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SHORT-DISTANCE BEHAVIOUR OF QUANTUM ELECTRODYNAMICS AND THE CALLAN-SYMANZIK EQUATION FOR THE PHOTON PROPAGATOR *)

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A B S T R A C T

The short-distance behaviour of the photon propagator is discussed within the context of the corresponding Callan-Symanzik equation. The Callan-Symanzik function $\beta(\alpha)$ is calculated in perturbation theory up to sixth order. We find

$$\beta(\alpha) = \frac{2}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 - \frac{121}{144}\left(\frac{\alpha}{\pi}\right)^3 + \mathcal{O}\left[\left(\frac{\alpha}{\pi}\right)^4\right] .$$

The simplicity of this result is to be contrasted with a corresponding perturbation theory calculation of the Gell-Mann, Low function $\psi(z)$, whose sixth order coefficient contains the transcendental $\zeta(3)$ (the Riemann zeta function of argument three). A mechanism of cancellations in the calculation of $\beta(\alpha)$ has been found, and we prove its validity to all orders in perturbation theory.

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1. INTRODUCTION

An appropriate way to study the short distance behaviour of Green's functions in perturbation theory is by means of their corresponding "broken scale invariance Ward identities", the so-called Callan-Symanzik equations [1)-3)]. These equations have been proved useful in a variety of applications [4)-10)]. In this paper we shall be concerned with the corresponding Callan-Symanzik equation for the asymptotic photon propagator. There is only one function, the Callan-Symanzik function $\beta(\alpha)$, which governs the short-distance behaviour of the photon propagator. In the case where $\beta(\alpha) = 0$ quantum electrodynamics becomes a finite theory. As we shall see, the well-known renormalization group constraints [11),12)] for the photon propagator are very easily obtained from the Callan-Symanzik equation. Furthermore, the function $\beta(\alpha)$ appears to be simpler than the Gell-Mann Low function $\psi(z)$ which governs the usual renormalization group analyses. A perturbation theory calculation of $\beta(\alpha)$, up to sixth order, shows that the coefficients of $(\frac{\alpha}{\pi})^n$ $n = 1,2,3$ are all rational numbers. This is to be contrasted with a corresponding perturbation theory calculation [13)] of $\psi(z)$, whose sixth order coefficient contains the transcendental $\zeta(3)$ (the Riemann zeta function of argument three).

We present a direct calculation of $\beta(\alpha)$, up to sixth order, which explicitly exhibits an interesting mechanism of cancellations. These cancellations are due to the fact that terms arising from the mass differentiation of fermion loop corrections to internal photon lines cancel with corresponding terms arising from the coupling constant differentiation. We give a formal proof of this type of cancellation to all orders in perturbation theory and show explicitly how it works up to eighth order.

The paper has been organized as follows. In the next section we discuss the Callan-Symanzik equation for the photon propagator, and from it we derive the well-known renormalization group constraints. We also discuss the relationship between the various functions which are relevant in the study of the asymptotic photon propagator. The results of previous perturbation theory calculations are summarized in Section 2c. Section 3 deals with the actual calculation of $\beta(\alpha)$ in perturbation theory. Although the second and fourth order terms are well known, we calculate them explicitly in order to illustrate the techniques used in higher order calculations. The general proof of the mechanism of cancellations mentioned above can be found in Section 4. An explicit discussion of the eighth order cancellations can also be found in Section 4. A summary of the results obtained and the conclusions appear in Section 5. The details of calculations and some technical points are discussed in the appendices.

2. THE CALLAN-SYMANZIK EQUATION FOR THE PHOTON PROPAGATOR

The general expression for the renormalized photon propagator $\alpha D_R^{\mu\nu}(q)$ is

$$\alpha D_R^{\mu\nu}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{\alpha d_R(q^2/m^2, \alpha)}{q^2} + \alpha(\xi - 1)\, i\,\frac{q^\mu q^\nu}{(q^2)^2} \tag{2.1}$$

where ξ is a parameter which expresses the gauge freedom in the free-field propagator. In the Landau gauge $\xi = 1$. The relation between $\alpha D_R^{\mu\nu}(q)$ and the renormalized photon proper self energy tensor $i\Pi_R^{\mu\nu}(q)$ (see Fig. 1) is as follows:

$$i\Pi_R^{\mu\nu}(q) = -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\,\alpha\,\Pi_R(q^2, m^2, \alpha)\,, \tag{2.2}$$

with (see Fig. 2):

$$d_R\left(\frac{q^2}{m^2}, \alpha\right) = \frac{1}{1 + \alpha\,\Pi_R(q^2, m^2, \alpha)} \tag{2.3}$$

The Callan-Symanzik equation [1)-3),5)] is a linear partial differential equation, with partial derivatives of $d_R((q^2/m^2), \alpha)$ with respect to the physical parameters in the theory: the renormalized mass m and the renormalized coupling constant α. The equation reads

$$\left\{ m\frac{\partial}{\partial m} + \beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha} - 1\right)\right\} d_R^{-1}\left(\frac{q^2}{m^2}, \alpha\right) = [1 + \delta(\alpha)]\,\alpha\,\Gamma_{\gamma\gamma S}\left(\frac{q^2}{m^2}, \alpha\right) \tag{2.4}$$

where $\Gamma_{\gamma\gamma S}((q^2/m^2),\alpha)$ is a cut off-independent "photon-photon-scalar vertex" with the scalar current carrying null four-momentum; $\beta(\alpha)$ and $\delta(\alpha)$ are functions of α alone, also cut-off independent. The definition of these functions in terms of renormalization constants is as follows:

$$\beta(\alpha) \equiv Z_3^{-1}\, m \,\frac{dZ_3}{dm} \tag{2.5}$$

$$1+\delta(\alpha) \equiv m_0^{-1}\, m\, \frac{dm_0}{dm} \tag{2.6}$$

and

$$\Gamma_{\gamma\gamma S}\left(\frac{q^2}{m^2},\alpha\right) \equiv m_0 \frac{\partial \Pi(\Lambda^2;q^2,m^2,\alpha)}{\partial m_0} \tag{2.7}$$

where m_0 is the unrenormalized mass of the electron; Z_3 the charge-renormalization constant; and $\Pi(\Lambda^2;q^2,m^2,\alpha)$ the unrenormalized photon proper self-energy, with Λ^2 expressing the cut-off dependence [1]. We recall that

$$Z_3 = 1 - \alpha\, \Pi(\Lambda^2;0,m^2,\alpha), \tag{2.8}$$

and

$$\Pi_R(q^2,m^2,\alpha) = \Pi(\Lambda^2;q^2,m^2,\alpha) - \Pi(\Lambda^2;0,m^2,\alpha) \tag{2.9}$$

We are interested in the high energy behaviour of the photon propagator, i.e., in the asymptotic behaviour of $\alpha\, d_R((q^2/m^2),\alpha)$ when $-q^2/m^2 > 0$ becomes large [2]. The asymptotic part of the photon propagator $\alpha d_R^{\infty}((q^2/m^2),\alpha)$ is defined as follows: at each order of the perturbation series in powers of α one takes the limit $-q^2/m^2 \to \infty$ and drops terms which vanish as $-q^2/m^2 \to \infty$; constant terms are kept. The expression for $\alpha d_R^{\infty}((q^2/m^2),\alpha)$ therefore has the form [3]

$$\alpha d_R^{\infty}\left(\frac{q^2}{m^2},\alpha\right) = q(\alpha) + p(\alpha)\log\left(\frac{-q^2}{m^2}\right) + r(\alpha)\log^2\left(-\frac{q^2}{m^2}\right) + \cdots \quad (2.10)$$

It can be shown [5], by direct application of Weinberg's theorem [14] on high energy behaviour, that the vertex $\Gamma_{\gamma\gamma S}((q^2/m^2),\alpha)$ vanishes at each order of perturbation theory when $-q^2/m^2 \to \infty$. The Callan-Symanzik equation for the asymptotic part of the photon propagator is then

$$\left\{ m\frac{\partial}{\partial m} + \beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha} - 1\right)\right\}\left\{ 1 + \alpha\Pi_R^{\infty}(q^2, m^2, \alpha)\right\} = 0 \quad (2.11)$$

We see from this equation that the high energy behaviour of the photon propagator is uniquely governed by the function $\beta(\alpha)$.

It can be seen from Eq. (2.5) that the charge renormalization constant Z_3 also obeys a linear partial differential equation of the same type as Eq. (2.11). Indeed, from Eq. (2.5) and taking into account that

$$m\frac{d}{dm} = m\frac{\partial}{\partial m} + \alpha\beta(\alpha)\frac{\partial}{\partial\alpha} \quad (2.12)$$

we have

$$\left\{ m\frac{\partial}{\partial m} + \beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha} - 1\right)\right\} Z_3\left(\frac{\Lambda^2}{m^2},\alpha\right) = 0 \quad (2.13)$$

This is the Callan-Symanzik equation for the charge renormalization constant [5),15)].

a. The renormalization group constraints

Let us examine Eq. (2.11) in perturbation theory. The asymptotic part of the photon proper self energy has the following expansion $(x \equiv -q^2/m^2)$

$$\alpha\Pi_R^{\infty}(-x,\alpha) = (a_1 + b_1\log x)\left(\frac{\alpha}{\pi}\right) + (a_2 + b_2\log x + \cdots)\left(\frac{\alpha}{\pi}\right)^2 + \\ + (a_3 + b_3\log x + c_3\log^2 x + \cdots)\left(\frac{\alpha}{\pi}\right)^3 + \\ + (a_4 + b_4\log x + c_4\log^2 x + d_4\log^3 x + \cdots)\left(\frac{\alpha}{\pi}\right)^4 + \cdots \quad (2.14)$$

The fact that no higher powers of $\log x$ appear in the coefficient of α/π in (2.14) may be verified by an explicit calculation. On the other hand $\beta(\alpha)$ in perturbation theory also has a power series expansion in α:

$$\beta(\alpha) = \beta_1\frac{\alpha}{\pi} + \beta_2\left(\frac{\alpha}{\pi}\right)^2 + \beta_3\left(\frac{\alpha}{\pi}\right)^3 + \cdots \quad (2.15)$$

Inserting these series in Eq. (2.11) and taking into account the fact that $\beta(\alpha)$ is a function of α alone, finite to each order in perturbation theory, we find order by order the following relations

$$\left(\frac{\alpha}{\pi}\right): \quad \beta_1 = -2b_1$$

$$\left(\frac{\alpha}{\pi}\right)^2: \quad \beta_2 = -2b_2\,; \qquad \text{no } \log^n x \text{ terms, } n>1$$

$$\left(\frac{\alpha}{\pi}\right)^3: \quad \beta_3 = -2b_3 - 2b_1a_2\,; \begin{cases} 2c_3 + b_1b_2 = 0 \\ \text{no } \log^n x \text{ terms, } n>2 \end{cases}$$

$$\left(\frac{\alpha}{\pi}\right)^4: \quad \beta_4 = -2b_4 - 2b_2a_2 - 4b_1a_3\,; \begin{cases} 2c_4 + (b_2)^2 + 2b_1b_3 = 0 \\ 3d_4 + 2b_1c_3 = 0 \\ \text{no } \log^n x \text{ terms, } n>3 \end{cases}$$

$$\cdots \quad (2.16)$$

The equations in the first column give the relation between the coefficients β_n and the coefficients of $\log x$ and constant terms in the perturbation series expansion of $\alpha \Pi_R^{\infty}(-x,\alpha)$. The other equations are precisely the well-known renormalization group constraints [4,16]. These are the constraints which in the perturbation series expansion of $\alpha \Pi_R^{\infty}(-x,\alpha)$, or equivalently $Z_3((\Lambda^2/m^2),\alpha)$ determine leading powers of $\log x$ or $\log \Lambda^2/m^2$, in terms of lower order terms. It is remarkable how easily these constraints are obtained from the Callan-Symanzik equation.

b. The Relation between $\beta(\alpha)$ and the Gell-Mann and Low function $\psi(z)$; the Johnson-Baker-Willey program and Adler's eigenvalue problem

The Gell-Mann, Low equation follows from the Lie equation of the group of renormalization transformations in the asymptotic region. The equation reads

$$\log x = \int_{q(\alpha)}^{\alpha d_R^{\infty}(-x,\alpha)} \frac{dz}{\psi(z)} \tag{2.17}$$

where we recall [see Eq. (2.10)] that

$$q(\alpha) = \alpha\, d_R^{\infty}(x=1,\alpha) \tag{2.18a}$$

and

$$\psi[q(\alpha)] = p(\alpha) \tag{2.18b}$$

The latter result follows from differentiating Eq. (2.17) with respect to x and setting $x = 1$. The function $\psi(z)$ is the Gell-Mann, Low function. By formal integration of the Callan-Symanzik equation it is possible to write its integral solution in the form of the Gell-Mann, Low equation and to obtain a functional relation between $\psi(z)$ and $\beta(\alpha)$. This relation is [5]

$$\psi[q(\alpha)] = \frac{1}{2}\alpha\beta(\alpha)\,\frac{dq(\alpha)}{d\alpha} \quad . \tag{2.19}$$

As $x \to \infty$ in Eq. (2.17), the upper limit of the integral [i.e., $\alpha d_R^{\infty}(-x,\alpha)$] will also tend to infinity unless $\psi(z)$ has a sufficiently strong zero at a value $z = \alpha_0$ that the integral diverges. In this case the asymptotic photon propagator is finite:

$$\lim_{x \to \infty} \left[\alpha d_R^{\infty}(-x,\alpha)\right] = \alpha_0 \tag{2.20}$$

and

$$\psi(\alpha_0) = 0. \tag{2.21}$$

This is the Gell-Mann, Low eigenvalue condition for the asymptotic coupling constant α_0. This possibility has been analyzed in much detail by Baker, Johnson, Willey and collaborators [13),17)-22)]. In particular, these authors Refs. 18), 13), have shown that the Gell-Mann, Low eigenvalue condition can be replaced by a much simpler condition

$$F^{[1]}(\alpha_0) = 0 \quad . \tag{2.22}$$

The function $F^{[1]}(\alpha_0)$ is the single fermion loop part of the coefficient of $\log(-q^2/m^2)$ in a power series expansion of $\Pi_R(q^2, m^2, \alpha_0)$ with full propagators for the internal photons equal to

$$\alpha \tilde{D}_R^{\mu\nu}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{\alpha_0}{q^2} + \text{gauge terms} \tag{2.23}$$

Recently, Adler [5] has shown that any such zero of the function $F[1]$ is a zero of infinite order. As a consequence, he has furthermore shown that there is then a possible solution where the fine structure constant α, instead of α_0, is the zero of $F[1]$. This intriguing possibility has triggered a new interest in the study of short distance behaviour of quantum electrodynamics [23),24)].

c. Review of previous perturbation theory calculations

Several coefficients in the perturbation series expansions introduced above are already known. The second order coefficients in Eq. (2.14) have been calculated by anyone lecturing on quantum electrodynamics:

$$b_1 = -1/3 \quad ; \quad a_1 = 5/9 \quad . \tag{2.24}$$

The calculation of the fourth order coefficient b_2 was first done by Jost and Luttinger [25)]:

$$b_2(\text{Jost-Luttinger}) = -1/4 \quad . \tag{2.25}$$

The corresponding fourth order constant term a_2 is already rather complicated. It has been calculated by Hagen and Samuel [26)], and independently by Lautrup and de Rafael [27)] as a by-product of a sixth order calculation of the anomalous magnetic moment of the muon [5]. Both groups used the fourth order vacuum polarization spectral function calculated by Källen and Sabry [28)]. The result is

$$a_2(\text{Hagen-Samuel; Lautrup-de Rafael}) = \frac{5}{24} - \zeta(3) , \tag{2.26}$$

where $\zeta(3)$ is Riemann's ζ function of argument 3,

$$\zeta(3) = \sum_{n=1}^{\infty} (1/n^3).$$

The presence of this transcendental number is an indication of the complexity of the calculation [6],29)-31)].

Up to and including fourth order, the Feynman diagrams which contribute to $\alpha \Pi_R$ contain a single fermion loop (see Fig. 1). Two fermion loops appear for the first time in sixth order (see Fig. 3). The contribution to b_3 from Feynman diagrams involving one fermion loop only (see Fig. 3a) which we shall denote by $b_3^{[1]}$, has been calculated by Rosner [32]. The motivation for this calculation was the JBW program, which requires the knowledge of the function $F^{[1]}(\alpha_0)$ introduced in Section 2b. Rosner found an extremely simple result

$$b_3^{[1]}(\text{Rosner}) = 1/32 \quad . \tag{2.27}$$

The mechanism of transcendental cancellations in this (a priori very complex) calculation is not yet fully understood [7]. The function $F^{[1]}(y)$ is therefore known up to sixth order in perturbation theory [8]:

$$-y\,F^{[1]}(y) = \frac{2}{3}\left(\frac{y}{2\pi}\right) + \left(\frac{y}{2\pi}\right)^2 - \frac{1}{4}\left(\frac{y}{2\pi}\right)^3 + \mathcal{O}\left\{\left(\frac{y}{2\pi}\right)^4\right\} . \tag{2.28}$$

The asymptotic form of the Gell-Mann-Low function is also known to sixth order [13]. With [8]

$$\psi(z) = z\left[\psi_1 \frac{z}{2\pi} + \psi_2\left(\frac{z}{2\pi}\right)^2 + \psi_3\left(\frac{z}{2\pi}\right)^3 + \cdots\right] , \tag{2.29}$$

the coefficients ψ_1 and ψ_2 can be immediately determined from the second and fourth order vacuum polarization calculations :

$$\psi_1 = -2b_1 = 2/3 \quad ; \tag{2.30}$$

$$\psi_2 = -4b_2 = 1 \quad . \tag{2.31}$$

The term ψ_3 has been directly evaluated by Baker and Johnson [13]. These authors find

$$\psi_3 = \frac{8}{3}\zeta(3) - \frac{101}{36} . \tag{2.32}$$

The presence of $\zeta(3)$ is an indication that the eigenvalue condition $F^{[1]}(\alpha_0)=0$, which involves a restricted class of Feynman diagrams, may be much simpler than the Gell-Mann, Low eigenvalue condition.

From the functional relation between $\beta(\alpha)$ and $\psi(z)$ [see Eqs. (2.19) and (2.10)] and their corresponding perturbation series expansions we have that

$$\beta_1 = 2/3 \quad ; \tag{2.33}$$

$$\beta_2 = 1/2 \quad ; \tag{2.34}$$

and

$$\beta_3 = -2b_1a_2 + 2b_2a_1 + \frac{1}{4}\psi_3 = -\frac{121}{144} , \tag{2.35}$$

i.e., the transcendental $\zeta(3)$ cancels in the determination of β_3! The Callan-Symanzik function $\beta(\alpha)$ has therefore a perturbation series expansion which, up to sixth order, only involves rational coefficients. The aim of the next sections is to gain further insight into this observation.

3. CALCULATION OF THE CALLAN-SYMANZIK FUNCTION $\beta(\alpha)$ IN PERTURBATION THEORY

As we have seen in the previous section, the asymptotic behaviour of the photon propagator is governed by the function $\beta(\alpha)$ through the Callan-Symanzik equation [see Eq. (2.11)] which we now write in the form

$$\beta(\alpha) = \frac{m(\partial/\partial m)\alpha\Pi_R^{\infty}(q^2, m^2, \alpha)}{1-\alpha^2(\partial/\partial\alpha)\Pi_R^{\infty}(q^2, m^2, \alpha)} . \tag{3.1}$$

In this section we shall present a direct calculation of $\beta(\alpha)$ up to sixth order. The main purpose of the calculation is to understand the mechanism of transcendental cancellations pointed out at the end of the previous section. At the same time, a direct sixth order calculation of $\beta(\alpha)$ provides a check [via Eq. (2.35)] of Baker and Johnson's sixth order calculation of $\psi(z)$. As we shall see, a number of interesting facts emerge from this calculation.

In the process of the calculation we shall often make use of the following result: with

$$\alpha \Pi^{\infty}(\Lambda^2;\ q^2, m^2, \alpha) = \frac{\alpha}{\pi} \Pi^{\infty}_{(2)}(\Lambda^2;\ q^2, m^2) + \left(\frac{\alpha}{\pi}\right)^2 \Pi^{\infty}_{(4)}(\Lambda^2;\ q^2, m^2) + \cdots \tag{3.2}$$

the expansion of the asymptotic unrenormalized photon self-energy in perturbation theory, it follows from the asymptotic Callan-Symanzik equation that

$$m \frac{\partial}{\partial m} \Pi^{\infty}_{(2)}(\Lambda^2;\ q^2, m^2) = 0 \tag{3.3}$$

and

$$m \frac{\partial}{\partial m} \Pi^{\infty}_{(4)}(\Lambda^2;\ q^2, m^2) = 0 \quad . \tag{3.4}$$

The proof is rather simple: inserting the definition of $\Pi^{\infty}_{R}(q^2, m^2, \alpha)$ [see Eq. (2.9)] into the asymptotic Callan-Symanzik equation [Eq. (2.11)] we have

$$\left\{ m \frac{\partial}{\partial m} + \beta(\alpha)\left(\alpha \frac{\partial}{\partial \alpha} - 1\right) \right\} \times \tag{3.5}$$

$$\times \left\{ 1 + \alpha \Pi^{\infty}(\Lambda^2;\ q^2, m^2, \alpha) - \alpha \Pi(\Lambda^2;\ 0, m^2, \alpha) \right\} = 0.$$

Then we use the fact that Z_3 [see Eq. (2.8)] also satisfies the same Callan-Symanzik equation. We are thus left with the simple equation

$$m \frac{\partial}{\partial m} \alpha \Pi^{\infty}(\Lambda^2; \; q^2, m^2, \alpha) = -\alpha^2 \beta(\alpha) \frac{\partial \Pi^{\infty}(\Lambda^2; \; q^2, m^2, \alpha)}{\partial \alpha} . \tag{3.6}$$

The right-hand side is at least of order α^3. Then, for the second and fourth order terms, Eqs. (3.3) and (3.4) follow [9].

An expansion similar to (3.2) for the renormalized photon self-energy may also be written:

$$\begin{aligned} \alpha \Pi_R^{\infty}(\; q^2, m^2, \alpha) = \frac{\alpha}{\pi} \Pi_{R(2)}^{\infty} (\; q^2, m^2) \quad + \\ + \; (\alpha/\pi)^2 \; \Pi_{R(4)}^{\infty} (\; q^2, \; m^2, \; \alpha) \; + \cdots \end{aligned} \quad . \tag{3.7}$$

a. <u>Second order calculation of $\beta(\alpha)$</u>

To lowest order in α we have from Eq. (3.1)

$$\beta_1 \; = \; m \frac{\partial}{\partial m} \; \Pi_{R(2)}^{\infty} (\; q^2, \; m^2) \quad .$$

There are many ways to calculate β_1, all of them simple. We choose one which will be useful to pave the way for higher order calculations. Using Eq. (2.2) we can write

$$\beta_1 \frac{\alpha}{\pi} = m \frac{\partial}{\partial m} \; Asp \; \frac{i}{3} \; g_{\mu\nu} \; \frac{1}{q^2} \; i \; \Pi_{R(2)}^{\mu\nu} (q, m^2) \; , \tag{3.8}$$

where the symbol Asp means asymptotic part when $-q^2/m^2$ is large, and the subscript (2) refers to second order. In terms of unrenormalized quantities we have

$$\beta_1 \frac{\alpha}{\pi} = m \frac{\partial}{\partial m} A_{sp} \frac{i}{3} g_{\mu\nu} \left\{ \frac{1}{q^2} i \Pi^{\mu\nu}_{(2)}(q, m^2; \Lambda^2) + \right.$$
$$\left. - \frac{1}{8} g_{\alpha\beta} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} i \Pi^{\mu\nu}_{(2)}(q, m^2; \Lambda^2)\Big|_{q=0} \right\} . \tag{3.9}$$

The first term in the bracket is the unrenormalized second order vacuum polarization tensor; the second term, the corresponding renormalization counterterm. The presence of second order derivatives with respect to the external momenta is due to the fact that $i \Pi^{\mu\nu}(q=0, m^2; \Lambda^2)$ vanishes because of gauge invariance. According to Eq. (3.3), the first term gives a null contribution, and we are left with the simple expression

$$\beta_1 \frac{\alpha}{\pi} = \frac{1}{24} m \frac{\partial}{\partial m} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} \Pi^{\mu\nu}_{(2)}(q, m^2; \Lambda^2)\Big|_{q=0} . \tag{3.10}$$

Differentiation of a fermion loop with respect to an external momentum acts like the insertion of zero energy momentum photons and decreases the degree of divergence by one. The twice differentiated loop is only logarithmically divergent and the spurious quadratic divergence has thereby been removed. From dimensional arguments it is clear that the mass dependence of the quantity

$$g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} \Pi^{\mu\nu}_{(2)}(q, m^2; \Lambda^2)\Big|_{q=0}$$

is only through the ratio Λ^2/m^2, and we have

$$\beta_1 \frac{\alpha}{\pi} = -\frac{1}{12} \frac{\Lambda^2}{m^2} \frac{\partial}{\partial\left(\frac{\Lambda^2}{m^2}\right)} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} \Pi^{\mu\nu}_{(2)}(q, m^2; \Lambda^2)\Big|_{q=0} . \tag{3.11}$$

The explicit calculation of the right-hand side is now very simple and we find the expected result

$$\beta_1 = \frac{2}{3} . \tag{3.12}$$

b. Fourth order calculation of $\beta(\alpha)$

To fourth order in the electric charge we still have that only the numerator in Eq. (3.1) contributes:

$$\beta_2 = m\frac{\partial}{\partial m} \Pi^{\infty}_{R(4)}(q^2, m^2; \alpha^2) \quad . \tag{3.13}$$

The calculation is entirely equivalent to Jost-Luttinger's calculation [25] of b_2. (Recall that $\beta_2 = -2b_2$.) We shall reproduce it here, using the method developed in the previous subsection. The relevant Feynman diagrams are the last three of Fig. 1. The analogue to Eq. (3.9) is now

$$\beta_2\left(\frac{\alpha}{\pi}\right)^2 = m\frac{\partial}{\partial m} Asp \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\epsilon} g_{\rho\sigma} g_{\mu\nu} \times$$

$$\times \left\{ \frac{-1}{3q^2} \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) + \frac{1}{24} g^{\alpha\beta} \frac{\partial^2}{\partial q^\alpha \partial q^\beta} \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) \right|_{q=0} \tag{3.14}$$

where the energy-momentum dependence of the internal photon propagator is explicitly exhibited. By gauge invariance, the tensor $\Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)$ vanishes when $q=0$, hence the presence of second order derivatives with respect to the external energy momentum in the vacuum polarization counterterm.

By definition

$$Asp \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\epsilon} g_{\rho\sigma} g_{\mu\nu} \frac{-1}{3q^2} \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) =$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \Pi^{\infty}_{(4)}(\Lambda^2;\ q^2, m^2) \quad , \tag{3.15}$$

and, according to Eq. (3.4)

$$m\frac{\partial}{\partial m} \Pi^{\infty}_{(4)}(\Lambda^2;\ q^2, m^2) = 0.$$

We are thus left with the expression

$$\beta_2\left(\frac{\alpha}{\pi}\right)^2 = m\frac{\partial}{\partial m}\int\frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2+i\epsilon}\,\frac{g_{\mu\nu}g_{\rho\sigma}g_{\alpha\beta}}{24}\,\frac{\partial^2\Pi^{\mu\nu\rho\sigma}(q,k,m^2,\Lambda^2)}{\partial q_\alpha\,\partial q_\beta}\Bigg|_{q=0} . \tag{3.16}$$

The actual evaluation of this integral can be most conveniently carried out if one first separates the mass renormalization counterterm which is implicit in the definition of $\Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)|_{q=0}$. We have in this way

$$\begin{aligned}\beta_2\left(\frac{\alpha}{\pi}\right)^2 &= \frac{\alpha}{\pi}\left(-\frac{2}{3}\right)m\frac{\partial}{\partial m}\frac{1}{m}\Sigma^{(2)}(m) \;+ \\ &+ m\frac{\partial}{\partial m}\int\frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2+i\epsilon}\,\frac{g_{\mu\nu}g_{\rho\sigma}g_{\alpha\beta}}{24}\,\frac{\partial^2\Pi_{(0)}^{\mu\nu\rho\sigma}(q,k,m^2)}{\partial q_\alpha\,\partial q_\beta}\Bigg|_{q=0} ,\end{aligned} \tag{3.17}$$

where $\Sigma^{(2)}(m)$ is the second order self mass of the electron,

$$m\frac{\partial}{\partial m}\frac{1}{m}\Sigma^{(2)}(m) = -\frac{3}{2}\frac{\alpha}{\pi} . \tag{3.18}$$

(The explicit separation of this mass-renormalization counterterm is done in detail in Appendix D.) The tensor $\Pi_{(0)}^{\mu\nu\rho\sigma}(q,k,m^2)$ is the lowest order light by light scattering amplitude corresponding to the forward scattering $q+k \to q+k$. The action of the q derivatives in Eq. (3.17) increases the convergence of the light by light fermion loop, thereby making the presence of a regularization cut-off unnecessary. From dimensional arguments we can write

$$\frac{g_{\mu\nu}g_{\rho\sigma}g_{\alpha\beta}}{24}\,\frac{\partial^2\Pi_{(0)}^{\mu\nu\rho\sigma}(q,k,m^2)}{\partial q_\alpha\,\partial q_\beta}\Bigg|_{q=0} = \frac{\alpha^2}{-k^2}\,\Xi\left(\frac{m^2}{-k^2}\right) . \tag{3.19}$$

The function $\Xi(m^2/-k^2)$ has been calculated by two different methods. The results and details of the calculations can be found in the Appendices. Inserting this function in Eq. (3.17) and doing the integration over k by Wick's rotation, we finally find

$$\beta_2 = 1 + \frac{1}{8}\,\Xi\left(\frac{m^2}{-k^2}\to 0\right) = \frac{1}{2}\,, \tag{3.20}$$

which is the Jost-Luttinger result [25]. One advantage of the method described above is that it completely avoids the problem of overlapping divergences that one encounters using the conventional rules [10].

c. Sixth order calculation of $\beta(\alpha)$

The interesting feature about the sixth order is that it is the lowest order in which there appear corrections to internal photon propagators. There are two classes of vacuum polarization diagrams in sixth order: those with one fermion loop (see Fig. 3a); and those with two fermion loops (see Fig. 3b). We shall denote by $\beta_3^{[1]}$ and $\beta_3^{[2]}$ their corresponding contributions to

$$m\frac{\partial}{\partial m}\,\alpha\Pi^{\infty}_{R(6)}(\,q^2, m^2;\ \alpha^2).$$

From Eq. (3.1) we have that

$$\beta_3 = \beta_3^{[1]} + \beta_3^{[2]} + \beta_1\,(a_2 + b_2\log x)\,, \tag{3.21}$$

where the last term comes from expanding the denominator in Eq. (3.1) to the required order. Note that β_3 cannot be a function of x, and hence the $\log x$ in the last term must be cancelled in some way. We shall see how this arises.

The value of $\beta_3^{[1]}$ can be directly obtained from Rosner's sixth order calculation [32],

$$\beta_3^{[1]} = -2 b_3^{[1]} = -\frac{1}{16} . \tag{3.22}$$

The term $\beta_3^{[2]}$ has to be calculated. We have

$$\beta_3^{[2]} = m \frac{\partial}{\partial m} \Pi^{\infty\,[2]}_{R(6)} (q^2, m^2; \alpha^2). \tag{3.23}$$

where $\Pi^{\infty\,[2]}_{R(6)}$ is the renormalized asymptotic proper photon self-energy contribution from sixth order Feynman diagrams with two-fermion loops (see Fig. 3b). Formally, this contribution can be obtained from the fourth order diagrams in Fig. 1 via the substitution

$$\frac{-i}{k^2+i\epsilon} g_{\rho\sigma} \rightarrow \frac{i}{k^2+i\epsilon}\left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right)\frac{\alpha}{\pi} \Pi_{R(2)}(k^2, m^2) , \tag{3.24}$$

for the internal photon propagator, and integrating over the four-momentum k. Thus, we have

$$\begin{aligned} \beta_3^{[2]}\left(\frac{\alpha}{\pi}\right)^2 &= m \frac{\partial}{\partial m} \, Asp \int \frac{d^4 k}{(2\pi)^4} \Pi_{R(2)}(k^2, m^2) \frac{i}{k^2+i\epsilon} \times \\ &\times \left\{ g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2} \right\} g_{\mu\nu} \left\{ \frac{-1}{3q^2} \Pi^{\mu\nu\rho\sigma}(q, k, m^2; \Lambda^2) + \right. \\ &\left. + \frac{1}{24} g_{\alpha\beta} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} \Pi^{\mu\nu\rho\sigma}(q, k, m^2; \Lambda^2)\Big|_{q=0} \right\} . \end{aligned} \tag{3.25}$$

We are only interested in the asymptotic (logarithmic) part of this integral when $-q^2/m^2$ is large. It is then sufficient to consider the contribution to the integral from the asymptotic part of the internal photon self-energy insertion $\alpha \Pi^{\infty}_{R(2)}(k^2, m^2)$ [11]. The mass differentiation in Eq. (3.25) acts on both the internal fermion loop and the external fermion loop. We can split the total contribution to $\beta_3^{[2]}$ accordingly:

$$\beta_3^{[2]} = \beta_3^{[2]}(\text{internal}) + \beta_3^{[2]}(\text{external}) . \tag{3.26}$$

Let us first discuss the term obtained from the mass differentiation of the internal fermion loop. Since

$$m \frac{\partial}{\partial m} \Pi^{\infty}_{R(2)}(k^2, m^2) = \beta_1 , \tag{3.27}$$

we have

$$\left(\frac{\alpha}{\pi}\right)^2 \beta_3^{[2]}(\text{internal}) = \beta_1 \, \mathrm{ASP} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon} g_{\rho\sigma} g_{\mu\nu} \times$$

$$\times \left\{ -\frac{1}{3q^2} \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) + \frac{g_{\alpha\beta}}{24} \frac{\partial^2 \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)}{\partial q_\alpha \partial q_\beta}\bigg|_{q=0} \right\} . \tag{3.28}$$

The integral multiplying β_1 is, up to a sign, the fourth order renormalized asymptotic photon self-energy; i.e.,

$$\beta_3^{[2]}(\text{internal}) = -\beta_1 \left(a_2 + b_2 \log x\right) . \tag{3.29}$$

When inserted in Eq. (3.21) this contribution exactly cancels with the last term, i.e., in sixth order, the contribution to $\beta(\alpha)$ from the mass differentiation of the internal fermion loop exactly cancels with a corresponding contribution from the α differentiation term in the Callan-Symanzik equation. Therefore, up to sixth order, we can simply write

$$\beta(\alpha) = m_e \frac{\partial}{\partial m_e} \alpha \Pi^{\infty}_{R}(q^2, m_e^2, m_i^2, \alpha)\bigg|_{m_e = m_i = m} , \tag{3.30}$$

with m_e = electron mass of the external fermion loop, m_i = electron mass of the internal fermion loop.

Let us now consider the contribution from the mass differentiation of the external fermion loop. This contribution can be written as the sum of the two terms:

$$\left(\frac{\alpha}{\pi}\right)^2 \beta_3^{[2]}(\text{external}) =$$

$$= A\,\mathrm{Sp}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\,\Pi^{\infty}_{R(2)}(k^2,m^2)\, m\frac{\partial}{\partial m}\, g_{\rho\sigma}\, g_{\mu\nu}\left(\frac{-1}{3q^2}\right)\Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)+$$

$$+ \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\,\Pi^{\infty}_{R(2)}(k^2,m^2)\, m\frac{\partial}{\partial m}\,\frac{g_{\rho\sigma}\,g_{\mu\nu}\,g_{\alpha\beta}}{24}\,\frac{\partial^2\Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)}{\partial q_\alpha\,\partial q_\beta}\Bigg|_{q=0}. \tag{3.31}$$

The first term vanishes as a consequence of Weinberg's power counting theorem [14]. Indeed, the self-energy insertion $\Pi^{\infty}_{R(2)}(k^2,m^2)$ only grows logarithmically, and we have proved [see Eq. (3.4)] that [9]

$$A\,\mathrm{Sp}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\, m\frac{\partial}{\partial m}\, g_{\rho\sigma}\, g_{\mu\nu}\left(-\frac{1}{3q^2}\right)\Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) =$$
$$= -m\frac{\partial}{\partial m}\,\Pi^{\infty}_{(4)}(\Lambda^2;\, q^2, m^2, \alpha^2) = 0 \tag{3.32}$$

In order to evaluate the second term in Eq. (3.31) we shall again separate the mass renormalization counterterm which is implicit in the definition of $\Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)|_{q=0}$. With m_e and m_i defined as in Fig. 4, and using Eq. (3.19), we have

$$\left(\frac{\alpha}{\pi}\right)^2 \beta_3^{[2]}(\text{external}) = \left(\frac{\alpha}{\pi}\right)\left(\frac{-2}{3}\right) m_e\frac{\partial}{\partial m_e}\,\frac{1}{m_e}\,\Sigma^{(4)}(m_e,m_i)\Big|_{m_e=m_i=m} +$$
$$+\alpha^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\,\Pi_{R(2)}(k^2,m^2)\, m\frac{\partial}{\partial m}\,\frac{1}{-k^2}\,\Xi(m^2/-k^2)\,. \tag{3.33}$$

The mass renormalization counterterm is evaluated in Appendix D. We find

$$m_e \frac{\partial}{\partial m_e} \frac{1}{m_e} \Sigma^{(4)}(m_e, m_i)\Big|_{m_e = m_i = m} = \frac{\alpha}{\pi}\left(\frac{5}{6} + \frac{1}{4}\right) . \tag{3.34}$$

Performing a Wick rotation over the four-momentum k variable in the integral of the right-hand side of Eq. (3.33) we have, with $z = m^2/-k^2$,

$$\alpha^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon} \Pi_{R(2)}(k^2, m^2)\, m \frac{\partial}{\partial m} \frac{1}{-k^2} \Xi\left(\frac{m^2}{-k^2}\right) =$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{-1}{8}\right) \left\{ a_1 \Xi\left(\frac{m^2}{-k^2} \to 0\right) + b_1 \int_0^\infty dz \log z \frac{d}{dz} \Xi(z) \right\} . \tag{3.35}$$

The evaluation of this integral is done in Appendices A and C. We find

$$\int_0^\infty dz \log z \frac{d}{dz} \Xi(z) = -8 . \tag{3.36}$$

Therefore, we finally have

$$\beta_3 = \beta_3^{[1]} + \beta_3^{[2]}(\text{external}) =$$

$$= -\frac{1}{16} + \left\{ \left(-\frac{5}{9} + \frac{-1}{6}\right) + \frac{5}{18} + \frac{-1}{3} \right\} = \frac{-121}{144} , \tag{3.37}$$

in agreement with the value (2.35) implied by the Baker-Johnson calculation [13] of the Gell-Mann, Low function.

The perturbation series expansion of the Callan-Symanzik function $\beta(\alpha)$ is thus

$$\beta(\alpha) = \frac{2}{3}\frac{\alpha}{\pi} + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 - \frac{121}{144}\left(\frac{\alpha}{\pi}\right)^3 + O\left\{\left(\frac{\alpha}{\pi}\right)^4\right\}. \tag{3.38}$$

We see that the first three coefficients in this power series are rational. At first sight the rational nature of the sixth order coefficient may seem surprising. A priori, from a simple analysis of the integrals involved in the diagrams, one expects the appearance of transcendental numbers. We have seen that, in this particular sixth order calculation, the cancellation between mass differentiation of the internal loop and the α differentiation term, which is at the origin of the renormalization group constraints, is also the reason for the absence of transcendentals in β_3, given their absence to sixth order [32] in $F^{[1]}(\alpha)$.

In the next section we shall prove that the mechanism of cancellations between mass differentiation of self energy corrections to internal photons and the α- differentiation terms is indeed valid to all orders in perturbation theory. Whether or not the coefficients in the power series expansion of $\beta(\alpha)$ continue to be rational in higher orders is a question to which we do not yet know the answer beyond sixth order.

4. HIGHER ORDER CONTRIBUTIONS TO $\beta(\alpha)$ FROM SELF ENERGY CORRECTIONS TO INTERNAL PHOTONS

a. Proof of the mechanism of cancellations to all orders in perturbation theory

Let us recall the general Callan-Symanzik equation (valid at all q^2 values) which we have written in Eq. (2.4)

$$\left\{m\frac{\partial}{\partial m} + \beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha} - 1\right)\right\}\left[1 + \alpha\Pi_R(q^2, m^2, \alpha)\right] = \left[1+\delta(\alpha)\right]\alpha\Gamma_{\gamma\gamma S}\left(\frac{q^2}{m^2}, \alpha\right) \tag{4.1}$$

where $\delta(\alpha)$ and $\Gamma_{\gamma\gamma S}((q^2/m^2), \alpha)$ have been defined in Eqs. (2.6) and (2.7). At $q^2 = 0$, $\Pi_R(0, m^2, \alpha) = 0$ by definition, and Eq. (4.1) gives the following relation:

$$\beta(\alpha) = -[1+\delta(\alpha)]\alpha\Gamma_{\gamma\gamma S}(0,\alpha) \quad . \tag{4.2}$$

At $-q^2 \to \infty$, Eq. (4.1) leads to the asymptotic Callan-Symanzik equation which we have been considering in the previous section [see Eq. (3.1)],

$$\beta(\alpha) = \frac{m\frac{\partial}{\partial m}\alpha\Pi_R^{\infty}(q^2,m^2,\alpha)}{1-\alpha^2\frac{\partial}{\partial\alpha}\Pi_R^{\infty}(q^2,m^2,\alpha)} \quad . \tag{4.3}$$

Matching the two determinations of $\beta(\alpha)$ we have

$$\frac{m\frac{\partial}{\partial m}\alpha\Pi_R^{\infty}(q^2,m^2,\alpha)}{1-\alpha^2\frac{\partial}{\partial\alpha}\Pi_R^{\infty}(q^2,m^2,\alpha)} = -[1+\delta(\alpha)]\alpha\Gamma_{\gamma\gamma S}(0,\alpha) \, . \tag{4.4}$$

Next, we shall introduce a precise definition of what we mean by "external" mass m_e and "internal" mass m_i. Assume that all internal photon self-energy parts in a vacuum polarization Feynman diagram are shrunk down to points. The mass in any fermion loop which is left after this shrinking procedure will be called external. All other fermion masses, in the initial Feynman diagram, will be called internal. According to this definition, when we write Eq. (4.4) order by order in perturbation theory we can split the left-hand side into three terms

$$m_e\frac{\partial}{\partial m_e}\alpha\Pi_R^{\infty}(q^2,m_e^2,m_i^2,\alpha)\Big|_{m_e=m_i=m} + m_i\frac{\partial}{\partial m_i}\alpha\Pi_R^{\infty}(q^2,m_e^2,m_i^2,\alpha)\Big|_{m_e=m_i=m} +$$

$$+ \text{"}\alpha\frac{\partial}{\partial\alpha}\text{ terms"} = -[1+\delta(\alpha)]\alpha\Gamma_{\gamma\gamma S}(0,\alpha) \, , \tag{4.5}$$

where by "$\alpha(\partial/\partial\alpha)$ terms" we mean terms which arise from the perturbation series expansion of the denominator up to the appropriate order. Let us now consider the first term in the left-hand side of Eq. (4.5). By definition we have

$$m_e \frac{\partial}{\partial m_e} \alpha \Pi_R^\infty(q^2, \ldots) = m_e \frac{\partial}{\partial m_e} \alpha \Pi^\infty(\Lambda^2; q^2, \ldots) - m_e \frac{\partial}{\partial m_e} \alpha \Pi(\Lambda^2; 0, \ldots). \tag{4.6}$$

The renormalization counterterm in this equation, i.e.,

$$m_e \frac{\partial}{\partial m_e} \alpha \Pi(\Lambda^2; 0, m_e^2, m_i^2, \alpha)$$

is precisely the same object as the right-hand side term in Eq. (4.5). This is because the dependence of $\Pi(\Lambda^2; 0, m_e^2, m_i^2, \alpha)$ on m_e is only through the unrenormalized mass m_0. We are thus left with the equation

$$m_e \frac{\partial}{\partial m_e} \alpha \Pi^\infty(\Lambda^2; q^2, m_e^2, m_i^2, \alpha)\Big|_{m_e = m_i = m} + m_i \frac{\partial}{\partial m_i} \alpha \Pi_R^\infty(q^2, m_e^2, m_i^2, \alpha)\Big|_{m_e = m_i = m} + $$

$$+ \text{"}\alpha \frac{\partial}{\partial \alpha} \text{ terms"} = 0 . \tag{4.7}$$

The next step consists in proving that

$$m_e \frac{\partial}{\partial m_e} \alpha \Pi^\infty(\Lambda^2; q^2, m_e^2, m_i^2, \alpha)\Big|_{m_e = m_i = m} = 0. \tag{4.8}$$

We have already seen that this is trivially the case in second and fourth order [see Eqs. (3.3) and (3.4)] where there are no internal masses. In fact, Adler and Bardeen [4)] have proved that in a theory with no internal masses, i.e., in a theory where all internal photon self energies are shrunk down to points, one also has

$$m_e \frac{\partial}{\partial m_e} \alpha \Pi^\infty(\Lambda^2; q^2, m_e^2, \alpha) = 0. \tag{4.9}$$

The difference in going from a theory without photon self-energy insertions to a complete theory is only the appearance of powers of $\log(k^2/m_i^2)$ in the loop integrations. An example of this has already appeared in the previous section [see the first line in Eq. (3.31)]. With only powers of logarithms appearing, Weinberg's theorem [14)] guarantees that there cannot be promotion of divergences in external masses, and we have that

$$m_e \frac{\partial}{\partial m_e} \alpha \Pi^\infty(\Lambda^2;\ q^2, m_e^2, m_i^2, \alpha)\Big|_{m_e = m_i = m} = 0 ; \tag{4.10}$$

i.e., <u>the only mass divergences which appear in the unrenormalized asymptotic photon self-energy are due to internal masses</u> [12]. From Eq. (4.7) we then have

$$m_i \frac{\partial}{\partial m_i} \alpha \Pi_R^\infty(q^2, m_e^2, m_i^2, \alpha)\Big|_{m_e = m_i = m} + \text{"}\alpha\frac{\partial}{\partial\alpha}\ \text{terms"} = 0. \tag{4.11}$$

A more precise way of putting this is

$$m_i \frac{\partial}{\partial m_i} \alpha \Pi_R^\infty\Big|_{m_e = m_i = m} = -\left\{ m_e \frac{\partial}{\partial m_e} \alpha \Pi_R^\infty \right\}\Big\|_{m_e = m_i = m} \alpha^2 \frac{\partial}{\partial\alpha} \Pi_R^\infty . \tag{4.12}$$

A specific example of this cancellation was found in sixth order in the previous section. Since Eqs. (4.7) and (4.10) hold order by order in perturbation theory, so does the mechanism of cancellations Eq. (4.12). Therefore, in terms of the renormalized asymptotic photon self-energy we can simply write

$$\beta(\alpha) = m_e \frac{\partial}{\partial m_e} \alpha \Pi_R^\infty(q^2, m_e^2, m_i^2, \alpha)\Big|_{m_e = m_i = m} . \tag{4.13}$$

b. Illustration of the mechanism of cancellations in the eighth order determination

We can distinguish four types of Feynman diagrams which, in principle, contribute to the eighth order photon self-energy: genuine one-loop type diagrams, Fig. 5a, two-loop type diagrams which after shrinkage of internal photon self-energies become one-loop type diagrams, Fig. 5b; genuine two-loop type diagrams, Fig. 5c; and three-loop type diagrams which after shrinkage become one-loop, Fig. 5d. Let us call their respective contributions to

$$m\frac{\partial}{\partial m}\alpha\Pi_R^{\infty}(q^2, m^2, \alpha)$$

as follows [13]:

$$\beta_4^{[1]} \; ; \; \beta_4^{[2,1]} \; ; \quad \beta_4^{[2,2]} \; ; \quad \beta_4^{[3,1]} \quad . \tag{4.14}$$

After expansion of the denominator in Eq. (3.1) up to eighth order, the total eighth order contribution to $\beta(\alpha)$ can be written as follows:

$$\begin{aligned}\beta_4 = \; & \beta_4^{[1]} + \beta_4^{[2,2]} + \beta_4^{[2,1]} + \\ & + \beta_2 (a_2 + b_2 \log x) + 2\beta_1 (a_3^{[1]} + b_3^{[1]} \log x) + \\ & + \beta_4^{[3,1]} + 2\beta_1 (a_3^{[2,1]} + b_3^{[2,1]} \log x + c_3 \log^2 x) \, ,\end{aligned} \tag{4.15}$$

where we have also introduced a loop-wise notation [13] for the coefficients a_3 and b_3.

Each of the terms $\beta_4^{[2,1]}$ and $\beta_4^{[3,1]}$ can be split into two parts corresponding to internal and external mass differentiation contributions, i.e.,

$$\left.\begin{aligned}\beta_4^{[3,1]} &= \beta_4^{[3,1]}(\text{internal}) + \beta_4^{[3,1]}(\text{external}) ; \\ &\text{and} \\ \beta_4^{[2,1]} &= \beta_4^{[2,1]}(\text{internal}) + \beta_4^{[2,1]}(\text{external}) .\end{aligned}\right\} \tag{4.16}$$

Let us write down the expression for $\beta_4^{[3,1]}$ (internal):

$$\beta_4^{[3,1]}(\text{internal}) = m_i \frac{\partial}{\partial m_i} \Pi_{R(8)}^{\infty\,[3,1]}(q^2; m_e^2, m_i^2)\Big|_{m_e=m_i=m} =$$

$$= A\,Sp \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\epsilon}\left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right) m \frac{\partial}{\partial m}\left(\left[\Pi_{R(2)}(k^2; m^2)\right]^2\right) g_{\mu\nu} \times$$

$$\times \left\{ \frac{-1}{3q^2} \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) + \frac{g_{\alpha\beta}}{24} \frac{\partial^2 \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)}{\partial q_\alpha \partial q_\beta}\Big|_{q=0} \right\} =$$

$$= 2\beta_1\, A\,Sp \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\epsilon}\left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right) \Pi_{(2)}(k^2; m^2)\, g_{\mu\nu} \times$$

$$\times \left\{ \frac{-1}{3q^2} \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) + \frac{g_{\alpha\beta}}{24} \frac{\partial^2 \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)}{\partial q_\alpha \partial q_\beta}\Big|_{q=0} \right\} =$$

$$= 2\beta_1\,(-1)\left(a_3^{[2,1]} + b_3^{[2,1]} \log x + c_3 \log^2 x\right), \tag{4.17}$$

a contribution which exactly cancels with the last term in Eq. (4.15).

Let us next consider $\beta_4^{[2,1]}$ (internal). Here we can further distinguish two types of contributions: those arising from the fourth order vacuum polarization insertions (see Fig. 5b1), and those arising from the second order vacuum polarization insertions (see Fig. 5b2). Accordingly, we have

$\beta_4^{[2,1]}$ (internal, fourth order vacuum polarization) =

$$= A\,Sp \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon}\left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right)\left[m \frac{\partial}{\partial m} \Pi_{R(4)}(k^2; m^2)\right] g_{\mu\nu} \times$$

$$\times \left\{ \frac{-1}{3q^2} \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2) + \frac{g_{\alpha\beta}}{24} \frac{\partial^2 \Pi^{\mu\nu\rho\sigma}(q,k,m^2;\Lambda^2)}{\partial q_\alpha \partial q_\beta}\bigg|_{q=0} \right\} =$$

$$= \beta_2(-1)\,(a_2 + b_2 \log x)\,; \tag{4.18}$$

and $\beta_4^{[2,1]}$ (internal, second order vacuum polarization) =

$$A\,Sp \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \left[\frac{-i}{k^2+i\epsilon} \frac{i}{k'^2+i\epsilon} g_{\rho\sigma} \left(g_{\gamma\delta} - \frac{k'_\gamma k'_\delta}{k'^2}\right)\left(m\frac{\partial}{\partial m} \Pi_{R(2)}(-k'^2, m^2)\right) + \right.$$

$$\left. + \frac{-i}{k'^2+i\epsilon} \frac{i}{k^2+i\epsilon} g_{\gamma\delta} \left(g_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2}\right)\left(m\frac{\partial}{\partial m} \Pi_{R(2)}(-k^2, m^2)\right)\right] g_{\mu\nu} \times$$

$$\times \left\{ \frac{-1}{3q^2} \Pi^{\mu\nu\rho\sigma\gamma\delta}(q,k,k',m^2;\Lambda^2) + \frac{g_{\alpha\beta}}{24} \frac{\partial^2 \Pi^{\mu\nu\rho\sigma\gamma\delta}(q,k,k',m^2;\Lambda^2)}{\partial q_\alpha \partial q_\beta}\bigg|_{q=0} \right\} = \tag{4.19}$$

$$= 2\,\beta_1(-1)\,(a_3^{[1]} + b_3^{[1]} \log x).$$

As expected, these contributions [Eqs. (4.18) and (4.19)] exactly cancel with corresponding ones in Eq. (4.15) which arise from the $\alpha(\partial/\partial\alpha)$ terms and we finally have that

$$\beta_4 = \beta_4^{[1]} + \beta_4^{[2,2]} + \beta_4^{[2,1]}(\text{external}) + \beta_4^{[3,1]}(\text{external})\,. \tag{4.20}$$

5. SUMMARY OF RESULTS AND CONCLUSIONS

We have made a systematic study of the function $\beta(\alpha)$ in the Callan-Symanzik equation in quantum electrodynamics to low orders in perturbation theory. Our main results are the following.

1) The coefficients of (α/π), $(\alpha/\pi)^2$ and $(\alpha/\pi)^3$ in $\beta(\alpha)$ are rational. This result is reminiscent of a similar one for the single-loop function $F^{[1]}(\alpha)$.

2) The function $\beta(\alpha)$ may be calculated in two distinct ways corresponding respectively to "hard-photon" and "soft-photon" limits:

$$"q^2 \to -\infty": \quad \beta(\alpha) = \frac{m\frac{\partial}{\partial m}\Pi_R^{\infty}}{1-\alpha^2\frac{\partial}{\partial\alpha}\Pi_R^{\infty}} \quad ; \tag{5.1}$$

$$"q^2 \to 0": \quad \beta(\alpha) = -\left[1+\delta(\alpha)\right]\alpha\,\Gamma_{\gamma\gamma S} \;. \tag{5.2}$$

While we have used the first method to calculate $\beta(\alpha)$ in the present work, our results indicate that the second form is in fact more direct.

3) The matching of Eqs. (5.1) and (5.2) allows us to prove a theorem about contributions of multi-loop diagrams to $\beta(\alpha)$. The theorem states that only differentiation of the renormalized asymptotic photon self-energy with respect to external masses m_e which are not associated with internal photon self-energy insertions actually contributes to $\beta(\alpha)$. The form of $\beta(\alpha)$ is then very simple:

$$\beta(\alpha) = m_e\frac{\partial}{\partial m_e}\Pi_R^{\infty}\Big|_{m_e=m_i=m} \;. \tag{5.3}$$

It is in fact the simple form of (5.3), and the cancellation of internal mass differentiations with contributions from the denominator in (5.1), which guarantees that $F^{[1]}(\alpha)$ and $\beta(\alpha)$ differ only by a rational coefficient of $(\alpha/\pi)^3$ to sixth order in the electric charge.

For completeness we give below the perturbation series expansions of the various functions discussed in the text up to the highest power where they have been calculated.

i) <u>The Johnson-Baker-Willey function $F^{[1]}(y)$</u>:

$$-yF^{[1]}(y) = \frac{2}{3}\frac{y}{2\pi} + \left(\frac{y}{2\pi}\right)^2 - \frac{1}{4}\left(\frac{y}{2\pi}\right)^3 + O\left\{\left(\frac{y}{2\pi}\right)^4\right\}. \tag{5.4}$$

ii) <u>The Callan-Symanzik function $\beta(\alpha)$</u>:

$$\beta(\alpha) = \frac{2}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 - \frac{121}{144}\left(\frac{\alpha}{\pi}\right)^3 + O\left\{\left(\frac{\alpha}{\pi}\right)^4\right\}. \tag{5.5}$$

iii) The Gell-Mann, Low function $\psi(z)$:

$$\psi(z) = z\left[\frac{2}{3}\left(\frac{z}{2\pi}\right) + \left(\frac{z}{2\pi}\right)^2 + \left(\frac{8}{3}\zeta(3) - \frac{101}{36}\right)\left(\frac{z}{2\pi}\right)^3 + \mathcal{O}\left\{\left(\frac{z}{2\pi}\right)^4\right\}\right]. \tag{5.6}$$

iv) The asymptotic part of the proper photon self-energy:

$$\begin{aligned}
\alpha\Pi_R^\infty(-x,\alpha) = {} & \left(\frac{\alpha}{\pi}\right)\left(\frac{5}{9} - \frac{1}{3}\log x\right) + \\
& + \left(\frac{\alpha}{\pi}\right)^2\left(\frac{5}{24} - \zeta(3) - \frac{1}{4}\log x\right) + \\
& + \left(\frac{\alpha}{\pi}\right)^3\left(a_3 + \left[\frac{47}{96} - \frac{1}{3}\zeta(3)\right]\log x - \frac{1}{24}\log^2 x\right) + \\
& + \left(\frac{\alpha}{\pi}\right)^4\left(a_4 + b_4\log x + \left[\frac{19}{144} - \frac{1}{9}\zeta(3)\right]\log^2 x + \right. \\
& \left. - \frac{1}{108}\log^3 x\right) + \cdots .
\end{aligned} \tag{5.7}$$

The question of the simplicity of the functions $F^{[1]}(\alpha)$ and $\beta(\alpha)$ to higher orders in perturbation theory remains unanswered. While the absence of transcendentals is not an a priori guarantee of the existence of a simple closed form for either function, we find it encouraging that, at least to sixth order, not one but two fundamental functions recommend themselves by their simplicity for use in the practical problem of formulating a finite theory of quantum electrodynamics. The eventual success of such a program of course rests on being able to go beyond the framework of perturbation theory used here.

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APPENDIX A

CALCULATION OF THE FUNCTION $\Xi(m^2/-k^2)$ USING FEYNMAN PARAMETRIZATION

The function $\Xi(m^2/-k^2)$ has been introduced in Eq. (3.19),

$$\frac{\alpha^2}{-k^2}\,\Xi\left(\frac{m^2}{-k^2}\right)=\frac{g_{\mu\nu}g_{\rho\sigma}g_{\alpha\beta}}{24}\left.\frac{\partial^2\Pi_{(0)}^{\mu\nu\rho\sigma}(q,k,m^2)}{\partial q_\alpha\,\partial q_\beta}\right|_{q=0} \quad \text{(A.1)}$$

In this Appendix we shall give the details of the calculation of this function, using standard Feynman techniques. The relevant Feynman diagrams are shown in Fig. 6. There are two self-energy correction diagrams of the type indicated in Fig. 6b but they give equal total contribution because of charge conjugation invariance. The particular routing of internal momenta which we choose is also indicated in the diagrams of Fig. 6. This choice has the particular advantage of giving very few terms when the second order differentiation with respect to the external photon momentum q is made.

With the notation

$$p_+ \equiv p + k/2 \quad \text{and} \quad p_- \equiv p - k/2 \quad \text{(A.2)}$$

the contribution to the tensor $\Pi_{(0)}^{\mu\nu\rho\sigma}(q,k,m^2)$ from the diagram in Fig. 6a is

$$i\,\Pi_{(a)}^{\mu\nu\rho\sigma}(q,-q,k,-k)=(-1)(-ie)^4\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left\{\gamma^\mu\frac{i}{\not{p}_- - m + i\epsilon}\times\right.$$

$$\left.\times\,\gamma^\rho\frac{i}{\not{p}_+ - m + i\epsilon}\gamma^\nu\frac{i}{\not{p}_+ - \not{q} - m + i\epsilon}\gamma^\sigma\frac{i}{\not{p}_- - \not{q} - m + i\epsilon}\right\}; \quad \text{(A.3)}$$

and from the diagram in Fig. 6b

$$i\Pi^{\mu\nu\rho\sigma}_{(b)}(q,-q,k,-k) = (-1)(-ie)^4 \int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\Big\{ \gamma^\mu \times$$

$$\times \frac{i}{\not p_- - m + i\epsilon}\,\gamma^\rho\, \frac{i}{\not p_+ - m + i\epsilon}\,\gamma^\sigma\, \frac{i}{\not p_- - m + i\epsilon}\,\gamma^\nu\, \frac{i}{\not p_- - \not q - m + i\epsilon}\Big\}. \tag{A.4}$$

After the second order differentiation with respect to q, we find

$$\frac{1}{24}\, g_{\mu\nu}\, g_{\rho\sigma}\, g_{\alpha\beta}\, \frac{\partial^2}{\partial q_\alpha \partial q_\beta}\, \Pi^{\mu\nu\rho\sigma}_{(0)}(q,k,m^2)\Big|_{q=0} = i\,\frac{4}{3}\,\pi^2\alpha^2 \int \frac{d^4p}{(2\pi)^4} \times$$

$$\times \Big\{ \frac{1}{D_+^3 D_-^3}\, \mathrm{Tr}\, \gamma^\mu(\not p_- + m)\gamma^\rho(\not p_+ + m)\gamma_\mu(\not p_+ + m)\gamma^\alpha(\not p_+ + m)\gamma_\rho(\not p_- + m)\gamma_\alpha(\not p_- + m) +$$

$$+ \frac{1}{D_+^4 D_-^2}\, \mathrm{Tr}\, \gamma^\mu(\not p_- + m)\gamma^\rho(\not p_+ + m)\gamma_\mu(\not p_+ + m)\gamma^\alpha(\not p_+ + m)\gamma_\alpha(\not p_+ + m)\gamma_\rho(\not p_- + m) +$$

$$+ \frac{1}{D_+^2 D_-^4}\, \mathrm{Tr}\, \gamma^\mu(\not p_- + m)\gamma^\rho(\not p_+ + m)\gamma_\mu(\not p_+ + m)\gamma_\rho(\not p_- + m)\gamma^\alpha(\not p_- + m)\gamma_\alpha(\not p_- + m) +$$

$$+ \frac{2}{D_+ D_-^5}\, \mathrm{Tr}\, \gamma^\mu(\not p_- + m)\gamma^\rho(\not p_+ + m)\gamma_\rho(\not p_- + m)\gamma_\mu(\not p_- + m)\gamma^\alpha(\not p_- + m)\gamma_\alpha(\not p_- + m)\Big\}. \tag{A.5}$$

where

$$D_+ \equiv p_+^2 - m^2 \quad \text{and} \quad D_- \equiv p_-^2 - m^2 \ . \tag{A.6}$$

The contributions from each trace in Eq. (A.5) can be represented symbolically by the diagrams of Fig. 7. These diagrams correspond to six-photon amplitudes with two pairs of photons carrying null four-momenta; the ordering (a), (b), (c), (d) in Fig. 7 corresponds to the four lines on the right-hand side of Eq. (A.5). In fact, the contributions from the second and third line corresponding to Figs. 7b and c are equal since they only differ by the transformation $k \to -k$ which is a symmetry transformation for the whole expression.

Combining the three types of denominators which appear in Eq. (A.5) with the help of one Feynman parameter, we get

$$\frac{1}{24} g_{\mu\nu} g_{\rho\sigma} g_{\alpha\beta} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} \Pi^{\mu\nu\rho\sigma}_{(0)}(q,k,m^2)\Big|_{q=0} = i\,\frac{40}{3}\pi^2\alpha^2 \times$$
$$\times \int_0^1 dx (1-x)^2 \int \frac{d^4p}{(2\pi)^4} \, \frac{3x^2\,\mathrm{Tr}(3,3) + 4x(1-x)\,\mathrm{Tr}(2,4) + (1-x)^2\,\mathrm{Tr}(1,5)}{[D_+ x + D_-(1-x)]^6} \ , \tag{A.7}$$

with the following result for the traces

$$\mathrm{Tr}(3,3) \equiv \mathrm{Tr}\,\gamma^\mu(\not{p}_- + m)\gamma^\rho(\not{p}_+ + m)\gamma_\mu(\not{p}_+ + m)\gamma^\alpha(\not{p}_+ + m)\gamma_\rho(\not{p}_- + m)\gamma_\alpha(\not{p}_- + m) =$$
$$= 32\Big\{ 2m^6 - 3m^4(p_-\cdot p_+) - 2m^2 p_-^2 p_+^2 + \tag{A.8}$$
$$+ 8m^2(p_-\cdot p_+)^2 - 4(p_-\cdot p_+)^3 + 3p_-^2 p_+^2 (p_-\cdot p_+) \Big\} ;$$

$$
Tr(2,4) \equiv Tr\,\gamma^{\mu}(\not{p}_- + m)\gamma^{\rho}(\not{p}_+ + m)\gamma_{\mu}(\not{p}_+ + m)\gamma_{\rho}(\not{p}_- + m)\gamma^{\alpha}(\not{p}_- + m)\gamma_{\alpha}(\not{p}_- + m) =
$$

$$
= 32\Big\{ -4m^6 + 2m^4 p_+^2 + 3m^4 p_-^2 + 10m^4 p_-\cdot p_+ +
$$

$$
-6m^2(p_-\cdot p_+)^2 - m^2 p_-^4 - 2m^2 p_-^2 p_-\cdot p_+ + 2p_-^2(p_-\cdot p_+)^2 \Big\} ; \qquad (A.9)
$$

and

$$
Tr(1,5) \equiv Tr\,\gamma^{\mu}(\not{p}_- + m)\gamma^{\rho}(\not{p}_+ + m)\gamma_{\rho}(\not{p}_- + m)\gamma_{\mu}(\not{p}_- + m)\gamma^{\alpha}(\not{p}_- + m)\gamma_{\alpha}(\not{p}_- + m) =
$$

$$
= 32\Big\{ 8m^6 - 4m^4 p_-^2 - 5m^4(p_-\cdot p_+) + 4m^2 p_-^4 +
$$

$$
+ 2m^2 p_-^2(p_-\cdot p_+) - p_-^4(p_-\cdot p_+) \Big\} . \qquad (A.10)
$$

The denominator in Eq. (A.7) can be rewritten in the following way

$$
D_+ x + D_-(1-x) \equiv (p-\ell)^2 - R^2 \qquad (A.11)
$$

where ℓ is the four-vector

$$
\ell_\mu = \frac{1}{2}(1-2x)\,k_\mu \qquad (A.12)
$$

and

$$\mathbb{R}^2 = -k^2 x(1-x) + m^2 . \qquad (A.13)$$

The integral in Eq. (A.7) is perfectly convergent and we can perform the shift of momentum variable

$$p - \ell \equiv \tilde{p} \qquad (A.14)$$

under which p_+ and p_- are transformed as follows

$$p_+ = \tilde{p} + k(1-x) \; ; \quad p_- = \tilde{p} - kx \quad . \qquad (A.15)$$

Once the shift has been performed in Eq. (A.7) the integrals over the $\tilde{p}$ variable can be trivially done and we obtain the result

$$\frac{g_{\mu\nu} g_{\rho\sigma} g_{\alpha\beta}}{24} \left. \frac{\partial^2 \Pi_{(0)}^{\mu\nu\rho\sigma}(q,k,m^2)}{\partial q_\alpha \partial q_\beta} \right|_{q=0} = \frac{\alpha^2}{-k^2} \Xi\left(\frac{m^2}{-k^2}\right) , \qquad (A.16)$$

with

$$\Xi\left(\frac{m^2}{-k^2}\right) = \frac{-4}{9} \int_0^1 dx (1-x)^2 \sum_{n=1}^{4} \left[\frac{m^2}{-k^2} + x(1-x)\right]^{-n} f_n\left(\frac{m^2}{-k^2}, x\right) ; \qquad (A.17)$$

where

$$f_1\left(\frac{m^2}{-k^2}, x\right) = -12 \, P_{10}(x) \; ;$$

$$f_2\left(\frac{m^2}{-k^2}, x\right) = 3\left[\frac{m^2}{-k^2} P_{20}(x) - P_{21}(x)\right] ;$$

$$f_3\left(\frac{m^2}{-k^2},x\right) = -2\left[\left(\frac{m^2}{-k^2}\right)^2 P_{30}(x) - \left(\frac{m^2}{-k^2}\right) P_{31}(x) + P_{32}(x)\right];$$

$$f_4\left(\frac{m^2}{-k^2},x\right) = 3\left[\left(\frac{m^2}{-k^2}\right)^3 P_{40}(x) - \left(\frac{m^2}{-k^2}\right)^2 P_{41}(x) + \right.$$

$$\left. + \frac{m^2}{-k^2} P_{42}(x) - P_{43}(x)\right]. \tag{A.18}$$

The polynomials $P_{ij}(x)$ $i = 1,2,3,4$; $j = 0,1,2,3$, $j < i$ are tabulated in Table 1.

For calculation purposes, it is convenient to express the function $\Xi(-k^2/m^2)$ in a slightly different way. With

$$z \equiv \frac{m^2}{-k^2} \tag{A.19}$$

and after reduction into partial fractions we have

$$\Xi(z) = \frac{-4}{3}\int_0^1 dx \sum_{n=1}^{4} \frac{x^{n-1}(1-x)^{n-1}}{[z+x(1-x)]^n}\left\{B_n x^2(1-x)^2 + E_n x(1-x)^3 + \right.$$
$$\left. + D_n (1-x)^4 \right\}, \tag{A.20}$$

where the numerical values of the coefficients B_n, E_n and D_n are;

$$\begin{array}{lll} B_1 = 42 & E_1 = -124 & D_1 = 24 \\ B_2 = -170 & E_2 = 289 & D_2 = -65 \\ B_3 = 190 & E_3 = -274 & D_3 = 62 \\ B_4 = -69 & E_4 = 90 & D_4 = -21 \end{array} \tag{A.21}$$

i) Calculation of $\Xi((m^2/-k^2)\to 0)$

As we have seen in the text, the calculation of the Jost-Luttinger term β_2 requires the knowledge of $\Xi((m^2/-k^2)\to 0)$ [see Eq. (3.20)]. With the parametrization given in Eqs. (A.20) and (A.21) we find

$$\Xi\left(z \equiv \frac{m^2}{-k^2} \to 0\right) = -4. \tag{A.22}$$

ii) The integral in Eq. (3.33)

This integral has appeared in the course of the sixth order calculation of $\beta(\alpha)$. After Wick rotation of the integration contour of the k variable, we have

$$I \equiv \alpha^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon} \Pi^{\infty}_{R(2)}(k^2; m^2)\, m\frac{\partial}{\partial m} \frac{1}{-k^2} \Xi\left(\frac{m^2}{-k^2}\right) =$$

$$= \alpha^2 \frac{i\pi^2}{(2\pi)^4} \int_0^{\infty} d(-k^2)(-k^2)\frac{-i}{-k^2}\left[a_1 + b_1 \log\left(\frac{-k^2}{m^2}\right)\right] \times \tag{A.23}$$

$$\times \frac{1}{-k^2}\, 2z \frac{d}{dz}\Xi(z),$$

where we have put

$$z = m^2/(-k^2). \tag{A.24}$$

Then

$$I = \left(\frac{\alpha}{\pi}\right)^2 \cdot \frac{1}{8} \int_0^\infty \frac{dz}{z} \left(a_1 - b_1 \log z\right) z \, \frac{d}{dz} \, \Xi(z) \quad = \tag{A.25}$$

$$= \left(\frac{\alpha}{\pi}\right)\left(-\frac{1}{8}\right)\left\{ a_1 \, \Xi\!\left(\frac{m^2}{-k^2} \to 0\right) + b_1 \int_0^\infty dz \log z \, \frac{d}{dz} \, \Xi(z) \right\},$$

which is the right-hand side we have written in Eq. (3.35).

Using the parametrization given in Eqs. (A.20), (A.21), the evaluation of the integral in Eq. (A.25) is straightforward. We find

$$\int_0^\infty dz \log z \, \frac{d}{dz} \, \Xi(z) = -8 \, . \tag{A.26}$$

APPENDIX B

THE GEGENBAUER POLYNOMIAL EXPANSION TECHNIQUE

The integrals encountered in Appendix A involve terms of the general form

$$\int \frac{d^4p}{(2\pi)^4} \frac{f(p,k,m)\, g(p^2,m)}{\left[(p-k)^2+m^2\right]^n} \quad . \tag{B.1}$$

Here the Euclidean metric is used; $f(p,k,m)$ is an invariant polynomial (usually of rather low order); $g(p^2,m)$ may be any function but here will consist of terms $\left[p^2+m^2\right]^{-k}$, and variables may clearly be chosen so that $n \leq 3$. In fact, as we shall see below, it is sufficient to consider the case $n \leq 2$.

The integral over p in (B.1) may be decomposed into an angular part and an integration over the magnitude of p:

$$\int \frac{d^4p}{(2\pi)^4} = \frac{1}{8\pi^2} \cdot \frac{1}{2} \int_0^\infty p^2 dp^2 \int \frac{d\Omega_p}{2\pi^2} \quad . \tag{B.2}$$

The angular integration may be performed trivially by expanding the denominator of (B.1) in Gegenbauer polynomials. A specific change of variables then becomes appropriate for the integration over p^2.

We start from the following properties [36),37)] of the Gegenbauer polynomials $C_n^1(z) = C_n(z)$:

$$\frac{1}{1+h^2-2hz} = \sum_{n=0} h^n C_n(z) \qquad (h<1) \quad ; \tag{B.3}$$

$$\int \frac{d\Omega_p}{2\pi^2} C_n(\hat{p}\hat{p}_0)\, C_m(\hat{p}\hat{p}_0) = \delta_{nm} \quad , \tag{B.4}$$

where $\widehat{pp_o}$ refers to the direction cosine of p with respect to any fixed vector p_o in Euclidean four-space. The recursion relation for the $C_n(z)$, namely

$$C_{n+1}(z) + C_{n-1}(z) = 2z\, C_n(z) \tag{B.5}$$

may be used to obtain explicit expressions:

$$\begin{aligned} C_0(z) &= 1 \\ C_1(z) &= 2z \\ C_2(z) &= 4z^2 - 1 \\ C_3(z) &= 8z^3 - 4z \quad , \text{etc.} \end{aligned} \tag{B.6}$$

It is also helpful to recall that

$$C_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \quad . \tag{B.7}$$

The denominator in (B.1) with $n = 1$ may be related to that on the left-hand side of (B.3) as follows:

$$[(p-k)^2 + m^2]^{-1} = N[1 + h^2 - 2h(p\cdot k)/pk]^{-1} , \tag{B.8}$$

where $p = \sqrt{p^2}$ and $k = \sqrt{k^2}$. Comparing terms, we find

$$p^2 + k^2 + m^2 = [1 + h^2]/N \tag{B.9}$$

and

$$1 = h/Npk \quad . \tag{B.10}$$

or

$$h = \frac{p^2+k^2+m^2}{2pk} - \sqrt{\left(\frac{p^2+k^2+m^2}{2pk}\right)^2 - 1} \tag{B.11}$$

and

$$N = h/pk \tag{B.12}$$

Then, we may write

$$\left[(p-k)^2+m^2\right]^{-1} = (pk)^{-1}\sum_{n=0}^{\infty} h^{n+1} C_n(\hat{p}k) , \tag{B.13}$$

with h given explicitly by (B.11).

The expansion of $[(p-k)^2+m^2]^{-n}$ is then obtained by differentiating both sides of (B.13) repeatedly with respect to m^2 for fixed p and k. For example,

$$\left[(p-k)^2+m^2\right]^{-2} = (pk)^{-1}\sum_{n=0}^{\infty}(n+1)h^n\left(\frac{-\partial h}{\partial m^2}\right)C_n(\hat{p}k),$$

and, since

$$-\partial h/\partial m^2 = \frac{1}{pk}\,\frac{h^2}{1-h^2} ,$$

we find

$$\left[(p-k)^2+m^2\right]^{-2} = \frac{1}{p^2k^2}\,\frac{1}{1-h^2}\sum_{n=0}^{\infty}(n+1)\,h^{n+2}\,C_n(\hat{p}k) . \tag{B.14}$$

Equations (B.13) and (B.14) reduce in the zero mass limit to simpler expressions quoted in Ref. 32).

As mentioned, the angular integrations may be performed using the definitions (B.6) of the Gegenbauer polynomials and their orthogonality properties (B.4). However, this leaves expressions containing powers of h. The following variable is closely related to h but may be expressed rather simply in terms of p^2. Set [14]

$$x \equiv ph/k \tag{B.15}$$

We then find, with the help of (B.11), that

$$p^2 = \frac{x}{1-x}\left[m^2 + k^2(1-x)\right] \quad . \tag{B.16}$$

Hence

$$dp^2 = \frac{dx}{(1-x)^2}\left[m^2 + k^2(1-x)^2\right] , \tag{B.17}$$

so that the change of variables introduces little complication. The typical denominators that one encounters are also fairly simple:

$$p^2 + m^2 = \left[m^2 + k^2 x(1-x)\right] / \left[1-x\right] \tag{B.18}$$

and

$$1 - h^2 = x\left[m^2 + k^2(1-x)^2\right] / p^2(1-x) \tag{B.19}$$

Note that (B.18) is reminiscent of a similar form that arises in a totally different way in Appendix A.

We summarize the angular averages and changes of variable in Table 2. The integrand used in the present calculation, as shown in Appendix C, requires only the angular averages shown.

APPENDIX C

CALCULATION OF THE FUNCTION $\Xi(m^2/-k^2)$ USING THE GEGENBAUER POLYNOMIAL TECHNIQUE

The technique of Appendix B is most suitable when the least possible powers of $(p-k)^2+m^2$ appear in the denominator of the Feynman integrals for

$$\mathcal{P} \equiv \frac{g_{\rho\sigma}\, g_{\mu\nu}\, g_{\alpha\beta}}{24} \left.\frac{\partial^2 \Pi^{\mu\nu\rho\sigma}_{(0)}(q,k,m^2)}{\partial q_\alpha\, \partial q_\beta}\right|_{q=0} . \tag{C.1}$$

We shall show how one may ensure that no powers higher than the second occur. The identities quoted in Appendix B will then be all that are needed to perform the required expansions.

Consider first the diagram of Fig. 8. Here the momentum q is routed symmetrically as in Appendix A. Differentiation with respect to q leads to four types of graph, shown in Fig. 9. Here a black dot denotes that a fermion propagator has been differentiated, e.g.,

$$\frac{\partial}{\partial q_\alpha} S\left(p+\frac{q}{2}\right) = -S\left(p+\frac{q}{2}\right)\frac{\gamma^\alpha}{2} S\left(p+\frac{q}{2}\right) , \tag{C.2}$$

and q has been set equal to zero after differentiation. (The vertices carrying q are denoted by clear dots.) The arrows are to remind the reader of the direction in which k has been routed.

Figures 9a and 9b clearly give rise to precisely two powers of $(p-k)^2+m^2$ in the Feynman integrals for $\mathcal{P}$. Similar graphs occur with both differentiations on the right; in that case the momentum k can be taken to run upwards on the left. This freedom is a consequence of the convergence of loop integrals with six fermion propagators. Denote the contribution to $\mathcal{P}$ of all graphs of the type illustrated in Figs. 9a and b by $\mathcal{P}^{(a,b)}$, respectively. Then

$$\mathcal{P}^{(a,b)} = e^4 \int \frac{d^4p}{(2\pi)^4} I^{(a,b)}(p,k,m) \quad , \tag{C.3}$$

where the integral denotes Euclidean metric and $I^{(a,b)}_{(p,k,m)}$ will be expressed accordingly. Notation similar to (C.3) will be used for other contributions to $\mathcal{P}$. Trace calculations then give

$$I^{(a)} = \frac{8}{3} D^{-4} D'^{-2} \left[4m^6 + m^4(10p\cdot p' + 2p'^2 + 3p^2) + \right.$$
$$\left. + m^2\left\{2p^2 p\cdot p' + p^4 + 6(p\cdot p')^2\right\} + 2p^2(p\cdot p')^2 \right\} \tag{C.4}$$

and

$$I^{(b)} = \frac{4}{3} D^{-4} D'^{-2} \left[2m^6 + m^4(-4p\cdot p' + p'^2 + 6p^2) + \right. \tag{C.5}$$
$$\left. + m^2\left\{4p^2 p\cdot p' + 4p^2p'^2 + 2p^4 - 4(p\cdot p')^2\right\} + p^4p'^2 \right]$$

Here

$$p' \equiv p-k; \quad D \equiv p^2+m^2 \; ; \quad D' \equiv p'^2+m^2 . \tag{C.6}$$

Figures 9c and 9d give rise to integrands involving $D^{-3}D'^{-3}$. Their sum is simpler, however:

$$I^{(c+d)} = 4p\cdot p' \; D^{-2} D'^{-2} . \tag{C.7}$$

Hence the graphs of Fig. 8 give rise to only two powers of $(p-k)^2+m^2$ in denominators.

The diagram of Fig. 10 (and a similar one with both photons carrying k attached to the lower fermion line) gives rise, upon differentiation, to the types of graph shown in Fig. 11. From the standpoint of the present method, the graph of Fig. 11d is undesirable, since it would lead to three powers of $(p-k)^2+m^2$ in the denominator.

One can show that the diagram of Fig. 10 is independent of the routing of q, once its logarithmically divergent part has been subtracted out. Specifically, one could route q through the upper fermion leg, in which case only graphs of the type in Figs. 11a-d would occur, or through the lower leg, in which only that of Fig. 11 would arise. Hence one may represent the contribution of Figs. 11a-d merely by doubling that of Fig. 11g. The total contributions of all graphs of the type e-g are then

$$I^{(e+g)} = -8\, D^{-4} D'^{-1} \left[p^2 + 2m^2 \right] \left[p \cdot p' + 2m^2 \right] \tag{C.8}$$

$$I^{(f)} = I^{(b)} . \tag{C.9}$$

Now, (C.8) is considerably simpler than either $I^{(e)}$ or $I^{(g)}$ alone. A similar simplification occurs in $I^{(a)}+I^{(b)}+I^{(f)} = I^{(a)}+2I^{(b)}$:

$$\begin{aligned} I^{(a)} + 2\, I^{(b)} = 8 D^{-3} D'^{-2} \Big[& \tfrac{2}{3} (p \cdot p')^2 + \tfrac{1}{3} p^2 p'^2 + \\ & + m^2 (p^2 + p'^2 + 2 p \cdot p') + 2 m^4 \Big] . \end{aligned} \tag{C.10}$$

Hence, not only has the presence of no power of D' higher than the second been insured in all denominators, but the integrand has been simplified considerably. It is

$$\begin{aligned} I^{(\text{total})} = \; & I^{(a)} + 2 I^{(b)} + I^{(c+d)} + \\ & + I^{(e+g)} . \end{aligned} \tag{C.11}$$

We shall sketch some steps in the integration of these contributions.

The angular average over $\hat{p}$ may be performed using the method of Appendix B. There then remains an integral over the magnitude of p^2, which it is convenient to write in terms of the variable x, also defined in Appendix B. Using the definitions and angular averages of Table 2, the integral for Ξ may now be expressed as

$$\Xi\left(\frac{1}{y}\right) = 8y\int_0^1 \frac{x\,dx\,(1-x)^2}{d_3^{\,4}}\left[-\frac{3}{2}y^2x(1-x)^2 + y\left(\frac{7}{2}x-3\right)-2\right] + + 4y\int_0^1 \frac{x^2\,dx}{\left[1+yx(1-x)\right]^2} \quad , \tag{C.12}$$

from which, it may be checked, one recovers the results (A.22) and (A.26).

APPENDIX D

CALCULATION OF THE MASS RENORMALIZATION COUNTERTERMS

In this Appendix we shall calculate the contributions to $\beta(\alpha)$ from the mass renormalization counterterms in fourth and sixth order.

The second order mass renormalization counterterm $\Sigma^{(2)}(m)$ gives a fourth order contribution to $\beta(\alpha)$ via Eq. (3.16). The contribution is simply

$$\beta_2(\text{counterterm})\left(\frac{\alpha}{\pi}\right)^2 = -\frac{1}{24} m \frac{\partial}{\partial m} \int \frac{d^4p}{(2\pi)^4} g_{\mu\nu} g_{\kappa\beta} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} \times$$

$$\times (-1)(-ie)^2\, 2\, \mathrm{Tr}\left\{ \gamma^\mu \frac{i}{\not{p}-m} \Sigma^{(2)}(m) \frac{i}{\not{p}-m} \gamma^\nu \frac{i}{\not{p}-\not{q}-m} \right\}, \tag{D.1}$$

where (using Pauli-Villars regularization, $\lambda_1 \equiv 0$; $\lambda_2 \equiv \Lambda$)

$$-i\Sigma^{(2)}(m) = (-ie)^2 \sum_i c_i \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2-\lambda_i^2} \gamma^\rho \frac{i}{\not{p}+\not{k}-m} \gamma_\rho \Bigg|_{\not{p}=m}. \tag{D.2}$$

There is a factor 2 in front of Tr in Eq. (D.1) because of the equal contribution from the two self-energy diagrams. The integral in Eq. (D.1) can be easily performed. We find

$$\beta_2(\text{counterterm})\left(\frac{\alpha}{\pi}\right)^2 = -\frac{2}{3}\frac{\alpha}{\pi} m \frac{\partial}{\partial m} \frac{\Sigma^{(2)}(m)}{m}, \tag{D.3}$$

which is the expression introduced in Eq. (3.17). The second order expression for $\Sigma^{(2)}(m)$ is well known:

$$\frac{1}{m}\Sigma^{(2)}(m) = \frac{\alpha}{\pi}\left[\frac{3}{4}\log\frac{\Lambda^2}{m^2} + \text{const.}\right]. \tag{D.4}$$

It follows then that

$$\beta_2\,(\text{counterterm}) \;=\; 1 \;. \tag{D.5}$$

There is a mass renormalization counterterm from the electron self-energy diagram in Fig. 4 which gives a sixth order contribution to $\beta(\alpha)$, more precisely to $\beta_3^{[2]}$(external), in Eq. (3.33). This contribution is

$$\left(\frac{\alpha}{\pi}\right)^2 \beta_3^{[2]}(\text{counterterm}) = \frac{\alpha}{\pi}\left(\frac{-2}{3}\right) m_e \frac{\partial}{\partial m_e} \frac{1}{m_e} \Sigma^{(4)}(m_e, m_i)\Big|_{m_e = m_i = m} . \tag{D.6}$$

The calculation of

$$m_e \frac{\partial}{\partial m_e} \frac{1}{m_e} \Sigma^{(4)}(m_e, m_i)$$

is simple. The only term in

$$\Sigma^{(4)}(m_e, m_i)$$

which gives a non-null contribution after external mass differentiation is the divergent part [the $\log(\Lambda^2/m_e^2)$ term]. In order to obtain this term it is then sufficient to consider the asymptotic part of the photon self-energy correction. Using standard Feynman parametrization one gets

$$\begin{aligned} m_e \frac{\partial}{\partial m_e} \frac{1}{m_e} \Sigma^{(4)}(m_e, m_i)\Big|_{m_e = m_i = m} &= \\ = (-1)\left(\frac{\alpha}{\pi}\right) \int_0^1 dx\,(x-2)\left(\frac{5}{9} - \frac{1}{3}\log\frac{[1-x]^2}{x}\right) &= \\ = \frac{\alpha}{\pi}\left(\frac{5}{6} + \frac{1}{4}\right). \end{aligned} \tag{D.7}$$

The first term arises from the constant part in the asymptotic photon propagator; the second term from the logarithmic term.

TABLE 1

COEFFICIENT	POLYNOMIAL	NOTATION
$\tilde{p}^6$	$-3x^2+8x(1-x)-(1-x)^2$	$P_{10}(x)$
$\tilde{p}^4m^2$	$6[3x^2-6x(1-x)+(1-x)^2]$	$P_{20}(x)$
$\tilde{p}^4k^2$	$x[(9/2)x^3+45x^2(1-x)-(55/2)x(1-x)^2+4(1-x)^3]$	$P_{21}(x)$
$\tilde{p}^2m^4$	$3[-3x^2+20x(1-x)-3(1-x)^2]$	$P_{30}(x)$
$\tilde{p}^2m^2k^2$	$x(1-x)[-84x^2+87x(1-x)-9(1-x)^2]$	$P_{31}(x)$
$\tilde{p}^2k^4$	$x^3(1-x)[-(5/2)x^2-57x(1-x)+(35/2)(1-x)^2]$	$P_{32}(x)$
m^6	$2[3x^2-8x(1-x)+4(1-x)^2]$	$P_{40}(x)$
m^4k^2	$x(1-x)[21x^2-44x(1-x)+13(1-x)^2]$	$P_{41}(x)$
m^2k^4	$2x^3(1-x)[-2x^2+15x(1-x)-13(1-x)^2]$	$P_{42}(x)$
k^6	$12x^5(1-x)^3$	$P_{43}(x)$

Polynomials $P_{ij}(x)$ in Eq. (A.18)

T A B L E 2

$d_1 \equiv m^2 + k^2(1-x)$	
$d_2 \equiv m^2 + k^2(1-x)^2$	$D' \equiv (p-k)^2 + m^2$
$d_3 \equiv m^2 + k^2 x(1-x)$	

Definition of x : $p^2 = (x/1-x)d_1$

Change of variables : $dp^2 = (dx/(1-x)^2)d_2$

Denominators without k : $p^2 + m^2 = (1/1-x)d_3$

Angular averages :

$$\int \frac{d\Omega_p}{2\pi^2}\,\frac{1}{D'} = \frac{1-x}{d_1}$$

$$\int \frac{d\Omega_p}{2\pi^2}\,\frac{p\cdot k}{D'} = \frac{k^2 x(1-x)}{2d_1}$$

$$\int \frac{d\Omega_p}{2\pi^2}\,\frac{1}{D'^2} = \frac{(1-x)^2}{d_1 d_2}$$

$$\int \frac{d\Omega_p}{2\pi^2}\,\frac{p\cdot k}{D'^2} = \frac{k^2 x(1-x)^2}{d_1 d_2}$$

$$\int \frac{d\Omega_p}{2\pi^2}\,\frac{(p\cdot k)^2}{D'^2} = \frac{k^2 x(1-x)\left[m^2 + k^2(1-x)(1+3x)\right]}{4d_1 d_2}$$

Changes of variables and angular averages in Gegenbauer polynomial expansions

FOOTNOTES

1] It is assumed that an appropriate gauge invariant regularization procedure has been adopted in order to define $\Pi(\Lambda^2; q^2, m^2, \alpha)$.

2] Our metric tensor is $g_{oo} = 1$, $g_{ij} = -\delta_{ij}$, $g_{oi} = 0$; $i = 1,2,3$. We are taking the limit in the spacelike region, i.e., for large negative q^2.

3] Implicit in what follows is the assumption that terms dropped at each order, when summed, do not dominate asymptotically the terms which have been kept.

4] For a discussion of these relations, obtained with the usual renormalization group methods, see e.g., K.E. Erickson's lectures at Cargèse, Ref. 16).

5] The calculation in question is the sixth order contribution from the fourth order vacuum polarization to the difference of the anomalous magnetic moments of muon and electron.

6] The transcendental $\zeta(3)$ appears in the evaluation of integrals of the type

$$\int_0^1 \frac{dx}{x} \log x \; \log(1-x) .$$

The complexity of transcendentals appearing in quantum electrodynamics calculations is expected to increase as one calculates higher and higher order terms. Some studies in this direction have been undertaken by Peterman (unpublished), and by Kölbig, Mignaco and Remiddi, Ref. 29). [See also Barbieri, Mignaco and Remiddi, Ref. 30); and Levine and Roskies, Ref. 31).]

7] Some considerations relevant to the cancellations have been discussed in Refs. 21 and 22).

8] Our definitions of $F^{[1]}(y)$ and $\psi(z)$ are the same as in Adler's paper, Ref. 5).

9] The order of vanishing of Eqs. (3.3) and (3.4) is clearly the same as the order of vanishing of the right-hand side in the general Callan-Symanzik equation [see Eq. (2.4)], which is given by Weinberg's theorem [14], i.e., [4]

$$\mathcal{O}\left\{ \left(\frac{m^2}{-q^2}\right)^{1/2} \log^n\left(\frac{-q^2}{m^2}\right) \right\} \quad .$$

10] For a detailed description of the Jost-Luttinger calculation, using conventional methods, see e.g., Bjorken and Drell's textbook, Relativistic Quantum Fields, Ref. 33); pp. 344-357. Another method which avoids overlapping divergences is Källen's dispersion method [see Ref. 28)].

11] Notice that only the asymptotically <u>logarithmic</u> part of $\Pi^{\infty}_{R(6)}{}^{[2]}(q^2,m^2)$ contributes to $\beta_3^{[2]}$ [see Eq. (3.21)]. As shown by Johnson, Baker and Willey, Refs. 18) and 13), this asymptotic logarithmic part is correctly obtained keeping only the asymptotic part of the internal photon self-energy.

12] This result may also be obtained using the mass singularities theorems of Kinoshita [see Ref. 34)]; and Lee and Nauenberg [see Ref. 35)]. In this connection see also Sirlin, Ref. 15).

13] This is a similar notation to the one introduced by Adler in his loop-wise summation method [see Ref. 5)].

14] The authors thank R. Roskies for suggesting this variable.

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FIGURE CAPTIONS

Figure 1 :

The renormalized photon proper self-energy tensor $i\Pi_R^{\mu\nu}(q)$.

Figure 2 :

The renormalized photon propagator $d_R((q^2/m^2),\alpha)$.

Figure 3 :

Sixth order Feynman diagrams contributing to $\alpha\Pi_R(q^2,m^2,\alpha)$. Diagrams (a) have one fermion loop only; diagrams (b) have two fermion loops.

Figure 4 :

Definition of external and internal masses m_e and m_i in fourth order mass renormalization with one-fermion loop.

Figure 5 :

Feynman diagrams contributing to eighth order photon self-energy.

a) Genuine one-loop diagrams.
b) Two-loop diagrams which reduce to one loop when internal photon self-energy insertions are shrunk to a point.
c) Genuine two-loop diagrams.
d) Three-loop diagrams which reduce to one loop after shrinkage of internal photon self-energy insertions.

Figure 6 :

Routing of momenta in Feynman diagrams contributing to the function $\Xi(m^2/(-k^2))$ in Eq. (3.19).

a) Term in Eq. (A.3).
b) Term in Eq. (A.4).

Figure 7 :

Representation of contributions to traces in Eq. (A.5). The external photon line is associated with Lorentz indices ρ; the black dots correspond to the indices α and the clear dots to the indices μ.

Figure 8 :

One diagram contributing to Eq. (C.1), with symmetric routing of q.

Figure 9 :

Contributions of the graph of Fig. 8 to Eq. (C.1) after differentiating with respect to q and setting $q = 0$. Black dots denote an index $\alpha = \beta$, clear dots denote an index $\mu = \nu$. a) and b) give rise to Eqs. (C.4) and (C.5), respectively, while c) and d) combine to give Eq. (C.7).

Figure 10 :

Second class of diagram (together with that of Fig. 8) contributing to Eq. (C.1)

Figure 11 :

Contributions of the graph of Fig. 10 to Eq. (C.1) after differentiating with respect to q and setting $q = 0$. Black dots: indices $\alpha = \beta$; clear dots: indices $\mu = \nu$. Integrands $I^{(e,f,g)}$ referred to in text are associated with graphs e), f), g), respectively. Contributions of graphs a)-d) are expressed in terms of that of graph g); see text.

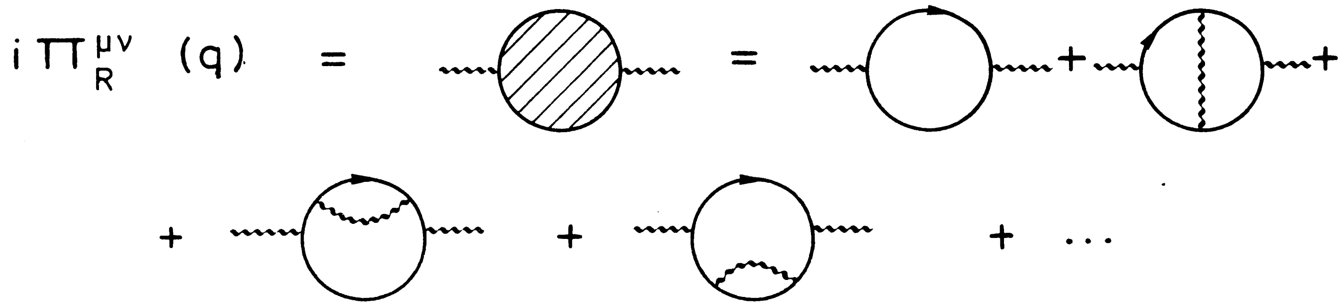

FIG. 1

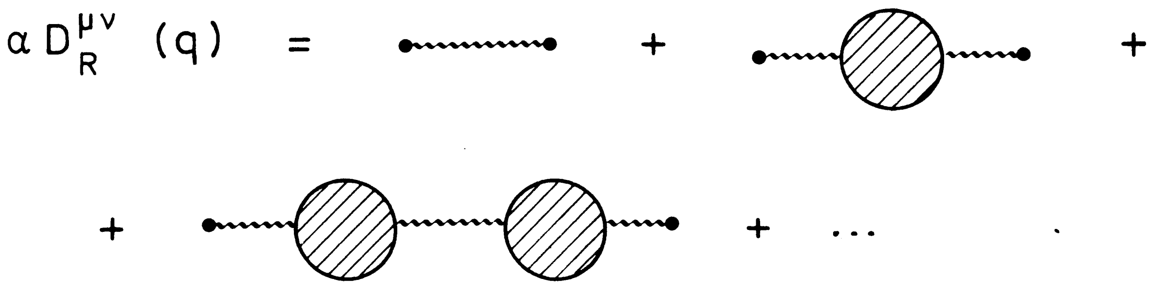

FIG. 2

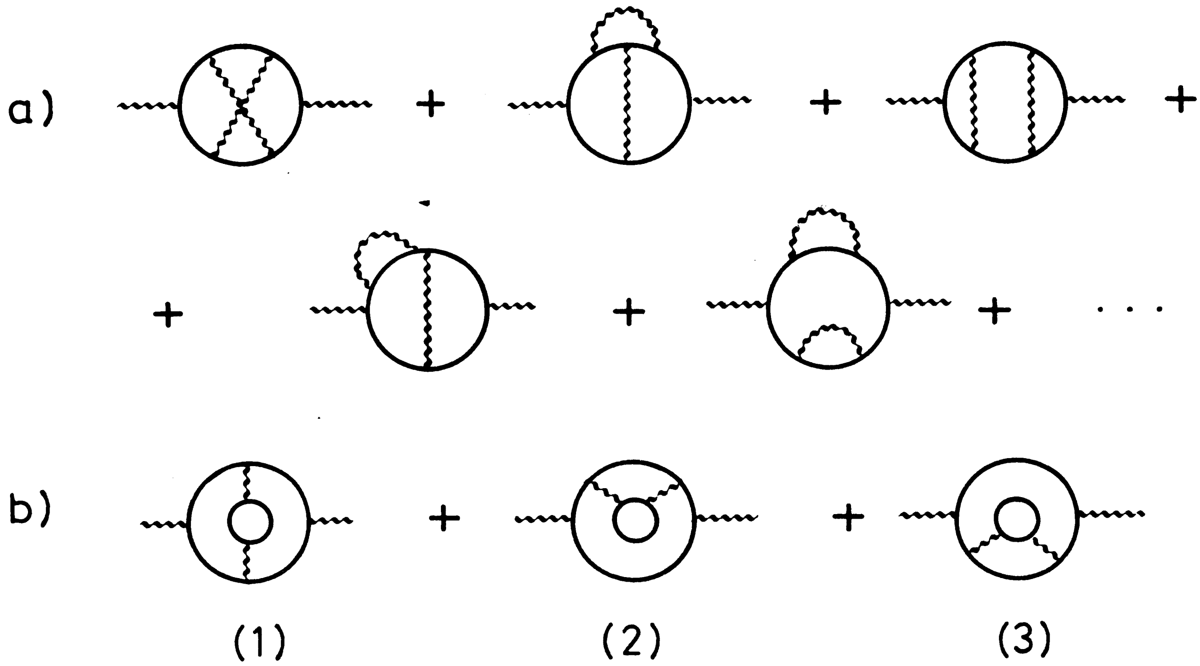

FIG. 3

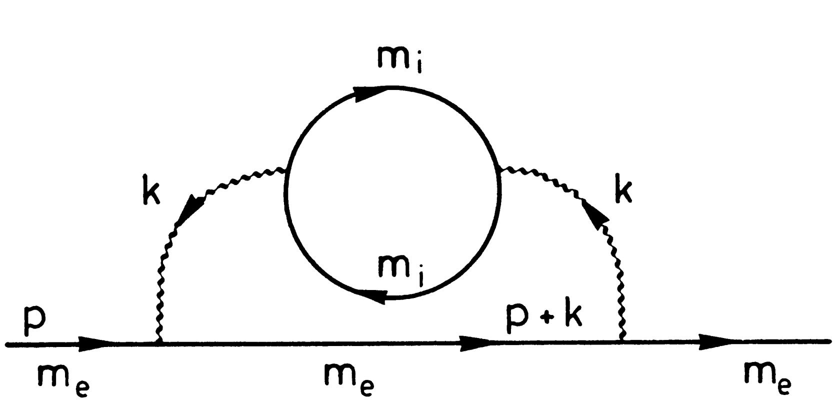

FIG. 4

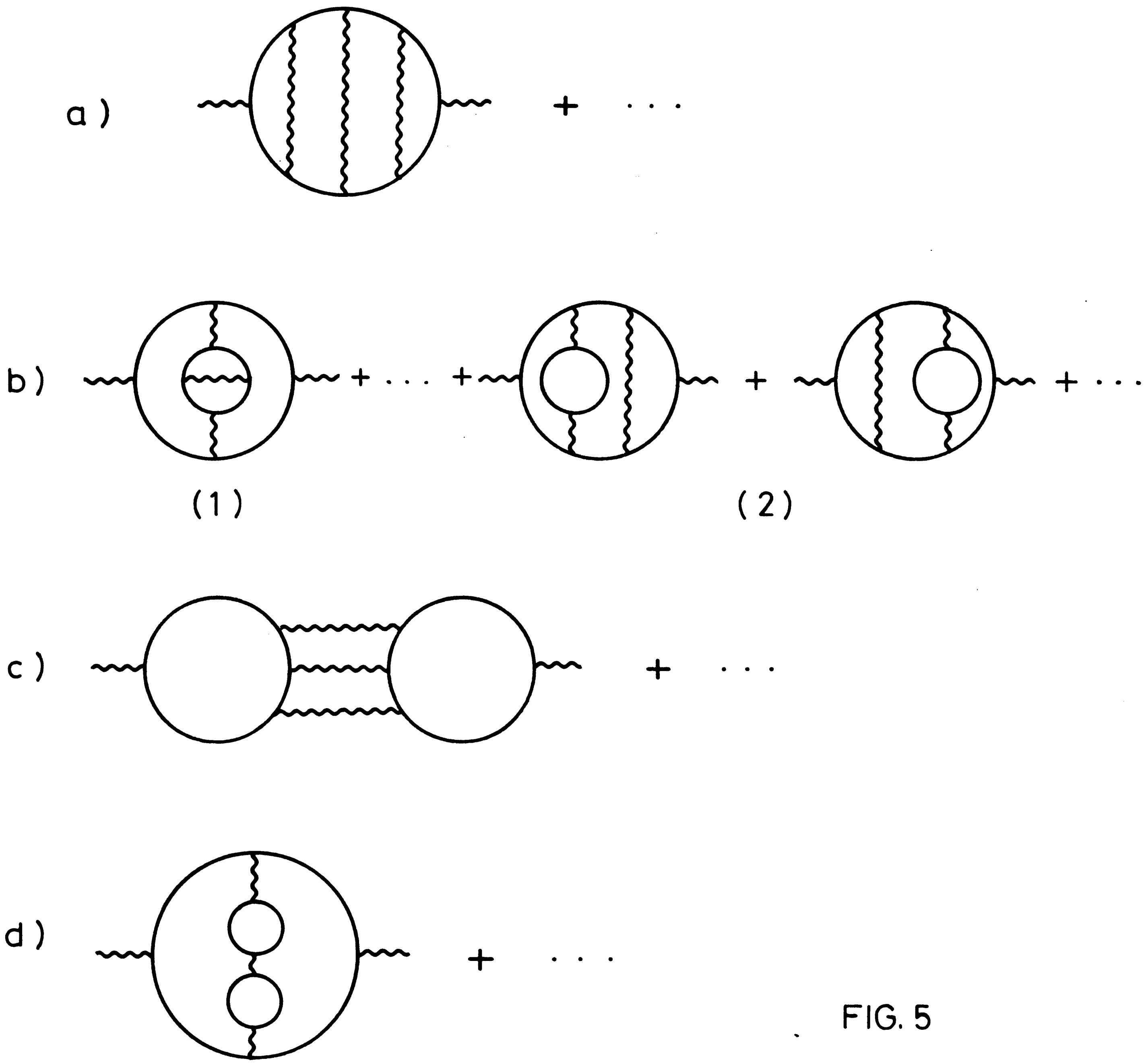

FIG. 5

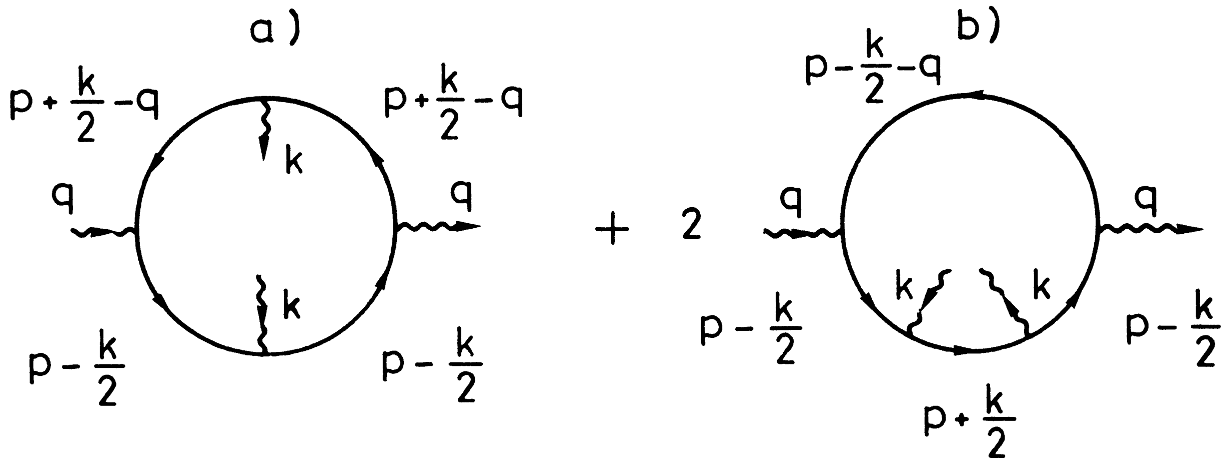
a)
b)
$p+\frac{k}{2}-q$
$p+\frac{k}{2}-q$
$p-\frac{k}{2}-q$
k
k
q
q
+ 2
q
q
k
k
$p-\frac{k}{2}$
$p-\frac{k}{2}$
$p-\frac{k}{2}$
$p-\frac{k}{2}$
$p+\frac{k}{2}$

FIG.6

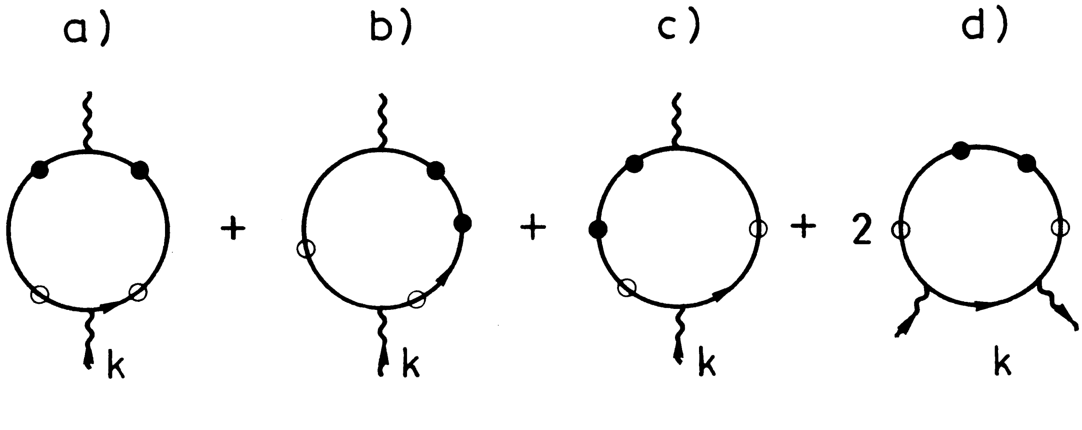
a)
b)
c)
d)
+
+
+ 2
k
k
k
k

FIG.7

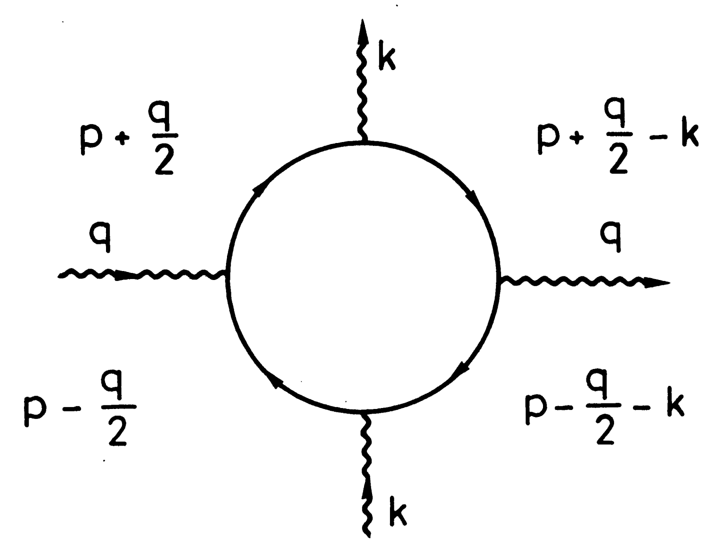

FIG. 8

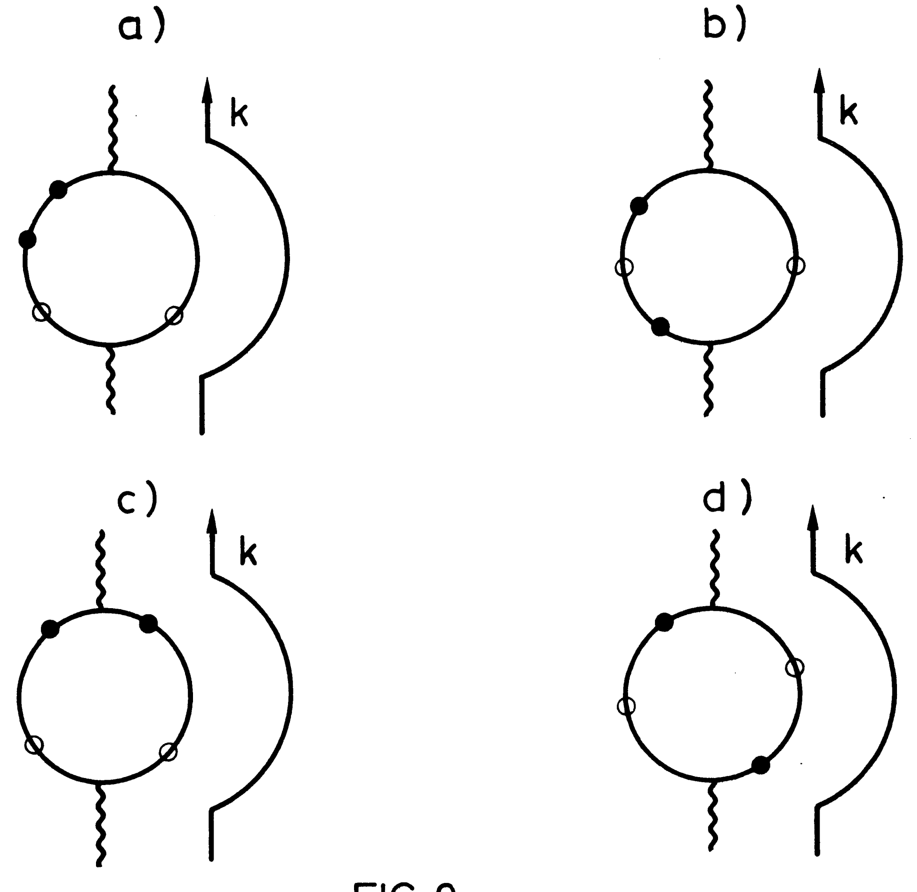

FIG. 9

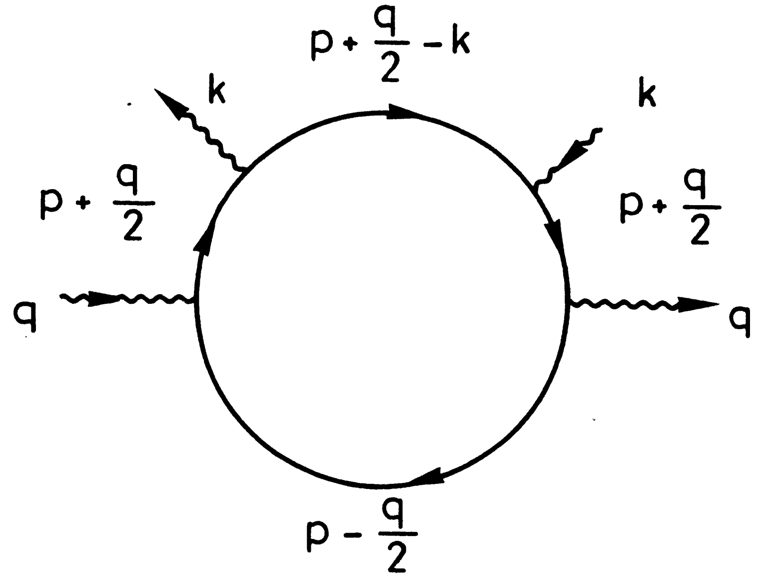

FIG. 10

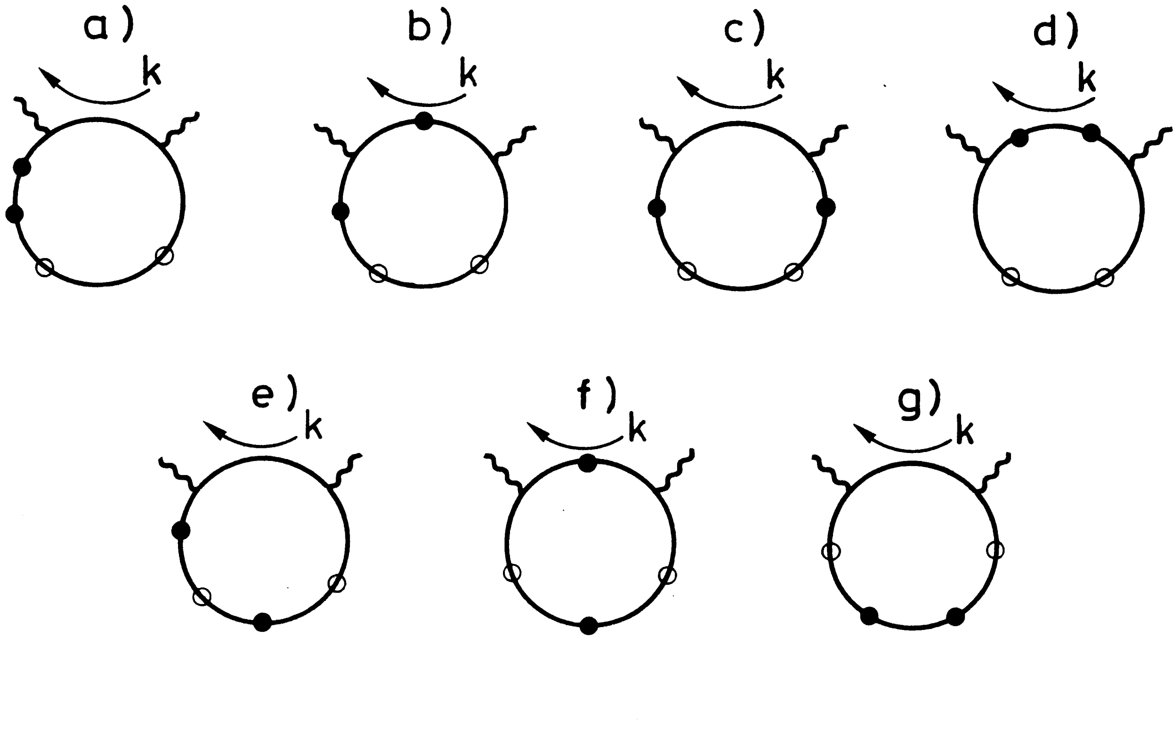

FIG. 11