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NEW DISPERSION RELATIONS AND THEIR APPLICATION TO PARTIAL WAVE AMPLITUDES

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A B S T R A C T

After reviewing the commonly used dispersion relations, a systematic investigation of more generalized dispersion relations on parametrized curves in the Mandelstam plane for $s-u$ crossing symmetric amplitudes is made with the aim of obtaining dispersion relations which receive contributions from all three channels, however, in such a way that knowledge of the absorptive parts is only required in regions well inside the various Lehmann ellipses. In addition we require that the dispersion relations receive no contributions from kinematic singularities arising from the parametrization and that they allow partial wave projections to be made in a relatively simple manner. It is found that dispersion relations on hyperbolic curves in the Mandelstam plane are a natural solution of the problem. The dispersion relations are written in a remarkably simple form similar to the usual fixed t dispersion relation but with an additional t channel contribution. As an interesting application, we derive generalized partial wave dispersion relations for elastic pion-nucleon scattering, where the left-hand cut contribution is explicitly given by convergent partial wave series in the crossed channels.

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1. - INTRODUCTION

> "The importance of dispersion relations in the study of elementary particle reactions is connected with the fact that we are dealing with phenomena for which no detailed theory exists."
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> A. Bohr, Boulder 1960

Lacking a detailed theory of elementary particle reactions, it is important to supplement the general properties of amplitudes as incorporated in dispersion relations with as much dynamical information as possible to obtain relationships between dynamical quantities. Relationships of this nature can be used to check proposed models, self-consistency of experimental data, or to calculate parameters unaccessible experimentally. Furthermore, such relationships can be viewed as a system of equations from which the dynamical quantities could be solved in some sort of bootstrap method.

Among the most useful of such relationships are partial wave dispersion relations [1)] which relate a given partial wave amplitude to an integral over unphysical cuts, and the more general partial wave relations [2)] which relate a given partial wave amplitude to integrals (involving definite kernels) over all partial wave amplitudes of the same channel and perhaps over partial wave amplitudes of the crossed channel.

The first partial wave relations were obtained more than fifteen years ago by Oehme [3)] from unsubtracted fixed momentum transfer dispersion relations, but only recently a detailed study has been carried out [4)] which avoids the approximations made in the earlier work. These partial wave relations can be reformulated in a form similar to that of the normal partial wave dispersion relations, but with the important difference that the generalized potential or driving force is now expressed as a convergent partial wave series in the direct channel [5)].

Expressing the subtraction functions that are necessary for some fixed momentum transfer dispersion relations by a fixed energy dispersion relation [6)] has the advantage of introducing partial wave amplitudes from the crossed channel into the partial wave relations. This is an important feature since a system of partial wave relations which does not involve partial wave amplitudes of the crossed channel, e.g., $\pi\pi \rightarrow N\bar{N}$, could not have an approximate solution involving a small number of low partial wave amplitudes, if there were important low mass resonances in the crossed channel. This results from the fact that the simulation of a nearby crossed channel pole requires many direct channel partial wave amplitudes. Consequently, in order to obtain partial wave relations that have approximate solutions containing as few partial wave amplitudes as possible, it is necessary to begin with dispersion relations which contain crossed channel as well as direct channel contributions.

A fixed angle dispersion relation is one example of a dispersion relation which contains contributions from both channels. In fact, such dispersion relations have already been written for the reactions $\pi N \rightarrow \pi N$, $\gamma N \rightarrow \pi N$ and $NN \rightarrow NN$ [7]. However, in projecting out partial wave amplitudes from these dispersion relations, one obtains terms containing integrations over double spectral regions. While it is true that these undesired integrals begin above the physical threshold in the crossed channel, e.g., $N\bar{N}$, the assumption that they can be neglected is dubious and reduces the resulting partial wave relations at best to approximate relationships.

Thus, our object is to find new dispersion relations which contain both direct and crossed channel contributions that can be expanded in convergent partial wave expansions. Mathematically, one is dealing with an n parameter family of curves in the Mandelstam plane along which dispersion relations are to be written. To avoid problems of convergence, this parametrization must be such that the curves needed at a given energy for partial wave projections do not pass through double spectral regions.

In order to have mathematically simple equations, two more constraints on the parametrization are required. First, it is important that the angular integration necessary for the partial wave projection results in relatively simple kernels. Second, it is important that amplitudes can easily be defined that do not contain kinematic cuts or singularities resulting from the parametrization of the usual s, t and u Mandelstam variables. The latter requirement is imposed to avoid the complicated multi-sheet structure encountered in some parametrizations, e.g., fixed angle.

This paper is organized as follows. In Section 2 we recall the situation for fixed momentum transfer and fixed angle dispersion relations. Then the mathematical implications of the proposed requirements are discussed and a parametrization with the desired properties is found which can be applied to elastic as well as to inelastic reactions. The parametrization turns out to describe hyperbolas in the Mandelstam plane. In Section 3 we write dispersion relations on these hyperbolas and investigate their general properties.

As a specific example, we treat in Section 4 the case of elastic pion-nucleon scattering. For this reaction, we also derive partial wave relations and generalized partial wave dispersion relations. Possible applications of the hyperbola dispersion relations, some concluding remarks and a short discussion for inelastic reactions are given in the last section. Whenever possible arguments and algebra not pertinent to the body of the paper have been put in the Appendices.

2. - KINEMATICS AND GENERALIZED CURVES IN THE MANDELSTAM PLANE

Throughout this paper, we take the direct or s channel reaction to be $a + b \to c + d$ and the t channel reaction to be $a + \bar{c} \to \bar{b} + d$ and use the standard s, t and u Mandelstam variables with

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \equiv \Sigma \ . \tag{2.1}$$

The cosines of the centre-of-mass scattering angles for the s and t channels are given by

$$\cos\theta_s \equiv z_s = \left[(t-u) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)/s\right] / \nu_s \tag{2.2a}$$

$$\cos\theta_t \equiv z_t = \left[(s-u) + (m_a^2 - m_c^2)(m_b^2 - m_d^2)/t\right] / \nu_t \quad , \tag{2.2b}$$

where $\nu_s \equiv 4p_s p_s'$ and $\nu_t \equiv 4p_t p_t'$ (p and p' are the initial and final centre-of-mass momenta in the appropriate channels).

We will be concerned with amplitudes which for fixed t have definite crossing properties under interchange of s and u, i.e., under change of sign of $\nu \equiv s - u$.

Before starting a general discussion of the parametrization of the integration path for dispersion relations, it is useful to recall the situation for fixed t and fixed angle dispersion relations.

For fixed t dispersion relations, t is considered as a parameter and the path of integration is described by solving $s - u = \nu$ and $s + u = \Sigma - t$ for s and u as functions of ν and the parameter t, i.e.,

$$\begin{aligned} 2\, s(\nu; t) &= (\Sigma - t) + \nu \\ 2\, u(\nu; t) &= (\Sigma - t) - \nu \quad . \end{aligned} \tag{2.3}$$

This, of course, is a trivial constraint for s and u and introduces no kinematic singularities.

For fixed angle (θ_s) dispersion relations, z_s is considered as a parameter describing the path of integration. The independent variable is taken to be s. Solving Eq. (2.1) and Eq. (2.2a) for t and u gives

$$\begin{aligned} 2\, t(s; z_s) &= (\Sigma - s) - (m_a^2 - m_b^2)(m_c^2 - m_d^2)/s + \nu_s z_s \\ 2\, u(s; z_s) &= (\Sigma - s) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)/s - \nu_s z_s \quad . \end{aligned} \tag{2.4}$$

Since $\nu_s = 4 p_s p_s'$ as a function of s has in general square root branch cuts, such a parametrization will introduce square root cuts and a possible pole at $s = 0$ into the amplitude. Except in the equal mass case, these kinematic cuts lead to a multi-sheet problem with its associated complications [7),*)]. In addition, as pointed out in the Introduction, a fixed z_s curve that lies inside the physical s channel region unfortunately lies completely outside the physical t channel region. Consequently, the variation of z_s needed for a partial wave projection introduces contributions from double spectral regions where partial wave expansions are not convergent.

As a final and illustrative example, consider fixed z_t dispersion relations. Solving Eq. (2.1) and Eq. (2.2b) for s and u as functions of t and the parameter z_t, gives

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*) It is worth while, however, to mention that, for reactions with $m_a = m_c$ or $m_b = m_d$, the kinematics simplifies in the special case of backward scattering, e.g., $z_s = -1$, and useful backward dispersion relations [8)] or more general "boundary" dispersion relations [9)] can be written.

$$2s(t;z_t) = (\Sigma - t) - (m_a^2 - m_c^2)(m_b^2 - m_d^2)/t + z_t \nu_t$$

$$2u(t;z_t) = (\Sigma - t) + (m_a^2 - m_c^2)(m_b^2 - m_d^2)/t - z_t \nu_t \, . \tag{2.5}$$

Analogous to the fixed θ_s case, the factor ν_t in this parametrization introduces kinematic cuts and a possible pole at $t = 0$ into the amplitudes. But, as recently observed [9] for reactions where $m_a = m_c$ or $m_b = m_d$, ν is directly proportional to ν_t and the assumed crossing properties of the amplitudes allow one to work with amplitudes which are even functions of ν and thus free of kinematic cuts. In particular, the set of amplitudes

$$\tilde{A} \equiv \begin{cases} A & \text{if} \quad A(\nu,t) = +A(-\nu,t) \\ \dfrac{A}{\nu} & \text{if} \quad A(\nu,t) = -A(-\nu,t) \end{cases} \tag{2.6}$$

are free of kinematic cuts. Consequently, this parametrization allows dispersion relations to be written for all amplitudes on a single kinematic sheet.

Unfortunately, when a partial wave projection is performed, fixed θ_t dispersion relations, like fixed θ_s dispersion relations have the undesirable feature of introducing contributions from double spectral regions. In addition, the resulting kernels for the s channel partial wave amplitudes are extremely complicated.

Thus, it is seen that the dispersion relations in common usage either do not contain contributions from both direct and crossed channels (fixed t dispersion relations) or contain contributions from double spectral regions where partial wave expansions are invalid (fixed θ_s and θ_t dispersion relations).

As a result of this discussion we consider it worth while to search for new dispersion relations written on more general curves in the Mandelstam plane. In addition to containing contributions from both channels but not from double spectral regions, it is of practical importance that the parametrization does not introduce kinematic cuts into the amplitudes and allows partial wave projections involving reasonably uncomplicated kernels. Consequently, the four tests for any proposed parametrized set of curves are :

a) curves pass through both direct and crossed channels ;

b) curves do not enter regions, e.g., double spectral regions, where partial wave expansions are not valid ;

c) amplitudes can be easily defined that do not contain any kinematic cuts introduced by the parametrization ;

d) the kernels resulting from the angular integration needed for the partial wave projection are reasonably simple.

The choice of variables used to describe a dynamical system is always very important. In order to present the dynamics in a manner as simple as possible, it is necessary to select variables which best describe the symmetry of the problem being considered. Since we are working with amplitudes which, for fixed t, have definite crossing properties under change of sign of $\nu \equiv s-u$, it is clear that ν and t are the natural choice of variables. Because any parametrization will reduce the number of independent kinematic variables from two to one, it is convenient to take t as the independent variable and to consider ν as a function of t and the curve parameters *).

Once t is accepted as the natural independent variable, conditions c) and d) can be put on a more mathematical footing by writing

$$\nu = F(t; a_0, a_1, \dots) \quad , \tag{2.7}$$

where F is a function of t and in addition contains the real parameters a_0, a_1, ... defining a curve in the Mandelstam plane.

The constraint b), imposing that in the Mandelstam plane the curves parametrized by F should not enter regions where a double spectral function is non-zero, restricts the asymptotic behaviour of F. Geometrically, it is apparent that

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*) This approach cannot be applied to fixed t dispersion relations but such dispersion relations do not satisfy condition a).

$$\lim_{t\to+\infty} \left|\frac{dF}{dt}\right| \leq 1$$

$$\lim_{t\to-\infty} \left|\frac{dF}{dt}\right| \geq 1 \tag{2.8}$$

are necessary conditions that the curves do not asymptotically enter double spectral regions. If $F(t)$ is assumed to be a rational function or a power of such a function, then (2.8) implies

$$\lim_{t\to\pm\infty} \left|\frac{dF}{dt}\right| = 1 \quad . \tag{2.8'}$$

From condition (2.8'), we infer

$$F \underset{|t|\to\infty}{\sim} t + \text{lower terms} \ , \tag{2.9}$$

which implies that we must restrict ourselves to such curves whose asymptotes (for $t \to \pm\infty$) are straight lines at constant s and constant u, respectively.

The simplest possibility consists in assuming that the function F is a rational function of t. This choice, however, is not preferable if one wants to build in explicitly the $s-u$ crossing symmetry which involves at fixed t the replacement $\nu \to -\nu$. To illustrate this point, let us consider the most trivial example, which satisfies the asymptotic condition, i.e.,

$$\nu = F(t;a_0) = a_0 + t \quad . \tag{2.10}$$

Obviously, the curves (2.10) define straight lines in the Mandelstam plane at constant s or u and therefore lead to the familiar fixed s or fixed u dispersion relations with their limited range of convergence *). Moreover, as is well known, the fixed s and fixed u dispersion relations are not explicitly crossing symmetric under change of sign of ν.

Thus we must look for a more general parametrization. Remembering that crossing symmetry permits us to work with the crossing even amplitudes (2.6), $\tilde{A}(\nu^2,t)$, which are functions of ν^2 and t, it is clear that one will avoid the introduction of kinematic cuts into the amplitudes if the only kinematic cuts are of the square root type, $F = \sqrt{f}$, i.e.,

$$\nu = \left(f(t; a_0, a_1, \dots) \right)^{1/2} , \qquad (2.11)$$

where now f - in order that the crossing properties be retained - cannot be the square of a rational function. Consequently, Eq. (2.11) represents a mathematical constraint on the parametrization, which, if satisfied, will ensure condition c). In terms of t and the parameters a_j contained in f, the variables s and u can then be written

$$\begin{aligned} 2\, s(t;f) &= (\Sigma - t) + \sqrt{f} \\ 2\, u(t;f) &= (\Sigma - t) - \sqrt{f} \quad . \end{aligned} \qquad (2.12)$$

We discuss in Appendix D the general type of angular integrations encountered in performing partial wave projections. Essentially condition d) demands that t, when considered as a function of s and the parameters a_j, must be linear in the parameter b which is varied in projecting the partial wave amplitudes for a given s, i.e.,

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*) For example, in the case of πN scattering the fixed s dispersion relations only converge in the small strip $-5m_\pi^2 \lesssim s \leq 4m_\pi^2$ (similarly for fixed u dispersion relations) [10]. Nevertheless they have been used by several authors [6),11]. (Of course, in the case of $\pi\pi$ scattering the fixed s or fixed u dispersion relations are as useful as the usual fixed t dispersion relations.)

$$t(s;a_j) = g(s)\, b + h(s) \quad ,$$

where g and h can depend on some of the a_j's but are independent of the combination of the parameters a_j which form b.

It turns out (for details see Appendix D) that the only acceptable function f satisfying the constraints a)-d) and the asymptotic condition (2.8') is given by

$$f = a_0 + a_1 t + t^2 \qquad (a_1^2 \neq 4a_0) \tag{2.13}$$

which together with (2.12) leads to

$$(s-a)(u-a) = b \tag{2.14}$$

and

$$t = g\, b + h \quad , \tag{2.15}$$

where

$$g = g(s,a) \equiv -(s-a)^{-1}$$

$$h = h(s,a) \equiv \Sigma - s - a \tag{2.16}$$

$$a \equiv \frac{1}{4}(a_1 + 2\Sigma)$$

$$b \equiv \frac{1}{16}(a_1^2 - 4a_0) . \tag{2.17}$$

Equation (2.14) defines hyperbolas in the Mandelstam plane with "hyperbola parameter" b and asymptotes $s = a$ and $u = a$.

Thus we conclude that hyperbolas appear as the natural solution when one attempts to find curves that lead to dispersion relations involving contributions from the direct as well as from the crossed channels but no contributions from double spectral regions, that are free of complications due to kinematic cuts and which result in partial wave relations with manageable kernels. [The last result follows from the linear relation between t and b, see Eq. (2.15).]

In terms of the parameters a and b we get

$$\nu(t;a,b) = \sqrt{(t-t_0)^2 - 4b} \tag{2.11'}$$

$$\begin{aligned} 2s(t;a,b) &= (\Sigma - t) + \sqrt{(t-t_0)^2 - 4b} \\ 2u(t;a,b) &= (\Sigma - t) - \sqrt{(t-t_0)^2 - 4b} \end{aligned}, \tag{2.12'}$$

where

$$t_0 = t_0(a) \equiv \Sigma - 2a . \tag{2.18}$$

It is seen from Eq. (2.11') that for

$$t_- \leq t \leq t_+$$

$$t_\pm = t_\pm(a,b) \equiv t_0(a) \pm 2\sqrt{b} \tag{2.19}$$

there is a kinematical cut such that ν is real for $t < t_-$ and $t > t_+$ and purely imaginary on the cut with $\mathrm{Im}\,\nu \leq 2\sqrt{b}$. The parametrization (2.11') therefore defines for $t < t_-$ and $t > t_+$ two branches of a hyperbola lying in the real Mandelstam plane, whereas for $t_- \leq t \leq t_+$ it defines an analytic continuation lying outside the real Mandelstam plane [above or below the line $\mathrm{Re}\,\nu = 0$, i.e., the line $s = u = \frac{1}{2}(\Sigma - t)$].

3. - DISPERSION RELATIONS ON HYPERBOLAS

It was seen in the last section that along hyperbolic paths in the Mandelstam plane, dispersion relations for crossing symmetric amplitudes can be written on one kinematic sheet from which partial wave relations can be projected involving both direct and crossed channel partial wave amplitudes with reasonably simple kernels.

Assuming that no subtractions are necessary and suppressing isospin indices, the hyperbola dispersion relation for a crossing even amplitude, $\tilde{A} = A_{even}$, can be written in the general form

$$A_{even}(t;a,b) = A^{B}_{even}(t;a,b)$$

$$+\frac{1}{\pi}\int_{t_{th}}^{\infty} dt' \, \frac{\mathrm{Im}\, A_{even}(t';a,b)}{t'-t} + \frac{1}{\pi}\int_{-\infty}^{\tilde{t}_{th}} dt' \, \frac{\mathrm{Im}\, A_{even}(t';a,b)}{t'-t} \quad , \tag{3.1}$$

where A^{B}_{even} represents possible Born terms and the first and second integrals represent the t and s channel contributions, respectively (see also Fig. 2).

In writing Eq. (3.1), the amplitude was considered as a function of t along a hyperbola defined by a and b according to Eq. (2.14). For many purposes, however, it is convenient to eliminate b by means of the usual Mandelstam variables, i.e., to consider b as a function of s and t for a given a, $b = b(s,t;a)$. This means that for given s and a, a family of hyperbolas will be considered, where each member of the family is uniquely defined by the value of t or z_s. After some algebraic manipulations, Eq. (3.1) can be written as

$$A_{even}(s,t,u;a) = A^{B}_{even} + \frac{1}{\pi}\int_{t_{th}}^{\infty} dt' \, \frac{\mathrm{Im}\, A_{even}(t', z'_t)}{t'-t} \tag{3.2a}$$

$$+\frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \, \mathrm{Im}\, A_{even}(s',t') \left[\left(\frac{1}{s'-s} + \frac{1}{s'-u} \right) - \frac{1}{s'-a} \right] ,$$

where

$$z'_t \equiv \left[(t'-2a-\Sigma)^2 - 4b(s,t;a)\right]^{1/2} \Big/ \nu_t(t') \tag{3.3a}$$

$$t' \equiv g(s',a)\, b(s,t;a) + h(s',a) \tag{3.3b}$$

and

$$b(s,t;a) \equiv (s-a)(\Sigma-a-s-t)\,. \tag{3.4}$$

The remarkably simple form of the dispersion relation (3.2a) is a direct consequence of condition d) or of Eq. (2.15), $t = gb + h$, respectively. It is interesting that the last integral of Eq. (3.2a) looks like a fixed t dispersion relation (with a subtraction only in the s channel at $s = a$) apart from the dependence of the absorptive part on t' *).

The dispersion relation (3.2a) is written for the crossing even amplitudes, $\tilde{A} = A_{even}$, which means that the corresponding dispersion relation for the crossing odd amplitudes, $A_{odd} = \nu\,\tilde{A}$, is obtained from Eq. (3.2a) by multiplying the integrands by the factor $\nu/\nu' \equiv \nu(s,t)/\nu(s',t')$, i.e.,

$$A_{odd}(s,t,u;a) = A^{B}_{odd} + \frac{1}{\pi}\int_{t_{th}}^{\infty} dt' \left(\frac{\nu}{\nu'}\right) \frac{\mathrm{Im}\, A_{odd}(t',z'_t)}{t'-t}$$
$$+ \frac{1}{\pi}\int_{s_{th}}^{\infty} ds'\, \mathrm{Im}\, A_{odd}(s',t') \left(\frac{\nu}{\nu'}\right)\left[\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right)-\frac{1}{s'-a}\right]. \tag{3.2c}$$

Using the fact that the two points (s,u) and (s',u') lie on the same hyperbola, i.e., $(s-a)(u-a) = (s'-a)(u'-a)$, the hyperbola dispersion relation for the odd amplitudes can be cast into the remarkably simple form

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*) This integral in the limit of $|a|$ going to infinity formally reduces to that of the fixed t dispersion relation.

$$A_{odd}(s,t,u;a) = \overset{B}{A}_{odd} + \frac{1}{\pi}\int_{t_{th}}^{\infty} dt' \left(\frac{\nu}{\nu'}\right) \frac{\operatorname{Im} A_{odd}(t', z'_t)}{t'-t}$$

$$+ \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \operatorname{Im} A_{odd}(s',t') \left[\frac{1}{s'-s} - \frac{1}{s'-u}\right] . \tag{3.2b}$$

We notice that the last integral has exactly the same structure as in the usual fixed t dispersion relation for a crossing odd amplitude.

Together with the partial wave relations which we shall derive in the next section, the hyperbola dispersion relations (3.2a,b) represent the main result of our paper *). An important new feature of these dispersion relations is that they incorporate both s and t channel contributions without any "double counting".

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*) After completion of this paper, we became aware that dispersion relations on hyperbolic curves have also been independently proposed by Greenberg and Sandusky [12]. The main differences between this paper and our work are the following. i) The authors of Ref. 12) did not realize the drastic simplification gained by eliminating the parameter b by means of the Mandelstam variables, which lead us to the remarkably simple representations (3.2a) and (3.2b). ii) Due to the improvements mentioned under i), we were able to derive relations for the partial wave amplitudes, Section 4, which have not been discussed in Ref. 12). iii) Our derivation of the hyperbola dispersion relations is more general showing the unique rôle played by these dispersion relations.

4. - PARTIAL WAVE RELATIONS AND MODIFIED PARTIAL WAVE DISPERSION RELATIONS

As an interesting application of the hyperbola dispersion relations, we consider in this section the derivation of the partial wave relations*) for the elastic reaction $\pi N \to \pi N$. In the case of πN scattering, we have

$$A_{\text{even}} = \begin{cases} A^+ \\ B^- \end{cases}$$
$$A_{\text{odd}} = \begin{cases} A^- \\ B^+ \end{cases} \quad , \tag{4.1}$$

where A and B are the usual invariant amplitudes and the crossing even and odd combinations are defined by

$$A^{\pm} = \frac{1}{2}\left[A(\pi^- p) \pm A(\pi^+ p)\right] \quad \text{etc.} \tag{4.2}$$

The only Born term contributing to the amplitudes is the nucleon exchange in the $B^{\pm}$ amplitudes. Thus, the hyperbola dispersion relations for the πN amplitudes read

$$A^+(s,t,u;a) = \frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty} ds' \, \mathrm{Im}\, A^+(s',t')\left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right]$$
$$+ \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \, \frac{\mathrm{Im}\, A^+(t',z_t')}{t'-t}$$

$$B^+(s,t,u;a) = G^2\left[\frac{1}{M^2-s} - \frac{1}{M^2-u}\right] + \frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty} ds' \, \mathrm{Im}\, B^+(s',t')\left[\frac{1}{s'-s} - \frac{1}{s'-u}\right]$$
$$+ \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \left(\frac{\nu}{\nu'}\right) \frac{\mathrm{Im}\, B^+(t',z_t')}{t'-t}$$

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*) For details of the πN kinematics, see Refs. 1) or 2).

$$A^-(s,t,u;a) = \frac{1}{\pi}\int\limits_{(M+m_\pi)^2}^{\infty} ds' \, \mathrm{Im}\, A^-(s',t') \left[\frac{1}{s'-s} - \frac{1}{s'-u}\right]$$

$$+ \frac{1}{\pi}\int\limits_{4m_\pi^2}^{\infty} dt' \left(\frac{\nu}{\nu'}\right) \frac{\mathrm{Im}\, A^-(t',z_t')}{t'-t}$$

$$B^-(s,t,u;a) = G^2\left[\frac{1}{M^2-s} + \frac{1}{M^2-u} - \frac{1}{M^2-a}\right]$$

$$+\frac{1}{\pi}\int\limits_{(M+m_\pi)^2}^{\infty} ds' \, \mathrm{Im}\, B^-(s',t')\left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right] + \frac{1}{\pi}\int\limits_{4m_\pi^2}^{\infty} dt' \frac{\mathrm{Im}\, B^+(t',z_t')}{t'-t} \tag{4.3}$$

where t' and z_t' are given in Eq. (3.3), M is the nucleon mass and G is the πN coupling constant, i.e., $G^2/4\pi \simeq 14.6$ [13].

Before we project partial wave relations from the hyperbola dispersion relations, let us briefly discuss the question of possible subtractions to these dispersion relations. It is usually assumed that the minimum number of subtractions can be inferred from the Regge pole model. Since the asymptote for hyperbola dispersion relations is the line $u = a$, the convergence is then determined by u channel Regge poles. This implies that in reactions involving nucleons, no subtractions are expected since the relevant trajectories have rather low intercepts. Obviously, this is a big advantage of the hyperbola dispersion relations over the fixed t dispersion relations where in some amplitudes subtractions are necessary.

In order to extract partial wave dispersion relations from the dispersion relations (4.3), we expand the absorptive parts in the s and t channels into the $\pi N \to \pi N$ and $\pi\pi \to N\bar{N}$ partial wave amplitudes, respectively, and then project out the s channel partial wave amplitudes $f_{\ell\pm}(W)$ (corresponding to total angular momentum $j = \ell \pm \frac{1}{2}$ and orbital angular momentum ℓ). As in our earlier work on fixed t dispersion relations [2),5)], we consider the partial wave amplitudes in the W plane ($W \equiv \sqrt{s}$ = total centre-of-mass energy) rather than in the s plane which is crucial in order to make use of the simplification due to the McDowell symmetry relation [14)]

$$f_{\ell+}(-W) = - f_{(\ell+1)-}(W) \quad , \tag{4.4}$$

where the partial wave amplitudes $f_{\ell\pm}(W)$ are defined by Eq. (A.1). We then obtain the following partial wave relations ($\ell = 0,1,2,\ldots$) *)

$$\begin{aligned}
f^{\pm}_{\ell+}(W) = N^{\pm}_{\ell+}(W) &+ \frac{1}{\pi}\int_{M+m_\pi}^{\infty} dW' \sum_{\ell'=0}^{\infty} \Big\{ K^{\pm}_{\ell\ell'}(W,W')\, \mathrm{Im}\, f^{\pm}_{\ell'+}(W') \\
&\qquad + K^{\pm}_{\ell\ell'}(W,-W')\, \mathrm{Im}\, f^{\pm}_{(\ell'+1)-}(W') \Big\} \\
&+ \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \sum_J \Big\{ G_{\ell J}(W,t')\, \mathrm{Im}\, f^J_+(t') \\
&\qquad + H_{\ell J}(W,t')\, \mathrm{Im}\, f^J_-(t') \Big\}
\end{aligned}$$

$$\begin{aligned}
f^{\pm}_{(\ell+1)-}(W) = N^{\pm}_{(\ell+1)-}(W) &- \frac{1}{\pi}\int_{M+m_\pi}^{\infty} dW' \sum_{\ell'=0}^{\infty} \Big\{ K^{\pm}_{\ell\ell'}(-W,W')\, \mathrm{Im}\, f^{\pm}_{\ell'+}(W') \\
&\qquad + K^{\pm}_{\ell\ell'}(-W,-W')\, \mathrm{Im}\, f^{\pm}_{(\ell'+1)-}(W') \Big\} \\
&- \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \sum_J \Big\{ G_{\ell J}(-W,t')\, \mathrm{Im}\, f^J_+(t') \\
&\qquad + H_{\ell J}(-W,t')\, \mathrm{Im}\, f^J_-(t') \Big\} \, .
\end{aligned} \tag{4.5}$$

---

*) In the following we do not write explicitly the dependence of the kernels on the parameter a, which is kept fixed.

Here the quantities $N^{\pm}_{\ell\pm}(W)$ represent the contributions to the partial wave amplitudes $f^{\pm}_{\ell\pm}(W)$ resulting from the nucleon Born terms in (4.3). Explicit expressions for the N's are given in Appendix A.

The kernels $K^{\pm}_{\ell\ell'}(W,W')$ couple each partial wave amplitude, $f_{\ell\pm}(W)$, to the imaginary parts of the other s channel partial wave amplitudes. The derivation of these kernels can be found in Appendix B.

Similarly, the kernels $G_{\ell J}(W,t')$ and $H_{\ell J}(W,t')$ couple each s channel partial wave amplitude to the imaginary parts of the t channel (i.e., $\pi\pi \to N\bar{N}$) partial wave amplitudes, $f^{J}_{\pm}(t)$, with total angular momentum J and relative nucleon-antinucleon helicity (+ = parallel, - = antiparallel helicities). According to Bose statistics (or crossing symmetry) the summations over J in Eq. (4.5) run over even values of J in the case of the crossing even amplitudes (upper index +, corresponding to isospin zero exchange in the t channel) and over odd values of J in the case of the crossing odd amplitudes (upper index -, corresponding to isospin one exchange in the t channel) *). A detailed discussion of the t channel kernels is given in Appendix C.

In the derivation of the partial wave relations (4.5), the hyperbola dispersion relations (4.3) have been evaluated for $-1 \leq z_s \leq 1$. Geometrically, this means that the dispersion relations have been evaluated on a family of hyperbolas parametrized by $b(s,z_s=1;a) \leq b \leq b(s,z_s=-1;a)$, where $b(s,z_s=\pm 1;a)$ define the hyperbolas that pass through the points $(s,z_s=1)$ and $(s,z_s=-1)$, respectively. Since we have expanded the absorptive parts in partial wave amplitudes, care must be taken that the hyperbolas do not enter a double spectral region. Actually we are concerned here with the large Lehmann ellipse [15], since it limits the region of convergence of the absorptive parts. The boundary of the ellipse can be encountered even at points where there are no double spectral functions due to the reflection or mirror curve in the z_s plane of the nearest double spectral boundary. Therefore, there is in general some energy, s_{max}, such that for $s > s_{max}$ there is no value of a for which the partial wave expansion converges. This implies that the partial wave relations (4.5) obtained from hyperbola dispersion relations are valid only in the energy region below s_{max}. It is easy to see that the value of s_{max} for a given reaction is always greater than that for a fixed t dispersion relation. Estimates for s_{max} are given in Appendix E.

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*) It is this fact which allows us to omit the upper indices ± on the t channel kernels.

In the special case of πN scattering, s_{max} is of the order of $105\ m_\pi^2$ (corresponding to a pion lab. energy of 480 MeV), where the optimal value of a is $-117\ m_\pi^2$.

Obviously, the fact that a given s channel partial wave amplitude in Eq. (4.5) is completely determined by the partial wave amplitudes of the s and t channels is an important improvement over the situation for the usual partial wave dispersion relations [1] where unphysical cut contributions occur. Nevertheless, the kernels $K_{\ell\ell'}$, $G_{\ell J}$ and $H_{\ell J}$ generate together all the well-known cuts in the complex s plane, i.e., the physical cut $s \geq (M+m_\pi)^2$, the u channel cut $s \leq (M-m_\pi)^2$ and the t channel cuts $|s| = M^2 - m_\pi^2$ and $s \leq 0$, as can be seen from the explicit expressions given in the Appendices B and C. In particular, we have the important result that the s channel cut is generated by the kernel $K_{\ell\ell'}$ in the following way

$$K_{\ell\ell'}(W,W') = \frac{\delta_{\ell\ell'}}{W'-W} + \overline{K}_{\ell\ell'}(W,W') \quad , \tag{4.6}$$

where the kernel $\overline{K}_{\ell\ell'}$ has only the u channel cut. This leads us immediately to a new kind of partial wave dispersion relations, which we call <u>generalized partial wave dispersion relations</u> *)

$$\mathrm{Re}\, f^{\pm}_{\ell+}(W) = \frac{1}{\pi}\int_{M+m_\pi}^{\infty} dW' \frac{\mathrm{Im}\, f^{\pm}_{\ell+}(W')}{W'-W} + V^{\pm}_{\ell+}(W) \quad , \tag{4.7}$$

where the <u>"left-hand cut contribution"</u> $V^{\pm}_{\ell+}$ is now explicitly given by

$$\begin{aligned} V^{\pm}_{\ell+}(W) \equiv N^{\pm}_{\ell+}(W) &+ \frac{1}{\pi}\int_{M+m_\pi}^{\infty} dW' \sum_{\ell'=0}^{\infty}\Big\{ \overline{K}^{\pm}_{\ell\ell'}(W,W')\, \mathrm{Im}\, f^{\pm}_{\ell'+}(W') \\ &+ \overline{K}^{\pm}_{\ell\ell'}(W,-W')\, \mathrm{Im}\, f^{\pm}_{(\ell'+1)-}(W') \Big\} \\ &+ \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \sum_J \Big\{ G_{\ell J}(W,t')\, \mathrm{Im}\, f^J_+(t') + H_{\ell J}(W,t')\, \mathrm{Im}\, f^J_-(t') \Big\} . \end{aligned} \tag{4.8}$$

---

*) Equation (4.7) may equally well be written in the s plane.

While having all the desired left-hand cuts, the "generalized potential" $V^{\pm}_{\ell+}$ is not given by integrals over these unphysical cuts but by integrals over the <u>physical</u> s and t channel regions.

Thus we conclude that by using the generalized partial wave dispersion relations (4.7) we are able to calculate uniquely, at least in principle, the left-hand cut contribution from a knowledge only on the physical cuts. This is not possible in the usual partial wave dispersion relations with their unavoidable convergence problems.

5. - <u>CONCLUDING REMARKS</u>

In this paper we have considered generalized dispersion relations for crossing symmetric amplitudes. After a discussion of the deficiencies of the commonly used fixed variable dispersion relations, we proposed a set of four properties which if satisfied would avoid such deficiencies while retaining the better features of previous dispersion relations.

A priori it is completely open whether there exists a solution at all or whether there exist several solutions to the posed problem. Surprisingly enough, it was found that the postulated constraints are restrictive enough to favour a single solution, namely the dispersion relations written on hyperbolic curves in the Mandelstam plane.

For a useful and powerful application to physical problems, it is crucial that these hyperbola dispersion relations can be written in a transparent form such as Eqs. (3.2a,b). The simplicity of Eqs. (3.2a,b) results from the hyperbolic parametrization satisfying the conditions given in Section 2, i.e., the parametrization introduces no kinematic cuts into the dispersion relation and results in the desired dependence of t on the partial wave projection parameter, b, as seen in Eq. (2.15). As a particular interesting feature, we point out the similarity of the s and u channel contributions to these dispersion relations with those to fixed t dispersion relations and the explicit occurrence of the t channel contributions to these dispersion relations that are not present in the usual fixed t dispersion relations. Besides these contributions, which are very desirable since they represent the physical exchange channels, there are no contributions, e.g., due to kinematical cuts, which could, if present, destroy the usefulness of the dispersion relations *). Another advantage of the hyperbola dispersion relations is that they explicitly maintain the $s-u$ crossing symmetry, in contrast to most other fixed variable dispersion relations. Concerning the important question of subtractions, the Regge pole model implies that no subtractions are necessary in hyperbola dispersion relations for reactions involving nucleons. Without doubt, the main advantage of these new dispersion relations over the previous dispersion relations consists in the large domain of convergence covering all three channels.

In the second part of this paper, we considered elastic pion-nucleon scattering in order to illustrate by an explicit example the useful properties of these dispersion relations on hyperbolas. In particular, using these dispersion relations, we obtained relations for partial wave amplitudes of the πN channel. These so-called "partial wave relations", which are given in Eq. (4.5), have the important property that the analyticity of the partial wave amplitudes is determined completely by the integral kernels, given in Appendices B and C. Thus, since the kernels are known, the analytic structure of the partial wave amplitudes can be inferred from these partial wave relations. Since all three channels are explicitly present, the correct analytic structure for the partial wave amplitudes is obtained within the region of convergence which is shown in Appendix E to be at least the interval $19\, m_{\pi}^2 \leq \mathrm{Re}\, s \leq 105\, m_{\pi}^2$. In particular, in any approximation involving a finite number of partial waves, one obtains not only the correct right-hand cut, $\mathrm{Re}\, s \geq (M+m_{\pi})^2$, but also the correct left-hand cut including the circle $|s| = M^2 - m_{\pi}^2$ **). Furthermore, the partial wave relations derived from the hyperbola dispersion relations contain $s-u$ crossing symmetry.

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*) Yu and Moravcsik [16] published recently a paper on dispersion relations for fixed transverse momentum squared, $q_{\perp}^2 \equiv q^2 \sin^2 \theta_s$; these dispersion relations illustrate nicely what happens if one discards conditions b)-d) of Section 2 ! [Notice the close connection with the "counter-example" discussed in Appendix D, since $q_{\perp}^2 = \varphi(s,t)/4sq^2$.]

**) Recall that the circle cut is normally absent in the partial wave relations derived from fixed t dispersion relations [5], although it could be produced by a "controlled" divergence of the direct channel partial wave expansion in the t channel, as realized in the Veneziano model [17]

This implies for example that the zero at $s \simeq M^2$ of the crossing-odd s wave amplitude, $f_{0+}^{-}(s)$, which is known to be a rigorous consequence of the s - u crossing symmetry [18], is produced by the kernels independent of the input imaginary parts.

The explicit knowledge of the kernels made it possible to rewrite the partial wave relations (4.5) in the form *)

$$\mathrm{Re}\, f_\ell(s) = \frac{1}{\pi} \int_{(M+m_\pi)^2}^{\infty} ds' \frac{\mathrm{Im}\, f_\ell(s')}{s'-s} + V_\ell(s) \quad , \tag{5.1}$$

which looks exactly like a partial wave dispersion relation [1]. However, while the so-called "potential" or "driving force", $V_\ell(s)$, in a partial wave dispersion relation is given by integrals over the <u>unphysical</u>, left-hand cuts, i.e.,

$$V_\ell^{\text{Partial wave D.R.}}(s) \equiv \frac{1}{2\pi i} \int_{\text{left-hand cuts}} ds' \frac{\mathrm{disc}\, f_\ell(s')}{s'-s}$$

$$= \frac{1}{\pi} \int_{S_1}^{S_2} ds' \frac{\mathrm{Im}\, f_\ell(s')}{s'-s} + \frac{1}{\pi} \int_{-\infty}^{(M-m_\pi)^2} ds' \frac{\mathrm{Im}\, f_\ell(s')}{s'-s} + \frac{1}{2\pi i} \oint_{|s|=M^2-m_\pi^2} ds' \frac{\mathrm{disc}\, f_\ell(s')}{s'-s} \tag{5.2}$$

($s_1 = (M - m_\pi^2/M)^2$, $s_2 = M^2 + 2m_\pi^2$), we obtained from the hyperbola dispersion relations the representation

$$V_\ell^{\text{Hyperbola D.R.}}(s) = \text{nucleon contr.} + \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' \sum_J \left\{ \text{kernels} \times \mathrm{Im}\, f^J(t') \right\}$$

$$+ \frac{1}{\pi} \int_{(M+m_\pi)^2}^{\infty} ds' \sum_{\ell'=0}^{\infty} \left\{ \text{kernels} \times \mathrm{Im}\, f_{\ell'}(s') \right\} \tag{5.3}$$

where one has to integrate only over <u>physical</u> cuts.

---

*) For simplicity, in the following discussion, we work in the s plane and use the shorthand notation $f_\ell(s)$ for $f_{\ell+}^{\pm}(W)$.

As mentioned in the previous paragraph, the relation (5.3) is only valid for values of s inside the region of convergence. In order to distinguish the relation (5.1) with V_ℓ given by Eq. (5.3) from the usual partial wave dispersion relations, we call it a "generalized" partial wave dispersion relation.

It is well known that in the usual partial wave dispersion relations there are a number of serious problems among which the most important are : i) the left-hand cut discontinuities cannot be expressed along all the cuts by convergent partial wave expansions ; ii) in general one has to introduce subtractions and thus unknown parameters ; iii) there exists no general proof of these relations. As a result of these difficulties, the best use of these relations is a semi-phenomenological approach, in which one tries to deduce information about the left-hand cut term, $V_\ell(s)$, from direct channel experimental phase shifts. By this method, which has been extensively used in the last decade by Hamilton and co-workers [1], one obtains information on the nearest singularities, the so-called "long range forces" (e.g., ρ and ε exchanges) and some global information about the remaining cut contributions, the so-called "short range forces". The critical point in this approach concerns the analytic continuation from the real axis, where one has the input information, to the circle cut, where one wants to extract the t channel amplitudes. Without a doubt, some progress has been made recently [19] by using the new analytic continuation techniques devised by Ciulli, Cutkosky and others [20]. Nevertheless, we believe that a Cauchy representation, i.e., a dispersion relation, still provides the best method for doing analytic continuations. Independent of whether our belief is correct or not, it is clear that the generalized partial wave dispersion relation given by Eq. (5.1) together with Eq. (5.3) is a powerful tool which can be used to calculate such quantities as the t channel discontinuity. Since the t channel kernels are explicitly known, one may hope to be able to disentangle the various partial waves in the $\pi\pi \rightarrow N\bar{N}$ channel. For instance, in the case of the crossing-even amplitudes, it is easy to form combinations of the direct channel partial wave amplitudes in which the $J = T = 0$ wave does not contribute, and one can essentially determine the $J = 2$, $T = 0$ amplitude.

Another possible use of the generalized partial wave dispersion relations consists in testing proposed $\pi\pi$ phase shifts. It is generally assumed that the elastic unitarity relation connecting the $\pi\pi \rightarrow N\bar{N}$ amplitudes with the $\pi\pi \rightarrow N\bar{N}$ amplitudes is valid not only up to $t = 16\ m_\pi^2$ but presumably up to $t_{max} \simeq 50\ m_\pi^2$. This then allows an approximate calculation of the t channel contribution to $V_\ell(s)$,

$$V_\ell^t(s) \equiv \frac{1}{\pi}\int_{4m_\pi^2}^{t_{max}} dt' \sum_J^{J_{max}} \left\{ \text{kernels} \times \mathrm{Im}\, f^J(t') \right\} , \qquad (5.4)$$

as a function of s. On the other hand, it follows from the generalized partial wave dispersion relations that $V_\ell^t(s)$ is also approximately given by

$$V_\ell^t(s) \simeq \mathrm{Re}\, f_\ell(s) - \frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty} ds' \frac{\mathrm{Im}\, f_\ell(s')}{s'-s} - \frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty} ds' \sum_{\ell'=0}^{\ell_{max}} \left\{ \text{kernels} \times \mathrm{Im}\, f_{\ell'}(s') \right\} - \text{nucleon contr.} \qquad (5.5)$$

which can be calculated from direct channel $\pi N \to \pi N$ phase shifts, the pion nucleon coupling constant and some high energy model. Thus a comparison of the two results provides a test for proposed $\pi\pi$ phase shifts.

Finally we mention that partial wave dispersion relations have been used during the last few years to constrain the energy dependence of the phase shifts in analyzing the experimental data [21]. Clearly, this requires a knowledge of the potential V_ℓ, and one needs a parametrization of V_ℓ which is not only sufficiently accurate but also reasonably simple. It has been found that such parametrizations require a large number of poles on the left-hand cut (the actual number was of the order of 10-20) [22]. Moreover, since the discontinuity along the left-hand cut cannot in general be expanded into partial waves, most of these poles have no simple physical interpretation. Considering this, it seems worth while to calculate $V_\ell(s)$ from Eq. (5.3) with its obvious physical meaning. For the higher waves, for example, one expects a saturation of Eq. (5.3) with a small number of narrow resonances.

Although in this paper we have illustrated the usefulness of dispersion relations on hyperbolas only for pion-nucleon scattering we must point out that the applicability of these dispersion relations is by no means restricted to this reaction. In fact, the general form of the hyperbola dispersion relations, as given in Eqs. (3.2a,b), holds for all binary reactions with the appropriate modifications ; e.g., for KN scattering, one has to include the unphysical cut running from the $\pi\Lambda$ to the KN threshold. Even for inelastic reactions like photoproduction, there is no essential complication. Explicit applications to other reactions than pion-nucleon scattering will be presented elsewhere.

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APPENDIX A - NUCLEON EXCHANGE

In order to calculate the nucleon exchange terms, $N_{\ell\pm}$, in the partial wave relations (4.5) derived from the hyperbola dispersion relations (4.3), we use the projection formula [2)]

$$f_{\ell\pm}(W) = \frac{1}{16\pi W}\Big\{(E+M)\big[A_\ell + (W-M)B_\ell\big] + (E-M)\big[-A_{\ell\pm1} + (W+M)B_{\ell\pm1}\big]\Big\}\,, \tag{A.1}$$

which expresses the πN partial wave amplitudes $f_{\ell\pm}$ by the projections of the invariant amplitudes

$$A_\ell(s) \equiv \int_{-1}^{1} dz_s\, P_\ell(z_s)\, A(s,t)\Big|_{t=-2q^2(1-z_s)} \tag{A.2}$$

(B_ℓ by analogy). The nucleon centre-of-mass energy, E, and the square of the centre-of-mass three-momentum, q^2, are given by

$$E = E(W) = \sqrt{M^2+q^2} = \frac{s+M^2-m_\pi^2}{2W} \tag{A.3}$$

$$q^2 = q^2(s) = \frac{\big[s-(M+m_\pi)^2\big]\big[s-(M-m_\pi)^2\big]}{4s}\,. \tag{A.4}$$

The result is

$$\begin{aligned} N^{+}_{\ell\pm} &= N^{+}_{\ell\pm,\mathrm{FTDR}} && (\ell=0,1,2,\dots)\\ N^{-}_{\ell+} &= N^{-}_{\ell+,\mathrm{FTDR}} + \frac{E+M}{W}\,\frac{2M^2f^2(W-M)}{a-M^2}\,\delta_{\ell 0} && (\ell=0,1,2,\dots)\\ N^{-}_{\ell-} &= N^{-}_{\ell-,\mathrm{FTDR}} + \frac{E-M}{W}\,\frac{2M^2f^2(W+M)}{a-M^2}\,\delta_{\ell 1} && (\ell=1,2,3,\dots) \end{aligned} \tag{A.5}$$

The subscript FTDR indicates that the corresponding expressions are identical to the nucleon exchange contributions to the partial wave relations derived from fixed t dispersion relations [2])

$$N^{\pm}_{\ell\pm,FTDR} = \frac{M^2 f^2}{W} \Big\{ (W-M)(E+M) \Big[\frac{2\delta_{\ell 0}}{M^2-s} \pm \frac{1}{q^2} Q_\ell(y) \Big] + (W+M)(E-M) \Big[\frac{2\delta_{\ell\pm1,0}}{M^2-s} \pm \frac{1}{q^2} Q_{\ell\pm1}(y) \Big] \Big\} \tag{A.6}$$

with

$$y = y(s) \equiv 1 - \frac{s - M^2 - 2m_\pi^2}{2q^2} \; . \tag{A.7}$$

Here f^2 is the reduced πN coupling constant, $G^2 \equiv 16\pi M^2 m_\pi^{-2} f^2$, with $f^2 \simeq 0.081$ [13]).

It is seen that the nucleon exchange contributions derived from the hyperbola dispersion relations and from the fixed t dispersion relations are identical, except for the f^-_{0+} and f^-_{1-} partial wave amplitudes, in which case we have additional "contact term" contributions depending not only on f^2 but also on a. (Notice that these "contact terms" vanish in the limit $|a| \to \infty$.)

Finally, we notice that the nucleon exchange terms (A.5) satisfy the McDowell symmetry relation (4.4), i.e.,

$$N_{\ell+}(-W) = -N_{(\ell+1)-}(W) \; . \tag{A.8}$$

APPENDIX B - s AND u CHANNEL EXCHANGE

For the derivation of the s and u channel exchange contributions to the partial wave relations (4.5), it is convenient, for the πN kinematics, to use a matrix notation used in Refs. 2) and 18). We define the two "vectors"

$$\underset{\sim}{A} = (A, B) \tag{B.1}$$

$$\underset{\sim}{f_\ell} = (f_{\ell+}, f_{(\ell+1)-}) \quad , \tag{B.2}$$

which allow us to cast the projection formulas (A.1) and (A.2) into the form ($\ell = 0,1,2,\ldots$)

$$\underset{\sim}{f_\ell}(W) = \int_{-1}^{1} dz_s \, \underset{\sim}{R}^{\ell}(W, z_s) \, \underset{\sim}{A}(s,t)\Big|_{t=-2q^2(1-z_s)} \tag{B.3}$$

where the projection kernel $\underset{\sim}{R}^{\ell}$ is defined by

$$\underset{\sim}{R}^{\ell}(W, z_s) = \begin{pmatrix} r^{I}_{\ell,\ell+1} & r^{II}_{\ell,\ell+1} \\ r^{I}_{\ell+1,\ell} & r^{II}_{\ell+1,\ell} \end{pmatrix} \tag{B.4}$$

and ($m,n = 0,1,2,\ldots$)

$$r^{I}_{mn}(W, z_s) = \frac{1}{16\pi W}\left\{(E+M)P_m(z_s) - (E-M)P_n(z_s)\right\}$$

$$r^{II}_{mn}(W, z_s) = \frac{1}{16\pi W}\left\{(E+M)(W-M)P_m(z_s) + (E-M)(W+M)P_n(z_s)\right\} . \tag{B.5}$$

We also need the inversion of Eq. (B.3), i.e., the partial wave expansion 2),18)

$$\underset{\sim}{A}(s,t)\Big|_{t=-2q^2(1-z_s)} = \sum_{\ell=0}^{\infty} \underset{\sim}{S}^{\ell}(W, z_s) \, \underset{\sim}{f_\ell}(W) \tag{B.6}$$

with

$$\underset{\sim}{S}^{\ell}(W,z_s) = \begin{pmatrix} S^{I}_{\ell,\ell+1} & -S^{I}_{\ell,\ell+1} \\ S^{II}_{\ell+1,\ell} & -S^{II}_{\ell+1,\ell} \end{pmatrix} \tag{B.7}$$

$$S^{I}_{mn}(W,z_s) = 4\pi\left\{\frac{W+M}{E+M}P'_m(z_s) + \frac{W-M}{E-M}P'_n(z_s)\right\}$$

$$S^{II}_{mn}(W,z_s) = 4\pi\left\{\frac{1}{E+M}P'_m(z_s) - \frac{1}{E-M}P'_n(z_s)\right\}. \tag{B.8}$$

For the subsequent derivation of the partial wave relations (4.5) it is crucial to notice that the kernels $\underset{\sim}{R}^{\ell}$ and $\underset{\sim}{S}^{\ell}$ are functions of W and z_s, rather than of s and t, satisfying the symmetry relations

$$\begin{aligned} \underset{\sim}{R}^{\ell}(-W,z_s) &= -\underset{\sim}{\sigma}_1\,\underset{\sim}{R}^{\ell}(W,z_s) \\ \underset{\sim}{S}^{\ell}(-W,z_s) &= -\underset{\sim}{S}^{\ell}(W,z_s)\,\underset{\sim}{\sigma}_1 \end{aligned} \tag{B.9}$$

in agreement with the McDowell symmetry relation (4.4) which can be written as

$$\underset{\sim}{f}_{\ell}(-W) = -\underset{\sim}{\sigma}_1\,\underset{\sim}{f}_{\ell}(W) \tag{B.10}$$

$$\underset{\sim}{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.11}$$

Considering only the s and u channel terms, the hyperbola dispersion relations (4.3) can be written in the form $(I = +$ or $-)$ *)

$$\underset{\sim}{A}^{I}(s,t)\Big|_{t=-2q^2(1-z_s)} = \frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty} ds'\,\underset{\sim}{h}^{I}_s(s,s';z_s)\,\mathrm{Im}\,\underset{\sim}{A}^{I}(s',t')\Big|_{t'=-2q'^2(1-z'_s)} \tag{B.12}$$

---

*) In the following, we suppress the dependence on a.

where the matrix $\underset{\sim}{h}{}_s^I$ is given by

$$\underset{\sim}{h}{}_s^{I} = h_1 \, \underset{\sim}{\sigma}_0 - \varepsilon^{I} h_2 \, \underset{\sim}{\sigma}_3 \tag{B.13}$$

with

$$h_1 = h_1(s,s') \equiv \frac{1}{s'-s} - \frac{1}{2}\,\frac{1}{s'-a}$$

$$h_2 = h_2(s,s';z_s) \equiv \frac{1}{2q^2}\,\frac{1}{z-z_s} + \frac{1}{2}\,\frac{1}{s'-a}$$

$$z = z(s,s') \equiv 1 - \frac{s'+s-2(M^2+m_\pi^2)}{2q^2} \tag{B.14}$$

$$\varepsilon^{I} = (\varepsilon^{+}, \varepsilon^{-}) = (1, -1)$$

$$\underset{\sim}{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \underset{\sim}{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

In order to derive partial wave relations from the hyperbola dispersion relations (B.12), we expand the absorptive part into s channel partial wave amplitudes by means of Eq. (B.6) and then project out the s channel partial wave amplitudes by using the projection formula (B.3). This leads us immediately to the partial wave relations

$$\underset{\sim}{f}{}_\ell^{I}(W) = \frac{1}{\pi}\int_{M+m_\pi}^{\infty} dW' \sum_{\ell'=0}^{\infty} \underset{\sim}{K}{}^{\ell\ell',I}(W,W')\, \mathrm{Im}\, \underset{\sim}{f}{}_{\ell'}^{I}(W') \; . \tag{B.15}$$

The s and u channel kernels are defined by

$$\underset{\sim}{K}{}^{\ell\ell',I}(W,W') \equiv 2W' \int_{-1}^{1} dz_s \, \underset{\sim}{R}{}^{\ell}(W,z_s)\, \underset{\sim}{h}{}_s^{I}(s,s';z_s)\, \underset{\sim}{S}{}^{\ell'}(W',z_s') \; , \tag{B.16}$$

where [see Eq. (3.3b)]

$$
\begin{aligned}
z_s' = z_s'(s,s';z_s) &\equiv 1 + \frac{t'}{2q'^2} = \alpha z_s + \beta \\
\alpha = \alpha(s,s') &\equiv \frac{q^2}{q'^2} \cdot \frac{s-a}{s'-a} \\
\beta = \beta(s,s') &\equiv 1 - \alpha + \frac{s'-s}{s'-a}\, \frac{2(M^2+m_\pi^2)-s-s'}{2q'^2} \\
q'^2 &\equiv q^2(s') .
\end{aligned}
\tag{B.17}
$$

Due to the symmetry relations (B.9) and the fact that $\underset{\sim}{h}_s^I$ depends on s and s' rather than on W and W' we obtain the symmetry relations

$$
\begin{aligned}
\underset{\sim}{K}^{\ell\ell',I}(-W,W') &= -\underset{\sim}{\sigma}_1\, \underset{\sim}{K}^{\ell\ell',I}(W,W') \\
\underset{\sim}{K}^{\ell\ell',I}(W,-W') &= -\underset{\sim}{K}^{\ell\ell',I}(W,W')\, \underset{\sim}{\sigma}_1 ,
\end{aligned}
\tag{B.18}
$$

which allow us to write the matrix $\underset{\sim}{K}^{\ell\ell',I}$ in the form

$$
\underset{\sim}{K}^{\ell\ell',I}(W,W') = \begin{pmatrix} K_{\ell\ell'}^I(W,W') & K_{\ell\ell'}^I(W,-W') \\ -K_{\ell\ell'}^I(-W,W') & -K_{\ell\ell'}^I(-W,-W') \end{pmatrix}
\tag{B.19}
$$

with

$$
K_{\ell\ell'}^I(W,W') \equiv 2W' \int_{-1}^{1} dz_s \left[\underset{\sim}{R}^{\ell}(W,z_s)\, \underset{\sim}{h}_s^I(s,s';z_s)\, \underset{\sim}{S}^{\ell'}(W',z_s') \right]_{11} .
\tag{B.20}
$$

[The subscript 11 indicates that we have to take the "11 element" of the matrix appearing in the bracket in Eq. (B.20).]

Using (B.19) we obtain immediately from Eq. (B.15) the partial wave relations ($\ell = 0,1,2,\ldots$)

$$f^{I}_{\ell+}(W)\Big|_{\substack{s \text{ and}\\ u \text{ channel}}} = \frac{1}{\pi}\int_{M+m_\pi}^{\infty} dW' \sum_{\ell'=0}^{\infty}\Big\{K^{I}_{\ell\ell'}(W,W')\,\mathrm{Im}\, f^{I}_{\ell'+}(W') + K^{I}_{\ell\ell'}(W,-W')\,\mathrm{Im}\, f^{I}_{(\ell'+1)-}(W')\Big\}$$

(B.21)

$$f^{I}_{(\ell+1)-}(W)\Big|_{\substack{s \text{ and}\\ u \text{ channel}}} = -\frac{1}{\pi}\int_{M+m_\pi}^{\infty} dW' \sum_{\ell'=0}^{\infty}\Big\{K^{I}_{\ell\ell'}(-W,W')\,\mathrm{Im}\, f^{I}_{\ell'+}(W') + K^{I}_{\ell\ell'}(-W,-W')\,\mathrm{Im}\, f^{I}_{(\ell'+1)-}(W')\Big\}$$

in agreement with the result already stated in Eq. (4.5).

Obviously, the s and u channel contributions (B.21) satisfy the McDowell symmetry relation (4.4) which reflects the fact that in the W plane there is only one independent kernel, $K^{I}_{\ell\ell'}$, which is given from Eqs. (B.20) and (B.13) by

$$K^{I}_{\ell\ell'}(W,W') = h_1(s,s')\int_{-1}^{1} dz_s\, 2W'\left[\underset{\sim}{R}^{\ell}(W,z_s)\,\underset{\sim}{S}^{\ell'}(W',z'_s)\right]_{11}$$

(B.22)

$$-\varepsilon^{I}\int_{-1}^{1} dz_s\, h_2(s,s';z_s)\, 2W'\left[\underset{\sim}{R}^{\ell}(W,z_s)\,\underset{\sim}{\sigma}_3\,\underset{\sim}{S}^{\ell'}(W',z'_s)\right]_{11}.$$

With the definition

$$\Phi\left[a_{mn}\,\middle|\, b(W,W')\right] \equiv \frac{W'}{W}\Big\{b(W,W')\,a_{m\,n+1} + b(W,-W')\,a_{mn} + b(-W,W')\,a_{m+1\,n+1} + b(-W,-W')\,a_{m+1\,n}\Big\} \qquad \text{(B.23)}$$

we derive from Eqs. (B.5) and (B.8) *)

$$2W'\left[\underset{\sim}{R}^{\ell}(W,z_s)\,\underset{\sim}{S}^{\ell'}(W',z'_s)\right]_{11} = \frac{1}{2}\,\Phi\left[P_\ell(z_s)P'_{\ell'}(z'_s)\,\middle|\,\delta(W,W')\right]$$

$$2W'\left[\underset{\sim}{R}^{\ell}(W,z_s)\,\underset{\sim}{\sigma_3}\,\underset{\sim}{S}^{\ell'}(W',z'_s)\right]_{11} = \frac{1}{2}\,\Phi\left[P_\ell(z_s)P'_{\ell'}(z'_s)\,\middle|\,\rho(W,W')\right] \qquad \text{(B.24)}$$

with

$$\delta(W,W') \equiv (W'+W)\,\frac{E+M}{E'+M}$$

$$\rho(W,W') \equiv (2M+W'-W)\,\frac{E+M}{E'+M}\,. \qquad \text{(B.25)}$$

Using (B.14) and defining the "angular kernels"

$$\mathcal{U}_{\ell\ell'} \equiv \frac{1}{2}\int_{-1}^{1} dz_s\, P_\ell(z_s)P'_{\ell'}(z'_s)$$

$$V_{\ell\ell'} \equiv \frac{1}{2}\int_{-1}^{1} dz_s\, \frac{P_\ell(z_s)P'_{\ell'}(z'_s)}{z-z_s} \qquad \text{(B.26)}$$

we finally get the following result for the <u>s and u channel kernel</u>

$$K^{\pm}_{\ell\ell'}(W,W') = \frac{\Phi\left[\mathcal{U}_{\ell\ell'}\,|\,\delta(W,W')\right]}{s'-s} \mp \frac{1}{2q^2}\,\Phi\left[V_{\ell\ell'}\,|\,\rho(W,W')\right] - \frac{\Phi\left[\mathcal{U}_{\ell\ell'}\,|\,\varkappa^{\pm}(W,W')\right]}{s'-a}$$

$$\equiv K^{\pm}_{\ell\ell'}(W,W')_{\mp TDR} - \frac{\Phi\left[\mathcal{U}_{\ell\ell'}\,|\,\varkappa^{\pm}(W,W')\right]}{s'-a}\,, \qquad \text{(B.27)}$$

where

$$\varkappa^{\pm}(W,W') \equiv \frac{1}{2}\left[\delta(W,W') \pm \rho(W,W')\right]$$

$$= \frac{E+M}{E'+M}\begin{pmatrix} W'+M \\ W-M \end{pmatrix}. \qquad \text{(B.28)}$$

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*) Notice that the second kernel of Eq. (B.24) is just the crossing kernel entering the partial wave crossing relations derived in Ref. 18).

The kernel with the subscript FTDR has exactly the same structure as the corresponding kernel derived from t dispersion relations [2], apart from the fact that the angular variable z_s' [defined in Eq. (B.17)] entering the angular kernels (B.26) is different in the case of fixed t dispersion relations, where one has [2]

$$z_s'(\mathrm{FTDR}) = \alpha(\mathrm{FTDR})\, z_s + \beta(\mathrm{FTDR}) \tag{B.29}$$

with

$$\alpha(\mathrm{FTDR}) = \frac{q^2}{q'^2}\,, \quad \beta(\mathrm{FTDR}) = 1 - \alpha(\mathrm{FTDR}). \tag{B.30}$$

Only in the limit $|a| \to \infty$ one obtains

$$z_s' \longrightarrow z_s'(\mathrm{FTDR}) \tag{B.31}$$

and therefore exactly the same kernels.

As in the case of the partial wave relations derived from fixed t dispersion relations [5] one infers that the only cuts generated by the partial wave relations (B.21) are caused by the denominator $(s'-s)^{-1}$ and the angular kernel $V_{\ell\ell'}$ occurring in the kernel (B.27). While the function $V_{\ell\ell'}$, Eq. (B.26), generates the cut $-1 \leq z(s,s') \leq 1$ and therefore [because of (B.14) and $s' \geq (M+m_\pi)^2$] the u channel cut $-\infty \leq s \leq (M-m_\pi)^2$, the denominator $(s'-s)^{-1}$ gives rise to the direct channel cut $s \geq (M+m_\pi)^2$. Thus we conclude that the partial wave amplitudes represented by the partial wave relations (B.21) have the correct cuts corresponding to s and u channel exchange.

Considering explicitly that part of the kernel (B.27) which generates the direct channel cut, we obtain from Eqs. (B.23) and (B.25)

$$K^{\pm}_{\ell\ell'}(W,W')\Big|_{\text{direct channel}} = \frac{\gamma_{\ell\ell'}(W,W')}{W'-W} + \frac{1}{W'+W}\cdot\frac{W'}{W}\left\{\frac{E+M}{E'-M}\,\mathcal{U}_{\ell\ell'} - \frac{E-M}{E'+M}\,\mathcal{U}_{\ell+1\,\ell'+1}\right\} \tag{B.32}$$

with

$$\gamma_{\ell\ell'}(W,W') \equiv \frac{W'}{W}\left\{\frac{E+M}{E'+M}\,\mathcal{U}_{\ell\,\ell'+1} - \frac{E-M}{E'-M}\,\mathcal{U}_{\ell+1\,\ell'}\right\} . \tag{B.33}$$

Obviously the direct channel cut is completely produced by the first term in Eq. (B.32) whose residue at the pole $W' = W$ is given by

$$\begin{aligned}\gamma_{\ell\ell'}(W,W) &= \left[\mathcal{U}_{\ell\,\ell'+1} - \mathcal{U}_{\ell+1\,\ell'}\right]_{W'=W} \\ &= \frac{1}{2}\int_{-1}^{1} dz_s \left[P_\ell(z_s)P'_{\ell'+1}(z_s) - P_{\ell+1}(z_s)P'_{\ell'}(z_s)\right] = \delta_{\ell\ell'} .\end{aligned} \tag{B.34}$$

[Here we used $z'_s = z_s$, which holds for $W' = W$ as can be seen from Eq. (B.17).]

As a consequence we derive the following important property of the kernel (B.27)

$$K^{\pm}_{\ell\ell'}(W,W') = \frac{\delta_{\ell\ell'}}{W'-W} + \overline{K}^{\pm}_{\ell\ell'}(W,W') , \tag{B.35}$$

where the kernel $\overline{K}^{\pm}_{\ell\ell'}$ contains only the u channel cut $s \leq (M - m_\pi)^2$.

To derive explicit expressions for the angular kernels (B.26), we start from the expansion [23]

$$\begin{aligned}P_\ell(z_s) &= \sum_{\lambda=0}^{\ell} a^\ell_\lambda\, x^\lambda \\ a^\ell_\lambda &= \frac{(-1)^\lambda}{(\lambda!)^2}\,\frac{(\ell+\lambda)!}{(\ell-\lambda)!} , \quad x \equiv \frac{1-z_s}{2} .\end{aligned} \tag{B.36}$$

Similarly

$$\begin{aligned}P'_{\ell'}(z'_s) &= -\frac{1}{2}\sum_{\lambda'=0}^{\ell'-1} (\lambda'+1)\, a^{\ell'}_{\lambda'+1}\, x'^{\lambda'} \\ x' &\equiv \frac{1-z'_s}{2} .\end{aligned} \tag{B.37}$$

From Eq. (B.17) we infer

$$x' = \omega + \alpha x$$

$$\omega \equiv \frac{1-(\alpha+\beta)}{2} = \frac{s'-s}{s'-a} \cdot \frac{s+s'-2(M^2+m_\pi^2)}{4q'^2} \tag{B.38}$$

and therefore

$$P'_{\ell'}(z'_s) = -\frac{1}{2}\sum_{\lambda'=0}^{\ell'-1}(\lambda'+1)a^{\ell'}_{\lambda'+1}\sum_{\mu=0}^{\lambda'}\binom{\lambda'}{\mu}\omega^{\lambda'-\mu}\,\alpha^{\mu}\,x^{\mu}\,. \tag{B.39}$$

Using (B.36) and (B.39) we obtain after a trivial integration and using the Saalschütz identity [24]

$$\sum_{\lambda=0}^{\ell}\frac{a^{\ell}_{\lambda}}{\lambda+(\mu+1)} = (-1)^{\ell}\frac{(\mu!)^2}{(\ell+\mu+1)!\,(\mu-\ell)!} \tag{B.40}$$

the final expression

$$U_{\ell\ell'} = \frac{(-1)^{\ell+1}}{2}\sum_{\lambda'=\ell}^{\ell'-1}\sum_{\mu=\ell}^{\lambda'}\binom{\lambda'}{\mu}\frac{(\lambda'+1)(\mu!)^2 a^{\ell'}_{\lambda'+1}}{(\ell+\mu+1)!\,(\mu-\ell)!}\,\omega^{\lambda'-\mu}\,\alpha^{\mu}\,. \tag{B.41}$$

By means of Eqs. (B.38) and (B.17), we deduce immediately the asymptotic relations

$$\begin{aligned} U_{\ell\ell'} &\sim q^{2\ell} + O(q^{2\ell+2}) && \text{for } s \to (M+m_\pi)^2 \\ U_{\ell\ell'} &\sim q'^{-2\ell'+2} + O(q'^{-2\ell'+4}) && \text{for } s' \to (M+m_\pi)^2 \\ U_{\ell\ell'} &\sim q'^{-2\ell} + O(q'^{-2\ell-2}) && \text{for } s' \to \infty \end{aligned} \tag{B.42}$$

which in conjuction with the asymptotic relations (B.46) ensure the correct threshold behaviour of the partial wave relations (B.21) and the convergence at the upper limits of the integrals occurring in (B.21).

In particular we obtain from (B.41)

$$\mathcal{U}_{\ell\ell'} = 0 \quad \text{for} \quad \ell' \leq \ell$$

$$\mathcal{U}_{\ell\,\ell+1} = \alpha^{\ell} \quad , \mathcal{U}_{\ell\,\ell+2} = (2\ell+3)\beta\,\alpha^{\ell} \tag{B.43}$$

$$\mathcal{U}_{\ell\,\ell+3} = \frac{\alpha^{\ell}}{2}\left[(2\ell+5)\alpha^2 + (2\ell+3)(2\ell+5)\beta^2 - (2\ell+3)\right].$$

Having evaluated the kernel $\mathcal{U}_{\ell\ell'}$, it is straightforward to obtain the following expression for the kernel $V_{\ell\ell'}$ [defined in Eq. (B.26)] *)

$$V_{\ell\ell'} = \frac{1}{\alpha} Q_{\ell}(z) \frac{d}{dz} P_{\ell'}(\alpha z+\beta)$$

$$+\begin{cases} 0 & \text{for } \ell' \leq \ell+1 \\ \sum_{n=\ell}^{\ell'-1} (2n+1)\,\mathcal{U}_{n\ell'}\left[P_{\ell}(z)W_{n-1}(z) - P_n(z)W_{\ell-1}(z)\right] & \text{for } \ell' > \ell+2 \end{cases} \tag{B.44}$$

where the functions $W_{n-1}(z)$ are polynomials of degree $n-1$ defined by [for explicit expressions, see Ref. 5)]

$$W_{n-1}(z) = \frac{1}{2}\int_{-1}^{1} dz_s \frac{P_n(z) - P_n(z_s)}{z - z_s}. \tag{B.45}$$

As before, one easily proves the asymptotic relations

$$\frac{1}{q^2} V_{\ell\ell'} \sim q^{2\ell} + O(q^{2\ell+2}) \quad \text{for } s \to (M+m_{\pi})^2$$

$$V_{\ell\ell'} \sim q^{1-2\ell'+2} + O(q^{1-2\ell'+4}) \quad \text{for } s' \to (M+m_{\pi})^2 \tag{B.46}$$

$$V_{\ell\ell'} \sim q^{1-2\ell-2} + O(q^{1-2\ell-4}) \quad \text{for } s' \to \infty .$$

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*) For a derivation of Eq. (B.44) see Appendix III of Ref. 5).

APPENDIX C - t CHANNEL EXCHANGE

Using again the matrix notation introduced in Appendix B we write the t channel terms of the hyperbola dispersion relations (4.3) in the following form $(I = +$ or $-)$

$$\left.\underset{\sim}{A}^{I}(s,t)\right|_{t=-2q^2(1-z_s)} = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt'\, \underset{\sim}{h}_t^{I}(s,t';z_s)\,\mathrm{Im}\,\underset{\sim}{A}^{I}(t',z_t') \tag{C.1}$$

with

$$\underset{\sim}{h}_t^{I}(s,t';z_s) = \frac{1}{2q^2}\frac{1}{x_t - z_s}\begin{pmatrix} \lambda_1^I & 0 \\ 0 & \lambda_2^I \end{pmatrix}$$

$$\lambda_1^I = \lambda_1^I(s,t';z_s) = \begin{cases} 1 & \text{for } I=+ \\ \frac{\nu}{\nu'} & \text{for } I=- \end{cases}, \quad \lambda_2^I = \lambda_2^I(s,t';z_s) = \begin{cases} \frac{\nu}{\nu'} & \text{for } I=+ \\ 1 & \text{for } I=- \end{cases} \tag{C.2}$$

and

$$x_t = x_t(s,t') \equiv 1 + \frac{t'}{2q^2}, \quad z_t' \equiv \frac{\nu'}{4p_t'q_t'} = (\gamma z_s + \delta)^{1/2}$$

$$\gamma = \gamma(s,t') \equiv \frac{q^2}{2p_t'^2 q_t'^2}(s-a), \quad p_t'^2 = \frac{t'}{4} - M^2, \quad q_t'^2 = \frac{t'}{4} - m_\pi^2 \tag{C.3}$$

$$\delta = \delta(s,t') \equiv \frac{(t'-\Sigma+2a)^2 - 4(s-a)(2q^2+\Sigma-s-a)}{16p_t'^2 q_t'^2}.$$

Introducing the "vector"

$$\underset{\sim}{f}^{J} = (f_+^J, f_-^J), \tag{C.4}$$

where $f_{\pm}^{J}(t)$ are the $\pi\pi \to N\bar{N}$ partial wave amplitudes, we may write the t channel partial wave expansion [25] in the matrix form

$$\boldsymbol{A}^{I}(t',z_t') = \sum_J \boldsymbol{T}^{J}(t',z_t') \, \boldsymbol{f}^{J}(t') \quad , \tag{C.5}$$

where the sum goes over even J for $I = +$ and over odd J for $I = -$, and

$$\boldsymbol{T}^{J}(t',z_t') = \zeta_J \begin{pmatrix} u_J & v_J \\ 0 & w_J \end{pmatrix}$$

$$\zeta_J = 4\pi (2J+1)(p_t' q_t')^{J-1} \, , \quad u_J = -\frac{q_t'}{p_t'} P_J(z_t') \tag{C.6}$$

$$v_J = \frac{q_t'}{p_t'} \frac{M}{\sqrt{J(J+1)}} z_t' P_J'(z_t') \, , \quad w_J = \frac{1}{\sqrt{J(J+1)}} P_J'(z_t') \, .$$

Expanding the t channel absorptive part of the hyperbola dispersion relations (C.1) into partial wave amplitudes by means of Eq. (C.5) and projecting out the s channel partial wave amplitudes by the projection formula (B.3) we obtain the partial wave relations

$$\boldsymbol{f}_\ell^{I}(W) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' \sum_J \boldsymbol{G}^{\ell J, I}(W,t') \, \mathrm{Im} \, \boldsymbol{f}^{J}(t') \quad , \tag{C.7}$$

where

$$\boldsymbol{G}^{\ell J, I}(W,t') \equiv \int_{-1}^{1} dz_s \, \boldsymbol{R}^{\ell}(W,z_s) \, \boldsymbol{h}_t^{I}(s,t';z_s) \, \boldsymbol{T}^{J}(t',z_t') \, . \tag{C.8}$$

Since $\boldsymbol{h}_t$ and z_t' are functions of s rather than of W, we obtain from (B.9) the symmetry relation

$$\boldsymbol{G}^{\ell J, I}(-W,t') = -\boldsymbol{\sigma}_1 \, \boldsymbol{G}^{\ell J, I}(W,t') \quad , \tag{C.9}$$

which guarantees that the t channel contribution to the s channel partial wave amplitudes also satisfies the McDowell symmetry (4.4). Equation (C.9) enables us to write

$$\underset{\sim}{G}^{\ell J,I}(W,t') = \begin{pmatrix} G_{\ell J}(W,t') & H_{\ell J}(W,t') \\ & \\ -G_{\ell J}(-W,t') & -H_{\ell J}(-W,t') \end{pmatrix} \tag{C.10}$$

and the partial wave relations (C.7) in the final form $(\ell = 0,1,2,\ldots)$

$$f^{\pm}_{\ell+}(W)\Big|_{t\ \text{channel}} = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \sum_J \left\{ G_{\ell J}(W,t')\,\mathrm{Im}\, f^J_+(t') + H_{\ell J}(W,t')\,\mathrm{Im}\, f^J_-(t') \right\}$$

$$f^{\pm}_{(\ell+1)-}(W)\Big|_{t\ \text{channel}} = -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \sum_J \left\{ G_{\ell J}(-W,t')\,\mathrm{Im}\, f^J_+(t') + H_{\ell J}(-W,t')\,\mathrm{Im}\, f^J_-(t') \right\} \tag{C.11}$$

in accordance with Eq. (4.5).

In order to calculate the kernels $G_{\ell J}$ and $H_{\ell J}$ let us define

$$\Psi[A_{\ell J}\,|\,g(W)] \equiv g(W)\,A_{\ell J} + g(-W)\,A_{\ell+1\ J}\ . \tag{C.12}$$

With this definition we obtain after some algebra the final result

$$G_{\ell J}(W,t') = -\eta_J\,\Psi[A_{\ell J}\,|\,E+M]$$

$$H_{\ell J}(W,t') = \frac{\eta_J}{\sqrt{J(J+1)}}\left\{ \frac{p_t'}{q_t'}\,\Psi[B_{\ell J}\,|\,(E+M)(W-M)] + M\,\Psi[C_{\ell J}\,|\,E+M] \right\}, \tag{C.13}$$

where

$$\eta_J = \eta_J(s,t') \equiv \frac{(2J+1)\,(p_t' q_t')^J}{4Wq^2 p_t'^2} \qquad , \tag{C.14}$$

and the "angular kernels" are defined by

$$A_{\ell J} \equiv \frac{1}{2}\int_{-1}^{1} dz_s\, \lambda_1^I(s,t';z_s)\, \frac{P_\ell(z_s)\, P_J(z_t')}{x_t - z_s}$$

$$B_{\ell J} \equiv \frac{1}{2}\int_{-1}^{1} dz_s\, \lambda_2^I(s,t';z_s)\, \frac{P_\ell(z_s)\, P_J'(z_t')}{x_t - z_s} \tag{C.15}$$

$$C_{\ell J} \equiv J A_{\ell J} + B_{\ell\, J-1} \qquad .$$

In the above expressions we have omitted the "crossing index" I on the left-hand side, since the full content of crossing symmetry or Bose statistics is already contained in the index J, which is explicitly seen by rewriting Eq. (C.2) in the form

$$\lambda_1^I = \begin{cases} 1 & \text{for} \quad J = 0,2,4,\ldots \\ \frac{\nu}{\nu'} & \text{for} \quad J = 1,3,5,\ldots \end{cases}$$

$$\lambda_2^I = \begin{cases} \frac{\nu}{\nu'} & \text{for} \quad J = 0,2,4,\ldots \\ 1 & \text{for} \quad J = 1,3,5,\ldots \end{cases} \qquad . \tag{C.16}$$

By virtue of the last statement and the decomposition

$$\frac{\nu}{\nu'} \cdot \frac{1}{x_t - z_s} = \frac{\mu_1}{z_t'} + \frac{\mu_2}{z_t'} \cdot \frac{1}{x_t - z_s}$$

$$\mu_1 = \mu_1(s,t') \equiv -\frac{q^2}{2p_t' q_t'} \tag{C.17}$$

$$\mu_2 = \mu_2(s,t') \equiv \frac{2s - 2(M^2 + m_\pi^2) + t'}{4 p_t' q_t'}$$

we finally get $(\ell = 0,1,2,\ldots)$

$$A_{\ell J} = \begin{cases} \frac{1}{2}\int_{-1}^{1} dz_s \frac{P_\ell(z_s)P_J(z_t')}{x_t - z_s} & \text{for } J = 0,2,4,\ldots \\ \frac{\mu_1}{2}\int_{-1}^{1} dz_s P_\ell(z_s)P_J'(z_t') + \frac{\mu_2}{2}\int_{-1}^{1} dz_s \frac{P_\ell(z_s)P_J'(z_t')}{x_t - z_s} & \text{for } J = 1,3,5,\ldots \end{cases}$$

(C.18)

$$B_{\ell J} = \begin{cases} \frac{\mu_1}{2}\int_{-1}^{1} dz_s P_\ell(z_s)\frac{P_J'(z_t')}{z_t'} + \frac{\mu_2}{2}\int_{-1}^{1} dz_s \frac{P_\ell(z_s)P_J'(z_t')/z_t'}{x_t - z_s} & \text{for } J = 0,2,4,\ldots \\ \frac{1}{2}\int_{-1}^{1} dz_s \frac{P_\ell(z_s)P_J'(z_t')}{x_t - z_s} & \text{for } J = 1,3,5,\ldots \,. \end{cases}$$

Remembering that $z_t' = (\gamma z_s + \delta)^{\frac{1}{2}}$, Eq. (C.3), it is important to notice that in the above integrals only even powers of z_t' occur, which is crucial in order to avoid a square root dependence on z_s which would lead not only to unwanted kinematic cuts but also to very complicated "angular kernels".

It is obvious that the t channel kernels (C.13) possess the cut $-1 \leq x_t(s,t') \leq 1$, generated by the angular kernels (C.18). Thus the partial wave relations (C.11) introduce in the s channel partial wave amplitudes the cut $-\infty \leq q^2 \leq -m_\pi^2$ in the q^2 plane, which translated into the s plane gives rise to the circle cut $|s| = M^2 - m_\pi^2$ and the cut $s \leq 0$.

For the phenomenologically interesting $\pi\pi \rightarrow N\bar{N}$ partial wave amplitudes $(J \leq 2)$ we explicitly obtain from Eqs. (C.13) and (C.18) $(\ell = 0,1,2,\ldots)$

$$G_{\ell 0} = \frac{-1}{4Wq^2 p_t'^2}\left\{(E+M)\,Q_\ell(x_t) - (E-M)\,Q_{\ell+1}(x_t)\right\}$$

$$G_{\ell 1} = \frac{3}{4}\left\{(2s-\Sigma+t')\,G_{\ell 0} + \frac{E+M}{2Wp_t'^2}\,\delta_{\ell 0}\right\}$$

$$G_{\ell 2} = \frac{5}{16}\left\{\left[(t'-\Sigma)^2 + 2(M^2-m_\pi^2)^2 + 6s(t'-\Sigma+s)\right]G_{\ell 0} + \frac{(s-a)(E+M)}{W\,p_t'^2}\,\delta_{\ell 0}\right\}$$

$$H_{\ell 0} = 0$$

$$H_{\ell 1} = \frac{3}{4\sqrt{2}}\left\{Z_\ell - M(2s-\Sigma+t')\,G_{\ell 0} - \frac{M(E+M)}{2Wp_t'^2}\,\delta_{\ell 0}\right\}$$

$$H_{\ell 2} = \frac{15}{16\sqrt{2}}\left\{(2s-\Sigma+t')\,Z_\ell - M\left[(t'-\Sigma)^2 + 4s(t'-\Sigma+s)\right]G_{\ell 0} - 2\,\frac{E+M}{W}\left[\frac{M(s-a)}{p_t'^2} + W - M\right]\delta_{\ell 0}\right\} \qquad (C.19)$$

with

$$Z_\ell \equiv \frac{1}{Wq^2}\left\{(E+M)(W-M)\,Q_\ell(x_t) + (E-M)(W+M)\,Q_{\ell+1}(x_t)\right\} . \qquad (C.20)$$

Due to the asymptotic expansion

$$\frac{1}{q^2}\,Q_\ell(x_t) = \left(\frac{2}{t'}\right)^{\ell+1}\cdot q^{2\ell} + O(q^{2\ell+2}) \quad , \qquad (C.21)$$

which is valid for $q^2 \to 0$, we see immediately that the t channel kernels (C.19) lead to the correct threshold behaviour of the s channel partial wave amplitudes for $s \to (M + m_\pi)^2$.

APPENDIX D - PARAMETRIZATION OF CURVES IN THE MANDELSTAM PLANE

In order to avoid the introduction of kinematic cuts, we utilize the crossing properties of the invariant amplitudes to write

$$\nu \equiv s - u = \left[f(t; a_0, a_1, \ldots) \right]^{1/2} , \tag{D.1}$$

where t is taken to be the independent variable, $-\infty \leq t \leq \infty$, and f is a meromorphic function of t depending on the parameters $\{a_j\}$. To ensure crossing symmetry explicitly, f is forbidden to be a perfect square.

From the condition that the curves (D.1) do not enter regions where double spectral functions are non-zero, we derived in Section 2 the asymptotic relation (2.8') which implies

$$\lim_{t\to\pm\infty} \left| \frac{\nu(t)}{t} \right|^2 = \lim_{t\to\pm\infty} \left| \frac{f(t)}{t^2} \right| = 1 . \tag{D.2}$$

Thus, for $t \to \pm\infty$, the curves defined by (D.1) have to be asymptotic to the lines of constant s or u.

It is the purpose of this Appendix to show that the simplest choice for f satisfying (D.2), i.e.,

$$f = a_0 + a_1 t + t^2 \tag{D.3}$$

which describes hyperbolic curves in the Mandelstam plane is the only solution that satisfies conditions a)-d) of Section 2.

Obviously the parametrization (D.1) together with the asymptotic condition (D.2) satisfies conditions a)-c). Consequently, we need to consider the restrictions implied by condition d), which states that the kernels resulting from the angular integration needed for the partial wave projection should be reasonably simple.

Previous investigations of fixed t dispersion relations [2),5)] and the hyperbola dispersion relations studied in this paper lead us to infer that the kernels occurring in the partial wave relations derived from dispersion relations on general curves in the Mandelstam plane will contain angular kernels of the form

$$\frac{1}{2}\int_{-1}^{1} dz_s\, G(z_s)\, P_\ell(z_s)\, P_{\ell'}(z_s') \quad , \tag{D.4}$$

where $z_s' \equiv z_s(s')$ is the corresponding angular variable along the curve and $G(z_s)$ is a rational function of z_s, e.g., in Eq. (B.26) $G(z_s) = (z - z_s)^{-1}$. In this integral s and s' are fixed and the variation of z_s and z_s', or equivalently t and t', is provided by a variation of one or a combination of the curve parameters $\{a_j\}$. If we call this combination of parameters b and consider b instead of z_s as the integration variable, it is clear that this integral will be simple only if b is a linear function of z_s. Thus condition d) can be written as

$$b = \alpha(s)\, z_s + \beta(s) = \alpha(s')\, z_s' + \beta(s') \quad , \tag{d'}$$

where the functions α and β depend on s and those parameters of the set $\{a_j\}$ which are held fixed during the angular integration.

It is perhaps useful for the sake of clearness to consider a redefinition of the parameters $\{a_0, a_1, \ldots, a_j\} \to \{b, c_1, \ldots c_j\}$ such that the coefficients α and β may be considered to depend on $\{c_j\}$ but not on b. If we also take into account the linear dependence of z_s on t, condition d') can be reformulated as

$$t(s; b, c_j) = g(s; c_j)\, b + h(s; c_j) \, . \tag{d''}$$

We will now show how this constraint supplemented with the condition that f is not a perfect square limits f to the form given in Eq. (D.3). To do this we write f as a power series in t of the form

$$f(t; a_0, a_1, \ldots) = t^2 + t \sum_{n=0} a_n t^{-n} \, . \tag{D.5}$$

If this expression is then substituted into the first line of Eq. (2.12) we obtain

$$S^2 + 2St = t \sum_{n=0} a_n(b,c_j)\, t^{-n} \quad , \tag{D.6}$$

where $S \equiv 2s - \Sigma$. For a given set of curve parameters $\{a_j\}$ or $\{b,c_j\}$ this equation relates s and t along the curve. The question is now what restrictions are imposed on the set $\{a_j\}$ if we assume a solution for t of the form given by condition d"). In the following we will consider S and b as independent variables and utilize the fact that g and h depend on S but not on b.

The proof is separated into two parts depending on whether the ratio h/g is dependent or independent of S. Let us first consider the situation when the ratio depends on S. Then Eq. (D.6) takes the form

$$S^2 + 2S(gb+h) = (gb+h) \sum_{n=0} a_n(b,c_j)\,(gb+h)^{-n} \quad . \tag{D.7}$$

If we rewrite the coefficient of $a_{m+1}(m \geq 1)$ as $g^{-m}(b+h/g)^{-m}$ and consider the point $b = -h/g$ for an arbitrary S it is clear that a_{m+1} must vanish at $b = -h/g$. But this ratio depends on S and a_{m+1} is independent of S, thus $a_{m+1}(m \geq 1)$ must be identically zero. Consequently, only a_o and a_1 can be non-zero and we have obtained the solution (D.3).

In the case that the ratio h/g is independent of S the proof is considerably more complicated. It is useful to notice that in this case we can assume h to be zero since we can always define a new b by $b' \equiv b + h/g$ and write $t = g(s)b'$. If we take this into account then Eq. (D.6) takes the form

$$S^2 + 2Sgb = \sum_{n=0} a_n(b,c_j)\,\big(g(s;c_j)\,b\big)^{-n+1} \quad . \tag{D.8}$$

If we consider any coefficient $a_{m+1}(m \geq 1)$, then the coefficients g^{-m} force each a_{m+1} separately to be proportional to b^m to cancel the pole at $b = 0$. Consequently we can write

$$a_{m+1}(b,c_j) \equiv \tilde{a}_{m+1}\, b^m \qquad (m \geq 1) \quad ,$$

where $\tilde{a}_{m+1}(b=0,c_j)$ is finite.

If in addition we realize that the left-hand side of Eq. (D.8) has only a zero and first order term in b, we can expand a_o, a_1 and the $\tilde{a}_n$'s in terms of new coefficients d_n and e_n which are independent of b. Thus we obtain

$$S^2 + 2Sgb = \sum_{n=0} \left(d_n(c_j) + 2b\, e_n(c_j)\right) g^{-n+1} , \tag{D.9}$$

which is equivalent to the relationship between S and g

$$S = \sum_{n=0} e_n g^{-n} \tag{D.10}$$

and

$$\sum_{n=0} d_n g^{-n+1} = \left(\sum_{n=0} e_n g^{-n}\right)^2 .$$

Having concluded that the sum in Eq. (D.6) must be of the form given in Eq. (D.9), we can use the relationships (D.10) to show in the case that h/g is independent of S, that f must be of the form

$$f = (bg)^2 + \sum_{n=0} (d_n + 2be_n) g^{-n+1}$$
$$= \left(bg + \sum_{n=0} e_n g^{-n+1}\right)^2 = \left(t + t\sum_{n=0} e_n (b/t)^n\right)^2 .$$

This, of course, violates the condition that f is not a perfect square. Consequently we must conclude that h/g must depend on S and that the only allowed solution is given by Eq. (D.3).

In order to illustrate the complications arising when the function f has not the form (D.3), we give a simple example. Starting from the Kibble boundary function [26]

$$\varphi(s,z_s) \equiv 4sq^4(1-z_s^2) = 4t\, p_t^2 q_t^2 (1-z_t^2) , \tag{D.11}$$

we consider those curves along which the quantity $t^n \varphi(s,z_s)$ is constant, i.e.,

$$t^n \varphi(s,z_s) = C \tag{D.12}$$

(c = constant). Using

$$z_t = \frac{\nu}{4 p_t q_t} \tag{D.13}$$

[see Eq. (C.3)] we obtain from (D.11) and (D.12)

$$f(t;c) = (t-4M^2)(t-4m_\pi^2) - \frac{4c}{t^{n+1}} \; . \tag{D.14}$$

Due to the asymptotic condition (D.2) only the values $n = -2,-1,0,1,2,\ldots$ are allowed. Obviously, for $n = -2$ and -1 f is a polynomial leading to hyperbolic curves, whereas for $n = 0,1,2,\ldots$ it has a pole of order $n+1$ at $t = 0$ leading to complicated curves in the Mandelstam plane.

Let us investigate the case $n = 0$ in more detail, i.e., the curves $\varphi(s,t) = c$ $(c>0)$. These curves consist of five different "branches" defined on the intervals $(-\infty,0)$, $(0,t_1)$, (t_1,t_2), (t_2,t_3) and (t_3,∞), where the t_i's $(i = 1,2,3)$ are the solutions of the equation $4tp_t^2q_t^2 = c$. The corresponding function f,

$$f(t;c) = (t-4M^2)(t-4m_\pi^2) - \frac{4c}{t} \tag{D.15}$$

changes its sign from interval to interval, however, in such a way that branch 1 lies between the asymptotes $u = -c$ and $t = 0$, i.e., completely inside the physical region, branch 2 lies outside the real Mandelstam plane ($\mathrm{Re}\,\nu = 0$, $\mathrm{Im}\,\nu \neq 0$), branch 3 lies again in the Mandesltam plane, branch 4 lies outside the Mandelstam plane ($\mathrm{Re}\,\nu = 0$, $\mathrm{Im}\,\nu \neq 0$), and finally branch 5 lies in the Mandelstam plane and starts at the point $(\nu,t) = (0,t_3)$ and goes asymptotically to the line $s = -c$. It is clear that a curve with such a complicated structure is not particularly suited for writing down dispersion relations. Furthermore, from the equality *)

$$\varphi(s,z_s) = \varphi(s',z_s') = c \tag{D.16}$$

---

*) Here (s,z_s) is the point where one wants to evaluate the dispersion relation, and (s',z_s') are the points over which one integrates in the dispersion relation written down on the curves defined by (D.15).

one obtains

$$z_s'^2 = 1 - \frac{s q^4}{s' q'^4} \left(1 - z_s^2\right) \quad , \tag{D.17}$$

and therefore condition d") is violated.

It is interesting to notice that the case $n = -2$ in Eqs. (D.12) and (D.14) corresponds to the t channel version of the Leader-Pennington variable [27)]

$$n_t^2 \equiv \frac{\varphi}{t^2} = \frac{4 p_t^2 q_t^2}{t} \left(1 - z_t^2\right) . \tag{D.18}$$

Thus keeping n_t^2 fixed, $n_t^2 = c$, we get the hyperbolas

$$\begin{aligned} &(s+c)(u+c) = b \\ &b = b(c) = \left(M^2 - m_\pi^2\right)^2 + c(c+\Sigma) \quad , \end{aligned} \tag{D.19}$$

which have the interesting property of passing through the threshold $s = (M + m_\pi)^2$ at $t = 0$.

Finally, we should point out that in this paper we have always considered the variable t in the interval $-\infty \leq t \leq \infty$. If we would have considered instead the <u>finite</u> interval $t_1 \leq t \leq t_2$, we would <u>not</u> have obtained the asymptotic condition (D.2). Consequently, if we allow the function f to have M poles at $t = \tau_m$, we get as a generalization of the curves (2.14)

$$(s-a)(u-a) \prod_{m=1}^{M} (t - \tau_m) = \sum_{n=0}^{N} b_n t^n \quad , \tag{D.20}$$

where N is now no more restricted to be equal to $M + 1$. In this case, the function f is given by

$$f(t; a, \tau_1, \ldots, b_0, \ldots) = (t - \Sigma + 2a)^2 - 4 \frac{\sum_{n=0}^{N} b_n t^n}{\prod_{m=1}^{M} (t - \tau_m)} . \tag{D.21}$$

Geometrically, one is dealing with curves which are closed in a finite region of the Mandelstam plane leading to "finite contour dispersion relations". The simplest case is obtained for $M = 1$, $N = 0$, $\tau_1 \equiv b$, $b_0 \equiv c$, i.e., *)

$$(s-a)(u-a)(t-b) = c \tag{D.22}$$

with

$$f(t;a,b,c) = (t-\Sigma+2a)^2 - \frac{4c}{t-b} . \tag{D.23}$$

As another example for a closed curve, we mention an ellipse in the (ν,t) plane $(-b \leq t \leq b)$

$$\left(\frac{\nu}{a}\right)^2 + \left(\frac{t}{b}\right)^2 = 1 \tag{D.24}$$

with

$$f(t;a,b) = \left(\frac{a}{b}\right)^2 (b^2-t^2) . \tag{D.25}$$

---

*) After having finished this paper we saw a preprint by Auberson and Khuri [28], where exactly the curves (D.22) are considered. The dispersion relations on these curves - given in Eq. (4.13) of their preprint - contain extremely complicated kernels, even for the invariant amplitudes ! This is just what we expect from the discussion in this paper.

APPENDIX E - CONVERGENCE OF THE PARTIAL WAVE RELATIONS

In the derivation of the partial wave relations (4.5) the hyperbola dispersion relations (4.3) have been evaluated on a family of hyperbolas,

$$(s-a)(u-a)=b \qquad , \tag{E.1}$$

where a is considered a fixed parameter and the "hyperbola parameter" b varies, at a given energy s, in the interval

$$b(s,z_s=1;a) \leq b \leq b(s,z_s=-1;a) \ . \tag{E.2}$$

In order that the partial wave expansions of the absorptive parts are convergent, it is necessary at all times to ensure that the hyperbolas defined by Eqs. (E.1) and (E.2) remain well inside the ellipse of convergence.

Assuming the validity of the Mandelstam representation *), the direct channel absorptive parts of the πN amplitudes are analytic (for given s) in the complex t plane, apart from cuts on the real axis for $t \geq \tau_+^M(s)$ and $t \leq \tau_-^M(s)$, where the "boundary curves" $\tau_\pm^M(s)$ are given in Ref. 25). Since for $s \geq (M+m_\pi)^2$ one has $\tau_+^M(s) \geq 4m_\pi^2$ and $\tau_-^M(s) \lesssim -86m_\pi^2$, the nearest boundary is $\tau_+^M(s)$,

$$\tau_+^M(s)=\begin{cases} 16m_\pi^2\left(1+\dfrac{m_\pi^2}{q^2}\right) & \text{for } (M+m_\pi)^2 \leq s \lesssim 80m_\pi^2 \\[2ex] 4m_\pi^2\left\{1+4m_\pi^2\dfrac{s+3(M^2-m_\pi^2)}{[s-(M+2m_\pi)^2][s-(M-2m_\pi)^2]}\right\} & \text{for } s \gtrsim 80m_\pi^2 \end{cases} \tag{E.3}$$

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*) Actually we only use "Mandelstam analyticity", i.e., we assume that the analytic properties of the amplitudes are exactly those given by the Mandelstam representation irrespective of the question of a finite number of subtractions, etc.

and therefore the ellipse of convergence of the absorptive parts is

$$E^M(s) = \left\{ 0, -4q^2 \,\middle|\, \tau_+^M(s) \right\} . \tag{E.4}$$

This implies convergence for real t in the interval

$$\tilde{\tau}_-^M(s) < t < \tau_+^M(s) , \tag{E.5}$$

where

$$\tilde{\tau}_-^M(s) \equiv -4q^2 - \tau_+^M(s) \tag{E.6}$$

and $\tilde{\tau}_-^M(s) \lesssim -26 m_\pi^2$ for $s \geq (M+m_\pi)^2$.

Consequently, for fixed a there exist two (uniquely defined) "limiting hyperbolas", characterized by their hyperbola parameters $b_\pm = b_\pm(a)$, which are tangent at some points to the boundaries $\tau_+^M(s)$ and $\tilde{\tau}_-^M(s)$, respectively. Thus the partial wave expansions of the absorptive parts converge on the whole set of hyperbolas satisfying

$$b_+(a) < b(s, z_s; a) < b_-(a) . \tag{E.7}$$

The two limiting hyperbolas define two energies, $s_{1,2} = s_{1,2}(a)$, determined as solutions of the equations *)

$$b(s_{1,2}, z_s = \pm 1; a) = b_\pm(a) , \tag{E.8}$$

with the property that for fixed a and $s \geq s_{max}(a) \equiv \min\{s_1(a), s_2(a)\}$ there does not exist a family of hyperbolas covering the <u>whole physical</u> angular range $-1 \leq z_s \leq 1$ and satisfying the "convergence condition" (E.7).

---

*) We only consider the solutions with $s_{1,2}(a) > (M+m_\pi)^2$.

The absolute highest energy, S_{max}, up to which the partial wave relations (4.5) converge is then obtained by varying the parameter a [geometrically, it is clear that a has to be smaller than $(M-m_\pi)^2$],

$$S_{max} \equiv \max_a \{ S_{max}(a) \} , \tag{E.9}$$

leading to an optimal value for a.

In order to calculate S_{max}, we have to find the two hyperbolas tangent to the boundaries $\tau_+^M(s)$ and $\tilde{\tau}_-^M(s)$. Demanding that a given hyperbola goes through the two points (s,t,u) and (s',t',u'), one can eliminate the parameter b with the result

$$a = \frac{su - s'u'}{t' - t} . \tag{E.10}$$

Thus the two hyperbolas going through the forward and backward points $(s, z_s = 1)$ and $(s, z_s = -1)$, respectively, and touching the boundaries at $(s_+, t_+ \equiv \tau_+^M(s_+), u_+ \equiv \Sigma - s_+ - t_+)$ and $(s_-, t_- \equiv \tilde{\tau}_-^M(s_-), u_- \equiv \Sigma - s_- - t_-)$, respectively, are given by *)

$$\begin{aligned} a_+ &= a_+(s) \equiv \frac{s(\Sigma - s) - s_+ u_+}{t_+} \\ a_- &= a_-(s) \equiv \frac{(M^2 - m_\pi^2)^2 - s_- u_-}{4q^2 + t_-} . \end{aligned} \tag{E.11}$$

To obtain an approximate value for S_{max}, we set **)

$$\begin{aligned} t_+ &= 12\, m_\pi^2 , \quad u_+ = 0 \\ s_- &= 85\, m_\pi^2 , \quad t_- = -25\, m_\pi^2 \end{aligned} \tag{E.12}$$

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*) Here we have used the identity $s(\Sigma - s + 4q^2) = (M^2 - m_\pi^2)^2$.

**) It is easily seen that the points (E.12) lie below and above the boundaries $\tau_+^M(s)$ and $\tilde{\tau}_-^M(s)$, respectively, thus leading to a <u>lower</u> estimate for S_{max}.

in Eq. (E.11) and solve the equation $a_+(s) = a_-(s)$. A graphical solution gives S_{max}(estimate) $\simeq 105\ m_\pi^2$, where the corresponding value of a is $\simeq -117\ m_\pi^2$.

Thus we conclude that the partial wave relations (4.5) are convergent at least up to $s = 105\ m_\pi^2$ which is somewhat larger than the corresponding value $S_{max} = 98\ m_\pi^2$ for the partial wave relations derived from fixed t dispersion relations [2].

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