariant quantities such as vacuum polarization loops ; this is indeed the case in massless QED where a convergent gauge invariant $\Pi_{\mu\nu}$ can be defined. Moreover, in this case, there is a gauge (the "canonical" or generalized Landau gauge [3),4)], in which the renormalized Fermion field has canonical dimension) which leads to a finite Z for arbitrary α. Consequently, in the canonical gauge, the bootstrap would really imply $\Pi_{\mu\nu} = 0$ and therefore $F(\alpha) = 0$. The problem is therefore to extend the bootstrap to the canonical gauge. This raises delicate convergence questions which are deeply related to gauge covariance ; indeed, according to Ref. 1), the canonical gauge is a singular one in which the bootstrap is not applicable a priori. The final result of the present paper is that such an extension is in fact possible. This extension provides an eigenvalue equation for α which is consistent with an essential zero of the function $F(\alpha)$.

The paper is organized as follows. In Section 2 we discuss the theory of an off-mass shell, massless Fermion interacting with a gauge field. The motivation for this rather academic exercise is to provide us with a test ground for studying the analyticity properties of the vertex bootstrap, in terms of a model where the exact solution can be written down in closed form.

Section 3, the heart of the paper, is devoted to a study of the vertex bootstrap equation in the vicinity of the canonical gauge. We show that in analogy to the model of Section 2, the limit of the bootstrap equation in the canonical gauge exists, if all integrals are evaluated as limits of integrals in nearby gauges. Furthermore, we show that under reasonable analyticity assumptions, this limiting procedure is valid for all values of α : the vertex bootstrap equation in the canonical gauge, viewed as a limiting equation, is an identity in α. These properties are proved to order zero and to order one in α by explicit evaluation of the relevant integrals. If we now impose, as a further condition, that the vertex bootstrap be valid in the canonical gauge independently of the limiting process, we obtain an eigenvalue equation for α ; what happens is that different pieces of the identity must separately vanish for a certain value of α. This equation is consistent with the general gauge covariance of the theory and expresses the vanishing of a certain renormalization constant calculated in the canonical gauge. It is written as $\widetilde{Z}_{1c}(\alpha)=0$. The function $\widetilde{Z}_{1c}(\alpha)$ can be expanded in powers of α via Feynman graphs.

In Section 4, we consider the relationship between $\widetilde{Z}_{1c}(\alpha)=0$ and $F(\alpha)=0$. We show that a zero of $\widetilde{Z}_{1c}(\alpha)$ implies a zero of $F(\alpha)$ and is also consistent with an essential zero of $F(\alpha)$. However, it is not easy to understand the nature of the zero of $\widetilde{Z}_{1c}(\alpha)$.

In Section 5, the condition $\widetilde{Z}_{1c}(\alpha)=0$ is interpreted in the context of the physical theory of massive QED. This is mainly an attempt to clarify the meaning of the various renormalization constants used in QED and to establish the connection between $\widetilde{Z}_{1c}$ and other related constants used in the literature.

Finally, in Section 6, we summarize the main features of the eigenvalue equation $\widetilde{Z}_{1c}(\alpha)=0$ and their possible relevance for an explicit evaluation of α.

All calculations are relegated to the Appendices.

2. - THE CONFORMAL GAUGE BOOTSTRAP

We shall consider here the interaction of a massless electron with its gauge field, described in terms of a gauge propagator $(-i)q_\mu q_\nu/q^4$ coupled to the bare vertex γ_μ with a coupling constant $\xi^{\frac{1}{2}}$; note that only $(\xi^{\frac{1}{2}})^2=\xi$ enters in the Feynman rules and so we shall take ξ to vary between $-\infty$ and $+\infty$. Throughout this paper we restrict ourselves to this class of gauges. The gauge interaction being renormalizable, we shall first calculate the renormalization constants $\widetilde{Z}_1^G$ and $\widetilde{Z}_2^G$. The latter will be defined through a subtraction procedure at finite momentum p, $p^2=-\kappa^2$ in order to avoid infra-red divergences ; for the same reason we shall consider only off-mass shell propagation.

The unrenormalized electron propagator S', dressed with gauge photons to all orders in ξ, obeys a differential equation [5] which can easily be derived using the Ward identity. In configuration space, this equation reads

$$\frac{d}{d\xi} S'(x;\xi) = [\mathcal{J}(x) - \mathcal{J}(0)]\, S'(x;\xi) \tag{2.1}$$

where formally

$$\mathcal{J}(x) = \frac{-i}{(2\pi)^4} \int \frac{e^{-iq\cdot x}}{q^4}\, d^4q \tag{2.2}$$

For $\xi = 0$, the electron is free and we have

$$S'(x;0) = \frac{i}{2\pi^2}\, \not{x}\, x^{-4} \tag{2.3}$$

The canonical norm chosen for the electron propagator implies $S'(p,0) = -i/\not{p}$ in momentum space ; the conventions and notations used throughout the paper as well as some basic Fourier transform relations are given in Appendix A. The integral (2.2) for $\mathcal{J}(x)$ is infra-red divergent, while $\mathcal{J}(0)$ is both infra-red and ultra-violet divergent ; the difference $\mathcal{J}(x) - \mathcal{J}(0)$ is, however, only ultra-violet divergent. If we evaluate $\mathcal{J}(0)$ with a high momentum cut-off at $p^2 = -\Lambda^2$ and regularize both $\mathcal{J}(0)$ and $\mathcal{J}(x)$ with a "photon mass" or with an anomalous dimension $(q^{-4} \to q^{-4+\epsilon})$ to control the infra-red divergence, we obtain [6] an expression valid for $|x| \gg 1/\Lambda$

$$\mathcal{J}(x) - \mathcal{J}(0) = -\frac{1}{16\pi^2} \log\left(-\frac{x^2\Lambda^2}{4}\right) - b \tag{2.4}$$

where b is a constant whose value depends on the regularization procedure used. The reason for the restriction $|x| \gg \Lambda^{-1}$ is that we have taken directly $\Lambda \to \infty$ in $\mathcal{J}(x)$, $x \neq 0$. From (2.1), (2.3) and (2.4) we obtain

$$\underset{|x| \gg \Lambda^{-1}}{S'(x,\Lambda^2;\xi)} = \frac{i}{2\pi^2}\, e^{-\xi b}\, \not{x}\, x^{-4} \left(-\frac{x^2\Lambda^2}{4}\right)^{-\xi/16\pi^2} \tag{2.5}$$

and in momentum space

$$i S'(p,\Lambda^2;\xi)\underset{|p|\ll\Lambda}{=} e^{-\xi b}\,\frac{\Gamma(l_F+\frac{\xi}{2})}{\Gamma(-l_F+\frac{1}{2})}\,\frac{\not p}{p^2}\left(-\frac{p^2}{\Lambda^2}\right)^{-l_F-\frac{3}{2}} \tag{2.6}$$

where we have defined

$$2l_F = -3 - \frac{\xi}{8\pi^2} \tag{2.7}$$

The renormalized electron propagator S is defined by a subtraction at $p^2 = -\kappa^2$ where κ is arbitrary but $|\kappa| \ll \Lambda$; explicitly, we have

$$-i S^{-1}(p,\kappa^2;\xi) = \not p - \Sigma^R(p,\kappa^2;\xi) \tag{2.8}$$

where the renormalized electron self-energy Σ^R has the form

$$\Sigma^R(p,\kappa^2;\xi) = \not p \lim_{\Lambda\to\infty}\left[f(p^2,\Lambda^2;\xi) - f(-\kappa^2,\Lambda^2;\xi)\right] \tag{2.9}$$

and therefore

$$-i S^{-1}(p^2=-\kappa^2) = \not p \tag{2.10}$$

When this subtraction procedure is consistently applied to all Feynman graphs contributing to $f(p^2,\Lambda^2;\xi)$, together with the corresponding subtractions in the vertex function Γ_μ as required by the Ward identity, one has realized the multiplicative renormalization and thus

$$S(p,\kappa^2;\xi) = (\tilde{Z}_2^q)^{-1}\left(\frac{\kappa^2}{\Lambda^2};\xi\right) S'(p,\Lambda^2;\xi) \tag{2.11}$$

$$\Gamma_\mu(p,\kappa^2;\xi) = \tilde{Z}_1^q\left(\frac{\kappa^2}{\Lambda^2};\xi\right)\Gamma'_\mu(p,\Lambda^2;\xi) \tag{2.12}$$

with

$$\tilde{Z}_1^G = \tilde{Z}_2^G = 1 + f(-\kappa^2, \Lambda^2; \xi) \tag{2.13}$$

In (2.6) we now write $(-p^2/\kappa^2)(\kappa^2/\Lambda^2)$ for $-p^2/\Lambda^2$ so that from (2.11) and (2.12) we obtain

$$\tilde{Z}_1^G = \tilde{Z}_2^G = e^{-\xi b} \frac{\Gamma(\ell_F + \frac{5}{2})}{\Gamma(-\ell_F + \frac{1}{2})} \left(\frac{\Lambda^2}{\kappa^2}\right)^{\ell_F + \frac{3}{2}} \tag{2.14}$$

From (2.14) we see that if $-5/2 < \ell_F < -3/2$, then $\tilde{Z}_1^G$ and $\tilde{Z}_2^G$ are zero in the limit $\Lambda \to \infty$ and thus we have a "bootstrap" situation for the vertex function, analogous to the one discussed in Ref. 1). The corresponding equations are

$$\Gamma_\mu = \int \Gamma^\sigma S \Gamma_\mu S \Gamma^\tau (\xi D^G_{\sigma\tau}) + \text{higher order irreducible skeleton graphs} \tag{2.15}$$

$$\Gamma_\mu(p,p) = -i \frac{\partial}{\partial p^\mu} S^{-1}(p) \tag{2.16}$$

provided that all integrals converge. The successive terms on the right-hand side of (2.15) are defined by the Feynman graphs of Fig. 2 ; we shall interpret them either in configuration space or in momentum space, according to which is more convenient ; $D^G_{\sigma\tau}$ is the gauge photon propagator $(-i)q_\sigma q_\tau/q^4$. Only the Ward identity for $q = 0$ is used in (2.16) as this will be sufficient for the complete determination of $\Gamma_\mu(p,p+q)$.

Before discussing the group theoretical and analyticity properties of (2.15) and (2.16) we shall make a direct evaluation of Γ_μ for <u>arbitrary</u> values of the gauge parameter ξ [or ℓ_F ; see Eq. (2.7)].

The unrenormalized three-point Green's function

$$G'_\mu(x_1, x_2; x_3) = \langle 0 | T\psi(x_1)\bar{\psi}(x_2) J_\mu(x_3) | 0 \rangle \tag{2.17}$$

obeys the differential equation (2.1) [5]. In the free theory, ($\xi = 0$, $\ell_F = -\frac{3}{2}$), using the same normalization as for the propagator (2.5), G'_μ is given by

$$G'^{(free)}_\mu = -\frac{1}{4\pi^4} \not{x}_{13}\, x_{13}^{-4}\, \gamma_\mu\, \not{x}_{32}\, x_{32}^{-4}$$

(where $x_{ij} = [(x_i - x_j)^2 - i\epsilon]^{\frac{1}{2}}$). Therefore for arbitrary ξ we have

$$G'_\mu(x_1, x_2; x_3) = -\frac{e^{-\xi\ell}}{4\pi^4} \not{x}_{13}\, x_{13}^{-4}\, \gamma_\mu \not{x}_{32}\, x_{32}^{-4} \left(-\frac{x_{12}^2 \Lambda^2}{4}\right)^{\ell_F + \frac{3}{2}} \tag{2.18}$$

The renormalized Green's function G_μ is obtained from (2.18) through multiplication by $(\tilde{Z}_2^G)^{-1}$, so that, by (2.14)

$$G_\mu(x_1, x_2; x_3) = -\frac{1}{4\pi^4} \frac{\Gamma(-\ell_F + \frac{1}{2})}{\Gamma(\ell_F + \frac{5}{2})} \not{x}_{13}\, x_{13}^{-4}\, \gamma_\mu \not{x}_{32}\, x_{32}^{-4} \left(-\frac{x_{12}^2 K^2}{4}\right)^{\ell_F + \frac{3}{2}} \tag{2.19}$$

which we write as

$$G_\mu(x_1, x_2; x_3) = -\frac{1}{4\pi^4} a(\ell_F) \not{x}_{13}\, x_{13}^{-4}\, \gamma_\mu \not{x}_{32}\, x_{32}^{-4}\, x_{12}^{2\ell_F + 3} \tag{2.20}$$

note that $a(-\frac{3}{2}) = 1$.

The renormalized vertex function Γ_μ can now be obtained from (2.20) by amputation of the external Fermion propagators. This is conveniently carried out in configuration space. We write the propagator (2.11) as

$$S(x) = \frac{i}{2\pi^2} a(\ell_F) \not{x}\, x^{2\ell_F - 1} \tag{2.21}$$

The inverse propagator is easily evaluated using the formulae in Appendix A. It turns out to be

$$S^{-1}(x) = \frac{-i \cdot 2\pi^2}{(2\pi)^4 a(\ell_F)} \left[(2\ell_F+7)(2\ell_F+5)(2\ell_F+3)(2\ell_F+1)\right] \not{x}\, x^{-2\ell_F - 9} \tag{2.22}$$

and therefore

$$\Gamma_\mu(x_1,x_2;x_3) = \int S^{-1}(x_{11'})\, G_\mu(x_{1'},x_{2'};x_3)\, S^{-1}(x_{2'2})\, d^4x_{1'}\, d^4x_{2'} \tag{2.23}$$

The evaluation of this integral is summarized in Appendix B1. The result is

$$\Gamma_\mu(x_1,x_2;x_3) = \frac{1}{(2\pi)^4 a(\ell_F)} \left[(2\ell_F+5)(2\ell_F+3)(2\ell_F+1)\right]\left\{(2\ell_F+3)A - 2(2\ell_F+5)B\right\} \tag{2.24}$$

with

$$A = \not{x}_{13}\, x_{13}^{-4}\, \gamma_\mu\, \not{x}_{32}\, x_{32}^{-4}\, x_{12}^{-2\ell_F-5} \tag{2.25}$$

$$\begin{aligned} B &= (x_{13}x_{32})^{-2} \frac{\partial}{\partial x_3^\mu}\left(\log \frac{x_{32}}{x_{13}}\right) \not{x}_{12}\, x_{12}^{-2\ell_F-7} \\ &= (x_{13}x_{32})^{-2}\left[\frac{x_{32,\mu}}{x_{32}^2} - \frac{x_{31,\mu}}{x_{31}^2}\right] \not{x}_{12}\, x_{12}^{-2\ell_F-7} \end{aligned} \tag{2.26}$$

In Appendix B2 we check that (2.24) satisfies the Ward identity (2.16). This in turn implies that the generalized Ward-Takahashi identity for non-zero momentum transfer is also verified, because of the functional form of form factors A and B [7]. In the limit $\ell_F \to -\frac{3}{2}$ the B part of (2.24) yields a finite contribution because a divergence in momentum space compensates the prefactor $(2\ell_F+3)$; the A part does not contribute in this limit. The resulting $\Gamma_\mu(p,p+q)$ is shown in Appendix B3 to be exactly γ_μ , for all external momenta, as it should.

Let us now turn to an analysis of the vertex bootstrap equations (2.15) and (2.16). The validity of these equations in the range $-5/2 < \ell_F < -3/2$ is clear : as discussed in Ref. 1), if one inserts a linear combination of the form factors A and B in the right-hand side of Eq. (2.15), then, in the range of ℓ_F given above, all integrals converge in the ordinary sense. We also note that because of (2.16), S(x) is defined in (2.15) up to an arbitrary constant, corresponding to the arbitrariness of the renormalization program ; we, of course, chose the constant such that the solution of the bootstrap is in fact S(x) as given by (2.21).

Let us check the group theoretical properties of the bootstrap equations (2.15) and (2.16). Because the photon propagator $D^{G}_{\sigma\tau}$ is conformally covariant [1], it follows from the Polyakov [8] - Migdal [9] theorem that the homogeneous equation (2.15) admits the conformal group $O(4,2)$ as its invariance group, and in fact we see that (2.21) and (2.24) are conformally covariant objects [7]. Herein lies the tremendous simplicity of the "gauge bootstrap" as compared with the realistic bootstrap written in terms of the true photon propagator $(-i)((g_{\mu\nu}/q^2)-\eta(q_\mu q_\nu/q^4))$. In the latter case, the $O(4,2)$ symmetry of the massless theory is concealed if one works in the usual set of covariant gauges [1),10)].

Next, let us discuss the extension of the gauge vertex bootstrap outside the range $-5/2 < \ell_F < -3/2$. The conformal form factors A and B will generate pole singularities in (2.15) at $\ell_F = -\frac{3}{2}+n$, n any integer, or zero [1]. Therefore, by analytic continuation, (2.15) is valid in nearly all gauges and its solution is always given by (2.24). Furthermore, since the solutions Γ_μ and S are well defined in momentum space in <u>all</u> gauges [as is immediately apparent from the fact that the Ward identity (2.16) is satisfied], they correctly yield the limit of the bootstrap solutions in the exceptional gauges $\ell_F = -\frac{3}{2}+n$. Hence, the bootstrap formulation is in fact valid in <u>all</u> gauges provided that in the case of an exceptional gauge one begins by evaluating all integrals in (2.15) in a nearby gauge and <u>then</u> takes the limit of their sum when $\ell_F \to -\frac{3}{2}+n$. From here on, whenever we talk of a vertex bootstrap in an exceptional gauge, we shall mean that the above limiting process has been used, unless the contrary is specifically stated.

A particular situation arises in the case $\ell_F = -\frac{3}{2}$ because this corresponds to the free theory $(\tilde{Z}_1^G = 1)$ and embodies the transition between the regions $\tilde{Z}_1^G = 0$ and $\tilde{Z}_1^G = \infty$ [see Eq. (2.14)]. We now want to show that not only is the bootstrap valid at this particular point in the sense of the preceding paragraph, but also that in this case each separate integral on the right-hand side of Eq. (2.15) has a well-defined limit for $\ell_F = -\frac{3}{2}$. Define

$$\epsilon = \ell_F + \frac{3}{2} \tag{2.27}$$

Using the analysis carried out by Mack [11] in a similar situation, we can see that in the neighbourhood of $\epsilon = 0$, each integral in (2.15) admits

readily the reason for this simple pole structure : in the limit $\epsilon \to 0$

each vertex Γ_μ in a given skeleton graph of Fig. 2 tends to a finite quantity γ_μ , and because of the irreducibility of the graphs considered, no divergent subdiagrams occur in this limit ; the over-all divergence then manifests itself through the $1/\epsilon$ singularity [12]. Because ξ is of order ϵ [see Eq. (2.7)], we see that as $\xi \to 0$ all integrals vanish except for the triangle graph in Fig. 2 which contains only a single factor of ξ . Therefore, Eq. (2.15) becomes

$$\lim_{\epsilon \to 0} \Gamma_\mu = \gamma_\mu = \lim_{\substack{\epsilon = \xi/16\pi^2 \\ \epsilon \to 0}} \int \Gamma_\sigma S \Gamma_\mu S \Gamma_\tau \left(\xi D^{G,\sigma\tau} \right) \tag{2.28}$$

The last integral is evaluated in Appendix B4, where it is shown to be indeed equal to γ_μ , thereby confirming the validity of the bootstrap for $\ell_F = -\frac{3}{2}$.

We close this section with a brief summary of its content. We have shown that the interaction of a massless electron with its gauge field is entirely described, in an arbitrary gauge within the class considered, by the vertex bootstrap equations (2.15) and (2.16) for the renormalized quantities S and Γ_μ . These quantities were constructed explicitly and the following properties of the bootstrap were exhibited :

a) the vertex bootstrap equation admits O(4,2) as its invariance group ;

b) the vertex bootstrap equation is singular in a discrete set of gauges which are conveniently parametrized by the Fermion field "dimension" ℓ_F ; these singular gauges occur for $\ell_F = -\frac{3}{2}+n$ where n is any integer or zero. The bootstrap is, however, valid in these gauges if all integrals are first evaluated in nearby gauges ; in particular for $\ell_F = -\frac{3}{2}$ each integral in Fig. 2 converges and is in fact zero in this limit except for the triangle graph which yields the correct free field result $\Gamma_\mu(p,p+q) = \gamma_\mu$; thus even the free field theory, where $\tilde{Z}_1^G$ is finite and equal to 1, can be reached through the bootstrap and was in fact obtained in a very simple way.

These features will serve as a useful guide in the investigation of the properties of the vertex bootstrap in the presence of true photons.

3. - THE VERTEX BOOTSTRAP FOR MASSLESS QED IN THE VICINITY OF THE CANONICAL GAUGE

3.1. - The renormalization constants $\tilde{Z}_1$ and $\tilde{Z}_2$

We now turn to the complete theory with photon propagator

$$D_{\mu\nu} = -i\left(g_{\mu\nu} - \eta\,\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2}$$

and coupling constant e^2 (or $\alpha = e^2/4\pi$). Let us write the vertex bootstrap equation of Ref. 1) in momentum space

$$\Gamma_\mu(p,p+q) = \int d^4k\, S(p+k)\Gamma_\mu(p+k,p+k+q)S(p+k+q)K(p,p+k;p+k+q,p+q) \tag{3.1}$$

$$\Gamma_\mu(p,p) = -i\frac{\partial}{\partial p^\mu}S^{-1}(p). \tag{3.2}$$

Equation (3.1) is depicted in Fig. 3a ; the skeleton expansion of the irreducible Bethe-Salpeter kernel K in terms of Γ_σ, S and $D_{\sigma\tau}$ is given in Fig. 3b [13]. One should keep in mind that in Eq. (3.1), closed Fermion loops have been omitted in the kernel K.

In Ref. 1), Eqs. (3.1) and (3.2) were established on the basis of an argument similar to the one used in Section 2 for the gauge bootstrap. Namely, these equations were first shown to hold in a restricted class of gauges ; it was then argued on grounds of convergence and analytic continuation that they are in fact valid in nearly all gauges. We recall that (3.1) has a hidden $O(4,2)$ symmetry in the sense that whereas S is conformally covariant, Γ_μ and $D_{\sigma\tau}$ are only limits of certain conformally covariant expressions depending on an external scalar field, when the latter is taken at infinity [for details, see Ref. 1)]. Thus, if α and η are given

$$S(p) = c\,\not{p}\,p^{-2\ell_F-5} \tag{3.3}$$

but Γ_μ contains form factors in addition to those given in (2.24) and (2.25). The presence of the latter precludes an explicit proof of the general validity of the bootstrap equations. However, it was argued in Ref. 1), on the basis of the self-consistency of the theory and from direct inspection that the new permissible form factors will not destroy the convergence properties of the bootstrap. If this is indeed so, then (3.1) and (3.2) are valid, as in Section 2, in nearly all gauges ; the exceptional gauges may still be parametrized by $\ell_F = -\frac{3}{2}+n$ with n any integer or zero. Note that ℓ_F is now a function of both η and α. As previously, we may extend the bootstrap to encompass these exceptional gauges, by demanding that all integrals be evaluated in nearby gauges before going to the limit of an exceptional gauge.

Equations (3.1) and (3.2) still determine S up to an arbitrary constant C ; we choose C such that

$$S(p) = \frac{-i}{\not{p}}\left(-\frac{p^2}{\kappa^2}\right)^{-\ell_F - \frac{3}{2}} \tag{3.4}$$

where κ is an arbitrary momentum and $\ell_F = \ell(\alpha, \eta)$. This normalization is consistent with the interpretation of S and Γ_μ as renormalized quantities through the subtraction procedure (2.8) and (2.9). (For a more detailed discussion of this point, see footnote 23 in Section 5.) The corresponding renormalization constants $\tilde{Z}_1$ and $\tilde{Z}_2$ then have the form

$$\tilde{Z}_1 = \tilde{Z}_2 = g(\alpha,\eta)\left(\frac{\Lambda^2}{\kappa^2}\right)^{\ell_F + \frac{3}{2}} \tag{3.5}$$

This equation follows from the Gell-Mann-Low factorization argument [14] or equivalently from the Callan-Symanzik [15] equations as used by Adler and Bardeen [6], with κ^2 now playing the rôle of the mass squared.

The gauge transformation properties of $g(\alpha, \eta)$ and $\ell_F = \ell_F(\alpha, \eta)$ follow immediately from the fact that the differential equation (2.1) remains valid if ξ is replaced by η and $\mathcal{J}(x)$ by $-e^2\mathcal{J}(x)$. This means that the ratio $S'(\xi')/S'(\xi)$ as obtained from (2.6) and (2.7) is equal to the ratio of $S'(\eta')/S'(\eta)$ in the present case, with the identification $-e^2\eta = \xi$; thus

$$\frac{\tilde{Z}_1^G(\xi,\ell_F)}{\tilde{Z}_1^G(\xi',\ell_F)} = \frac{\tilde{Z}_1(\eta,\ell_F)}{\tilde{Z}_1(\eta',\ell_F)} \quad ; \quad (-e^2\eta = \xi) \tag{3.6}$$

This implies

$$\ell_F(\alpha,\eta') - \ell_F(\alpha,\eta) = \frac{\alpha}{4\pi}(\eta'-\eta) \tag{3.7}$$

and

$$\frac{g(\alpha,\eta')}{g(\alpha,\eta)} = e^{4\pi\alpha(\eta'-\eta)b}\,\frac{\Gamma(\ell'_F+\frac{5}{2})}{\Gamma(-\ell'_F+\frac{1}{2})}\,\frac{\Gamma(-\ell_F+\frac{1}{2})}{\Gamma(\ell_F+\frac{5}{2})} \tag{3.8}$$

Let $\eta_0(\alpha)$ be the gauge for which $\ell_F = -\frac{3}{2}$; this we call the canonical gauge (or the generalized Landau gauge[3),4)]) ; Eq. (3.5) may now be written in the form

$$\tilde{Z}_1 = \tilde{Z}_2 = g(\alpha,\eta)\left(\frac{\Lambda^2}{K^2}\right)^{\frac{\alpha}{4\pi}[\eta-\eta_0(\alpha)]} \tag{3.9}$$

where the gauge dependence of the exponent $\ell_F + \frac{3}{2}$ has been explicitly exhibited.

3.2. - The canonical identity

It is clear, both from (3.9) and from (3.1) and (3.2) that in general the vertex bootstrap does not determine α . From (3.7), (3.8) and (3.9) we see that if we are in the defining range $-5/2 < \ell_F < -3/2$ then $\tilde{Z}_1$ and $\tilde{Z}_2 \to 0$ as $\Lambda \to \infty$ for any value of α . Thus, the bootstrap equations (3.1) and (3.2) must be valid for arbitrary α in this range of gauges and analytic continuation should not modify this feature. What the bootstrap in fact does is to provide one relation between α, ℓ_F and η , as can be seen by direct inspection [1)]. Thus, if we write (3.7) in the form

$$\ell_F + \frac{3}{2} = \frac{\alpha}{4\pi}[\eta - \eta_0(\alpha)] \tag{3.10}$$

we see that the vertex bootstrap in an arbitrary gauge can only determine the function $\eta_0(\alpha)$, with α arbitrary.

Now consider Eqs. (3.1) and (3.2) in the vicinity of the canonical gauge, namely for $\delta\eta = \eta - \eta_0(\alpha)$ arbitrarily small but not zero ; let us study the limit of the bootstrap equations as $\delta\eta \to 0$. As $\delta\eta \to 0$, $\Gamma_\mu(\alpha, \eta) \to \Gamma_\mu(\alpha, \eta_0)$, and because $\widetilde{Z}_1(\alpha, \eta_0)$ is finite no divergences should appear on the right-hand side of (3.1) <u>if the integrals are evaluated with $\eta = \eta_0$ in the photon propagator</u>. In principle, this is true for the sum of all graphs but we assume it also holds to any finite order in α. More precisely we assume that $\Gamma_\mu(\alpha, \eta_0)$ can be expressed as a power series in α, and that $\eta_0(\alpha)$ can be determined to any finite order in α by requiring convergence of the integrals at $\eta = \eta_0$.

On the other hand, in carrying out the limiting process $\ell_F + \frac{3}{2} \to 0$, we must remember that $\ell_F = -\frac{3}{2}$ is an exceptional gauge ; this means that if η is kept fixed at a value different from η_0 in $D_{\mu\nu}$, while taking the canonical limit in Γ_μ and S, all integrals will diverge in the limit $\ell_F = -\frac{3}{2}$, however, small $\delta\eta = \eta - \eta_0$ may be. Therefore even though we may calculate $\eta_0(\alpha)$ from the above convergence requirement, we cannot vary ℓ_F and η independently but we must restrict their variation according to (3.10). As in Section 2, the divergences of integrals (at fixed η) should appear as simple poles in $\epsilon = \ell_F + \frac{3}{2}$; indeed our previous argument (Section 2) to this effect, based on the irreducibility of the graphs considered is still applicable in spite of the presence of additional form factors [12]. We conclude therefore that, to any order in α, in the limit $\ell_F \to -\frac{3}{2}$, the right-hand side of (3.1) is the sum of two terms :

1) a "regular" contribution which arises by actually setting $\eta = \eta_0(\alpha)$ in all photon propagators, $\eta_0(\alpha)$ being determined by the convergence requirement ;
2) a "pole" contribution stemming from the $\delta\eta = \eta - \eta_0(\alpha)$ corrections to the photon propagators ; these "pole" terms are in fact finite ; they are proportional to $\delta\eta/\epsilon$, which is finite in the limit $\epsilon \to 0$ because of (3.10) ; this feature is already familiar from Section 2, where the "pole" contribution (2.28) in the gauge bootstrap was shown to be finite in the "canonical" free limit $\ell_F = -\frac{3}{2}$.

Let us now apply the above analysis to see what information can be obtained from the bootstrap in the canonical limit. If we set $q = 0$ in (3.1), then because of (3.2), the left-hand side is simply γ_μ ; the right-hand side, which depends explicitly on α, will yield $\varphi(\alpha)\gamma_\mu$ where $\varphi(\alpha)$ is a function which contains contributions both from "regular" terms, $\varphi_R(\alpha)$, and from "pole" terms, $\varphi_P(\alpha)$; thus

$$1 = \varphi_R(\alpha) + \varphi_P(\alpha) \tag{3.11}$$

Clearly, this equation contradicts our previous assertion that (3.1) and (3.2) do not determine α, unless it is an identity in α. We shall argue that such is indeed the case. Before presenting the general argument, it will be useful to evaluate first all the contributions to $\varphi_R(\alpha)$ and $\varphi_P(\alpha)$ of order zero and one in α, and confirm the identity to order α.

To this end it is convenient to define a quantity $\Gamma_\mu^{(1)}(\alpha;\eta)$ by

$$\Gamma_\mu(\alpha;\eta) = \Gamma_\mu(0;\delta\eta) + \Gamma_\mu^{(1)}(\alpha;\eta) \tag{3.12}$$

where $\Gamma_\mu(0;\delta\eta)$ is by definition the solution of the gauge bootstrap with gauge parameter $\xi = -e^2\delta\eta$; we then substitute (3.12) into (3.1). Since

$$\lim_{\delta\eta\to 0} \Gamma_\mu(0;\delta\eta) = \gamma_\mu$$

we see that :

$$\lim_{\delta\eta\to 0} \Gamma_\mu^{(1)}(\alpha;\eta) \equiv \Gamma_\mu^{(1)}(\alpha;\eta_0)$$

begins at first order in α. The reason for using $\Gamma_\mu(0;\delta\eta)$ in (3.12) instead of simply γ_μ, is that both the complete $\Gamma_\mu(\alpha;\eta)$ and $\Gamma_\mu(0;\delta\eta)$, and therefore also $\Gamma_\mu^{(1)}(\alpha;\eta)$, give rise to integrals with the same behaviour under a scale transformation, and this is crucial for evaluating correctly the residues at the poles in $1/\epsilon$, as seen in Appendices B and C. Essentially, in order to handle correctly the pole singularities, we prepare our power series expansion in α by first dressing all propagators and vertices by gauge photons with coupling constant $\xi = -e^2\delta\eta$ to all orders in ξ ; afterwards we shall retain only terms with a non-vanishing limit as $\xi \to 0$, $\epsilon \to 0$, ξ/ϵ finite.

Since we are only interested in terms of order zero and one in α, we take $\Gamma_\mu^{(1)}(\alpha;\eta)$ to order α. In order to sort out the orders in α we separate each diagram into a regular part R and its remaining pole part P. This is because, by (3.10), the ϵ denominator starts at order α and therefore the pole term of each diagram will be of one order less in α than its regular part. In the limiting gauge η_o, (3.1) yields

$$\lim_{\delta\eta \to 0} \left[\Gamma_\mu(0;\delta\eta) + \Gamma_\mu^{(1)}(\alpha;\eta) \right] = \gamma_\mu + \Gamma_\mu^{(1)}(\alpha;\eta_o) =$$

$$= P_{0\mu} + R_{1\mu} + P_{1\mu}^{(1)} + P_{1\mu}^{(2)} + P_{1\mu}^{(3)} + P_{1\mu}^{(4)} \tag{3.13}$$

where the diagrams $P_{0\mu}$, $R_{1\mu}$, $P_{1\mu}^{(i)}$ $(i=1,2,3,4)$ are represented in Fig. 4. We have kept only terms with a non-vanishing limit as $\delta\eta \to 0$; $P_{0\mu}$ is of order 0 in α, all other terms being of order 1.

The calculation is performed as follows.

1) $R_{1\mu}$ is evaluated at vanishing momentum transfer in Appendix C1. We take η arbitrary but fixed ; in the limiting process we then choose η such that the triangle graph is non-singular. In this way we determine $\eta_o(\alpha)$ (to order zero) as explained in the preceding analysis. This yields the well-known results [3),4)]

$$R_{1\mu}(p,p) = \frac{3\alpha}{8\pi}\gamma_\mu \tag{3.14}$$

$$\eta_o^{(o)}(\alpha) = 1 \qquad (\text{Landau gauge}) \tag{3.15}$$

In the same Appendix we also evaluate $R_{1\mu}(p,0)$, which will be needed below ; we find

$$R_{1\mu}(p,0) = -\frac{\alpha}{8\pi}\gamma_\mu + \frac{\alpha}{2\pi}\frac{p_\mu \not{p}}{p^2} \tag{3.16}$$

2) The pole terms may be evaluated by using the following recipe. Because these terms arise from the $\delta\eta$ corrections in the photon propagator and since all diagrams considered have no divergent subdiagrams in the limit $\epsilon \to 0$, we may neglect $\delta\eta$ in all subintegrations when taking the canonical limit in a subdiagram. Thus, the pole term of a graph

arises entirely from the last integration ; we may therefore <u>in the last integration</u> replace the photon propagator by a gauge photon with coupling $\xi = -e^2 \delta\eta$ and perform the integral. One obtains in this manner

$$P_{o\mu} = \gamma_\mu \tag{3.17}$$

$$P^{(1)}_{1\mu} = -\frac{3\alpha}{8\pi}\gamma_\mu \tag{3.18}$$

Equation (3.17) is obvious because the method described above reduces the evaluation of $P_{o\mu}$ to the calculation of the triangle graph in the gauge bootstrap. A direct check of (3.17) is given in Appendix C1, Eq. (C.14). $P^{(1)}_{1\mu}$ is evaluated in Appendix C2. Note that both (3.17) and (3.18) are independent of the external momenta, a general feature of pole terms.

3) To calculate $P^{(2)}_{1\mu}$ and $P^{(3)}_{1\mu}$ (or $P^{(4)}_{1\mu}$) which depend explicitly on $\Gamma^{(1)}_\mu(\alpha;\eta)$, one may consider (3.13) as an inhomogeneous linear integral equation for $\Gamma^{(1)}_\mu(\alpha;\eta)$ and solve it by iteration, the inhomogeneity being $R_{1\mu} + P^{(1)}_{1\mu}$; [from (3.17) we see that $P_{o\mu}$ has dropped out of (3.13)]. Now in calculating the pole terms we may put the external momenta to zero (and use an infra-red cut-off) because the integral is dominated by the ultra-violet region of integration (see, for instance, Appendices B3 and C2). Thus $R_{1\mu} + P^{(1)}_{1\mu}$ has to be taken at zero momentum transfer in $P^{(2)}_{1\mu}$ and at momentum transfer equal to the integration variable q in $P^{(3)}_{1\mu}$ ($P^{(4)}_{1\mu}$). Therefore from (3.14), (3.16) and (3.18) we obtain the relevant kernels, (3.19) and (3.20) respectively, for $P^{(2)}_{1\mu}$ and $P^{(3)}_{1\mu}$ (or $P^{(4)}_{1\mu}$)

$$R_{1\mu}(p,p) + P^{(1)}_{1\mu} = 0 \tag{3.19}$$

$$R_{1\mu}(q,0) + P^{(1)}_{1\mu} = -\frac{\alpha}{2\pi}\left(\gamma_\mu - \frac{q_\mu \not{q}}{q^2}\right) \tag{3.20}$$

The kernel (3.19) gives zero contribution to $P_{1\mu}^{(2)}$; the contribution of the kernel (3.20) to $P_{1\mu}^{(3)}$ and $P_{1\mu}^{(4)}$ is also zero because the transverse vertex (3.20) is coupled to the longitudinal gauge propagator $\delta\eta\,(q_\mu q_\nu/q^2)$. Thus, the iteration process stops here and

$$P_{1\mu}^{(2)} = P_{1\mu}^{(3)} = P_{1\mu}^{(4)} = 0 \tag{3.21}$$

From (3.14), (3.17), (3.18) and (3.21) we see that to order α , (3.11) is identically satisfied in α with the identification

$$\varphi_R(\alpha) = \frac{3\alpha}{8\pi} \tag{3.22}$$

$$\varphi_P(\alpha) = 1 - \frac{3\alpha}{8\pi} \tag{3.23}$$

Finally, from (3.13) and the convergence requirement it is possible to obtain the value of $\eta_0(\alpha)$ to first order in α ; this is done in Appendix C3 and yields [16]

$$\eta_0(\alpha) = 1 - \frac{3\alpha}{8\pi} \tag{3.24}$$

which agrees with the earlier evaluation of Johnson, Baker and Willey [3].

To order α , the above calculation explicitly exhibits a possible mechanism through which (3.11) could be an identity ; namely, the cancellation of the regular terms by the pole terms. We shall argue that this cancellation phenomenon ought to be true to any order in α and that therefore (3.11) is indeed an identity, which we shall call the <u>canonical identity</u>. Note that to order zero in α , the canonical identity simply reproduces the result of Section 2, namely that free field theory is correctly obtained as the "canonical" limit of the gauge bootstrap.

The reason why one expects (3.11) to be an identity is that the validity of the bootstrap in the canonical limit depends on the validity of the following equation for the vertex renormalization constant

$$\lim_{\delta\eta\to 0}\ \lim_{\Lambda^2/K^2\to\infty} \tilde{Z}_1 = 0 \tag{3.25}$$

where we have chosen $\delta\eta < 0$ to ensure convergence of all bootstrap integrals in the ordinary sense. The order of the limits in (3.25) is crucial ; it is only because we <u>first</u> take $(\Lambda^2/\kappa^2) \to \infty$ that the bootstrap is valid in a nearby gauge $\delta\eta \neq 0$. From (3.9) we then see that (3.25) is indeed satisfied for arbitrary values of α and therefore Eq. (3.11) must be an identity.

In objection to the above argument, one may say that $\eta_o(\alpha)$ as determined from the convergence requirement need not be the same expression as $\eta_o(\alpha)$ determined from the bootstrap in a different gauge. But clearly if $\eta_o(\alpha)$ can be determined from perturbation theory as an analytic function of α [as was already assumed in Ref. 3], then both determinations of $\eta_o(\alpha)$ should yield the same function. Thus if the rather weak analyticity assumptions postulated in writing down (3.11) are correct, then (3.11) is indeed an identity in α .

3.3. - The eigenvalue equation $\widetilde{Z}_{1c}(\alpha) = 0$

Up to this point, when we studied a bootstrap equation in an exceptional gauge (e.g., the canonical gauge), we always meant the limit of the bootstrap equation from nearby gauges. The above analysis suggests, however, the possibility of imposing a stronger convergence condition in the canonical gauge ; namely that the bootstrap be valid in the canonical gauge independent of the limiting process. For this to be true it suffices to impose

$$\varphi_P(\alpha) = 0 \tag{3.26}$$

so that the bootstrap in the canonical gauge is valid even if $\delta\eta$ is set equal to zero at the outset. By (3.11), Eq. (3.26) implies

$$1 = \varphi_R(\alpha) \tag{3.27}$$

The reason why we now get an eigenvalue equation for α is clear if we look at the expression (3.9) for the renormalization constants. Our stronger convergence condition (3.26) means that in (3.25) we have the right to permute the limits and therefore, from (3.5), we must have

$$\tilde{Z}_{1c}(\alpha) = g(\alpha, \eta_0(\alpha)) = 0 \tag{3.28}$$

where the index c refers to the canonical gauge. Equations (3.28) and (3.27) are in fact the same equation ; indeed by definition

$$\tilde{Z}_{1c}(\alpha) = 1 - \Lambda^R(\kappa, \kappa) \tag{3.29}$$

where $\gamma_\mu \Lambda^R(\kappa, \kappa)$ is the sum of all radiative corrections to $\Gamma_\mu(\kappa, \kappa)$ evaluated in renormalized perturbation theory. From the Ward identity, $\Lambda^R(\kappa, \kappa)$ is a pure number and can therefore be evaluated at arbitrary momentum. But by definition $\varphi_R(\alpha)$ is just $\Lambda^R(p,p)$; this proves our statement. For the sake of completeness we show, in Appendix D, how the renormalized perturbation expansion for $\tilde{Z}_{1c}$ can be derived directly from the bootstrap equations. The lowest order contributions to (3.29) are depicted in Fig. 5. In carrying out these integrals, however, one should work in a general gauge in order to avoid ambiguities. More precisely, one has to use a gauge covariant cut-off in an unspecified gauge : this means that to a given order in α one should use, for instance, the Pauli-Villars regularization method and keep all graphs needed to ensure gauge covariance up to that order. Then all terms in $\log^n(\Lambda^2/\kappa^2)$, which may lead to an ambiguous Λ independent constant, will vanish simultaneously from (3.5) for a well defined value of η. This procedure thus fixes the canonical gauge and at the same time determines uniquely the Λ independent term in that gauge.

Finally, we wish to emphasize that it is the canonical identity which allows us to write down the eigenvalue equation (3.29). Indeed, if (3.11) had been an equation for α and not an identity, it would most probably be incompatible with (3.27) and hence also with (3.28). Therefore, if we then decided to retain (3.28) but give up (3.11), we would have a discontinuity at the canonical gauge and thus a violation of gauge covariance for non-vanishing Γ_μ. This inconsistency would show up in practice because the evaluation of Γ_μ directly in the canonical gauge is ambiguous (see Section 5) and so is the eigenvalue equation. It is only because we have the possibility of working in a nearby gauge that these ambiguities are absent. This is clear in the bootstrap calculation of the eigenvalue equation in the form (3.26) or (3.27) ; it is also clear in the perturbation expansion of (3.28) from the preceding discussion. To summarize, we may say that it is the canonical identity which permits the equation $\tilde{Z}_{1c}(\alpha) = 0$ to be defined and to have a gauge invariant content.

4. - THE SELF-ENERGY CONDITION

In addition to the vertex bootstrap equations (3.1) and (3.2), the self-consistent formulation of massless QED contains a self-energy condition expressing the vanishing of the renormalized photon self-energy $\Pi^R_{\mu\nu}(q)$.

$$\Pi^R_{\mu\nu}(q) = (g_{\mu\nu}q^2 - q_\mu q_\nu)\left[\Pi(q^2) - \Pi(-\kappa'^2)\right] = 0 \qquad (4.1)$$

where κ' is an arbitrary momentum. This is, of course, Adler's [2] eigenvalue condition

$$F(\alpha) = 0 \qquad (4.2)$$

because, as is well known [4]

$$\Pi(q^2) = G(\alpha) + F(\alpha) \log \frac{q^2}{\Lambda^2} \qquad (4.3)$$

where $G(\alpha)$ is a finite constant. From now on it should be understood, unless otherwise stated, that graphs containing closed Fermion loops have been left out of $\Pi(q^2)$ [and of $F(\alpha)$].

For arbitrary α the unsubtracted self-energy

$$\Pi_{\mu\nu}(q) = (g_{\mu\nu}q^2 - q_\mu q_\nu)\,\Pi(q^2) \qquad (4.4)$$

is finite in configuration space for non-coincident external points. As this point is crucial, we recall why it is so. Take the Fourier transform of (4.4) after rewriting (4.3) in the form

$$\Pi(q^2) = \left[G(\alpha) - 2F(\alpha)\frac{\Lambda^\epsilon}{\epsilon}\right] + 2F(\alpha)\frac{q^\epsilon}{\epsilon} \tag{4.5}$$

where ϵ is infinitesimal ; then, for $x \neq 0$

$$\Pi_{\mu\nu}(x) = \frac{i}{4\pi^2} F(\alpha)\left(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\frac{1}{x^4} \tag{4.6}$$

which is finite.

Now, instead of expressing $\Pi_{\mu\nu}(x)$ in terms of the bootstrap solutions for Γ_μ and S, as was done in Ref. 1), we shall use the relation

$$\Pi_{\mu\nu}(x) = \tilde{Z}_1 \int \Gamma_\mu S S \gamma_\nu \tag{4.7}$$

It is important to notice that it is not legitimate to insert into (4.7) the bootstrap solutions for Γ_μ and S while keeping the cut-off Λ in $\tilde{Z}_1$. This is so not only because the integral may diverge but also because an arbitrary cut-off regularization of the integral <u>after</u> Γ_μ and S have been replaced by their value at $\Lambda = \infty$ (as is the case in the bootstrap) will violate the gauge invariance of $\Pi_{\mu\nu}$. This arises from the fact that the gauge invariance of (4.7) is due to the cancellation of the cut-off dependence in the short-distance behaviour of $S\Gamma_\mu S = G_\mu$ by the cut-off dependence of $\tilde{Z}_1$; indeed, for small Δx, $G'_\mu = \tilde{Z}_1 . S(x) . \Gamma_\mu . S(x+\Delta x)$ contains the factor $\eta(\Delta x) - \eta(0)$ as is seen from (2.1). Since $\eta(0)$ is ultra-violet divergent it is therefore not legitimate in the present case to evaluate $\eta(\Delta x)$ with $\Lambda \to \infty$ because according to (2.4) this would yield a divergent answer for $\eta(\Delta x) - \eta(0)$ in the limit $\Delta x \to 0$, instead of zero.

It is interesting to note that this difficulty in expressing $\Pi_{\mu\nu}(x)$ in terms of the bootstrap solutions Γ_μ and S, is not overcome by retaining the inhomogeneity $\tilde{Z}_1 \gamma_\mu$ in the Schwinger-Dyson equation (3.1) and eliminating it from (4.7), in favour of Γ_μ's, before going to the bootstrap limit. This would lead to

$$\Pi_{\mu\nu} = \int \Gamma_\mu S S \Gamma_\nu - \int \Gamma_\mu S S K S S \Gamma_\nu \tag{4.8}$$

which is depicted in Fig. 6, using the expansion of K given in Fig. 3b. If both integrals on the right-hand side converged, then (3.1) would imply $\Pi_{\mu\nu} = 0$, assuming Γ_μ and S were taken to be the bootstrap solutions ; this vanishing of $\Pi_{\mu\nu}$ would be the reflection in (4.8) of the formal statement $\tilde{Z}_1 = 0$ used in establishing the bootstrap. But in any conformal bootstrap both terms in (4.8) separately diverge and any reasonable regularization procedure yields a finite result for the difference [17]. This is of course why the self-energy condition is not automatically satisfied in a bootstrap theory as was claimed before [18], and also why the Mack-Symanzik [19] recipe of expressing $x^\sigma \Pi_{\mu\nu}(x)$, instead of $\Pi_{\mu\nu}(x)$, in terms of bootstrap solutions was used in Ref. 1).

In the present case, however, the equation $\tilde{Z}_{1c} = 0$, namely the strong convergence condition of the bootstrap in the canonical gauge, does in fact imply $\Pi_{\mu\nu}(x) = 0$. This is because :

1) Equation (4.7) is valid without introducing a cut-off in Γ_μ and S since $\tilde{Z}_1$ is cut-off independent in the canonical gauge ;
2) the evaluation of cut-off independent functions Γ_μ and S in the canonical gauge can be performed unambiguously for any value of α because of the canonical identity as discussed at the end of Section 3.3.

The conclusion of this rather lengthy discussion is that the gauge invariant content of the equation $\tilde{Z}_{1c}(\alpha) = 0$ is indeed $F(\alpha) = 0$ as indicated by the naïve argument outlined in the introduction and in Ref. 18). This fact, however, hinges crucially upon the validity of the canonical identity (3.11).

We next discuss the nature of the possible zero of $\tilde{Z}_{1c}(\alpha)$ and its relationship to the induced zero of $F(\alpha)$.

First, despite the fact that we have not proved that a zero of $F(\alpha)$ implies a zero of $\tilde{Z}_{1c}(\alpha)$, we can easily show that <u>a zero of $\tilde{Z}_{1c}(\alpha)$ is consistent with an essential zero of $F(\alpha)$</u>. Suppose indeed that we replace K in Eq. (3.1) by K^T, where K^T contains all closed Fermion loops, except vacuum polarization loops. We can then repeat our whole argument and obtain an eigenvalue equation $\tilde{Z}^T_{1c}(\alpha) = 0$, in which closed Fermion loop contributions have been retained. This would lead to $F^T(\alpha) = 0$ (with closed loop contributions included) and hence by Adler's [2] analysis, unitarity (that is the Federbush-Johnson theorem [20]) would imply an essential zero in $F(\alpha)$ and the vanishing of all closed loop contributions in K^T. Therefore $\tilde{Z}_{1c}(\alpha)$ would have a zero at the same value of α as $\tilde{Z}^T_{1c}(\alpha)$, namely at the value where $F(\alpha)$ would possess an essential zero. In symbolic notation

$$\begin{array}{l} \tilde{Z}^T_{1c}(\alpha) = 0 \Longrightarrow F^T(\alpha) = 0 \Longrightarrow F(\alpha) = 0 \text{ essential zero} \\ \qquad\qquad\qquad\qquad\qquad\quad \searrow \tilde{Z}_{1c}(\alpha) = 0 \end{array} \tag{4.9}$$

The nature of the zero of $\tilde{Z}_{1c}(\alpha)$ is more difficult to understand. The following considerations illustrate the delicacy of the problem. If we write (4.7) in the canonical gauge as

$$\Pi_{\mu\nu}(x) = \tilde{Z}_{1c} \int \Gamma^c_\mu S^c S^c \gamma_\nu \tag{4.10}$$

and take the derivative of both sides with respect to α, then we have from $(\partial F(\alpha)/\partial\alpha) = 0$, at the zero of $F(\alpha)$

$$0 = \alpha \frac{\partial}{\partial\alpha} \tilde{Z}_{1c} \int \Gamma^c_\mu S^c S^c \gamma_\nu + \tilde{Z}_{1c}\, \alpha \frac{\partial}{\partial\alpha} \left[\int \Gamma^c_\mu S^c S^c \gamma_\nu \right] \tag{4.11}$$

where the partial derivative is taken at fixed η. The first integral is convergent but the second one is <u>not</u> because the canonical gauge $\eta_0(\alpha)$ is determined by the cancellation of divergences between terms of different orders in α. Thus, to nth order in α, the term corresponding to Γ^c_μ will appear n times in $\alpha(\partial/\partial\alpha)\Gamma^c_\mu$, so that the cancellation of divergences in $\int \Gamma^c_\mu S^c S^c \gamma_\nu$ is not realized in the derivative ; this is easily checked in a second order calculation. Therefore, when $\tilde{Z}_{1c}$ is put equal to zero, the second term in (4.11) is of the undetermined form $0 \times \infty$. From previous experience [17] on this type of ambiguity in conformal integrals of the type (4.11), there appears to be no reason why this undetermined form, when correctly regularized (for instance through the Mack and Symanzik recipe [19]), should yield zero. Therefore, barring a possible accident, $(\partial/\partial\alpha)\tilde{Z}_{1c}$ should be non-vanishing in order to cancel the second term. Nevertheless, this does not necessarily imply $(d/d\alpha)\tilde{Z}_{1c}(\alpha) \neq 0$: if one varies η in order to remain in the canonical gauge as α varies, the divergence in the second integral in (4.11) should cancel in principle through the readjustment of the gauge. On the other hand, this cancellation depends on the limiting process used in evaluating the integrals in the canonical gauge. Further work is needed to ascertain whether the zero of the function $\tilde{Z}_{1c}(\alpha)$ is of finite or infinite order.

5. - INTERPRETATION OF $\tilde{Z}_{1c}(\alpha) = 0$ IN MASSIVE QED

One might be tempted to relate $\tilde{Z}_1$ and $\tilde{Z}_2$ to the conventional renormalization constants Z_1 and Z_2 obtained by a subtraction procedure at $p^2 = m^2$ in massive QED. This section is aimed at avoiding any such confusion ; we shall show indeed that $\tilde{Z}_2$ is not directly related to Z_2 but rather to a different scaling constant which has appeared repeatedly in the literature. In the following discussion we always assume, unless otherwise stated, that the self-consistent coupling constant of massless QED is indeed the physical charge, that is we restrict our discussion to Adler's type I solution [2] of the Gell-Mann - Low equation, as summarized by the condition

$$\beta(\alpha) = m \frac{d}{dm} \log Z_3 = 0 \tag{5.1}$$

As shown by Adler, Eq. (5.1) implies, via the Callan-Symanzik equation [15], the following asymptotic behaviour for the renormalized electron propagator (for $m_o = 0$)

$$S^{-1}(p) \xrightarrow[p^2 \gg m^2]{} f(\alpha)\, C\left(\alpha; \frac{\mu^2}{m^2}\right) \not{p} \left(-\frac{p^2}{m^2}\right)^{\gamma(\alpha)/2} \tag{5.2}$$

$$Z_1 = Z_2 = C\left(\alpha; \frac{\mu^2}{m^2}\right) \left(\frac{\Lambda^2}{m^2}\right)^{\gamma(\alpha)/2} \tag{5.3}$$

Here, μ is a photon mass which has to be introduced because the renormalization program at $p^2 = m^2$ induces infra-red divergences.

We cannot yet make the connection with massless QED because, when $m \neq 0$, there are Fermion loop contributions which have no counterpart in massless QED. However, if one omits all closed loops, the forms of (5.2) and (5.3) are obviously maintained [21] though not the values of the constants $f(\alpha)$ and $c(\alpha; \mu^2/m^2)$ which now refer to a truncated theory. We call the new constants of this truncated theory $f^0(\alpha)$ and $c^0(\alpha; \mu^2/m^2)$; the corresponding renormalization constants are Z_1^0 and Z_2^0. Note that $Z_1^0 = Z_2^0$ because the approximation considered respects the gauge invariance of the massive theory.

The crucial point in the following discussion is that in the no-closed-loop approximation, the asymptotic behaviour of the unrenormalized electron propagator $S'(p)$ is the same in massive QED as it is in the massless case. It then follows from (5.2) and (5.3) that

$$S'(p) \underset{p^2 \gg m^2}{\sim} [f^0(\alpha)]^{-1} \left(-\frac{p^2}{\Lambda^2}\right)^{\gamma(\alpha)/2} \frac{1}{\not p} \qquad (5.4)$$

$$\text{(no-loop approximation)}$$

Comparing (5.4) with (3.4) and (3.5) we conclude that

$$[f^0(\alpha)]^{-1} = g(\alpha; \eta) \qquad (5.5)$$

$[f^0(\alpha)]^{-1}$ is indeed a gauge dependent quantity whose gauge transformation properties are given in Adler and Bardeen's work [6]. They are identical to the gauge transformation properties of $g(\alpha; \eta)$ given by (3.8) with the identification $\gamma = \ell_F + \frac{3}{2}$. If we now call $C_L(\alpha)$ the value of $[f^0(\alpha)]^{-1}$ in the canonical gauge, we have

$$\tilde{Z}_{1c}(\alpha) = C_L(\alpha) \equiv [f_0(\alpha)]^{-1}_{\text{canonical gauge}} \qquad (5.6)$$

Thus the equation $\tilde{Z}_{1c}(\alpha) = 0$ means that in the no-loop approximation to massive QED, the dressed unrenormalized propagator vanishes asymptotically. This property is not true in the full theory because $f^0(\alpha) \neq f(\alpha)$. We also see that $\tilde{Z}_{1c}(\alpha) = 0$ does not at all imply $Z_1^0(\alpha) = 0$ in the canonical gauge and a fortiori not $Z_1(\alpha) = 0$. Physically this is reasonable, because $\tilde{Z}_{1c}$ should be related, in the massive theory, to a renormalization for $K^2 \gg m^2$ (and for a restricted set of graphs) ; it therefore has to do with asymptotic scaling properties, as does $C_L(\alpha)$, and not with low momentum effects which are present in Z_1 [22].

The function $C_L(x)$ has been used earlier in a different context. It appeared in the evaluation of the asymptotic behaviour of the unrenormalized electron propagator in the canonical gauge by Baker, Johnson and Willey [3),4)]. These authors, and others since then [10),23)], identified x to α_0, the solution of the Gell-Mann - Low eigenvalue equation, thus working in the context of what Adler calls type II behaviour [2)]. Recently, Schnitzer [23)] has considered the possibility that the equation $C_L(\alpha_0)=0$, together with $Z_2(\alpha_0)=0$ (in the canonical gauge), is consistent with the short distance behaviour of finite QED. As we are not certain to know the correct asymptotic behaviour of the electron propagator in the case of type II behaviour, it is difficult to understand the relation of Z_2 and $C_L(\alpha)$ in this case ; it should be kept in mind, however, that in type I behaviour, these two quantities appear to be completely different objects, as follows from the above discussion.

Before closing this section, we remark that it has been stated earlier that the evaluation of $C_L(x)$ gives rise to ambiguous integrals. These ambiguities arose in fact when $C_L(x)$ was evaluated directly in the canonical gauge or, equivalently, when calculations using the conformal group were performed in the unrenormalized theory with canonical dimension for the Fermion field [10),23)]. According to the discussion at the end of Section 3.3 no ambiguities will arise in our approach.

6. - CONCLUDING REMARKS

The proposed eigenvalue equation for α, $\tilde{Z}_{1c}(\alpha)=0$, has the following properties, proved under rather weak analyticyty assumptions :

1) it is consistent with the gauge covariance of QED ;
2) $\tilde{Z}_{1c}(\alpha)$ admits a perturbation expansion in α where the coefficient of α^n is unambiguously determined ;
3) its roots are also roots of Adler's eigenvalue equation $F(\alpha)=0$;

These appealing features should not mask the fact that it is not easy to understand why such an equation should really admit a root which is as small a number as 1/137. If this really turns out to be the case (without, of course, modifying the eigenvalue equation), then this result should clearly arise from high order contributions in the power series expansion in α . Under such circumstances, further study of the nature of the asymptotic electron propagator may show that the restriction to some subclass of terms is more meaningful for $\tilde{Z}'_{1c}(\alpha)$ than for $F(\alpha)$.

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APPENDIX A - FOURIER TRANSFORMS, CONVENTIONS AND NOTATION

The following conventions are used throughout the text.

$$F f(x) = \bar{f}(p) = \int e^{ip.x} f(x)\, d^4x \tag{A.1}$$

$$F^{-1} \bar{f}(p) = f(x) = \frac{1}{(2\pi)^4} \int e^{-ip.x} \bar{f}(p)\, d^4p \tag{A.2}$$

In particular

$$F\,(-x^2+i\epsilon)^{\alpha} = -i\,(4\pi)^2\, 4^{\alpha}\, \frac{\Gamma(\alpha+2)}{\Gamma(-\alpha)}\, (-p^2-i\epsilon)^{-\alpha-2} \tag{A.3}$$

$$F^{-1}(-p^2-i\epsilon) = i\,\pi^{-2}\, 4^{\alpha}\, \frac{\Gamma(\alpha+2)}{\Gamma(-\alpha)}\, (-x^2+i\epsilon)^{-\alpha-2} \tag{A.4}$$

The metric is $g_{oo}=1$; $g_{11}=g_{22}=g_{33}=-1$; γ_o is Hermitian and γ_1, γ_2, γ_3 are anti-Hermitian. We also use the shorthand notation $x_{ij}=x_i-x_j$, and

$$x^{\gamma} = (x^2-i\epsilon)^{\gamma/2} \tag{A.5}$$

$$p^{\gamma} = (p^2+i\epsilon)^{\gamma/2} \tag{A.6}$$

When writing Green's functions in momentum space it will always be understood that the over-all momentum conserving factor $(2\pi)^4\delta^4(\sum_i p_i)$ has been divided out.

For Green's functions and renormalization constants, we have applied the following notation conventions throughout the text.

- Unrenormalized Green's functions are denoted by primed quantities ($D'_{\mu\nu}$, S', G'_{μ}, Γ'_{μ}).
- Renormalized Green's functions are denoted by unprimed quantities ($D_{\mu\nu}$, S, G_{μ}, Γ_{μ}).

- Renormalized self-energies (for which the renormalization is subtractive) are written with an explicit superscript R $(\Sigma^R , \Pi_{\mu\nu}^R)$.
- Renormalization constants defined by a subtraction procedure at $p^2 = m^2$ are denoted by capital letters ; a superscript o has been added whenever we refer to a truncated theory (Z_1, Z_2, Z_1^o, Z_2^o).
- Renormalization constants defined by a subtraction procedure at $p^2 = -\kappa^2$ (in massless theories) carry in addition a tilde symbol $(\tilde{Z}_1^G, \tilde{Z}_2^G, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_{1c}, \tilde{Z}_{2c})$.

APPENDIX B - CALCULATIONS IN THE GAUGE BOOTSTRAP

B.1. - Evaluation of the integral (2.23)

The essential tool here is the "vertex identity" of Parisi, Peliti, D'Eramo [24], which in our case reads

$$\int d^4 x_1 \, x_{12}^{a} x_{13}^{b} x_{14}^{c} = -i\pi^2 N(a) N(b) N(c) \, x_{23}^{-4-c} x_{34}^{-4-a} x_{42}^{-4-b} \tag{B.1}$$

under the restriction $a+b+c=-8$ with

$$N(x) = \Gamma\left(\frac{x+4}{2}\right) / \Gamma\left(-\frac{x}{2}\right)$$

One way to derive (B.1) is to obtain first the functional form of the right-hand side using infinitesimal conformal transformations ; the constant $-i\pi^2 N(a)N(b)N(c)$ is then evaluated by using the Fourier transforms of x_{12}^{a}, x_{13}^{b}, x_{14}^{c}.

Because of the spinology in (2.23), one needs the following formulae, which can readily be obtained from (B.1), essentially by differentiation

$$\begin{aligned} \int d^4 x_1 \, \not{x}_{12} \not{x}_{13} \, x_{12}^{a} x_{13}^{b-2} x_{14}^{c} &= \\ &= -i\pi^2 N(a) N(b) N(c) \frac{(4+a)}{b} \not{x}_{24} \not{x}_{34} \, x_{23}^{-4-c} x_{34}^{-6-a} x_{42}^{-4-b} \end{aligned} \tag{B.2}$$

$$\begin{aligned} \int d^4 x_1 \, x_{12}^{\mu} x_{12}^{a-2} x_{13}^{b} x_{14}^{c} &= \\ &= -i\pi^2 N(a) N(b) N(c) \frac{1}{a} \Big[(4+c) x_{23}^{\mu} x_{23}^{-6-c} x_{34}^{-4-a} x_{42}^{-4-b} - \\ &\quad - (4+b) x_{42}^{\mu} x_{23}^{-4-c} x_{34}^{-4-a} x_{42}^{-6-b} \Big] \end{aligned} \tag{B.3}$$

$$\int d^4x_1\, x_{12}^{\mu}\, \not{x}_{12}\, \not{x}_{13}\, x_{12}^{a-2}\, x_{13}^{b-2}\, x_{14}^{c} = i\pi^2 N(a)\,N(b)\,N(c)\cdot\frac{1}{ab}\Big[$$

$$(4+a)(4+b)\, x_{42}^{\mu}\, \not{x}_{24}\, \not{x}_{34}\, x_{23}^{-4-c}\, x_{34}^{-6-a}\, x_{42}^{-6-b} - (4+a)(4+c)\cdot$$

$$x_{23}^{\mu}\, \not{x}_{24}\, \not{x}_{34}\, x_{23}^{-6-c}\, x_{34}^{-6-a}\, x_{42}^{-4-b} + (4+c)\cdot \tag{B.4}$$

$$\cdot\,\gamma^{\mu}\, \not{x}_{23}\, x_{23}^{-6-c}\, x_{34}^{-4-a}\, x_{42}^{-4-b}\Big]$$

With these identities, the evaluation of (2.23) is straightforward, though long and tedious, and yields (2.24).

B.2. - Ward identity

Using the formulae in Appendix A, we find for the zero momentum transfer Fourier transforms of the form factors A and B given in (2.25) and (2.26), the following expression [the factor $(2\pi)^4\delta^4(p_1 - p_2 + q)$ has been omitted]

$$A_\mu(p,p) = B_\mu(p,p) = -(2\pi)^4\, 2^{-2l_F-7}\, \frac{\Gamma(-l_F-\frac{3}{2})}{\Gamma(l_F+\frac{9}{2})}\cdot\Big[\gamma_\mu + (2l_F+3)\, p_\mu \not{p}/p^2\Big]\, p^{2l_F+3} \tag{B.5}$$

From (B.5) and (2.24)

$$\Gamma_\mu(p,p) = \Big[\gamma_\mu + (2l_F+3)\, p_\mu \not{p}/p^2\Big]\,(-p^2/\kappa^2)^{l_F+\frac{3}{2}} \tag{B.6}$$

The renormalized $S^{-1}(p)$ is defined from (2.6) and (2.14) to be

$$S^{-1}(p) = i\not{p}\left(-\frac{p^2}{\kappa^2}\right)^{l_F+\frac{3}{2}} \tag{B.7}$$

We see from (B.6) and (B.7) that the Ward identity (2.16) is indeed satisfied in an arbitrary gauge.

B.3. - Canonical limit ($\ell_F = -\frac{3}{2}$) of (2.24)

In the limit $\ell_F = -\frac{3}{2}$ only the form factor B [cf. Eq. (2.26)] contributes. We first rewrite B in the Migdal form [9] [see Ref. 1), Appendix A]

$$B = \frac{1}{16} \gamma^\alpha \not{x}_{13}\, x_{13}^{-4}\, \gamma_\mu \not{x}_{32}\, x_{32}^{-4}\, \gamma^\beta \, \mathrm{Tr}\left(\gamma_\alpha \not{x}_{12} \gamma_\beta \not{x}_{12}\right) x_{12}^{-2\ell_F - 7} \tag{B.8}$$

The Fourier transform of (B.8) for arbitrary momentum transfer can be represented by the integral

$$B_\mu(p_1,p_2) = -i\,(2\pi)^2\, 2^{-2\ell_F-7} \frac{\Gamma(-\ell_F - \frac{1}{2})}{\Gamma(\ell_F + \frac{7}{2})} \int d^4 q \,\cdot$$
$$\frac{\gamma^\alpha (\not{p}_1 + \not{q}) \gamma_\mu (\not{p}_2 + \not{q}) \gamma^\beta}{(p_1+q)^2 (p_2+q)^2} \left[(2\ell_F + 1) \frac{q_\alpha q_\beta}{q^2} - (\ell_F + \frac{3}{2}) g_{\alpha\beta} \right] q^{2\ell_F + 1} \tag{B.9}$$

The term in $g_{\alpha\beta}$ in (B.9) vanishes in the limit $\ell_F \to -\frac{3}{2}$; we write the remaining contribution $B^0_\mu(p_1,p_2)$ as

$$B^0_\mu(p_1,p_2) = -2^{-2\ell_F - 6} (2\pi)^6 \frac{\Gamma(-\ell_F + \frac{1}{2})}{\Gamma(\ell_F + \frac{7}{2})} I_\mu(p_1,p_2) \tag{B.10}$$

where

$$I_\mu(p_1,p_2) = -\frac{i}{(2\pi)^4} \int d^4 q \left[\gamma^\alpha \frac{(\not{p}_1 + \not{q})}{(p_1+q)^2} \gamma_\mu \frac{(\not{p}_2 + \not{q})}{(p_2+q)^2} \gamma^\beta \right] q_\alpha q_\beta\, q^{-4 + (2\ell_F + 3)} \tag{B.11}$$

The integral (B.11) is a conventional Feynman diagram with a gauge photon having an anomalous dimension $2\epsilon = 2\ell_F + 3$; it is depicted in Fig. 7. In the limit $\epsilon \to 0$ this integral is independent of p_1 and p_2 as can be seen from the following argument. Take $\epsilon < 0$ and add to (B.11) the convergent integral

$$\frac{-i}{(2\pi)^4} \int d^4q \, \not{p}_1 \frac{(\not{p}_1+\not{q})}{(p_1+q)^2} \gamma_\mu \frac{(\not{p}_2+\not{q})}{(p_2+q)^2} \not{q} \, q^{-4+2\epsilon}$$

to obtain

$$-\frac{i}{(2\pi)^4} \int d^4q \, \gamma_\mu \frac{(\not{p}_2+\not{q})\not{q}}{(p_2+q)^2} \, q^{-4+2\epsilon}$$

This convolution is straightforwardly evaluated by the Fourier transform expressions in Appendix A and yields

$$\gamma_\mu \left(\frac{1}{16\pi^2}\right) \Gamma(-\ell_F - \tfrac{3}{2}) \, p_2^{2\ell_F+3}$$

so that when $\epsilon \to 0$

$$I_\mu(p_1,p_2) \longrightarrow -\frac{1}{16\pi^2} \gamma_\mu \frac{1}{\epsilon} \tag{B.12}$$

which holds for $\epsilon > 0$ by analytic continuation via the Γ function. Using (B.12), (B.10) and (2.24) we find that indeed $\Gamma_\mu(p_1,p_2) \to \gamma_\mu$ in the limit $\epsilon \to 0$.

A simpler way to derive (B.12) which we shall use repeatedly in similar calculations is to notice that as $\epsilon \to 0$ (B.11) is dominated by large loop momentum, so that in order to evaluate the residue at the $1/\epsilon$ pole, one may neglect p_1 and p_2 in (B.11) and use an infra-red cut-off momentum ℓ for $\epsilon < 0$. Thereupon (B.11) becomes

$$\gamma_\mu \frac{i}{(2\pi)^4} \int_\ell^\infty d^4q \, q^{-4+2\epsilon}$$

and by Wick rotation, for $\epsilon \to 0$, one obtains

$$\gamma_\mu \int_\ell^\infty d^4q \, q^{-4+2\epsilon} \longrightarrow -\frac{i\pi^2}{\epsilon} \gamma_\mu \tag{B.13}$$

so that (B.12) is recovered. For future use we also quote the following result which may be evaluated in the same way

$$\int_{\ell}^{\infty} d^4 q \, q_\mu \not{q} \, q^{-6+2\epsilon} \longrightarrow -\frac{i\pi^2}{4\epsilon}\gamma_\mu \tag{B.14}$$

B.4. - Evaluation of (2.28)

To prepare for the canonical limit, we use the fact that the total dimension (in units of momentum) of the triangle graph of Fig. 2 is 2ϵ, to redefine the gauge propagator as $q^\sigma q^\tau q^{-4+2\epsilon}$. This recipe, due to Mack [11], guarantees the correct evaluation of the numerical coefficient of the $1/\epsilon$ term when all propagators and vertex functions within the graph are put equal to their canonical value. The integral in (2.28) now becomes simply (B.11), multiplied by ξ. From (B.12) and (2.7) we thus have

$$\lim_{\epsilon\to 0}\Gamma_\mu = \lim_{\substack{\epsilon = -\xi/16\pi^2 \\ \epsilon\to 0}}\left[-\frac{\xi}{16\pi^2}\gamma_\mu\frac{1}{\epsilon}\right] = \gamma_\mu \tag{B.15}$$

APPENDIX C - THE CANONICAL IDENTITY TO ORDER α

C.1. - Evaluation of $R_{1\mu}$ and $P_{o\mu}$

We first calculate $P_{o\mu} + R_{1\mu}(p,p)$. As in Appendix B.4, we redefine the photon propagator $D_{\sigma\tau}$ as $-i(g_{\sigma\tau} - \eta\, q_\sigma q_\tau / q^2) q^{-2+2\epsilon}$, and put all propagators and vertex functions in the graphs equal to their canonical value. According to Mack's theorem [11], which should be applicable to the present case, this procedure guarantees the correct evaluation of the $1/\epsilon$ term ; it also provides us with a cut-off for the determination of $R_{1\mu}(p,p)$. Thus

$$P_{o\mu} + R_{1\mu}(p,p) = \lim_{\epsilon\to 0} \frac{-ie^2}{(2\pi)^4} \int d^4q\, \gamma^\sigma (\not p + \not q)\gamma_\mu (\not p + \not q)\gamma^\tau \cdot (p+q)^{-4} q^{-2+2\epsilon} (g_{\sigma\tau} - \eta q_\sigma q_\tau / q^2) \tag{C.1}$$

or

$$P_{o\mu} + R_{1\mu}(p,p) = \frac{ie^2}{(2\pi)^4} \frac{\partial}{\partial p^\mu} I(p) \tag{C.2}$$

with

$$I(p) = \int d^4q\, \gamma^\sigma \frac{(\not p + \not q)}{(p+q)^2} \gamma^\tau \left(g_{\sigma\tau} - \eta \frac{q_\sigma q_\tau}{q^2}\right) q^{-2+2\epsilon} \tag{C.3}$$

Using the anticommutation rules of the Dirac matrices, we have

$$I(p) = I_1(p) + I_2(p) + I_3(p) \tag{C.4}$$

$$I_1(p) = (-2+\eta) \int d^4q \frac{\not p + \not q}{(p+q)^2} q^{-2+2\epsilon} \tag{C.5}$$

$$I_2(p) = -\eta \int d^4q \frac{1}{(p+q)^2} \not q\, q^{-2+2\epsilon} \tag{C.6}$$

$$I_3(p) = \eta\, p^2 \int d^4q \frac{1}{(p+q)^2} \not q\, q^{-4+2\epsilon} \tag{C.7}$$

These three integrals are convolutions and so can easily be evaluated with the aid of the Fourier transform relations of Appendix A. We obtain, as $\epsilon \to 0$

$$I_1(p) = -i\pi^2(\eta-2)\frac{1}{\epsilon(\epsilon+1)(\epsilon+2)}\, \not{p}\, p^{2\epsilon} \tag{C.8}$$

$$I_2(p) = -i\pi^2\eta\, \frac{1}{\epsilon(\epsilon+2)}\, \not{p}\, p^{2\epsilon} \tag{C.9}$$

$$I_3(p) = i\pi^2\eta\, \frac{1}{(\epsilon+1)(\epsilon-1)}\, \not{p}\, p^{2\epsilon} \tag{C.10}$$

From (C.4) and (C.2) we find, keeping only terms of order $1/\epsilon$ and 1 in the limit $\epsilon \to 0$,

$$P_{o\mu} + R_{1\mu}(p,p) = \lim_{\epsilon\to 0}\left[\frac{\alpha}{4\pi}(\eta-1)\frac{1}{\epsilon} + \frac{3\alpha}{8\pi}\right]\frac{\partial}{\partial p^\mu}(\not{p}\, p^{2\epsilon}) \tag{C.11}$$

From (C.11), we obtain :

1) $\eta_o(\alpha)$ to order zero, by the convergence requirement, namely the vanishing of the residue at the $1/\epsilon$ pole ; clearly,

$$\eta_o^{(0)} = 1 \tag{C.12}$$

2) the regular part $R_{1\mu}$ of the diagram, by setting $\eta = \eta_o^{(0)} = 1$ to eliminate the pole term and then performing the limit $\epsilon \to 0$; this yields

$$R_{1\mu}(p,p) = \frac{3\alpha}{8\pi}\gamma_\mu \tag{C.13}$$

3) the pole part $P_{o\mu}$ of the diagram by retaining the most singular part in ϵ and using (3.10), namely

$$\epsilon = \frac{\alpha}{4\pi}(\eta-\eta_o) = \frac{\alpha}{4\pi}(\eta-1)$$

to order one in α ; upon taking the limit $\epsilon \to 0$ with the constraint $\epsilon = (\alpha/4\pi)(\eta-1)$, one obtains, in addition to (C.13), the contribution

$$P_{o\mu} = \gamma_\mu \tag{C.14}$$

The evaluation of $P_{0\mu} + R_{1\mu}(p,0)$ follows the same pattern. One starts from

$$P_{0\mu} + R_{1\mu}(p,0) = \lim_{\epsilon \to 0} \frac{(-i)e^2}{(2\pi)^4} \int d^4q \, \gamma^\sigma (\not p + \not q) \gamma_\mu \not q \gamma^\tau \cdot$$
$$(g_{\sigma\tau} - \eta q_\sigma q_\tau / q^2)(p+q)^{-2} q^{-4+2\epsilon} \tag{C.15}$$

and calculates all integrals with the help of the Fourier transform formulae (A.3) and (A.4). To order $1/\epsilon$ and 0 one thus obtains for the integral in (C.15)

$$\left\{ \frac{\alpha}{4\pi}(\eta - 1)\frac{1}{\epsilon}\gamma_\mu + \left[-\frac{\alpha}{8\pi}\gamma_\mu + \frac{\alpha}{2\pi} \not p p_\mu / p^2 \right] \right\} p^{2\epsilon} \tag{C.16}$$

This, of course, reproduces (C.12) and (C.14) ; it yields, in addition

$$R_{1\mu} = (p,0) = -\frac{\alpha}{8\pi}\gamma_\mu + \frac{\alpha}{2\pi} \frac{\not p p_\mu}{p^2} \tag{C.17}$$

As noted by Johnson [25], it can be seen by direct inspection of the integral (C.15) that its finite part $R_{1\mu}(p,0)$, in the Landau gauge $\eta_0 = 1$, must satisfy the relation

$$\gamma^\mu R_{1\mu}(p,0) = 0$$

Our result (C.17) indeed obeys this equation.

C.2. - Evaluation of $P_{1\mu}^{(1)}$

As previously, in order to perform correctly the successive subintegrations in the canonical limit, we multiply the photon propagator involved in the last integration by a factor $q^{2\epsilon}$. Thus

$$P_{1\mu}^{(1)}(p,p) = \lim_{\epsilon \to 0} \frac{-e^2}{(2\pi)^4} \int d^4q \, \tilde{D}_{\sigma\tau}(q) \gamma^\sigma \frac{1}{\not q + \not p} \frac{\partial}{\partial q^\mu} \Gamma^\tau(p+q, p) \tag{C.18}$$

with

$$\Gamma^{\tau}(p+q,p) = \frac{-ie^2}{(2\pi)^4}\int d^4q_1\,\gamma^{\eta}\frac{1}{\not q+\not q_1+\not p}\gamma^{\tau}\frac{1}{\not q_1+\not p}\gamma^{\rho}\cdot \left(g_{\eta\rho}-\eta\, q_{1\eta}q_{1\rho}/q_1^2\right)q_1^{-2+2\epsilon'} \tag{C.19}$$

and

$$\tilde{D}_{\sigma\tau}(q) = i\,\delta_\eta\, q_\sigma q_\tau\, q^{-4+2\epsilon} \tag{C.20}$$

Note that an infinitesimal ϵ' is needed in (C.19) in order to obtain a meaningful answer in this intermediate step of the calculation. This does not destroy the validity of (C.20) because we shall put ϵ' equal to zero before taking the limit $\epsilon \to 0$.

Since in the region $\epsilon \sim 0$, the dominant contributions arise from loop momenta q much larger than p, we may evaluate the $1/\epsilon$ contribution of the integral by setting $p=0$, using $\epsilon > 0$ and an infrared cut-off ℓ as in Appendix B.3. Thus we need to know only $(\partial/\partial q^\mu)\,\Gamma^\tau(q,0)$ [as in Ref. 25) for a similar problem]. From (C.16) we then get, in the limit $\epsilon' \to 0$,

$$\frac{\partial}{\partial q^\mu}\Gamma^\tau(q,0) = \frac{\alpha}{2\pi}\left[(\eta-1)\gamma^\tau q_\mu q^{-2} + q^\tau\gamma_\mu q^{-2} + \delta^\tau_\mu \not q\, q^{-2} - 2\not q\, q^\tau q_\mu\, q^{-4}\right] \tag{C.21}$$

In (C.21) we may now use the value of η to order zero in α, so that $\eta - 1 = \eta_0 - 1 = 0$. Inserting (C.21) into (C.18) we obtain

$$P^{(1)}_{1\mu} = \lim_{\epsilon\to 0}(-i)\frac{\alpha}{4\pi^3}\cdot\frac{\alpha}{2\pi}\delta_\eta\int_\ell^\infty d^4q\, q^{-4+2\epsilon}\left\{\gamma_\mu - \not q\, q_\mu/q^2\right\} \tag{C.22}$$

Using (B.13) and (B.14), this is

$$P^{(1)}_{1\mu} = \lim_{\epsilon\to 0}\left(-\frac{3\alpha}{8\pi}\right)\frac{\alpha}{4\pi}\frac{\delta_\eta}{\epsilon} \tag{C.23}$$

and from (3.10)

$$P_{1\mu}^{(1)} = -\frac{3\alpha}{8\pi} \tag{C.24}$$

C.3. - Evaluation of $\eta_0(\alpha)$ to first order in α

In calculating $P_{1\mu}^{(1)}$ we did not need to know $\eta_0(\alpha)$ to order α but only to order one [in (C.21)]. This is because we used the recipe of replacing, in the last integration, the photon propagator by the gauge photon (C.20), instead of evaluating directly the pole term as in Appendix C.1.

Of course, one may also evaluate $P_{1\mu}^{(1)}$ by calculating the coefficient of $1/\epsilon$ in the corresponding graph. To this effect, it suffices to use, instead of (C.20), the propagator

$$\tilde{D}_{\sigma\tau} = -i\left(g_{\sigma\tau} - \eta\, q_\sigma q_\tau / q^2\right) q^{-2+2\epsilon} \tag{C.25}$$

and to replace η by $\eta_0(\alpha)$ in (C.21). After repeating the calculation, one obtains, in place of (C.23)

$$P_{1\mu}^{(1)} = \lim_{\epsilon\to 0} \left(\frac{\alpha}{4\pi}\right)^2 \left\{4 - [\eta + \eta_0(\alpha)] - \frac{1}{2}\eta\eta_0(\alpha)\right\} \frac{1}{\epsilon}\gamma_\mu \tag{C.26}$$

Combining this with the singular term in (C.16) one gets the total singular contribution of the set of graphs in Fig. 4 [recall that $P_{1\mu}^{(2)} = P_{1\mu}^{(3)} = P_{1\mu}^{(4)} = 0$], which is

$$\gamma_\mu \frac{1}{\epsilon}\frac{\alpha}{4\pi}\left\{(\eta-1) + \frac{\alpha}{4\pi}\left[4 - (\eta + \eta_0(\alpha)) - \frac{1}{2}\eta\,\eta_0(\alpha)\right]\right\} \tag{C.27}$$

The convergence requirement which determines $\eta_0(\alpha)$ is that at $\eta = \eta_0(\alpha)$, (C.27) must vanish for fixed ϵ ; this yields to order α

$$\eta_0(\alpha) = 1 - \frac{3\alpha}{8\pi} \tag{C.28}$$

Note that if we now take the limit $\epsilon \to 0$, using the constraint (3.10), we indeed recover the contributions (C.14) and (C.24).

APPENDIX D - RENORMALIZED PERTURBATION THEORY FROM THE BOOTSTRAP EQUATIONS

The Ward identity (3.2), in conjunction with the expression (3.4) for the Fermion propagator, yields

$$\Gamma_\mu(p,p)\Big|_{p^2=-\kappa^2} = \gamma_\mu - 2\left(\ell_F + \frac{3}{2}\right)\frac{p_\mu \not{p}}{\kappa^2} \tag{D.1}$$

$$\Gamma_\mu(p,p)\Big|_{p^2=-\kappa^2} = \gamma_\mu \qquad (\text{if } \epsilon = 0) \tag{D.2}$$

For notational simplicity consider first the canonical case $\epsilon = 0$. If we subtract from Eq. (3.1) the same equation evaluated at $p^2 = -\kappa^2$ with $q = 0$, we get

$$\begin{aligned}\Gamma_\mu(p,p+q) = \gamma_\mu + \Big\{ & \int d^4k\, S(p+k)\,\Gamma_\mu(p+k,p+k+q)\,S(p+k+q)\cdot \\ & \cdot K(p,p+k;\,p+k+q,\,p+q) \\ & - \Big[\int d^4k\, S(p+k)\,\Gamma_\mu(p+k,p+k)\,S(p+k)\cdot \\ & K(p,p+k;\,p+k,\,p)\Big]_{p^2=-\kappa^2}\Big\}\end{aligned} \tag{D.3}$$

To obtain the usual renormalized perturbation expansion for $\Gamma_\mu(p,p+q)$ we simply solve (D.3) by iteration using the Ward identity (3.2) for S. If we then substitute the result of this expansion to a given order in α into the equation

$$\begin{aligned}\gamma_\mu \Lambda(\kappa,\kappa) = \Big[& \int d^4k\, S(p+k)\,\Gamma_\mu(p+k,p+k)\,S(p+k)\cdot \\ & K(p,p+k;\,p+k,\,p)\Big]_{p^2=-\kappa^2}\end{aligned} \tag{D.4}$$

we recover, from (3.29), the perturbation expansion for $\widetilde{Z}_{1c}(\alpha)$; the subsidiary condition in (D.3) arising from the subtracted equation is simply the eigenvalue equation (3.28).

Equations (D.3) and (D.4) are valid outside the canonical limit if we subtract from (3.1) only the form factor which is proportional to γ_μ at $p^2 = -\kappa^2$, which is then used in the right-hand side of (D.3) and (D.4). It is then apparent that a cut-off has to be used in evaluating (D.4) and as an intermediate step in (D.3) as long as ϵ is not zero.

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2) S.L. Adler - Phys.Rev. D5, 3021 (1972).

3) K. Johnson, M. Baker and R. Willey - Phys.Rev. 136, B1111 (1964).

4) K. Johnson, M. Baker and R. Willey - Phys.Rev. 163, 1699 (1967) ;
M. Baker and K. Johnson - Phys.Rev. 183, 1292 (1969), and D3, 2516, 2541 (1971).

5) See, for instance :
K.C. Johnson and B. Zumino - Phys.Rev.Letters 3, 351 (1959).
It is easy to check the validity of Eq. (2.1) for any Green's function involving a single Fermion line and for any value of the coupling to the $g_{\mu\nu}\, q^{-2}$ term in the photon propagator, including zero : one merely takes the derivative with respect to ξ of the corresponding Feynman graphs and uses the Ward identity.

6) S.L. Adler and W.A. Bardeen - Phys.Rev. D4, 3045 (1971).
These authors use a photon mass regularization in the infra-red. One can readily check that regularization with an anomalous dimension in the gauge propagator, $q_\mu q_\nu / q^{4-\epsilon}$, $\epsilon > 0$, leads to the same result (2.4) but with a different finite constant b.

7) A and B are the conformal form factors (III.5) and (III.6) of Ref. 1) (with $\ell = -3$) which do satisfy the functional form of the Ward-Takahashi identity [Eq. (VI.5) of Ref. 1)]. For a more general discussion on the relationship between zero and non-zero momentum transfer generalized Ward identities in the conformal group, see Ref. 19).

8) A.M. Polyakov - JETP Letters 12, 538 (1970).

9) A.A. Migdal - Phys.Letters 37B, 98 and 386 (1971).

10) R.A. Abdellatif - "Quantum Electrodynamics with no Self-Energy Insertions", University of Washington Dissertation (1970).

11) G. Mack - "Expansion Around Canonical Dimensions in Conformal Invariant Quantum Field Theory", Bern Report (1972), unpublished, and International Summer Institute on Theoretical Physics in Kaiserslautern (1972), Eds. W. Rühl and A. Vancura, Springer, Berlin (1973), p. 300.

12) A similar situation arises when the renormalization program is carried out by the methods of G. 't Hooft and M. Veltman, Nuclear Phys. B44, 189 (1972). It would be interesting to relate the bootstrap to this method of renormalization instead of the more conventional approach followed in the present paper. This is so not only because of the formal analogy between our integrals and theirs, but also because their method might provide a suitable tool for studying the canonical identity of Section 3 to all orders in α .

13) Note that the present conventions differ from the ones in Ref. 1) where a factor of e was included in the definition of Γ_μ .

14) M. Gell-Mann and F.E. Low - Phys.Rev. 95, 1300 (1954).

15) C.G. Callan - Phys.Rev. D2, 1541 (1970) ;
K. Symanzik - Commun.Math.Phys. 18, 227 (1970).

16) At this stage the fact that $\varphi_P(\alpha) = \eta_o(\alpha)$ appears to be accidental.

17) G. Parisi - Nuovo Cimento Letters 4, 777 (1972).

18) F. Englert and C. De Dominicis - Nuovo Cimento 53A, 1021 (1968). See also Ref. 1), footnote 17.

19) G. Mack and K. Symanzik - DESY 72/20, to appear in Commun.Math.Phys.

20) P.G. Federbush and K. Johnson - Phys.Rev. 120, 1926 (1960).

21) A similar discussion is to be found in S.L. Adler and W.A. Bardeen, Erratum to Ref. 6), Phys.Rev. D6, 734 (1972). In the case treated by these authors, however, the value of f did not necessarily change when one switched from a complete theory to a truncated one. This was because they refer to a model admitting type II asymptotic behaviour [2].

22) The non-vanishing of Z_1 in the canonical gauge was implicitly assumed in Ref. 1). Indeed (3.1) was established from the no-closed-loop approximation in massive QED at asymptotic momenta. The dependence of (3.1) on the electron mass dropped out, except as a scale to measure momentum ; this is because $C^o(\alpha ; \mu^2/m^2)$ cancelled out of the equations, leaving an infra-red finite theory. The latter was then interpreted in the language of renormalized massless QED. Of course, one could also obtain (3.1) in massless QED by renormalizing directly at $p^2 = -\kappa^2$, as in Section 2 of this paper. The consistency between these two approaches rests, however, on the assumption that $C^o(\alpha ; \mu^2/m^2)$ (which is equal to Z_1^o in the canonical gauge) is indeed different from zero.

23) H.J. Schnitzer - "Constraints and Anomalies in Finite Quantum Electro-Dynamics", Brandeis Preprint (January 1973).

24) M. D'Eramo, G. Parisi and L. Peliti - Nuovo Cimento Letters 2, 878 (1971).

25) K. Johnson - Lectures on Particles and Field Theory, Brandeis Summer Institute in Theoretical Physics (1964), Prentice-Hall (1965), Volume II, p. 1.

FIGURE CAPTIONS

Figure 1 Equation (1.1)

 $\equiv \Gamma_\mu$; $\equiv S$

Figure 2 Vertex bootstrap equation (2.15) for a gauge field

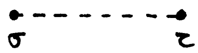
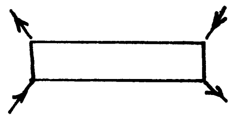
 $\equiv D^{G}_{\sigma\tau}$

Figure 3 Vertex bootstrap equation (3.1)

 $\equiv$ Irreducible Bethe-Salpeter kernel

 $\equiv D_{\sigma\tau}$

Figure 4 Equation (3.13). The e^2 dependence of each graph has been indicated explicitly. The symbol R (or P) in front of a graph means regular part (or pole part).

$$\mu\,\delta\eta \equiv \Gamma_\mu(0;\delta\eta)$$
$$\mu \otimes \equiv \Gamma_\mu^{(1)}(\alpha;\eta)$$

Figure 5 Graphs contributing to the perturbation expansion of $\widetilde{Z}_{1c}(\alpha)$. All external Fermion lines are at $p^2 = -\kappa^2$

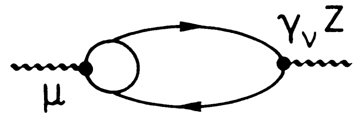
$$\overset{\bullet}{\mu} \equiv \gamma_\mu$$

$\square$ $\equiv$ subtracted subdiagram at $p^2 = -\kappa^2$

Figure 6 Formal expression for $\Pi_{\mu\nu}$ in terms of the Green's functions entering the bootstrap equations.

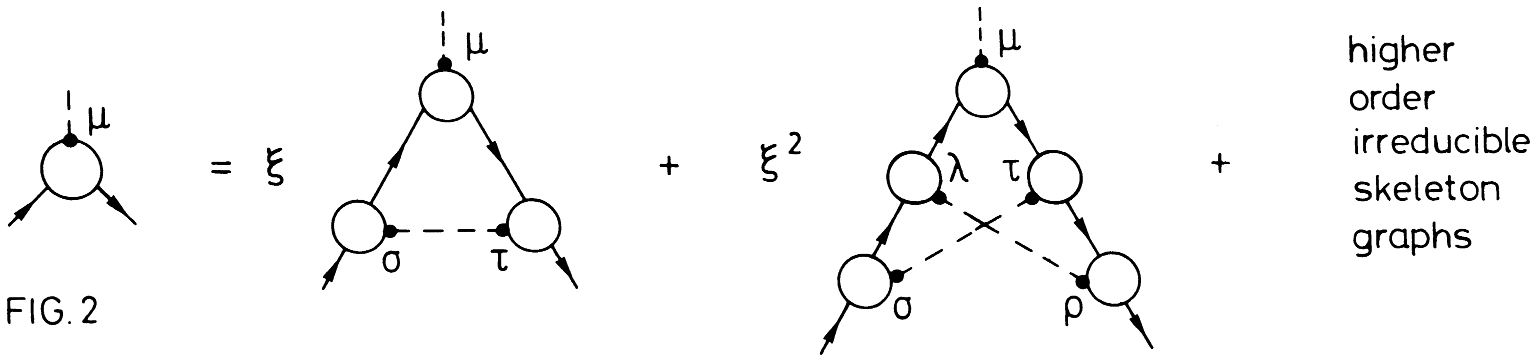
Figure 7 Triangle diagram representing (B.11)

 $\equiv$ anomalous gauge propagator $q_\alpha q_\beta q^{-4+2\epsilon}$.



$\gamma_v Z$
μ

FIG.1



μ
$= \xi$
μ
σ
τ
$+$
ξ^2
μ
λ
τ
σ
ρ
$+$
higher
order
irreducible
skeleton
graphs

FIG. 2

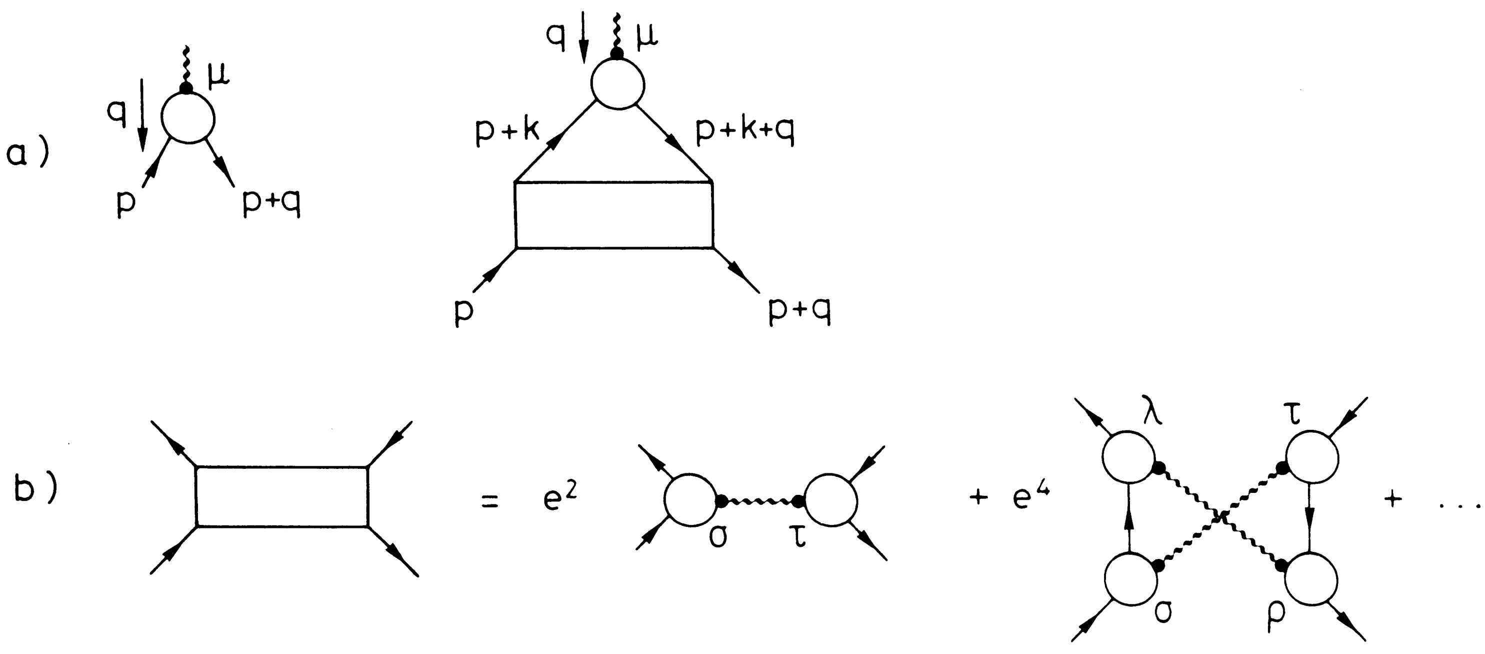
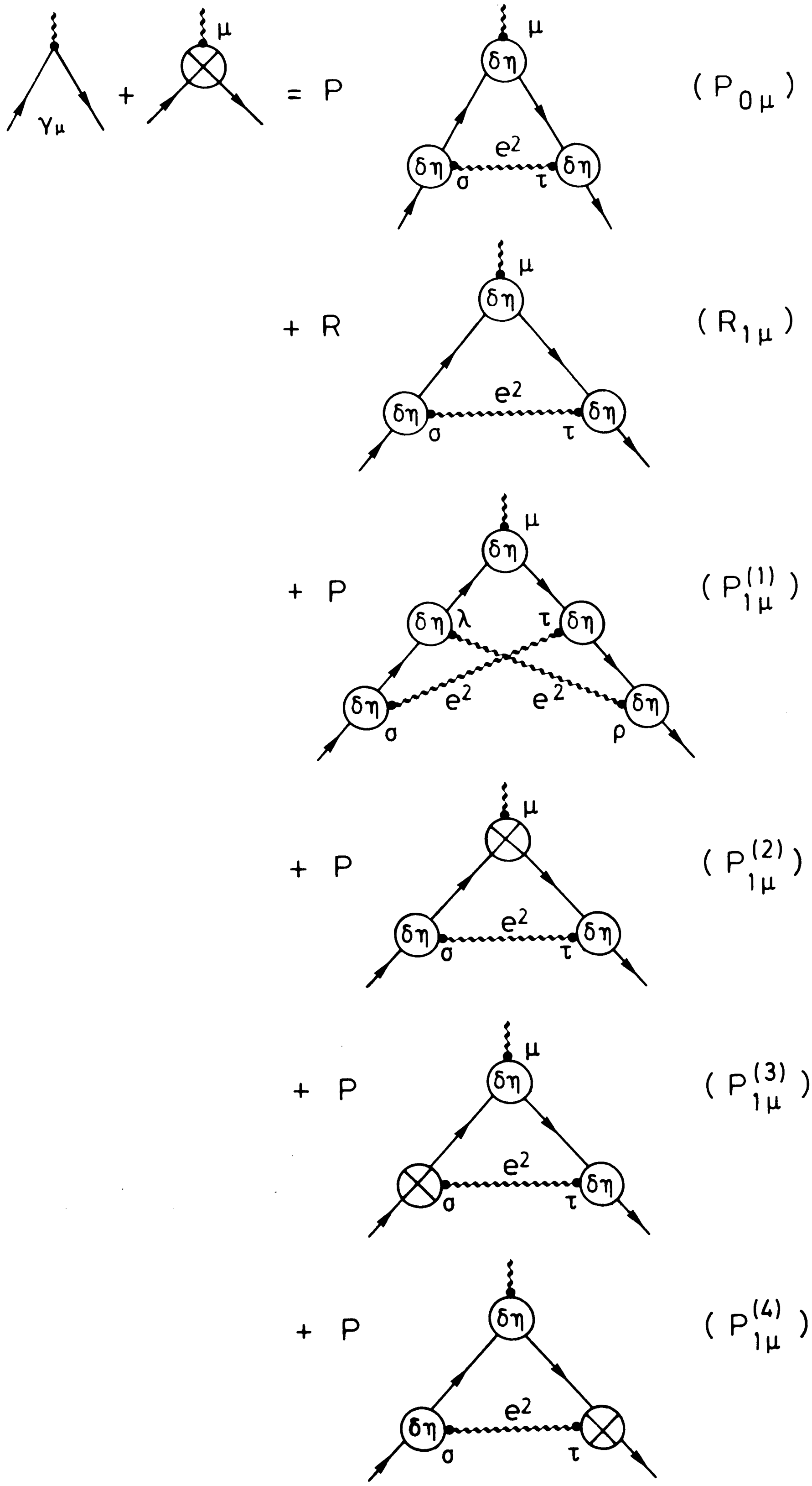


FIG. 3

FIG. 4



γ_μ
+
μ
= P
$\delta\eta$
e^2
σ
τ
$(P_{0\mu})$
+ R
$(R_{1\mu})$
+ P
λ
ρ
$(P^{(1)}_{1\mu})$
+ P
$(P^{(2)}_{1\mu})$
+ P
$(P^{(3)}_{1\mu})$
+ P
$(P^{(4)}_{1\mu})$

FIG. 5

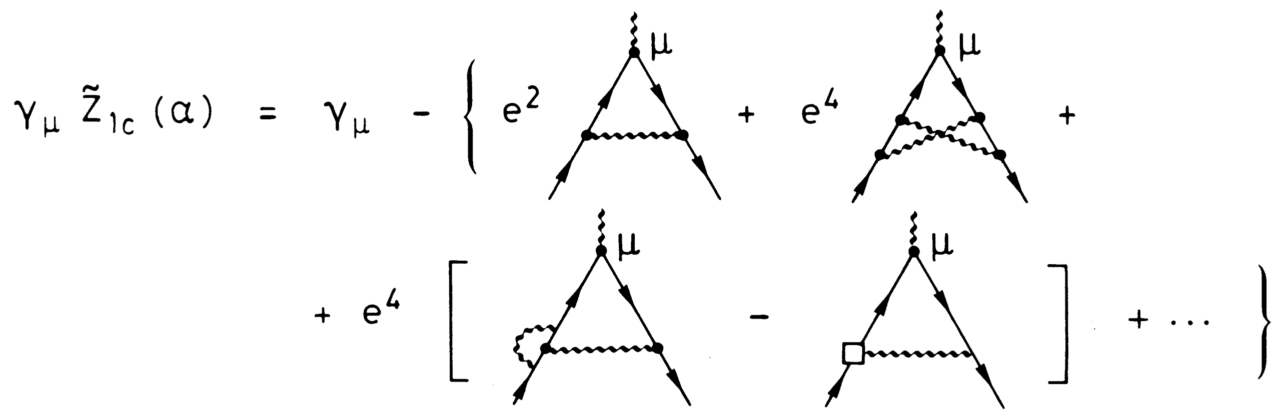


FIG. 6

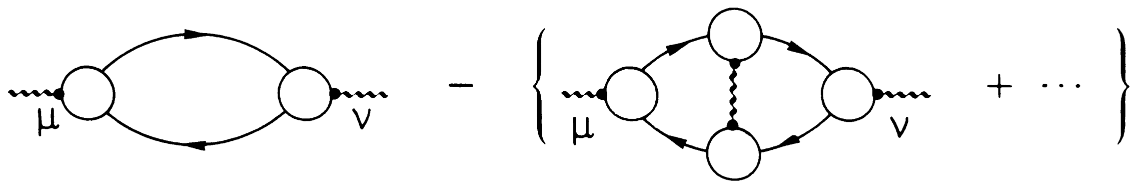
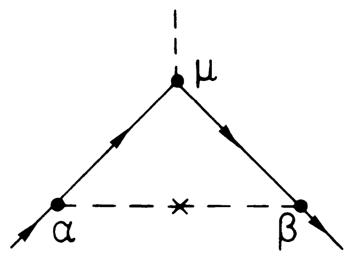


FIG. 7



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VERTEX BOOTSTRAP FOR THE FINE STRUCTURE CONSTANT

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E R R A T U M

The missing line (second line from the bottom of p. 9) reads :

"a Laurent expansion in ϵ (with ξ kept fixed). One understands"

Ref.TH.1671-CERN
18 June 1973