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DEUTERON-NUCLEUS COLLISIONS IN THE MULTI-GeV REGION

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A B S T R A C T

Elastic and inelastic collisions of relativistic deuterons with medium and heavy nuclei are treated in the framework of Glauber's high energy multiple scattering theory. For the actual calculation of the collision amplitudes we develop a new and efficient approximation method which works for all but the lightest nuclei. The application to elastic and quasi-elastic scattering, to diffractive dissociation as well as to production and stripping reactions, is studied.

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1. - INTRODUCTION

The diffractive dissociation and scattering of fast deuterons by a complex nucleus were discussed many years ago by Glauber [1], Feinberg [2] and Akhieser and Sitenko [3], and in more detail by Akhieser and Sitenko [4]. In these investigations the target nucleus is treated as a completely absorbing sphere of radius R. The problem is then reduced to a proper truncation of the deuteron wave function. The elastic scattering of deuterons by a complex nucleus has also been investigated by means of phenomenological p nucleus and n nucleus potentials [5].

Our approach is quite different. It is based on a generalization of Glauber's high energy multiple scattering theory [6], where the d nucleus elastic and quasielastic scattering amplitudes are expressed in terms of the elastic nucleon-nucleon amplitudes and the target and deuteron wave functions. Although Glauber's original work only deals with nucleon-nucleus collisions, the generalization to nucleus-nucleus collisions is straightforward and has been described at length by several authors [7].

We will concentrate on collisions of relativistic deuterons with medium and heavy nuclei. The new feature in our treatment is the introduction and application of a new approximation method. In the past one used to expand the nucleus-nucleus amplitude in a series of nucleon-nucleon or nucleon-nucleus amplitudes. The disadvantage of such an expansion is that the series converges rather slowly. We suggest a new method where the zero order term does not correspond to the impulse approximation, but to a configuration where proton and neutron come in with the same impact parameter, the impact parameter of the deuteron c.m. motion. The higher order contributions then correct for the true impact parameters. This approach necessitates the introduction of a new type of effective nucleon numbers. Numerical values for all pertinent nucleon numbers are given in an Appendix.

One main advantage of our approximation method is the remarkable stability over the whole nuclear spectrum. The first order correction is usually about 10%, whereas in conventional schemes the correction increases with atomic number and reaches about 60% for heavy nuclei. This attractive feature makes us believe that our method is a good starting point for realistic calculations on deuteron-nucleus collisions.

In Section 4, we discuss the deuteron-nucleus total cross-section in some detail. The reason is that it offers an interesting opportunity to test the accuracy of our method. Indeed, it turns out that the total cross-section can be calculated exactly within our model. Comparing with the predictions of our approximation scheme we find that the first and zero order terms together reproduce the exact result to within 2% for heavy nuclei and 3% for light nuclei. The stability of our method is also clearly exhibited.

Encouraged by the good agreement obtained for the total cross-section we have investigated some consequences of our approximation method. It works also for some complicated problems where a direct calculation seems difficult. Thus we treat coherent elastic scattering (Section 3) and incoherent elastic scattering (Sections 5 and 6), but we have also tried to estimate the coherent dissociation cross-section (Section 7) and incoherent dissociation cross-section (Section 8). The coherent dissociation cross-section comes out as a difference between two big numbers and is very sensitive to the deuteron wave function. Here our predictions must be considered with some caution.

Particle production is treated in Sections 9 and 10. The proton-nuclear case has been considered in detail by Kölbig and Margolis [8]. Even if the extension to deuteron collisions is trivial the actual treatment of this problem is much more involved. We restrict ourselves to a study of some simple cases.

Up to now only one experiment on collisions of relativistic deuterons with heavier nuclei has been reported [9]. In this experiment the deuteron stripping cross-section on Al, Cu and Pb was measured for deuterons at 3.54 GeV/c. In Table I we have compared the results of our calculations with this experiment. Since the neutrons were not detected we must add the diffractive dissociation cross-section to the shell nucleus and neighbouring non-closed shell nuclei. Indeed in an experiment of Fink et al. [12] with low energy deuterons one observed an enhancement of the deuteron break-up on ^{208}Pb as compared to neighbouring nuclei.

2. - GLAUBER THEORY OF DEUTERON COLLISIONS

Let $|i\rangle = |dt\rangle$ denote the product of the deuteron and target ground states and $|f\rangle = |d't'\rangle$ the product of the deuteron and target final states. The state $|d'\rangle$ may represent a deuteron state, a proton-nucleon scattering state or a state with one excited nucleon. We shall restrict ourselves to final states where the outgoing particles form two groups, d' and t', such that the momentum transfer $\bar{q} = \bar{k}_d - \bar{k}_{d'}$ to the group d' is small. For this case we can use the impact parameter representation for the scattering amplitude [6]

$$F_{fi}(\bar{q}) = \frac{ik_d}{2\pi} \int d^2b \, e^{i\bar{q}\bar{b}} \cdot \langle d't' | \Gamma(\bar{b}, \bar{r}, \bar{r}_1, \ldots, \bar{r}_A) | dt \rangle \quad , \tag{2.1}$$

where $\bar{b}$ is the two-dimensional impact parameter between the deuteron and the target nucleus (see the Figure). The factor $e^{i\bar{q}\bar{b}}$ comes from the wave functions for the c.m. motion of d and d',

$$e^{i\bar{k}_d\bar{R}} \cdot e^{-i\bar{k}_{d'}\bar{R}} \approx e^{i\bar{q}\bar{b}} \quad , \tag{2.2a}$$

$$\bar{R} = \frac{1}{2}(\bar{r}_p + \bar{r}_n) \quad . \tag{2.2b}$$

Furthermore, $\bar{r} = \bar{r}_p - \bar{r}_n$ is the internal deuteron co-ordinate and $\bar{r}_1, \ldots, \bar{r}_A$ are the internal co-ordinates for the target nucleons. The explicit form of the matrix element in (2.1) is proper stripping cross-section (reactions where the neutron suffers inelastic collisions) before comparing with experiment.

target	$\sigma_{p,\ strip}$	σ(diff. diss.) coherent	incoherent	sum (theor.)	sum (exp.)
^{27}Al	227.6	21.2	32.8	281.6	290
^{64}Cu	301.8	34.8	40.1	376.7	550
^{208}Pb	439.2	62.9	54.0	556.1	950

Table I : Cross-sections in mb for stripped protons. The first column contains the proper stripping and the following two columns the diffractive dissociation contributions. Experimental errors $\sim$ 25%.

The agreement between theory and experiment is satisfactory only for Al. For the heavier elements the experimental yield is larger than the theoretical yield. Lander et al. [9] compared their result with estimates of Serber [10] and proposed that the disagreement for heavier elements could be due to the assumption of a sharply defined, opaque nucleus and to the contribution from diffractive dissociation. As we used wave functions of the Woods-Saxon form in our calculations we conclude that a smoothening of the nuclear surface causes only small changes. Also the diffractive dissociation comes out relatively small.

Qualitatively an increase in the diffuseness of the nuclear surface will increase the stripping and diffraction dissociation yields. In particular Nemets et al. [11] have suggested that due to differences in diffuseness the deuteron break-up might be quite different in a closed

$$\langle d't' | \Gamma(\bar{b},\bar{r},\bar{r}_1,\ldots,\bar{r}_A) | dt \rangle \equiv$$

$$\equiv \int d^3r\, d^3r_1 \cdots d^3r_A \cdot \varphi'^{*}(\bar{r})\, \Psi'^{*}(\bar{r}_1,\ldots,\bar{r}_A) \cdot \Gamma(\bar{b},\bar{r},\bar{r}_1,\ldots,\bar{r}_A) \times \times \varphi(\bar{r})\, \Psi(\bar{r}_1,\ldots,\bar{r}_A) \quad , \qquad (2.3)$$

where φ and Ψ , φ' and Ψ' are the deuteron and target internal wave functions for the initial and final states respectively. The normalization of the amplitude F is such that

$$\frac{d\sigma}{d\Omega} = | F(\bar{q}) |^2 \quad , \qquad (2.4a)$$

$$\sigma = \frac{1}{k_d^2} \int d^2q\, | F(\bar{q}) |^2 \quad . \qquad (2.4b)$$

We start with a discussion of deuteron elastic scattering and diffractive dissociation. The profile function Γ of (2.1) is then given by

$$\Gamma(\bar{b},\bar{r},\bar{r}_1,\ldots,\bar{r}_A) = 1 - \exp\left[i \chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A) \right] \quad , \qquad (2.5)$$

where the total phase shift function χ_{tot} only depends on the projections $\bar{s},\bar{s}_1,\ldots,\bar{s}_A$ of the co-ordinates $\bar{r},\bar{r}_1,\ldots,\bar{r}_A$ on the plane perpendicular to $\bar{k}_d$. Following Glauber [6] we assume that the total phase χ_{tot} is the sum of the phases for scattering on the individual nucleons in the target,

$$\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A) = \sum_{t=1}^{A} \left\{ \chi_p(\bar{b}+\tfrac{\bar{s}}{2}-\bar{s}_t) + \chi_n(\bar{b}-\tfrac{\bar{s}}{2}-\bar{s}_t) \right\} \quad . \qquad (2.6)$$

As a consequence our profile function $\Gamma(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)$ can be expressed in terms of single particle profile functions as

$$\Gamma(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A) = 1 - \prod_{t=1}^{A} \left[1-\Gamma_p(\bar{b}+\tfrac{\bar{s}}{2}-\bar{s}_t)\right] \cdot \left[1-\Gamma_n(\bar{b}-\tfrac{\bar{s}}{2}-\bar{s}_t)\right] \quad . \qquad (2.7)$$

The single particle profile functions are related to the elastic nucleon-nucleon scattering amplitudes through

$$\Gamma_p(\bar{b}) = \frac{1}{2\pi i k_p} \cdot \int d^2q \cdot e^{-i\bar{q}\bar{b}} \cdot f_p(\bar{q}) \quad . \tag{2.8}$$

In the following we shall always take

$$f_p(\bar{q}) = f_n(\bar{q}) = \frac{i+\alpha}{4\pi} \cdot \sigma \cdot k_p \cdot e^{-a\bar{q}^2/2} \quad , \tag{2.9}$$

so that

$$\Gamma_p(\bar{b}) = \Gamma_n(\bar{b}) = \eta \cdot e^{-\bar{b}^2/2a} \quad , \tag{2.10a}$$

$$\eta = \frac{1-i\alpha}{4\pi} \cdot \frac{\sigma}{a} \quad , \tag{2.10b}$$

where σ is the total cross-section at momentum $k_p \approx k_n \approx k_d/2$ and α the ratio between real and imaginary parts of the nucleon-nucleon amplitude. In the following when discussing magnitudes of different contributions we shall use the values

$$\sigma = 40 \text{ mb} \quad , \tag{2.11a}$$

$$a = 7 \text{ GeV}^{-2} \quad , \tag{2.11b}$$

which give

$$\eta = 1.17 \quad . \tag{2.11c}$$

For the special form (2.9) of the nucleon-nucleon amplitude the profile function (2.7) can be written

$$\begin{aligned} \Gamma(\bar{b},\bar{r},\bar{r}_1,\ldots,\bar{r}_A) = 1 - \prod_{t=1}^{A} \Big\{ & 1 - \eta \exp\left[-\tfrac{1}{2a}\left(\bar{b}+\tfrac{\bar{s}}{2}-\bar{s}_t\right)^2\right] \\ & - \eta \exp\left[-\tfrac{1}{2a}\left(\bar{b}-\tfrac{\bar{s}}{2}-\bar{s}_t\right)^2\right] \\ & + \eta^2 \exp\left[-\tfrac{1}{a}\left(\bar{b}-\bar{s}_t\right)^2\right] \cdot \exp\left[-\tfrac{\bar{s}^2}{4a}\right] \Big\} \quad . \end{aligned} \tag{2.12}$$

The last term in the bracket corresponds to scattering of both incident nucleons on the same target nucleon. We will refer to such terms as "eclipse terms". They contain an additional damping factor $\exp[-\bar{s}^2/4a]$ which simply reflects the fact that if both nucleons scatter on the same target nucleon they must be closer to each other than the range of the nuclear forces. The contribution from the eclipse term will thus generally be much smaller than the contribution from the terms proportional to η.

For the nuclear ground state $|t\rangle$ we shall use a separable wave function

$$|\Psi(\bar{r}_1,\dots,\bar{r}_A)|^2 = \prod_{t=1}^{A} \rho(\bar{r}_t) \quad . \tag{2.13}$$

For light nuclei $\rho(\bar{r}_t)$ is approximately of Gaussian shape. For heavy nuclei we shall use a single particle density of the Woods-Saxon form [13]

$$\rho(\bar{r}) = \frac{\rho_0}{1+\exp[(r-c)/d]} \quad , \tag{2.14a}$$

$$c = 1.14 \cdot A^{1/3}\ \mathrm{fm} \quad , \quad d = 0.545\ \mathrm{fm} . \tag{2.14b}$$

For the deuteron we shall use a wave function of Gaussian form [14]

$$\varphi_d(\bar{r}) = (\pi R_d^2)^{-3/4} \cdot \exp\left[-\frac{\bar{r}^2}{2R_d^2}\right] \quad , \tag{2.15a}$$

$$R_d = 2.281\ \mathrm{fm} . \tag{2.15b}$$

We note that for this kind of wave function

$$\langle \bar{r}^2 \rangle = \frac{3}{2} R_d^2 \quad . \tag{2.16}$$

We shall often encounter the two-dimensional part of this wave function. We do not introduce a special notation for this case but simply write $\varphi_d(\bar{s})$, always reserving the letter $\bar{s}$ for the component of $\bar{r}$ orthogonal to $\bar{k}_d$.

The evaluation of (2.3) has now been reduced to an evaluation of single particle integrals. This is done in the target thickness approximation [6] which exploits the fact that

$$R_t^2 \gg a \quad , \quad R_d^2 \gg a \tag{2.17}$$

to approximate

$$\int d^3r_t \cdot \rho(\bar{r}_t) \cdot \Gamma(\bar{b}-\bar{s}_t) \approx \int d^2s_t \cdot \Gamma(\bar{b}-\bar{s}_t) \cdot \int dz_t \cdot \rho(\bar{b},z_t) \equiv \frac{2\pi a}{A} \cdot \eta \cdot T(\bar{b}), \tag{2.18}$$

where $T(\bar{b})$ is the two-dimensional target thickness

$$T(\bar{b}) = A \cdot \int_{-\infty}^{\infty} dz \cdot \rho(\bar{b},z) \quad . \tag{2.19}$$

The reason for the additional factor A in the definition of $T(\bar{b})$ is that $A\rho(\bar{r})$ for heavier nuclei is roughly constant as A increases. For products of several profile functions we use similar approximations. When all profile functions are identical we put

$$A \cdot \int d^3r_t \cdot \prod_{i=1}^{N} \Gamma(\bar{b}_i-\bar{s}_t)\, \rho(\bar{r}_t) \approx$$
$$\approx \frac{2\pi a}{N} \cdot \eta^N \cdot T\Big(\frac{1}{N}\sum_{i=1}^{N} \bar{b}_i\Big) \cdot \exp\Big[-\frac{1}{2a}\cdot\sum_{i=1}^{N} \bar{b}_i^2 + \frac{1}{2aN}\cdot\Big(\sum_{i=1}^{A} \bar{b}_i\Big)^2\Big] \quad . \tag{2.20}$$

We now turn to inelastic collisions of the type $dt \to xnt'$ where the incident proton collides inelastically. The same impact parameter $\bar{b} + \frac{\bar{s}}{2}$ is used for the incident proton and the outgoing x particle. Let us assume that the production takes place on nucleon number j and that the position of this nucleon is $\bar{r}_j = (\bar{s}_j, z_j)$. The profile function Γ_{px} for the $p \to x$ production can in many cases be taken in a form analogous to elastic scattering, i.e.,

$$f_{px}(\bar{q}) = k_p \cdot a_{px} \cdot \eta_{px} \cdot e^{-a_{px} \cdot \bar{q}^2/2} \tag{2.21}$$

$$\Gamma_{px}(\bar{b}) = \eta_{px} \cdot e^{-\bar{b}^2/2 \cdot a_{px}} \quad . \tag{2.22}$$

In the elastic scattering itself we must now distinguish between elastic scattering before and after the production. For target nucleon number t the scattering is a proton-nucleon scattering if the scattering takes place before the production, i.e., when $z_t < z_j$, and an x nucleon scattering if the scattering takes place after the production, i.e., when $z_t > z_j$. The elastic profile function for nucleon t is thus

$$\Gamma_p(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_t) \cdot \theta(z_j - z_t) + \Gamma_x(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_t) \cdot \theta(z_t - z_j) \quad , \tag{2.23}$$

where the elastic x nucleon profile function is assumed to have the form

$$\Gamma_x(\bar{b}) = \eta_x \cdot e^{-\bar{b}^2/2a_x} \quad . \tag{2.24}$$

Neglecting double inelastic collisions of the proton the final profile function becomes

$$\Gamma_{if}(\bar{b}, \bar{r}, \bar{r}_1, \ldots, \bar{r}_A) = \sum_{j=1}^{A} \Gamma_{px}(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_j) \cdot e^{i q_L \cdot (z_j - \frac{z}{2})} \cdot [1 - \Gamma_m(\bar{b} - \tfrac{\bar{s}}{2} - \bar{s}_j)] \times$$

$$\times \prod_{t \neq j} [1 - \Gamma_m(\bar{b} - \tfrac{\bar{s}}{2} - \bar{s}_t)] \cdot [1 - \Gamma_p(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_t) \cdot \theta(z_j - z_t) + \Gamma_x(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_t) \cdot \theta(z_t - z_j)] \tag{2.25}$$

Here we have also taken into account the possibility of a non-vanishing longitudinal momentum transfer q_L, where production on different nucleons has to be multiplied by different phase factors $\exp[i q_L (z_j - \frac{z}{2})]$.

3. - COHERENT ELASTIC SCATTERING

In coherent elastic scattering the nucleus remains in its ground state. In our formula (2.3) we thus put $\Psi = \Psi'$ and $\varphi = \varphi'$ and get

$$F_{c,el}(\bar{q}) = \frac{ik_d}{2\pi}\int d^2b \cdot e^{i\bar{q}\bar{b}} \int d^3x \cdot |\varphi(\bar{x})|^2 \cdot f_{c,el}(\bar{b},\bar{s}) \quad , \tag{3.1a}$$

$$f_{c,el}(\bar{b},\bar{s}) = 1 - \langle t | \exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\dots,\bar{s}_A)] | t \rangle \quad . \tag{3.1b}$$

We now introduce the quantity $Z_1(\bar{b},\bar{s})$, which in the following will play an important role, through the definition

$$\exp[Z_1(\bar{b},\bar{s})] = \langle t | \exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\dots,\bar{s}_A)] | t \rangle \quad . \tag{3.2}$$

Using separable wave functions (2.13) and the target thickness approximation (2.18) we have

$$\begin{aligned}\exp[Z_1(\bar{b},\bar{s})] &= [1 - \tfrac{1}{A}\cdot 2\pi a\eta \cdot T_2(\bar{b},\bar{s}) + \tfrac{1}{A}\pi a\eta^2 \cdot T(\bar{b})\cdot e^{-\bar{s}^2/4a}]^A \\ &\approx \exp[-2\pi a\eta\, T_2(\bar{b},\bar{s}) + \pi a\eta^2\, T(\bar{b})\cdot e^{-\bar{s}^2/4a}] \quad ,\end{aligned} \tag{3.3}$$

where we have defined the deuteron target thickness $T_2(\bar{b},\bar{s})$ as

$$T_2(\bar{b},\bar{s}) = T(\bar{b}+\tfrac{\bar{s}}{2}) + T(\bar{b}-\tfrac{\bar{s}}{2}) \quad . \tag{3.4}$$

We thus have

$$Z_1(\bar{b},\bar{s}) = -2\pi a\eta\, T_2(\bar{b},\bar{s}) + \pi a\eta^2\, T(\bar{b})\cdot e^{-\bar{s}^2/4a} \quad . \tag{3.5}$$

It is convenient to rewrite (3.1) in the form

$$F_{c,el}(\bar{q}) = F_0(\bar{q}) - F_e(\bar{q}) \quad , \tag{3.6a}$$

$$F_0(\bar{q}) = \frac{ik_d}{2\pi}\cdot\int d^2b\, e^{i\bar{q}\bar{b}} \int d^3x\, |\varphi(\bar{x})|^2 \cdot \left\{1 - \exp[-2\pi a \eta T_2(\bar{b},\bar{s})]\right\} , \tag{3.6b}$$

$$F_e(\bar{q}) = \frac{ik_d}{2\pi}\cdot\int d^2b\, e^{i\bar{q}\bar{b}} \int d^3x\, |\varphi(\bar{x})|^2 \cdot \exp[-2\pi a \eta\, T_2(\bar{b},\bar{s})] \times$$
$$\times \left\{\exp[\pi a \eta^2\, T(\bar{b})\cdot e^{-\bar{s}^2/4a}] - 1\right\} . \tag{3.6c}$$

These expressions are rather complicated and a direct evaluation is very difficult. We need an approximation scheme in which at least some of the integrations can be performed analytically. The method which was used in the past was to expand the amplitude in a multiple scattering series. This works very well for eclipse terms but for other terms great difficulties are encountered. The series is slowly convergent and the calculations of higher order terms is numerically difficult. We therefore propose a new approximation scheme where the zero order term does not correspond to the impulse approximation but rather to the passage of the deuteron as a whole at impact parameter $\bar{b}$, i.e., both proton and neutron have this impact parameter. The higher order correction terms then correct for the true impact parameters of proton and neutron.

We now proceed to a detailed calculation of $F_0(\bar{q})$ using our method. We neglect the imaginary part of η and put $4\pi a \eta = \sigma$. We expand the deuteron target thickness $T_2(\bar{b},\bar{s})$ around the point $\bar{s} = 0$. First order derivatives vanish and the expansion up to second order gives

$$T_2(\bar{b},\bar{s}) = 2T(\bar{b}) + \frac{1}{4}(\bar{s}\bar{\nabla})^2\cdot T(\bar{b}) + \ldots \tag{3.7}$$

We furthermore expand the exponential in (3.6b) and get

$$\exp[-2\pi a \eta T_2(\bar{b},\bar{s})] = \exp[-\sigma T(\bar{b})]\cdot\left\{1 - \frac{\sigma}{8}\cdot(\bar{s}\bar{\nabla})^2 T(\bar{b}) + \ldots\right\} . \tag{3.8}$$

The first term does not depend on the deuteron co-ordinates and in the second term the dependence is very simple. It can be evaluated for any spherically symmetric deuteron wave function,

$$\int d^3x \, |\varphi(\bar{x})|^2 \cdot (\bar{s}\bar{\nabla})^2 T(\bar{b}) = \frac{1}{2} \cdot \Delta T(\bar{b}) \cdot \frac{2}{3} \langle \bar{r}^2 \rangle \quad , \tag{3.9}$$

where Δ is the two-dimensional Laplace operator

$$\Delta T(\bar{b}) = \frac{d^2 T(b)}{d b^2} + \frac{1}{b} \cdot \frac{d T(b)}{d b} \quad . \tag{3.10}$$

For a wave function of Gaussian shape as in (2.15) we have

$$\int d^3x \, |\varphi(\bar{x})|^2 \cdot (\bar{s}\bar{\nabla})^2 T(\bar{b}) = R_d^2 \cdot \frac{1}{2} \Delta T(\bar{b}) \quad . \tag{3.11}$$

We are now in a position to evaluate the main contributions to $F_0(\bar{q})$. After some simplifications we get

$$F_0(\bar{q}) = F_1(\bar{q}) + F_2(\bar{q}) + \cdots \quad , \tag{3.12a}$$

$$F_1(\bar{q}) = i k_d \int_0^\infty b \, db \cdot J_0(qb) \cdot \{1 - \exp[-\sigma T(b)]\} \quad , \tag{3.12b}$$

$$F_2(\bar{q}) = i k_d \cdot \frac{\sigma R_d^2}{16} \cdot \int_0^\infty b \, db \cdot J_0(qb) \cdot e^{-\sigma T(b)} \cdot \Delta T(b) \quad . \tag{3.12c}$$

The remaining integration must be done numerically. If higher accuracy should be needed then one has to include higher order terms in the expansions (3.7) and (3.8).

We now come to the evaluation of the eclipse term $F_0(\bar{q})$ in (3.6c). As was already discussed in Section 2, this term comes from scattering processes where both the proton and the neutron in the deuteron scatter on the same target nucleon. Such a contribution can be significant only when proton and neutron are close to each other and within the range of the nucleon forces. This is brought about by the additional factor $\exp[-\bar{s}^2/4a]$, which tells us that the last factor in the integral (3.6c) is zero unless $\bar{s}^2 \lesssim 4a$. In the deuteron target thickness $T_2(\bar{b},\bar{s})$ it is therefore not necessary to use an expansion of the type (3.7), but we can neglect the $\bar{s}$ dependence altogether and put $T_2(\bar{b},\bar{s}) \approx 2T(\bar{b})$. This approximation is similar to the target thickness approximation of (2.18). We expand the exponential in (3.6c) and perform the integral over the deuteron wave function. For a Gaussian wave function

$$\int d^2s\, |\varphi(\bar{s})|^2 \cdot e^{-\bar{s}^2/4a} = \frac{1}{1+R_d^2/4a} \quad , \tag{3.13}$$

and thus

$$F_e(\bar{q}) = i k_d \cdot \frac{\eta}{4} \cdot \frac{1}{1+R_d^2/4a} \cdot \int_0^\infty b\,db \cdot J_0(qb) \cdot \sigma T(b) \cdot e^{-\sigma T(b)} + \cdots \tag{3.14}$$

The three terms (3.12b), (3.12c) and (3.14) should suffice for most practical purposes.

The integrated coherent cross-section $\sigma_{C,el}$ can conveniently be calculated with our method. From (2.4) and (3.1) we get

$$\sigma_{c,el} = \int d^2b \cdot \left| \int d^2s\, |\varphi(\bar{s})|^2 \cdot \{1 - \exp[Z_1(\bar{b},\bar{s})]\} \right|^2 \equiv \tag{3.15}$$
$$\equiv \int d^2b \cdot \sigma_{c,el}(\bar{b}) \quad .$$

This expression is treated as those above. The deuteron target thickness in Z_1 is expanded as in (3.7) and (3.8) and the eclipse term in Z_1 is expanded as in (3.6c). This gives

$$\sigma_{c,el}(\bar{b}) = \left| \{1 - e^{-\sigma T(\bar{b})}\} + e^{-\sigma T(\bar{b})} \cdot \left\{ \frac{\sigma R_d^2}{16} \cdot \Delta T(\bar{b}) - \frac{\eta}{4} \cdot \frac{1}{1+R_d^2/4a} \cdot \sigma T(\bar{b}) \right\} + \ldots \right|^2$$
$$= \{1 - e^{-\sigma T(\bar{b})}\}^2 + \{e^{-\sigma T(\bar{b})} - e^{-2\sigma T(\bar{b})}\} \times \tag{3.16}$$
$$\times \left\{ \frac{\sigma R_d^2}{8} \cdot \Delta T(\bar{b}) - \frac{\eta}{2} \cdot \frac{1}{1+R_d^2/4a} \cdot \sigma T(\bar{b}) \right\} + \ldots$$

The result is most elegantly expressed in terms of "effective nucleon numbers" defined by

$$N_0(\sigma) = \frac{1}{\sigma} \int d^2b \cdot \{1 - e^{-\sigma T(\bar{b})}\} \quad , \tag{3.17a}$$

$$N_k(\sigma) = \frac{1}{k!}\cdot\int d^2b\cdot(\sigma T(\bar{b}))^{k-1}\cdot T(\bar{b})\cdot e^{-\sigma T(\bar{b})} \quad , \tag{3.17b}$$

$$M_k(\sigma) = \frac{1}{(k-1)!}\cdot\frac{R_d^2}{16}\cdot\int d^2b\cdot(\sigma T(\bar{b}))^{k-1}\cdot\Delta T(\bar{b})\cdot e^{-\sigma T(\bar{b})} \quad . \tag{3.17c}$$

N_o and N_k were already introduced by Glauber in his discussion of proton-nucleon collisions. M_k is a new nucleon number particular for our problem. The numerical values for the nucleon numbers relevant for our discussion are given in the Tables in the Appendix. The leading terms in the coherent cross-section (3.15) can now be written as

$$\begin{aligned}\sigma_{c,el} &= 2\sigma\,[N_0(\sigma) - N_0(2\sigma)] + 2\sigma\,[M_1(\sigma) - M_1(2\sigma)] \\ &\quad - 2\sigma\cdot\frac{\eta}{4}\cdot\frac{1}{1+R_d^2/4a}\cdot[N_1(\sigma) - N_1(2\sigma)] \quad .\end{aligned} \tag{3.18}$$

The two correction terms to the zero order term are very small. The first one is about 6% and the eclipse correction only about 2%. Our expansion technique thus gives a rapidly convergent result.

We conclude this Section with a few remarks on the connection with the usual multiple scattering approach. In our formula (3.6b) for F_o we make use of the identity

$$\begin{aligned}&1 - \exp[-\tfrac{\sigma}{2}\{T(\bar{b}+\tfrac{\bar{s}}{2}) + T(\bar{b}-\tfrac{\bar{s}}{2})\}] \\ &\equiv \{1 - \exp[-\tfrac{\sigma}{2}T(\bar{b}+\tfrac{\bar{s}}{2})]\} + \{1 - \exp[-\tfrac{\sigma}{2}T(\bar{b}-\tfrac{\bar{s}}{2})]\} \\ &\quad - \{1 - \exp[-\tfrac{\sigma}{2}T(\bar{b}+\tfrac{\bar{s}}{2})]\}\cdot\{1 - \exp[-\tfrac{\sigma}{2}T(\bar{b}-\tfrac{\bar{s}}{2})]\} \quad ,\end{aligned} \tag{3.19}$$

to decompose the amplitude F_o into three terms

$$F_o = F_p + F_m - \delta F \quad . \tag{3.20}$$

In this way each term gets a direct physical interpretation in terms of proton-nucleus and neutron-nucleus scattering processes. F_p and F_n are single scattering terms whereas δF is a double scattering term. However, as the target in a composite system, these single and double scattering terms do not give the whole amplitude. The eclipse term F_e must be added. This must always be kept in mind when calculating deuteron-nucleus amplitudes from phenomenological proton-nucleus amplitudes and some kind of multiple scattering theory.

After some simplifications the following expressions for the amplitudes of (3.20) are obtained

$$F_p = F_n = \exp\left[-\frac{\bar{q}^2 R_d^2}{16}\right] \cdot \frac{ik_d}{2\pi} \cdot \int d^2b \cdot J_0(qb) \left\{1 - \exp\left[-\frac{\sigma}{2} T(b)\right]\right\}, \tag{3.21}$$

$$\delta F = \frac{2ik_d}{R_d^2} \cdot \int_0^\infty b_+ db_+ \int_0^\infty b_- db_- \cdot J_0\left(\tfrac{1}{2} q \sqrt{b_+^2 + b_-^2}\right) \cdot I_0\left(\frac{2 b_+ b_-}{R_d^2}\right) \times$$
$$\times \exp\left[-\frac{1}{R_d^2}(b_+^2 + b_-^2)\right] \cdot \left\{1 - \exp\left[-\frac{\sigma}{2} T(b_+)\right]\right\} \cdot \left\{1 - \exp\left[-\frac{\sigma}{2} T(b_-)\right]\right\}, \tag{3.22}$$

where I_0 is the modified Bessel function of order zero. Both terms can be treated numerically and it is thus possible to calculate F_0 without approximations. It is more difficult to calculate the integrated coherent cross-section in this approach. Here the single scattering terms give rise to double integrals and the interference terms between single and double scattering terms give rise to triple integrals and so on. Our approximation method is therefore better suited for this problem.

4. - THE TOTAL CROSS-SECTION

The optical theorem relates the total cross-section to the imaginary part of the amplitude for coherent elastic scattering in the forward direction. It reads

$$\sigma_{tot} = \frac{4\pi}{k_d} \cdot \mathrm{Im}\, F_{c,el}(0) \quad , \tag{4.1}$$

and combining it with Eq. (3.1) we get

$$\sigma_{tot} = 2\, \mathrm{Re} \int d^2b \int d^3x\, |\varphi(\bar{x})|^2 \cdot \{1 - \exp[Z_1(\bar{b},\bar{z})]\} \ . \tag{4.2}$$

This formula is rewritten as in Section 3. We separate the eclipse term from the direct scattering term and get

$$\sigma_{tot} = \sigma_o - \sigma_e \quad , \tag{4.3a}$$

$$\sigma_o = 2\, \mathrm{Re} \int d^2b \int d^3x\, |\varphi(\bar{x})|^2 \cdot \{1 - \exp[-2\pi a \eta\, T_2(\bar{b},\bar{z})]\} \quad , \tag{4.3b}$$

$$\sigma_e = 2\, \mathrm{Re} \int d^2b \int d^3x\, |\varphi(\bar{x})|^2 \cdot \exp[-2\pi a \eta\, T_2(\bar{b},\bar{z})] \times \times \{\exp[\pi a \eta^2\, T(\bar{b}) \cdot e^{-z^2/4a}] - 1\} \quad . \tag{4.3c}$$

For the calculation of σ_o we can now continue on two lines just as in Section 3. In the old-fashioned multiple scattering approach we encounter single and double integrals only which can be treated numerically. As a consequence a direct comparison between our own method, which uses single integrals only, and the exact result is possible. The aim of the present Section is to discuss in some detail the accuracy and the advantage of our method over the conventional multiple scattering expansion. In this discussion we shall neglect the imaginary part of η and shall put $4\pi a \eta = \sigma$.

We start off with a discussion of the old-fashioned multiple scattering expansion. We make use of the identity (3.19) to divide the direct scattering term σ_o into single and double scattering terms. After some simplifications of the double scattering term we arrive at

$$\sigma_o = \sigma_p + \sigma_n - \delta\sigma \qquad , \tag{4.4a}$$

$$\sigma_p = \sigma_n = 2\int d^2b \cdot \{1 - \exp[-\tfrac{\sigma}{2} T(\bar{b})]\} = \sigma\, N_o(\tfrac{1}{2}\sigma) \; , \tag{4.4b}$$

$$\delta\sigma = \frac{8\pi}{R_d^2} \cdot \int_0^\infty b_+ db_+ \int_0^\infty b_- db_- \cdot I_o\left(\frac{2b_+ b_-}{R_d^2}\right) \cdot \exp\left[-\frac{1}{R_d^2}(b_+^2 + b_-^2)\right] \times$$

$$\times \{1 - \exp[-\tfrac{\sigma}{2} T(b_+)]\} \cdot \{1 - \exp[-\tfrac{\sigma}{2} T(b_-)]\} \tag{4.4c}$$

$$\equiv 2\sigma \cdot \delta N(\tfrac{1}{2}\sigma) \qquad .$$

Here σ_p and σ_n are the total cross-sections for the scattering of a proton and a neutron respectively on the nucleus at hand. The term $\delta\sigma$, which is positive, corrects for the presence of double scattering contributions. As a result

$$\sigma_o = 2\sigma\,[N_o(\tfrac{1}{2}\sigma) - \delta N(\tfrac{1}{2}\sigma)] \qquad . \tag{4.5}$$

The relevant effective nucleon numbers N_o and δN are given in the Tables in the Appendix. It is immediately seen that the double scattering is very important and that its importance increases with the atomic number. For heavy nuclei it is as much as 60% of σ_o itself.

The eclipse term σ_e is treated as in Section 3. The presence of the additional exponential $\exp[-\bar{s}^2/4a]$ in (4.3c) implies that the contributions to the integral predominantly come from a region of small s values, $s \ll R_d$. The s dependence in the deuteron target thickness can be neglected and we can put $T_2(\bar{b},\bar{s}) \approx T_2(\bar{b},0)$. Upon expanding the exponential the eclipse term contributes

$$\sigma_e = \pi a \eta^2 \cdot \frac{1}{1 + R_d^2/4a} \cdot 2\int d^2b\; T(\bar{b}) \cdot e^{-\sigma T(\bar{b})} + \ldots$$

$$= \sigma \cdot \frac{\eta}{2} \cdot \frac{1}{1 + R_d^2/4a} \cdot N_1(\sigma) + \ldots \qquad . \tag{4.6}$$

Thus we find that the eclipse term is in fact a rather small correction to the main term σ_0. Its magnitude is in the region 1-4% and it is more important for light nuclei than for heavy nuclei. For a correction term of this magnitude the very small error committed in the approximation of the deuteron target thickness is certainly negligible.

We now turn to our own approximation method for σ_0. Expanding as in (3.7) and (3.8) the first non-vanishing terms become

$$\sigma_0 = 2\int d^2b \left\{1 - \exp[-\sigma T(\vec{b})]\cdot\left(1 - \sigma\frac{R_d^2}{16}\cdot\Delta T(\vec{b}) + \dots\right)\right\}$$
$$= 2\sigma\left[N_0(\sigma) + M_1(\sigma) + \dots\right] \quad . \tag{4.7}$$

From the numerical values in the Appendix we conclude that M_1 is a 10% correction to N_0 independent of atomic number. In order to get the total cross-section we must add, to (4.7), the eclipse contribution (4.6) just as in the previous approach.

In Table II, the predictions of our formula (4.7) are compared with the exact results (4.5). It is clearly seen that in the conventional multiple scattering approach the double scattering gives a substantial contribution $\delta N(\frac{1}{2}\sigma)$ which increases from 30% for light nuclei to 60% for heavy nuclei. On the other hand, in our approach the first correction term $M_1(\sigma)$ stays constant about 10%. Comparing the total result we see that our approximation (4.7) is as good as 2% for heavy nuclei and 3% for light nuclei. Our figures are always below the exact values. This is not so astonishing as the neglected terms give positive contributions. They include higher order terms in the MacLaurin expansion (3.7) as well as higher order terms in the expansion of the exponential (3.8).

The good agreement obtained for the total cross-section indicates that our approximation scheme is a good starting point for treating deuteron-nucleus collisions.

A = 27		A = 64	
exact	approx.	exact	approx.
17.04	12.23	32.84	21.51
-3.81	1.33	-9.64	2.21
13.28	13.56	23.20	23.72

A = 108		A = 208	
exact	approx.	exact	approx.
48.06	29.98	76.34	45.24
-15.87	2.89	-28.11	3.93
32.19	32.87	48.23	49.17

Table II : Comparison between the exact formulae $N_o(\frac{1}{2}\sigma) - \delta N(\frac{1}{2}\sigma)$ and our approximation $N_o(\sigma) + M_1(\sigma)$ for $\sigma = 40$ mb.

5. - INCOHERENT ELASTIC SCATTERING : DIFFERENTIAL CROSS-SECTION

At small scattering angles the elastically scattered deuterons come mainly from coherent scattering, where the nucleus remains in its ground state. This part of the amplitude falls off very fast with increasing angle and the slope is determined by the nuclear form factor. At larger angles the incoherent scattering will take over, i.e., scattering processes where the state of the nucleus is changed. The incoherent amplitude has a structure which differs from the corresponding proton-nucleus amplitude. In the deuteron case we first find an intermediate angular region where the slope of the amplitude is determined by the deuteron form factor and then for larger angles the region where the slope is determined by the slope of the nucleon-nucleon amplitude. The aim of the present Section is to show how this qualitative behaviour is obtained from our model. As differential cross-sections have not so far been measured we do not aim at a complete treatment.

We first consider elastic deuteron scattering summed over all target states as obtained from (2.1)

$$\frac{d\sigma_{el}}{d\Omega} = \frac{k_d^2}{4\pi^2}\int d^2b \int d^2b' \cdot e^{i\bar{q}(\bar{b}-\bar{b}')} \cdot \sum_{t'} \langle dt|\Gamma^*(\bar{b},\bar{r}',\bar{r}_1',\dots,\bar{r}_A')|dt'\rangle\langle dt'|\Gamma(\bar{b},\bar{r},\bar{r}_1,\dots,\bar{r}_A)|dt\rangle. \tag{5.1}$$

When the average energy transferred to the target is small we can use the closure approximation

$$\sum_{t'} |t'\rangle\langle t'| = \sum_{t'} \Psi'^*(\bar{r}_1',\dots,\bar{r}_A')\cdot\Psi'(\bar{r}_1,\dots,\bar{r}_A) = \delta(\bar{r}_1'-\bar{r}_1)\dots\delta(\bar{r}_A'-\bar{r}_A) \tag{5.2}$$

to simplify this expression. The incoherent cross-section is obtained upon subtracting the coherent part as given by (3.1). We get

$$\frac{d\sigma_{I,el}}{d\Omega} = \frac{k_d^2}{4\pi^2}\int d^2b\int d^2b'\, e^{i\bar{q}(\bar{b}-\bar{b}')}\int d^3x\, |\varphi(\bar{x})|^2 \int d^3x'\, |\varphi(\bar{x}')|^2 \cdot f_I(\bar{b},\bar{b}';\bar{s},\bar{s}'), \tag{5.3a}$$

$$f_I(\bar{b},\bar{b}';\bar{s},\bar{s}') = \langle t|\Gamma^*(\bar{b}',\bar{r}',\bar{r}_1,\dots,\bar{r}_A)\,\Gamma(\bar{b},\bar{r},\bar{r}_1,\dots,\bar{r}_A)\,|t\rangle - \langle t|\Gamma^*(\bar{b}',\bar{r}',\bar{r}_1',\dots,\bar{r}_A')\,|t\rangle\langle t|\Gamma(\bar{b},\bar{r},\bar{r}_1,\dots,\bar{r}_A)|t\rangle. \tag{5.3b}$$

Now, invoking the formula (2.5) for the profile function we see that terms linear in $\exp[i\chi_{tot}]$ cancel due to the normalization property of the nuclear wave functions. We are left with two terms

$$f_I(\vec{b},\vec{b}';\vec{s},\vec{s}') = \langle t|\exp[-i\chi^*_{tot}(\vec{b}',\vec{s}',\vec{s}_1,\ldots,\vec{s}_A)]\cdot\exp[i\chi_{tot}(\vec{b},\vec{s},\vec{s}_1,\ldots,\vec{s}_A)]|t\rangle$$
$$-\langle t|\exp[-i\chi^*_{tot}(\vec{b}',\vec{s}',\vec{s}'_1,\ldots,\vec{s}'_A)]|t\rangle\cdot\langle t|\exp[i\chi_{tot}(\vec{b},\vec{s},\vec{s}_1,\ldots,\vec{s}_A)]|t\rangle \tag{5.4}$$

The last term comes from the coherent scattering and was already calculated in Section 3 where we introduced the notation

$$\langle t|\exp[i\chi_{tot}(\vec{b},\vec{s},\vec{s}_1,\ldots,\vec{s}_A)]|t\rangle = \exp[Z_1(\vec{b},\vec{s})] \ . \tag{5.5}$$

The first term which originates from the incoherent part is more complicated but is calculated in the same way as (5.5). Feeding in Eq. (2.12) we get

$$\langle t|\exp[-i\chi^*_{tot}(\vec{b}',\vec{s}',\vec{s}_1,\ldots,\vec{s}_A)]\cdot\exp[i\chi_{tot}(\vec{b},\vec{s},\vec{s}_1,\ldots,\vec{s}_A)]|t\rangle$$
$$= \exp[Z_1^*(\vec{b}',\vec{s}')+Z_1(\vec{b},\vec{s})]\cdot\exp[Z_2(\vec{b},\vec{s};\vec{b}',\vec{s}')] \ , \tag{5.6}$$

where

$$Z_2(\vec{b},\vec{s};\vec{b}',\vec{s}') = A\int d^3r_t\,\rho(\vec{r}_t)\cdot\Big[\eta^*\cdot e^{-\frac{1}{2a}(\vec{b}'+\frac{\vec{s}'}{2}-\vec{s}_t)^2}+\eta^*\cdot e^{-\frac{1}{2a}(\vec{b}'-\frac{\vec{s}'}{2}-\vec{s}_t)^2}$$
$$+\eta^{*2}\cdot e^{-\frac{1}{a}(\vec{b}'-\vec{s}_t)^2-\vec{s}'^2/4a}\Big]\times\Big[\eta\cdot e^{-\frac{1}{2a}(\vec{b}+\frac{\vec{s}}{2}-\vec{s}_t)^2}$$
$$+\eta\cdot e^{-\frac{1}{2a}(\vec{b}-\frac{\vec{s}}{2}-\vec{s}_t)^2}+\eta^2\, e^{-\frac{1}{a}(\vec{b}-\vec{s}_t)^2-\frac{\vec{s}^2}{4a}}\Big] \ . \tag{5.7}$$

This expression is evaluated in the target thickness approximation (2.18). In order to clearly expose the symmetry in the primed and unprimed variables we introduce two new shorthand notations,

$$X(\vec{b},\vec{s};\vec{b}',\vec{s}') = T\left(\frac{\vec{b}+\vec{b}'}{2}+\frac{\vec{s}+\vec{s}'}{4}\right)\cdot\exp\left[-\frac{1}{4a}\left(\vec{b}-\vec{b}'+\frac{\vec{s}-\vec{s}'}{2}\right)^2\right] \ , \tag{5.8}$$

$$Y(\vec{b},\vec{s};\vec{b}',\vec{s}') = T\left(\frac{2\vec{b}+\vec{b}'}{3}+\frac{\vec{s}}{6}\right)\cdot\exp\left[-\frac{1}{3a}\left(\vec{b}'-\vec{b}-\frac{\vec{s}}{2}\right)^2-\frac{1}{4a}\vec{s}'^2\right] \ , \tag{5.9}$$

and can then write

$$Z_2 = Z_{11} + Z_{12} + Z_{21} + Z_{22} \quad , \tag{5.10a}$$

$$Z_{11} = a\pi\,|\eta|^2 \cdot \{X(\bar b,\bar s;\bar b',\bar s') + X(\bar b,\bar s;\bar b',-\bar s') + X(\bar b,-\bar s;\bar b',\bar s') + X(\bar b,-\bar s;\bar b',-\bar s')\} \,, \tag{5.10b}$$

$$Z_{12} = -\frac{2}{3}\pi a|\eta|^2 \eta^* \cdot \{Y(\bar b,\bar s;\bar b',\bar s') + Y(\bar b,-\bar s;\bar b',\bar s')\} \quad , \tag{5.10c}$$

$$Z_{21} = -\frac{2}{3}\pi a|\eta|^2 \eta \cdot \{Y(\bar b',\bar s';\bar b,\bar s) + Y(\bar b',-\bar s';\bar b,\bar s)\} \quad , \tag{5.10d}$$

$$Z_{22} = \frac{a}{2}\pi\,|\eta|^4 \cdot T\left(\frac{\bar b+\bar b'}{2}\right) \cdot \exp\left[-\frac{1}{2a}(\bar b-\bar b')^2 - \frac{1}{4a}(\bar s^2+\bar s'^2)\right] \quad . \tag{5.10e}$$

Here index 2 refers to eclipse terms. Using this artillery of notation the incoherent elastic cross-section can be written

$$\frac{d\sigma_{I,el}}{d\Omega} = \frac{k_d^2}{4\pi^2} \cdot \int d^2b\, d^2b' \cdot e^{i\bar q(\bar b-\bar b')} \cdot \int d^2s\,|\varphi(\bar s)|^2 \int d^2s'\,|\varphi(\bar s')|^2 \cdot$$
$$\cdot \exp[Z_1^*(\bar b',\bar s') + Z_1(\bar b,\bar s)]\{\exp[Z_2(\bar b,\bar s;\bar b',\bar s')] - 1\} \quad . \tag{5.11}$$

This is a rather complicated expression. Nevertheless we want to show that through series expansions and approximations it is possible to calculate the multidimensional integrals and essentially boil them down to the effective nucleon numbers introduced in Section 3. This is achieved by expanding the whole exponential Z_2 and the eclipse term of Z_1. We calculate these contributions term by term.

A - First order in Z_2. Zero order in eclipse term.

This term is given by

$$\frac{d\sigma^{(1)}_{I,el}}{d\Omega} = \frac{k_d^2}{4\pi^2} \int d^2b\, d^2b' \cdot e^{i\bar q(\bar b-\bar b')} \int d^2s\,|\varphi(\bar s)|^2 \int d^2s'\,|\varphi(\bar s')|^2 \times \tag{5.12}$$
$$\times \exp[-2\pi a\eta^* T_2(\bar b',\bar s') - 2\pi a\eta T_2(\bar b,\bar s)] \cdot Z_2(\bar b',\bar s';\bar b,\bar s) \quad .$$

We make a further subdivision and calculate separately each of the terms in our decomposition (5.10) of Z_2, and start with the biggest term.

1) <u>The term Z_{11}</u>

We first note that $T_2(\bar{b},\bar{s}) = T_2(\bar{b},-\bar{s})$. Therefore the four terms in the decomposition (5.10b) can all be transformed to the first one. We get

$$S_{11} = \frac{k_d^2}{4\pi^2} \cdot 4\pi a |\eta|^2 \int d^2b\, d^2b' \cdot e^{i\bar{q}(\bar{b}-\bar{b}')} \cdot \int d^2s\, |\varphi(\bar{s})|^2 \int d^2s' \, |\varphi(\bar{s}')|^2 \times$$
$$\times \exp\left[-\frac{1}{4a}\left(\bar{b}-\bar{b}'+\frac{\bar{s}-\bar{s}'}{2}\right)^2\right] \cdot \Xi \quad , \tag{5.13a}$$

$$\Xi = \exp\left[-2\pi a \eta^* T_2(\bar{b}',\bar{s}') - 2\pi a \eta\, T_2(\bar{b},\bar{s})\right] \cdot \frac{1}{2}\left\{T\left(\frac{\bar{b}+\bar{b}'}{2}+\frac{\bar{s}+\bar{s}'}{4}\right) + T\left(\frac{\bar{b}+\bar{b}'}{2}-\frac{\bar{s}+\bar{s}'}{4}\right)\right\} . \tag{5.13b}$$

We now want to show that the main contribution to the integral (5.13a) comes from the region $\bar{s} \approx s'$ and $\bar{b} \approx \bar{b}'$, a fact which allows us to perform some of the integrations explicitly. We first note that due to the exponential damping factor in (5.13a) the $\bar{b}$ integral can be performed in the target thickness approximation. In Ξ we thus put $\bar{b}' = \bar{b} + (\bar{s}-\bar{s}')/2$ and get after integration

$$S_{11} = \frac{k_d^2}{4\pi^2} \cdot |4\pi a \eta|^2 \cdot \exp[-a\bar{q}^2] \cdot \int d^2b \int d^2s\, |\varphi(\bar{s})|^2 \int d^2s' \, |\varphi(\bar{s}')|^2 \cdot$$
$$\cdot \exp\left[-i\bar{q}\cdot\frac{\bar{s}-\bar{s}'}{2}\right] \cdot \Xi \quad . \tag{5.14}$$

We now come to the second approximation and then first note that

$$|\varphi(\bar{s})|^2 \cdot |\varphi(\bar{s}')|^2 \equiv \left|\varphi\left(\frac{\bar{s}+\bar{s}'}{\sqrt{2}}\right)\right|^2 \cdot \left|\varphi\left(\frac{\bar{s}-\bar{s}'}{\sqrt{2}}\right)\right|^2 \quad . \tag{5.15}$$

The damping factor in $(\bar{s}-\bar{s}')$ is not as powerful as before. We only have the damping provided by the deuteron wave function. But we also have an oscillating factor $\exp[-i\bar{q}\cdot(\bar{s}-\bar{s}')/2]$. Thus we conclude that for large values of $\bar{q}$, i.e., in the incoherent region, the main contribution for (5.14) will come from the region $\bar{s} \approx \bar{s}'$. In Ξ we therefore put $\bar{s} = \bar{s}'$, and combining this with our previous findings we also have $\bar{b} = \bar{b}'$. Performing the integral over $\bar{s} - \bar{s}'$ we are left with

$$S_{11} = \frac{t_d^2}{4\pi^2} \cdot |4\pi a \eta|^2 \cdot \exp[-\bar{q}^2 \cdot (a + \frac{R_d^2}{8})] \cdot \Sigma_1' \tag{5.16}$$

$$\Sigma_1' = \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \exp[-4\pi \, \mathrm{Re}\,\eta \cdot T_2(\bar{b}, \frac{\bar{s}}{\sqrt{2}})] \cdot \frac{1}{2} T_2(\bar{b}, \frac{\bar{s}}{\sqrt{2}}) \cdot \tag{5.17}$$

Using the methods of (3.7) and (3.8) we have

$$\Sigma_1' \approx N_1(2\sigma) + \frac{1}{2}[M_1(2\sigma) - M_2(2\sigma)] \quad . \tag{5.18}$$

All the relevant effective nucleon numbers are found in the Tables in the Appendix.

2) The terms Z_{21} and Z_{12}

We use the symmetry property $T_2(\bar{b},\bar{s}) = T_2(\bar{b},-\bar{s})$ to write the contribution from Z_{12} as

$$S_{12} = -\frac{t_d^2}{4\pi^2} \cdot \frac{4}{3} \pi a |\eta|^2 \eta^* \int d^2b \, d^2b' . e^{i\bar{q}(\bar{b}-\bar{b}')} \cdot \int d^2s \, |\varphi(\bar{s})|^2 \int d^2s' \, |\varphi(\bar{s}')|^2 \cdot$$
$$\cdot \exp[-\frac{1}{3a}(\bar{b}' - \bar{b} - \frac{\bar{s}}{2})^2 - \frac{1}{4a}\bar{s}'^2] \cdot \Xi \quad , \tag{5.19}$$

where this time

$$\Xi = \exp[-2\pi a \eta^* T_2(\bar{b}', \bar{s}') - 2\pi a \eta \, T_2(\bar{b}, \bar{s})] \cdot T(\frac{2\bar{b}+\bar{b}'}{3} + \frac{\bar{s}}{6}) \quad . \tag{5.20}$$

The contribution from Z_{21} is related to S_{12}. A variable transformation and a simultaneous conjugation show that

$$S_{21} = S_{12}^* \quad . \tag{5.21}$$

The two exponential damping factors in (5.19) allow us to perform the $\bar{s}'$ and $\bar{b}'$ integrals in the target thickness approximation. We get

$$S_{12} = -\frac{k_d^2}{4\pi^2}\cdot\frac{1}{4}\cdot|4\pi a\eta|^2\cdot\eta^*\cdot\frac{1}{1+R_d^2/4a}\cdot\exp[-\tfrac{3}{4}a\bar{q}^2]\cdot$$
$$\cdot\int d^2b\int d^2s\,|\varphi(\bar{s})|^2\cdot\exp[-\tfrac{i}{2}\bar{q}\bar{s}]\cdot\Xi\quad, \tag{5.22}$$

where in the new function Ξ we have put $\bar{s}' = 0$ and $\bar{b}' = \bar{b} + \bar{s}/2$. The $\bar{s}$ integration is similar to the integration over $\bar{s}-\bar{s}'$ in (5.14). Due to the oscillating exponential and the damping effect of the deuteron wave function the integral (5.22) receives its main contribution from the region $\bar{s} \approx 0$. In this approximation

$$S_{12} = -\frac{k_d^2}{4\pi^2}\cdot\frac{1}{4}|4\pi a\eta|^2\cdot\eta^*\cdot\frac{1}{1+R_d^2/4a}\cdot\exp[-\bar{q}^2(\tfrac{3}{4}a+\tfrac{R_d^2}{16})]\cdot$$
$$\cdot\int d^2b\,\exp[-8\pi a\,\mathrm{Re}\,\eta\cdot T(\bar{b})]\cdot T(\bar{b})\quad. \tag{5.23}$$

Collecting the contributions from S_{12} and S_{21} we get

$$S_{12}+S_{21} = -\frac{k_d^2}{4\pi^2}\cdot|4\pi a\eta|^2\cdot\frac{\mathrm{Re}\,\eta}{2}\cdot\frac{1}{1+R_d^2/4a}\cdot N_1(2\sigma)\cdot\exp[-\bar{q}^2(\tfrac{3}{4}a+\tfrac{R_d^2}{16})]\quad. \tag{5.24}$$

For the values $a = 7\ \mathrm{GeV}^{-2}$ and $\sigma = 40$ mb of (2.11) we have

$$\frac{\mathrm{Re}\,\eta}{2}\cdot\frac{1}{1+R_d^2/4a} = 0.10 \tag{5.25}$$

and therefore $S_{12} + S_{21}$ is, in the region $|t| \lesssim 0.1\ \mathrm{GeV}^2$, at most a 25% correction to the main term S_{11} in (5.16).

3) <u>The term</u> Z_{22}

We now have

$$S_{22} = \frac{k_d^2}{4\pi^2}\,\frac{a}{2}\,\pi|\eta|^4\int d^2b\int d^2b'\,e^{i\bar{q}(\bar{b}-\bar{b}')}\int d^2s\,|\varphi(\bar{s})|^2\int d^2s'\,|\varphi(\bar{s}')|^2\cdot$$
$$\cdot T(\tfrac{\bar{b}+\bar{b}'}{2})\cdot\exp[-2\pi a\eta^* T_2(\bar{b}',\bar{s}') - 2\pi a\,T_2(\bar{b},\bar{s})]\cdot \tag{5.26}$$
$$\cdot\exp[-\tfrac{1}{2a}(\bar{b}-\bar{b}')^2 - \tfrac{1}{4a}(\bar{s}^2+\bar{s}'^2)]\quad.$$

This is straightforward. The integrals $\bar{b}'$, $\bar{s}$ and $\bar{s}'$ can be performed in the target thickness approximation. In the deuteron target thickness we thus put $\bar{b}' = \bar{s} = \bar{s}' = 0$. After integration

$$S_{22} = \frac{k_d^2}{4\pi^2} \cdot |4\pi a \eta|^2 \cdot \frac{|\eta|^2}{16} \cdot \left[\frac{1}{1+R_d^2/4a}\right]^2 \cdot N_1(2\sigma) \cdot \exp\left[-\frac{1}{2} a \bar{q}^2\right] . \qquad (5.27)$$

For $a = 7\ \text{GeV}^{-2}$ and $\sigma = 40$ mb we have

$$\frac{|\eta|^2}{16} \cdot \left[\frac{1}{1+R_d^2/4a}\right]^2 = 0.0026 \quad . \qquad (5.28)$$

As a consequence, in the region $|t| \lesssim 0.1\ \text{GeV}^2$ we find that S_{22} is much smaller than S_{11}. For larger angles the situation changes radically. S_{22} depends only on the nucleon-nucleon amplitudes whereas S_{11}, S_{12} and S_{21} depend also on the deuteron form factor. In fact already at $-t = 0.3\ \text{GeV}^2$ we find that S_{22} is of the same magnitude as these amplitudes.

We now come to the higher order terms in our expansions. They can be calculated with the techniques described above.

B - <u>First order in</u> Z_2. <u>First order in eclipse term.</u>

This term is obtained from (5.12) through the replacement

$$Z_2(\bar{b}',\bar{s}';\bar{b},\bar{s}) \longrightarrow$$
$$Z_2(\bar{b}',\bar{s}';\bar{b},\bar{s}) \cdot \left\{\pi a \eta^2 T(\bar{b}) \cdot \exp\left[-\frac{\bar{s}^2}{4a}\right] + \pi a \eta^{*2} T(\bar{b}') \cdot \exp\left[-\frac{\bar{s}'^2}{4a}\right]\right\} . \qquad (5.29)$$

As above we consider separately the contributions from Z_{11}, $Z_{12} + Z_{21}$ and Z_{22}.

1) <u>The term</u> Z_{11}

We first note that the additional factor in (5.29) is invariant under the transformation $\bar{b} \rightleftarrows \bar{b}'$, $\bar{s} \rightleftarrows \bar{s}'$ and a simultaneous complex conjugation. As it is also invariant under the change $\bar{s} \to -\bar{s}$ or $\bar{s}' \to -\bar{s}'$ we get the contribution

$$S_{11}^{e} = \frac{k_d^2}{4\pi^2}\cdot 8\cdot|\pi a\eta|^2\cdot \mathrm{Re}\,\eta^2 \int d^2b \int d^2b'\cdot e^{i\bar{q}(\bar{b}-\bar{b}')} \cdot$$
$$\cdot \int d^2s\,|\varphi(\bar{s})|^2 \int d^2s'\,|\varphi(\bar{s}')|^2\cdot \exp\left[-\frac{1}{4a}\left(\bar{b}-\bar{b}'+\frac{\bar{s}-\bar{s}'}{2}\right)^2 - \frac{\bar{s}^2}{4a}\right]\cdot \Xi \,, \qquad (5.30)$$

$$\Xi = T(\bar{b})\cdot T\left(\frac{\bar{b}+\bar{b}'}{2}+\frac{\bar{s}+\bar{s}'}{4}\right)\cdot \exp\left[-2\pi a\eta T_2(\bar{b},\bar{s}) - 2\pi a\eta^* T_2(\bar{b}',\bar{s}')\right] . \qquad (5.31)$$

A translation in $\bar{b}'$ gives

$$S_{11}^{e} = \frac{k_d^2}{4\pi^2}\cdot|4\pi a\eta|^2\cdot\frac{1}{2}\,\mathrm{Re}\,\eta^2\cdot\int d^2b\, d^2b'\, \exp\left[i\bar{q}\bar{b}' - \frac{\bar{b}'^2}{4a}\right]\cdot$$
$$\cdot \int d^2s\,|\varphi(\bar{s})|^2 \int d^2s'\,|\varphi(\bar{s}')|^2\cdot \exp\left[-\frac{\bar{s}^2}{4a} - i\bar{q}\cdot\frac{\bar{s}-\bar{s}'}{2}\right]\cdot \Xi \,. \qquad (5.32)$$

From this expression we immediately conclude that the main contribution to the integral comes from the region $\bar{b}' = 0$, or in the original formula (5.30) from the region $\bar{b}' = \bar{b}$ and $\bar{s} = \bar{s}' = 0$. Exploiting this fact all integrations are straightforward in our approximation. For real η we get the result

$$S_{11}^{e} = \frac{k_d^2}{4\pi^2}\cdot|4\pi a\eta|^2\cdot\frac{\eta}{2}\cdot\frac{1}{1+R_d^2/4a}\cdot N_2\,(2\sigma)\,\cdot$$
$$\cdot \exp\left[-\bar{q}^2\left\{a+\frac{R_d^2}{16}+\frac{a}{4(1+4a/R_d^2)}\right\}\right] . \qquad (5.33)$$

2) <u>The terms</u> Z_{12} <u>and</u> Z_{21}

We first note that $Z_{21} \to Z_{12}$ through a simultaneous complex conjugation and interchange of $\bar{b}$ and $\bar{b}'$ and of $\bar{s}$ and $\bar{s}'$. As the new factor in (5.29) is invariant under this transformation, we get

$$S_{12}^{e} + S_{21}^{e} = -\frac{k_d^2}{4\pi^2}\cdot\frac{8}{3}\cdot|\pi a\eta|^2\cdot \mathrm{Re}\,\eta^* \int d^2b\, d^2b'\, e^{i\bar{q}(\bar{b}-\bar{b}')} \cdot$$
$$\cdot \int d^2s\,|\varphi(\bar{s})|^2 \int d^2s'\,|\varphi(\bar{s}')|^2\cdot \exp\left[-\frac{1}{3a}\left(\bar{b}'-\bar{b}-\frac{\bar{s}}{2}\right)^2 - \frac{1}{4a}\bar{s}'^2\right]\cdot \qquad (5.34)$$
$$\cdot \left[\eta^2 T(\bar{b})\, e^{-\bar{s}^2/4a} + \eta^{*2} T(\bar{b}')\, e^{-\bar{s}'^2/4a}\right]\cdot \Xi \,,$$

$$\Xi = T\left(\frac{2\bar{b}+\bar{b}'}{3}+\frac{\bar{s}}{6}\right)\cdot \exp\left[-2\pi a\eta^{*}\,T_2(\bar{b}',\bar{s}') - 2\pi a\eta\,T_2(\bar{b},\bar{s})\right] \; . \qquad (5.35)$$

The first term in (5.34) contains exponential damping factors in $\bar{s},\bar{s}'$ and $\bar{b}-\bar{b}'$, whereas the second term contains exponential damping factors in $\bar{s}'$ and $\bar{b}-\bar{b}'$ but an oscillating damping factor in $\bar{s}$ (in addition there is a damping from the deuteron wave function). For both terms we conclude that the approximation scheme developed above can be applied and we can put $\bar{b}=\bar{b}'$ and $\bar{s}=\bar{s}'=0$ in the Ξ of (5.35). The integrations are then straightforward and we get for real η

$$S^{e}_{12}+S^{e}_{21} = -\frac{k_d^2}{4\pi^2}\cdot|4\pi a\eta|^2\cdot\frac{|\eta|^2}{8}\cdot N_2(2\sigma)\cdot\exp\left[-\frac{3a}{4}\bar{q}^2\right]\cdot$$

$$\cdot\left\{\frac{1}{(1+R_d^2/4a)^2}\cdot\exp\left[-\bar{q}^2\cdot\frac{a}{4(1+4a/R_d^2)}\right]+\frac{1}{1+R_d^2/4a}\cdot\exp\left[-\bar{q}^2\cdot\frac{R_d^2}{16}\right]\right\} \qquad (5.36)$$

3) <u>The term</u> Z_{22}

We now get

$$S^{e}_{22} = \frac{k_d^2}{4\pi^2}\cdot\frac{a}{2}\pi|\eta|^4\cdot 2\,\mathrm{Re}\,\pi a\eta^2\int d^2b\,d^2b'\,e^{i\bar{q}(\bar{b}-\bar{b}')}\int d^2s\,|\varphi(\bar{s})|^2\int d^2s'\,|\varphi(\bar{s}')|^2\cdot$$

$$\cdot\,\Xi\cdot\exp\left[-\frac{1}{2a}(\bar{b}-\bar{b}')^2-\frac{1}{4a}(\bar{s}^2+\bar{s}'^2)-\frac{1}{4a}\bar{s}^2\right] \quad , \qquad (5.37)$$

$$\Xi = T(\bar{b})\cdot T\left(\frac{\bar{b}+\bar{b}'}{2}\right)\cdot\exp\left[-2\pi a\eta^{*}\,T_2(\bar{b}',\bar{s}')-2\pi a\eta\,T_2(\bar{b},\bar{s})\right] \; . \qquad (5.38)$$

We have exponential damping factors in $\bar{b}-\bar{b}'$, $\bar{s}$ and $\bar{s}'$ and therefore Ξ in (5.38) can be evaluated at $\bar{b}=\bar{b}'$, $\bar{s}=\bar{s}'=0$. The integration gives for real η the result

$$S^{e}_{22} = \frac{k_d^2}{4\pi^2}\cdot|4\pi a\eta|^2\cdot\frac{\eta^3}{32}\cdot\frac{1}{(1+R_d^2/2a)\cdot(1+R_d^2/4a)}\cdot N_2(2\sigma)\,\cdot \qquad (5.39)$$

$$\cdot\exp\left[-a\bar{q}^2/2\right] \; .$$

Again we have a term where the slope depends only on the nucleon-nucleon amplitude and which therefore is important for large t values. It is comparable in magnitude to S^e_{11}, S^e_{12} and S^e_{21} at $-t = 0.35\ \mathrm{GeV}^2$. The slope for S^e_{22} is the same as for S_{22}, but as S^e_{22} is an eclipse correction to S_{22} its absolute magnitude is smaller.

C - <u>Second order in Z_2. Zero order in eclipse term.</u>

Upon expansion we have

$$e^{Z_2} - 1 = Z_2 + \frac{1}{2} Z_2^2 + \cdots \qquad , \tag{5.40}$$

where the contribution to be calculated in this Section comes from Z_2^2. We have

$$\frac{1}{2} Z_2^2 = \frac{1}{2}\left(Z_{11} + Z_{12} + Z_{21} + Z_{22}\right)^2$$

$$= \frac{1}{2}\left\{Z_{11}^2 + 2Z_{11}(Z_{12}+Z_{21}) + (Z_{12}+Z_{21})^2 + 2Z_{11}Z_{22} + 2(Z_{12}+Z_{21})Z_{22} + Z_{22}^2\right\} \tag{5.41}$$

We shall only consider the term Z_{11}^2, which is the most important one. Due to the symmetry of Z_{11} in the variables $\bar{s}$ and $\bar{s}'$ we have

$$S_{11\times 11} = \frac{1}{2}\cdot 4\pi^2 a^2 |\eta|^4 \cdot \frac{k_d^2}{4\pi^2}\cdot \mathrm{Re}\int d^2b\, d^2b'\, e^{i\bar{q}(\bar{b}-\bar{b}')} \int d^2s\, |\varphi(\bar{s})|^2 \int d^2s'\, |\varphi(\bar{s}')|^2 \cdot$$
$$\cdot \exp\left[-2\pi a\,\eta^* T_2(\bar{b}',\bar{s}') - 2\pi a\,\eta\, T_2(\bar{b},\bar{s})\right]\cdot T\left(\frac{\bar{b}+\bar{b}'}{2} + \frac{\bar{s}+\bar{s}'}{4}\right)\cdot \exp\left[-\frac{1}{4a}\left(\bar{b}-\bar{b}'+\frac{\bar{s}-\bar{s}'}{2}\right)^2\right]\cdot$$
$$\cdot\left\{T\left(\frac{\bar{b}+\bar{b}'}{2} + \frac{\bar{s}+\bar{s}'}{4}\right)\cdot \exp\left[-\frac{1}{4a}\left(\bar{b}-\bar{b}'+\frac{\bar{s}-\bar{s}'}{2}\right)^2\right] + T\left(\frac{\bar{b}+\bar{b}'}{2} + \frac{\bar{s}+\bar{s}'}{4}\right)\cdot\right.$$
$$\left.\cdot \exp\left[-\frac{1}{4a}\left(\bar{b}-\bar{b}'-\frac{\bar{s}-\bar{s}'}{2}\right)^2\right] + 2T\left(\frac{\bar{b}+\bar{b}'}{2} + \frac{\bar{s}-\bar{s}'}{4}\right)\cdot \exp\left[-\frac{1}{4a}\left(\bar{b}-\bar{b}'+\frac{\bar{s}+\bar{s}'}{2}\right)^2\right]\right\} . \tag{5.42}$$

The three terms present have different structure. In the first one we first evaluate the $\bar{b}'$ integral in the target thickness approximation and then the $(\bar{s}-\bar{s}')$ integral by exploiting (5.15) and the oscillating exponential factor. For real η this gives

$$I_1 = \frac{k_d^2}{4\pi^2}\cdot |4\pi a\eta|^2 \cdot \frac{\eta}{16}\cdot \exp\left[-\bar{q}^2\left(\frac{a}{2} + \frac{R_d^2}{8}\right)\right]\cdot P_1 \qquad , \tag{5.43}$$

$$P_1 = \sigma \int d^2b \int d^2s\, |\varphi(\bar{s})|^2 \cdot T^2(\bar{b}) \cdot \exp[-\sigma T(\bar{b}) - \sigma T(\bar{b} - \tfrac{\bar{s}}{\sqrt{2}})] \,. \tag{5.44}$$

The integral defining P_1 does not exhibit the nice symmetry in $\bar{s}$ found in all previous cases but it can of course be treated with our expansion methods. We get

$$\begin{aligned} P_1 &= \sigma \int d^2b\, T^2(\bar{b})\, e^{-2\sigma T(\bar{b})} \int d^2s\, |\varphi(\bar{s})|^2 \cdot \\ &\quad \cdot \left\{ 1 - \tfrac{\sigma}{4} (\bar{s}\bar{\nabla})^2 T + \tfrac{\sigma^2}{4} [(\bar{s}\bar{\nabla})^2 T]^2 + \cdots \right\} \\ &= N_2(2\sigma) - M_3(2\sigma) + \cdots \end{aligned} \tag{5.45}$$

We now come to the middle term in (5.42). This is the most interesting one. The $\bar{b}'$ and $(\bar{s}-\bar{s}')$ integrals are performed in the target thickness approximation. We also use (5.15) and find for real η

$$I_2 = \frac{k_d^2}{4\pi^2} \cdot |4\pi a \eta|^2 \cdot \frac{\eta}{16} \cdot \frac{1}{1+R_d^2/4a} \cdot \exp[-a\bar{q}^2/2]\; P_2 \quad , \tag{5.46}$$

$$P_2 = \sigma \int d^2b \int d^2s\, |\varphi(\bar{s})|^2 \cdot \exp[-\sigma T_2(\bar{b},\bar{s})]\, T(\bar{b}+\tfrac{\bar{s}}{2}) \cdot T(\bar{b}-\tfrac{\bar{s}}{2}) \,. \tag{5.47}$$

Also P_2 has a new structure. The usual methods give

$$P_2 = N_2(2\sigma) + M_2(2\sigma) - M_3(2\sigma) + \cdots \tag{5.48}$$

Finally, in the last term of (5.42) the $\bar{b}'$ and $\bar{s}'$ integrals are performed in the target thickness approximation and the $\bar{s}$ integral with the aid of the oscillating exponential factor, yielding

$$I_3 = \frac{k_d^2}{4\pi^2} \cdot |4\pi a \eta|^2 \cdot \frac{\eta}{8} \cdot \frac{1}{1+R_d^2/8a} \cdot N_2(2\sigma)\, \exp[-\bar{q}^2(\tfrac{a}{2} + \tfrac{R_d^2}{16})] \,. \tag{5.49}$$

Collecting all the terms

$$S_{\parallel \times \parallel} = I_1 + I_2 + I_3 \,. \tag{5.50}$$

The most interesting term in (5.50) is I_2. It does not contain the deuteron form factor and is therefore the only term in (5.50) which survives at large momentum transfers. It has the same slope as S_{22} and S^e_{22} of (5.27) and (5.39) respectively, but it is in fact about four times bigger than S^e_{22}.

We also remark that terms containing the deuteron form factor decrease rather fast with increasing momentum transfer. Already at $-t = 0.5\ \mathrm{GeV}^2$ we find that I_2 is ten times bigger than those contributions to the cross-section which contain the deuteron form factor. However, for such large momentum transfers one should also look at higher order contributions before comparing with experimental data.

6. - INCOHERENT ELASTIC SCATTERING : TOTAL CROSS-SECTION

Integrating the differential cross-section (5.11) over solid angle we get the total cross-section

$$\sigma_{I,el} = \int d^2b \int d^2s\, |\varphi(\bar{s})|^2 \int d^2s'\, |\varphi(\bar{s}')|^2 \exp\left[Z_1^*(\bar{b},\bar{s}') + Z_1(\bar{b},\bar{s}) \right] \cdot \\ \cdot \left\{ \exp\left[Z_2(\bar{b},\bar{s};\bar{b},\bar{s}') \right] - 1 \right\} , \tag{6.1}$$

where the notations were explained in Section 5. The evaluation of this expression goes as before. We expand $\exp[Z_2]$ and the eclipse part of $\exp[Z_1]$. In this way each term in the series obtained can be expressed in effective nucleon numbers. In the following we will not give the detailed evaluation of every term in the expansion. For those terms where the calculation is straightforward we only give the results whereas terms with new features are given in some detail. We calculate terms down to contributions of 2%.

A - <u>First order in</u> Z_2. <u>Zero order in eclipse terms.</u>

Straightforward integration gives, when the imaginary part of η is neglected,

$$\sigma_{11} = \sigma \cdot \frac{\eta}{1+R_d^2/8a} \cdot \Sigma_1' \quad , \tag{6.2}$$

$$\begin{aligned} \Sigma_1' &= \frac{1}{2} \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot T_2(\bar{b}, \frac{\bar{s}}{\sqrt{2}}) \cdot \exp[-\sigma T_2(\bar{b}, \frac{\bar{s}}{\sqrt{2}})] \\ &\approx N_1(2\sigma) + \frac{1}{2}[M_1(2\sigma) - M_2(2\sigma)] \quad , \end{aligned} \tag{6.3}$$

$$\sigma_{12} + \sigma_{21} = -\sigma \cdot \frac{2}{3} \cdot \eta^2 \cdot \frac{1}{1+R_d^2/12a} \cdot \frac{1}{1+R_d^2/4a} \cdot N_1(2\sigma) \quad , \tag{6.4}$$

$$\sigma_{22} = \sigma \cdot \frac{\eta^3}{8} \cdot \left[\frac{1}{1+R_d^2/4a}\right]^2 \cdot N_1(2\sigma) \quad . \tag{6.5}$$

It is interesting to note that these partial cross-sections are equal to the corresponding integrated partial differential cross-sections given in Section 5.

B - <u>First order in</u> Z_2. <u>First order in eclipse terms.</u>

These terms are the eclipse corrections to those listed under A. We have

$$\begin{aligned} \sigma_{11}^e = {} & 4\pi^2 a^2 \eta^3 \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \int d^2s' \, |\varphi(\bar{s}')|^2 \cdot \\ & \cdot \exp[-\frac{\sigma}{2} T_2(\bar{b}, \bar{s}') - \frac{\sigma}{2} T_2(\bar{b}, \bar{s})] \cdot \exp[-\frac{\bar{s}^2}{4a}] \cdot \\ & \cdot \exp[-\frac{1}{4a}(\frac{\bar{s}-\bar{s}'}{2})^2] \cdot T(\bar{b}) \cdot \{T(\bar{b} + \frac{\bar{s}+\bar{s}'}{4}) + T(\bar{b} - \frac{\bar{s}+\bar{s}'}{4})\} \quad . \end{aligned} \tag{6.6}$$

Here we argue that due to the second exponential damping factor $\bar{s} \approx \bar{s}'$ and due to the first one $\bar{s} \approx 0$. Thus we can put $\bar{s} = \bar{s}' = 0$ in the deuteron target thickness. This yields

$$\sigma_{11}^e = \sigma \cdot \frac{\eta^2}{2} \cdot \frac{1}{[1+\frac{R_d^2}{8a}]^2 + \frac{R_d^2}{8a}} \cdot N_2(2\sigma) \quad . \tag{6.7}$$

In the remaining eclipse terms the damping factors appear separately in $\bar{s}$ and $\bar{s}'$. We find

$$\sigma_{12}^{e} + \sigma_{21}^{e} = -\sigma \cdot \frac{\eta^3}{6} \cdot N_2(2\sigma) \cdot \left[\frac{1}{1+R_d^2/3a} \cdot \frac{1}{1+R_d^2/4a} + \frac{1}{1+R_d^2/12a} \cdot \frac{1}{1+R_d^2/2a} \right] . \tag{6.8}$$

C - <u>Second order in</u> Z_2. <u>Zero order in eclipse terms.</u>

The first term becomes after rearrangement

$$\begin{aligned} \sigma_{11\times 11} = (\pi a \eta^2)^2 \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \int d^2s' \, |\varphi(\bar{s}')|^2 \cdot \exp\left[-\tfrac{\sigma}{2} T_2(\bar{b},\bar{s}') - \tfrac{\sigma}{2} T_2(\bar{b},\bar{s})\right] \cdot \\ \cdot \Big\{ T_2^2\left(\bar{b}, \tfrac{\bar{s}+\bar{s}'}{2}\right) \cdot \exp\left[-\tfrac{1}{2a}\left(\tfrac{\bar{s}-\bar{s}'}{2}\right)^2\right] + T_2\left(\bar{b}, \tfrac{\bar{s}+\bar{s}'}{2}\right) \cdot T_2\left(\bar{b}, \tfrac{\bar{s}-\bar{s}'}{2}\right) \cdot \\ \cdot \exp\left[-\tfrac{1}{4a}\left(\tfrac{\bar{s}+\bar{s}'}{2}\right)^2 - \tfrac{1}{4a}\left(\tfrac{\bar{s}-\bar{s}'}{2}\right)^2\right] \Big\} . \end{aligned} \tag{6.9}$$

Here the second term is trivial

$$\sigma_{11\times 11} = \sigma \frac{\eta^2}{4} \cdot \left[\frac{1}{1+R_d^2/4a} \cdot \Sigma_2' + \left[\frac{1}{1+R_d^2/8a}\right]^2 \cdot N_2(2\sigma) \right] , \tag{6.10}$$

$$\begin{aligned} \Sigma_2' &= \frac{\sigma}{4} \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot T_2^2\left(\bar{b}, \tfrac{\bar{s}}{\sqrt{2}}\right) \cdot \exp\left[-\sigma T_2\left(\bar{b}, \tfrac{\bar{s}}{\sqrt{2}}\right)\right] \\ &\approx N_2(2\sigma) + \tfrac{1}{2}\left[M_2(2\sigma) - M_3(2\sigma)\right] . \end{aligned} \tag{6.11}$$

The following terms are again straightforward

$$\sigma_{11\times(12+21)} = -\sigma \frac{2}{3} \eta^3 \cdot \frac{1}{\left[1+\frac{11}{48}\cdot\frac{R_d^2}{a}\right]^2 - \left[\frac{5}{48}\cdot\frac{R_d^2}{a}\right]^2} \cdot N_2(2\sigma) , \tag{6.12}$$

$$\sigma_{(12+21)\times(12+21)} = \sigma \frac{\eta^4}{9} \cdot \left[\frac{1}{1+R_d^2/6a} \cdot \frac{1}{1+R_d^2/2a} + \frac{1}{[1+R_d^2/3a]^2} \right] \cdot N_2(2\sigma) . \tag{6.13}$$

D - <u>Second order in</u> Z_2. <u>First order in eclipse terms.</u>

Here only the eclipse correction to (6.10) is of importance,

$$\sigma_{11\times 11}^{e} = \sigma \frac{3}{16} \eta^3 \cdot \left[\frac{1}{[1+R_d^2/4a]^2 - R_d^4/32a^2} + \frac{1}{1+R_d^2/8a} \cdot \frac{1}{1+3R_d^2/8a} \right] \cdot N_3(2\sigma) . \tag{6.14}$$

E - Third order in Z_2. Zero order in eclipse terms.

Only $\sigma_{11\times 11\times 11}$ is of interest and it requires special attention. After some trivial rearrangements it can be written

$$\sigma_{11\times 11\times 11} = \frac{1}{3}(\pi a \eta^2)^3 \cdot \int d^2b \int d^2s\, |\varphi(\bar{s})|^2 \int d^2s'\, |\varphi(\bar{s}')|^2 \cdot \exp\left[-\frac{\sigma}{2} T_2(\bar{b},\bar{s}') - \frac{\sigma}{2} T_2(\bar{b},\bar{s})\right] \cdot$$

$$\cdot \left\{ T_2^3\left(\bar{b}, \frac{\bar{s}+\bar{s}'}{2}\right) \cdot \exp\left[-\frac{3}{4a}\left(\frac{\bar{s}-\bar{s}'}{2}\right)^2\right] + 3\, T_2^2\left(\bar{b}, \frac{\bar{s}+\bar{s}'}{2}\right) \cdot T_2\left(\bar{b}, \frac{\bar{s}-\bar{s}'}{2}\right) \cdot \right. \tag{6.15}$$

$$\left. \cdot \exp\left[-\frac{1}{2a}\left(\frac{\bar{s}-\bar{s}'}{2}\right)^2 - \frac{1}{4a}\left(\frac{\bar{s}+\bar{s}'}{2}\right)^2\right] \right\} .$$

The last term is trivial but the first one needs special care. We get

$$\sigma_{11\times 11\times 11} = \sigma \frac{\eta^3}{16} \cdot \left[\frac{1}{1+3R_d^2/8a} \cdot \Delta_3' + \frac{3}{1+R_d^2/4a} \cdot \frac{1}{1+R_d^2/8a} \cdot N_3(2\sigma) \right] , \tag{6.16}$$

$$\Delta_3' = \frac{\sigma^2}{12} \int d^2b \int d^2s\, |\varphi(\bar{s})|^2\, T_2^3\left(\bar{b}, \frac{\bar{s}}{\sqrt{2}}\right) \cdot \exp\left[-\sigma T_2\left(\bar{b}, \frac{\bar{s}}{\sqrt{2}}\right)\right]$$

$$\approx N_3(2\sigma) + \frac{1}{2}\left\{ M_3(2\sigma) - M_4(2\sigma) \right\} . \tag{6.17}$$

Finally we comment on the relative magnitude of the different terms. The leading one is of course σ_{11}. Next we have $\sigma_{12} + \sigma_{21}$ and σ^e_{11}, each in the region of 13%. As they have opposite sign only a few per cent survive in the difference. Next we have $\sigma_{11\times 11}$ which is about 10% but is much diminished by the joint action of σ^e_{12+21} and $\sigma_{11\times(12+21)}$, that each contributes about 5% but with a sign opposite to that of $\sigma_{11\times 11}$. The remaining terms are smaller than 3% of σ_{11}. Adding all terms together we find that the true correction to σ_{11} is in fact much smaller than 10%.

Comparing the total incoherent cross-section with the total coherent cross-section we see that the incoherent one is much smaller than the coherent one. For light nuclei it is about 5% of the coherent one and for heavy nuclei it is only 2%.

7. - COHERENT DIFFRACTIVE DISSOCIATION

In the coherent dissociation process the deuteron disintegrates into a proton and a neutron whereas the nucleus remains in its ground state. Such processes are expected to give small cross-sections. In our approach the coherent dissociation cross-section comes out as a difference between two rather big numbers and much care is therefore necessary. We shall show that our approximation scheme allows a calculation of the magnitude of the coherent dissociation errors to a reasonable accuracy.

First we use the closure approximation to calculate the sum of the coherent dissociation and coherent elastic cross-section. Then we subtract the coherent elastic part to get the coherent dissociation part. The amplitude for a process which changes the deuteron state from $|d\rangle$ to $|d'\rangle$ but leaves the nucleus intact is according to (2.1) and (2.5)

$$F_{d'd}(\bar{q}) = \frac{ik_d}{2\pi} \cdot \int d^2b \, e^{i\bar{q}\bar{b}} \langle d't | \{1 - \exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)]\} | dt \rangle$$

$$= \frac{ik_d}{2\pi} \int d^2b \, e^{i\bar{q}\bar{b}} \langle d' | \{1 - \exp[Z_1(\bar{b},\bar{s})]\} | d \rangle \,, \qquad (7.1)$$

where the last step follows from the definition (3.2). Using (2.4) we obtain the integrated cross-section as

$$\sigma_{c,el} + \sigma_{c,diss} = \sum_{d'} \int d^2b \, \langle d | \{1 - \exp[Z_1^*(\bar{b},\bar{s}')]\} | d' \rangle \cdot$$

$$\cdot \langle d' | \{1 - \exp[Z_1(\bar{b},\bar{s})]\} | d \rangle \quad . \qquad (7.2)$$

We now apply the closure relation

$$\sum_{d'} \varphi_{d'}^*(\bar{r}') \, \varphi_{d'}(\bar{r}) = \delta(\bar{r} - \bar{r}') \quad , \qquad (7.3)$$

and get

$$\sigma_{c,el} + \sigma_{c,diss} = \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot |1 - \exp[Z_1(\bar{b},\bar{s})]|^2 \quad . \qquad (7.4)$$

We now subtract the coherent elastic cross-section as given by (3.5). Terms linear in $\exp[Z_1]$ cancel due to the normalization property of the nuclear wave functions and we are left with

$$\sigma_{C,diss} = \int d^2b \left[\int d^2s\, |\varphi(\bar{s})|^2 \cdot \left| \exp[Z_1(\bar{b},\bar{s})] \right|^2 - \left| \int d^2s\, |\varphi(\bar{s})|^2 \cdot \exp[Z_1(\bar{b},\bar{s})] \right|^2 \right] . \quad (7.5)$$

This expression is certainly positive but not easy to evaluate.

In our attempt to estimate $\sigma_{C,diss}$ we neglect as usual the real part of the nucleon-nucleon amplitudes. We also introduce a few new notations

$$Z_1(\bar{b},\bar{s}) = -\sigma T(\bar{b}) + \delta Z_1(\bar{b},\bar{s}) \quad , \quad (7.6)$$

$$\delta Z_1(\bar{b},\bar{s}) = -\frac{\sigma}{2}\, \delta T_2(\bar{b},\bar{s}) + \sigma \frac{\eta}{4}\, T(\bar{b})\, e^{-\bar{s}^2/4a} \quad , \quad (7.7)$$

$$\delta T_2(\bar{b},\bar{s}) = T_2(\bar{b},\bar{s}) - 2T(\bar{b}) \quad . \quad (7.8)$$

It is then important to realize that if we expand the exponentials in the small quantity δZ_1, then the linear terms in (7.5) will cancel identically, i.e., independent of any approximation scheme. The second order terms in this expansion give

$$\sigma_{C,diss} = \int d^2b\, e^{-2\sigma T(\bar{b})} \cdot \left[\int d^2s\, |\varphi(\bar{s})|^2 \{\delta Z_1(\bar{b},\bar{s})\}^2 - \{ \int d^2s\, |\varphi(\bar{s})|^2\, \delta Z_1(\bar{b},\bar{s}) \}^2 \right] . \quad (7.9)$$

We lump together the eclipse terms in $\sigma^e_{C,diss}$ and obtain after some calculations

$$\sigma_{C,diss} = \sigma^0_{C,diss} + \sigma^e_{C,diss} \quad , \quad (7.10)$$

$$\sigma^0_{C,diss} = \frac{\sigma^2}{4} \int d^2b\, e^{-2\sigma T(\bar{b})} \cdot \left[\int d^2s\, |\varphi(\bar{s})|^2 \{\delta T_2(\bar{b},\bar{s})\}^2 - \{ \int d^2s\, |\varphi(\bar{s})|^2\, \delta T_2(\bar{b},\bar{s}) \}^2 \right] , \quad (7.11)$$

$$\sigma^{e}_{C,diss} = \sigma \frac{\eta}{4} \frac{1}{1+R_d^2/4a} M_2(2\sigma) + \sigma \frac{\eta^2}{16} N_2(2\sigma) \left[\frac{1}{1+R_d^2/2a} - \frac{1}{[1+R_d^2/4a]^2} \right] . \quad (7.12)$$

In evaluating (7.12) we have used the fact that when δT_2 and eclipse terms appear on the same time in a deuteron integral the result is zero. Putting in numerical values for the effective nucleon numbers we find that $\sigma^{e}_{C,diss}$ is about 4 mb for Pb and 1 mb for Al, indeed a small contribution.

We shall now try to give a reasonable estimate of $\sigma^{o}_{C,diss}$. In $\delta T_2(\bar{b},\bar{s})$ we then keep the first term in the MacLaurin expansion

$$\delta T_2(\bar{b},\bar{s}) = \frac{1}{4} (\bar{s}\bar{\nabla})^2 T(\bar{b}) \quad . \quad (7.13)$$

The second term in (7.9) has already been calculated in this approximation, but in the first one we meet a new kind of average, namely

$$\int d^2s \, |\varphi(\bar{s})|^2 \cdot \left[(\bar{s}\bar{\nabla})^2 T(\bar{b}) \right]^2 = \langle \bar{s}^4 \rangle \cdot \frac{1}{4} \left[\{\Delta T(\bar{b})\}^2 + \frac{1}{2} \left\{ \frac{d^2T(\bar{b})}{db^2} + \frac{1}{b} \cdot \frac{dT(\bar{b})}{db} \right\}^2 \right] . \quad (7.14)$$

In our Gaussian wave function

$$\langle \bar{s}^4 \rangle = 2 R_d^4 \quad . \quad (7.15)$$

If we introduce a new effective nucleon number

$$\delta M_0(\sigma) = \sigma \left(\frac{R_d^2}{16} \right)^2 \int d^2b \, e^{-\sigma T(\bar{b})} \left[\left\{ \frac{d^2T(\bar{b})}{db^2} \right\}^2 + \left\{ \frac{1}{b} \cdot \frac{dT(\bar{b})}{db} \right\}^2 \right] , \quad (7.16)$$

we can write

$$\sigma^{o}_{C,diss} = \sigma \cdot \delta M_0(2\sigma) \quad . \quad (7.17)$$

Numerical values of $\delta M_0(2\sigma)$ are given in the Appendix. They clearly indicate that the coherent dissociation is not completely negligible as $\delta M_0(2\sigma)$ is about 0.4 for Al and raises to about 1.3 for Pb. Therefore the eclipse correction as given by (7.12) is very small, being only 6% of $\sigma^{o}_{C,diss}$. A word of caution is also necessary. The coherent dissociation cross-section comes out as a difference between two big numbers and becomes very sensitive to the deuteron wave function. Use of a more realistic wave function can therefore affect our predictions.

The diffractive dissociation due to the Coulomb interaction was calculated by Akhieser and Sitenko [4]. At high energies they found it to be much smaller than the coherent dissociation.

8. - INCOHERENT DIFFRACTIVE DISSOCIATION

In an incoherent dissociation process the deuteron disintegrates into a proton and a neutron and at the same time the nucleus changes its state. Such processes are expected to be more important than the corresponding coherent processes and their magnitude is more easily estimated in our approximation scheme. The method to obtain the incoherent dissociation cross-section is similar to the one used for the coherent case. We first calculate the sum of the incoherent dissociation and incoherent elastic cross-sections by means of the closure approximation. Then we subtract the incoherent elastic part to obtain the incoherent dissociation part.

The amplitude for a process where the deuteron state changes from $|d\rangle$ to $|d'\rangle$ and the target state from $|t\rangle$ to $|t'\rangle$ is according to (2.1)

$$F_{d't',dt}(\bar{q}) = \frac{ik_d}{2\pi}\int d^2b\, e^{i\bar{q}\bar{b}} \langle d't'|\{1-\exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\dots,\bar{s}_A)]\}|dt\rangle . \qquad (8.1)$$

We now sum over final deuteron states and all target states different from the ground state and get after integration

$$\sigma_{I,diss} + \sigma_{I,el} =$$

$$\sum_{d't'} \int d^2b \langle dt | \{1-\exp[-i\chi^*_{tot}(\bar{b},\bar{s},\bar{s}'_1,\ldots,\bar{s}'_A)]\} | d't' \rangle \langle d't' | \{1-\exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)]\} | dt \rangle \tag{8.2}$$

$$-\sum_{d'} \int d^2b \langle dt | \{1-\exp[-i\chi^*_{tot}(\bar{b},\bar{s}',\bar{s}'_1,\ldots,\bar{s}'_A)]\} | d't \rangle \langle d't | \{1-\exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)]\} | dt \rangle$$

Applying the closure relation for the deuteron states (7.3) as well as for the nuclear states we obtain

$$\sigma_{I,diss} + \sigma_{I,el} =$$

$$\int d^2b \langle dt | \{1-\exp[-i\chi^*_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)]\} \{1-\exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)]\} | dt \rangle \tag{8.3}$$

$$-\int d^2b \langle dt | \{1-\exp[-i\chi^*_{tot}(\bar{b},\bar{s},\bar{s}'_1,\ldots,\bar{s}'_A)]\} | t \rangle \langle t | \{1-\exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)]\} | dt \rangle \quad .$$

Here terms linear in $\exp[i\chi_{tot}]$ cancel due to the normalization property of the nuclear wave functions yielding

$$\sigma_{I,diss} = \sigma_0 - \sigma_{I,el} \quad , \tag{8.4}$$

$$\sigma_0 = \int d^2b \, [\langle dt | \exp[-i\chi^*_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)] \cdot \exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)] | dt \rangle$$

$$- \langle dt | \exp[-i\chi^*_{tot}(\bar{b},\bar{s},\bar{s}'_1,\ldots,\bar{s}'_A)] | t \rangle \langle t | \exp[i\chi_{tot}(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A)] | dt \rangle$$

$$= \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \exp[Z^*_1(\bar{b},\bar{s}) + Z_1(\bar{b},\bar{s})] \cdot \{\exp[Z_2(\bar{b},\bar{s};\bar{b},\bar{s})] - 1\} , \tag{8.5}$$

where in the last step use was made of the definitions (3.2) and (5.6).

The incoherent elastic cross-section $\sigma_{I,el}$ was calculated in Section 6 and the calculation of σ_0 of (8.5) is similar, but simpler in the respect that only one deuteron integration is present. Therefore we shall here confine ourselves to listing the contributions to σ_0 bigger than 2%. The notations for the partial cross-sections are the same as in Section 6. We get

A - First order in Z_2. Zero order in eclipse terms.

$$\sigma_{11} = \sigma \cdot \frac{\eta}{2} \left[\Sigma_1 + N_1(2\sigma) \cdot \frac{1}{1+R_d^2/4a} \right] , \qquad (8.6)$$

$$\Sigma_1 = \frac{1}{2} \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot T_2(\bar{b},\bar{s}) \, e^{-\sigma T_2(\bar{b},\bar{s})}$$

$$\approx N_1(2\sigma) + M_1(2\sigma) - M_2(2\sigma) , \qquad (8.7)$$

$$\sigma_{12+21} = -\sigma \cdot \frac{2}{3} \eta^2 \cdot \frac{1}{1+R_d^2/3a} \cdot N_1(2\sigma) , \qquad (8.8)$$

$$\sigma_{22} = \sigma \, \frac{\eta^3}{8} \cdot \frac{1}{1+R_d^2/2a} \cdot N_1(2\sigma) . \qquad (8.9)$$

B - First order in Z_2. First order in eclipse terms.

$$\sigma_{11}^{e} = \sigma \cdot \frac{\eta}{4} \cdot N_2(2\sigma) \cdot \left[\frac{1}{1+R_d^2/4a} + \frac{1}{1+R_d^2/2a} \right] , \qquad (8.10)$$

$$\sigma_{12+21}^{e} = -\sigma \, \frac{\eta^3}{3} \cdot \frac{1}{1+7R_d^2/12a} \cdot N_2(2\sigma) . \qquad (8.11)$$

C - Second order in Z_2. Zero order in eclipse terms.

$$\sigma_{11\times 11} = \sigma \, \frac{\eta^2}{8} \left[\Sigma_2 + N_2(2\sigma) \left\{ \frac{2}{1+R_d^2/4a} + \frac{1}{1+R_d^2/2a} \right\} \right] , \qquad (8.12)$$

$$\Sigma_2 = \frac{\sigma}{4} \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \{ T_2(\bar{b},\bar{s}) \}^2 \, e^{-\sigma T_2(\bar{b},\bar{s})}$$

$$\approx N_2(2\sigma) + M_2(2\sigma) - M_3(2\sigma) , \qquad (8.13)$$

$$\sigma_{11\times(12+21)} = -\sigma \, \frac{\eta^3}{3} \, N_2(2\sigma) \cdot \left[\frac{1}{1+R_d^2/3a} + \frac{1}{1+7R_d^2/12a} \right] , \qquad (8.14)$$

$$\sigma_{(12+21)\times(12+21)} = \sigma \, \frac{2}{9} \, \eta^4 \, \frac{1}{1+2R_d^2/3a} \, N_2(2\sigma) . \qquad (8.15)$$

D - Second order in Z_2. First order in eclipse terms.

$$\sigma^{e}_{11\times 11} = \sigma \frac{3}{32} \eta^3 \left[\frac{1}{1+R_d^2/4a} + \frac{2}{1+R_d^2/2a} + \frac{1}{1+3R_d^2/4a} \right] \cdot N_3(2\sigma) \cdot \quad (8.16)$$

E - Third order in Z_2. Zero order in eclipse terms.

$$\sigma_{11\times 11\times 11} = \sigma \frac{\eta^3}{32} \left[\Sigma_3 + \left\{ \frac{3}{1+R_d^2/4a} + \frac{3}{1+R_d^2/2a} + \frac{1}{1+3R_d^2/4a} \right\} N_3(2\sigma) \right] \quad (8.17)$$

$$\Sigma_3 = \frac{\sigma^2}{12} \int d^2b \int d^2s \, |\varphi(\vec{s})|^2 \cdot \{T_2(\vec{b},\vec{s})\}^3 e^{-\sigma T_2(\vec{b},\vec{s})}$$

$$\approx N_3(2\sigma) + M_3(2\sigma) - M_4(2\sigma) \quad . \quad (8.18)$$

The dominant term is of course σ_{11}. Next in magnitude we have σ_{12+21} and $\sigma_{11\times 11}$. Each of these terms is about 15% of σ_{11}. As they appear with opposite sign they tend to cancel each other leaving a rest of only a few per cent. Going down in magnitude we next find σ^{e}_{11} and $\sigma_{11\times(12+21)}$, each about 6%. Also these terms have opposite sign and therefore give a very small rest. All the remaining terms are 3% or smaller. In conclusion we find that including all the terms listed the correction to the leading term σ_{11} is in fact only about 10%.

We are now in a position to give the incoherent diffractive dissociation cross-section. According to our formula (8.4) we have to subtract $\sigma_{I,el}$ from σ_o. We will certainly not give all the terms obtained but only the leading one

$$\sigma_{I,diss} = \sigma \frac{\eta}{2} N_1(2\sigma) \left\{ 1 - \frac{2}{1+R_d^2/8a} + \frac{1}{1+R_d^2/4a} \right\} \quad . \quad (8.19)$$

In particular we note that an appreciable part of σ_o survives also after the subtraction of $\sigma_{I,el}$. If this would not have been the case our approximation method would have been rather useless.

Comparing our predictions for the incoherent and coherent dissociation cross-sections we see that the coherent one increases more rapidly with atomic number than does the incoherent one. For light nuclei the incoherent dissociation cross-section is almost twice as large as the coherent one whereas for heavy nuclei the coherent dissociation cross-section is slightly bigger than the incoherent one.

9. - COHERENT PRODUCTION AND STRIPPING

We shall now make some comments on coherent production processes. We shall then consider the case where, e.g., the proton causes the production whereas the neutron is scattered elastically. In (2.22) we used Γ_{px} to denote the profile function for production on a single nucleon, and in (2.25) we gave the total profile function for such reactions. In a coherent production where the nuclear state does not change we thus obtain the production amplitude

$$F_{ii}(\bar{q}) = \frac{ik_d}{2\pi} \sum_{j=1}^{A} \int d^2b\, e^{i\bar{q}\bar{b}} \langle d' t | e^{iq_L(z_j - \frac{z}{2})} \Gamma_{px}(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_j) \cdot [1 - \Gamma_n(\bar{b} - \tfrac{\bar{s}}{2} - \bar{s}_j)] \cdot$$

$$\cdot \prod_{t \neq j} [1 - \Gamma_n(\bar{b} - \tfrac{\bar{s}}{2} - \bar{s}_t)] \cdot [1 - \Gamma_p(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_j) \cdot \theta(z_j - z_t) \qquad (9.1)$$

$$- \Gamma_x(\bar{b} + \tfrac{\bar{s}}{2} - \bar{s}_j) \cdot \theta(z_t - z_j)] | d t \rangle .$$

Here $|d'\rangle$ is the final x neutron state. When final state interactions can be neglected we can use the simple plane wave approximation

$$\varphi'(\bar{x}) = \frac{1}{(2\pi)^{3/2}} \exp[-i\bar{x}(\bar{k}_n - \bar{k}_x)/2] \qquad (9.2)$$

As long as we do not distinguish between the different nucleon-nucleon amplitudes and use separable wave functions for the nucleus then all terms in (9.1) are in fact equal. Due to the complications introduced by the longitudinal momentum transfer and the θ functions the integrations over the nuclear wave functions cannot be reduced to target thickness functions. Only the integrations over the transverse part of the nucleon variables can be performed. In the usual approximation we get for the production factor in (9.1)

$$f(\bar{b},\bar{s},z_j) \equiv \int d^2s_j \, |\Psi(\bar{x}_j)|^2 \cdot \Gamma_{px}(\bar{b}+\tfrac{\bar{s}}{2}-\bar{s}_j)\cdot[1-\Gamma_n(\bar{b}-\tfrac{\bar{s}}{2}-\bar{s}_j)]$$

$$\approx 2\pi a_{px}\eta_{px}\Big\{\varrho(\bar{b}+\tfrac{\bar{s}}{2},z_j) - \frac{a}{a+a_{px}}\eta\cdot\varrho(\bar{b}+\tfrac{\bar{s}}{2}\cdot\frac{a-a_{px}}{a+a_{px}},z_j)\exp[-\frac{\bar{s}^2}{2(a+a_{px})}]\Big\} \quad (9.3)$$

Similar expressions arise from the elastic scattering factors. In order to get manageable expressions we define

$$-\Sigma_{nx}(\bar{b},\bar{s},z_t) \equiv \int d^2s_t\,|\Psi(\bar{r}_t)|^2\,\{[1-\Gamma_n(\bar{b}-\tfrac{\bar{s}}{2}-\bar{s}_t)]\cdot[1-\Gamma_x(\bar{b}+\tfrac{\bar{s}}{2}-\bar{s}_t)]-1\} \quad (9.4)$$

which yields

$$\Sigma_{nx}(\bar{b},\bar{s},z_t) = 2\pi a_x\eta_x\cdot\varrho(\bar{b}+\tfrac{\bar{s}}{2},z_t) + 2\pi a\eta\cdot\varrho(\bar{b}-\tfrac{\bar{s}}{2},z_t)$$
$$-2\pi\frac{a a_x}{a+a_x}\cdot\eta\eta_x\cdot\varrho(\bar{b}+\tfrac{\bar{s}}{2}\cdot\frac{a-a_x}{a+a_x},z_t)\cdot\exp[-\frac{\bar{s}^2}{2(a+a_x)}] \quad . \quad (9.5)$$

We also introduce the symbol $\Sigma_{np}(z_t)$, which is obtained from (9.5) by the replacement $a_x \to a$, $\eta_x \to \eta$. Our amplitude (9.1) now takes the compact form

$$F_{ii}(\bar{q}) = \frac{ik_\ell}{2\pi}\int d^2b\, e^{i\bar{q}\bar{b}}\int d^3r\,\varphi'^*(\bar{r})\varphi(\bar{r})\int_{-\infty}^{\infty}dz_j\, e^{iq_L(z_j-\frac{z}{2})}\, A\, f(\bar{b},\bar{s},z_j)\cdot$$
$$\cdot\exp[-(A-1)\int_{-\infty}^{z_j}\Sigma_{np}(\bar{b},\bar{s},z_t)\,dz_t - (A-1)\int_{z_j}^{\infty}\Sigma_{nx}(\bar{b},\bar{s},z_t)\,dz_t\,] \quad . \quad (9.6)$$

This expression is rather complicated. It must also be stressed that it applies only to a very limited number of production reactions. We shall not try to analyze it further. Instead we discuss a few special cases.

In order to exhibit the connection with proton-nuclear collisions we neglect all eclipse terms. If we furthermore neglect the difference between A-1 and A we obtain

$$F_{ii}(\bar q) \simeq \frac{ik_d}{2\pi}\int d^2b\, e^{i\bar q\bar b}\int d^3r\,\varphi'^*(\bar r)\varphi(\bar r)\int_{-\infty}^{\infty} dz_j\, e^{iq_L(z_j-\frac{z}{2})}\, A\cdot 2\pi a_{px}\eta_{px}\cdot$$

$$\cdot\rho(\bar b+\tfrac{\bar s}{2}, z_j)\cdot \exp[-2\pi a\eta\, T(\bar b-\tfrac{\bar s}{2})]\cdot \tag{9.7}$$

$$\cdot \exp[-A\cdot 2\pi a\eta\int_{-\infty}^{z_j} dz_t\,\rho(\bar b+\tfrac{\bar s}{2}, z_t) - A\cdot 2\pi a_x\eta_x\int_{z_j}^{\infty} dz_t\,\rho(\bar b+\tfrac{\bar s}{2}, z_t)\,] \quad .$$

The origin of the different factors is easy to trace through their dependence on the impact parameters. The connection with proton-nuclear collisions is also clear. The factors containing the proton impact parameter are exactly those encountered in the proton-nuclear case. The new elements are the last exponential describing the simultaneous elastic scattering of the neutron, and of course, the weighting over the deuteron wave function. It is the scattering of the neutron which distorts the neutron momentum distribution. Only the longitudinal component is unaffected by the interaction. Qualitatively, it is clear that the transverse momentum distribution will be narrower than in the neutron spectator model, because stripping is larger for deuteron configurations with large $\bar s$ than for configurations with small $\bar s$, since for large $\bar s$ the probability that one particle collides while the other one escapes is larger. But large $\bar s$ in configuration space implies small $\bar k_n$ in momentum space.

For high energy processes where the longitudinal momentum transfer can be neglected an additional simplification arises. The integration over z_j in (9.7) can be performed explicitly and yields

$$F_{ii}(\bar q) = \frac{ik_d}{2\pi}\int d^2b\, e^{i\bar q\bar b}\int d^3r\,\varphi'^*(\bar r)\varphi(\bar r)\cdot\frac{f_{px}(0)}{f_p(0)-f_x(0)}\cdot$$

$$\cdot\left\{\exp[-2\pi a_x\eta_x T(\bar b+\tfrac{\bar s}{2})] - \exp[-2\pi a\eta\, T(\bar b+\tfrac{\bar s}{2})]\right\}\cdot \exp[-2\pi a\eta\, T(\bar b-\tfrac{\bar s}{2})] \quad . \tag{9.8}$$

Of course, also this amplitude has a direct counterpart in proton-nuclear collisions.

Finally we consider the case $\Gamma_p(\bar b) = \Gamma_x(\bar b)$, i.e., $a_x = a$, $\eta_x = \eta$. This is not an entirely unrealistic case if we remember that x is an excited nucleon. The exponential in (9.6) now reduces to the well known $Z_1(\bar b, j)$ of (3.5) and we get

$$F_{ii}(\vec{q}) \simeq \frac{ik_d}{2\pi} \int d^2b\, e^{i\vec{q}\vec{b}} \int d^3r\, \varphi'^*(\vec{r})\, \varphi(\vec{r}) \int_{-\infty}^{\infty} dz_j\, e^{iq_L(z_j - \frac{z}{2})} \cdot$$
$$\cdot A\, f(\vec{b},\vec{s},z_j) \exp[Z_1(\vec{b},\vec{s})] \quad . \tag{9.9}$$

When the longitudinal momentum transfer can be neglected the integration over z_j gives target thickness functions. The zero order approximation becomes very simple

$$F_{ii}(\vec{q}) \simeq \frac{ik_d}{2\pi} 2A f_{px}(0) \int d^2b\, e^{i\vec{q}\vec{b}} \int d^3r\, \varphi'^*(\vec{r})\, \varphi(\vec{r}) \cdot$$
$$\cdot T(\vec{b}+\frac{\vec{s}}{2}) \cdot \exp[-2\pi a \eta\, T_2(\vec{b},\vec{s})] \quad , \tag{9.10}$$

which is nothing else but the limit $\eta_x \to \eta$, $a_x \to a$ of (9.8).

The sum of all coherent production processes of the type $dt \to xnt$ constitutes the coherent stripping. This summation is difficult to perform in our approach for several reasons. Firstly, the amount of coherence depends on the longitudinal momentum transfer in the inelastic p nucleon collision. Only for $q_L \cdot R_t \ll 1$ coherence will be complete and this will not be the case for all production reactions. Secondly, we have several times used the requirement that the differential production cross-sections are steep, which again is true only for a limited number of processes.

10. - INCOHERENT PRODUCTION AND STRIPPING

Large angle incoherent production cross-sections are considerably simplified when summed over final states of the target. The production can then be considered as taking place incoherently from the individual target nucleons. Here we shall only treat a simplified case and leave the general one to the interested reader. Thus we neglect the longitudinal momentum transfer and assume equal elastic scattering in the initial and final states, i.e., $\eta_x = \eta$, $a_x = a$, where η_x

and a_x refer to the produced particle. Then the amplitude for a production process where the nuclear state changes from $|t\rangle$ to $|t'\rangle$ will be

$$F_{fi}(\bar{q}) = \frac{ik_d}{2\pi} \sum_{j=1}^{A} \int d^2b \, e^{i\bar{q}\bar{b}} \langle d't' | \Gamma_j(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A) | dt \rangle \quad , \tag{10.1}$$

$$\begin{aligned} \Gamma_j(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A) = \Gamma_{px}(\bar{b}+\tfrac{\bar{s}}{2}-\bar{s}_j) \cdot [1-\Gamma_n(\bar{b}-\tfrac{\bar{s}}{2}-\bar{s}_j)] \cdot \\ \cdot \prod_{t \neq j} [1-\Gamma_n(\bar{b}-\tfrac{\bar{s}}{2}-\bar{s}_t)] \cdot [1-\Gamma_p(\bar{b}+\tfrac{\bar{s}}{2}-\bar{s}_t)] \quad . \end{aligned} \tag{10.2}$$

Summing over all final target states and using the closure relation (5.2) we obtain

$$\begin{aligned} \sum_f |F_{fi}(\bar{q})|^2 = \frac{k_d^2}{4\pi^2} \sum_{j=1}^{A} \sum_{j'=1}^{A} \int d^2b \int d^2b' \, e^{i\bar{q}(\bar{b}-\bar{b}')} \quad . \\ \cdot \langle dt | \Gamma_{j'}^{*}(\bar{b}',\bar{s}',\bar{s}_1,\ldots,\bar{s}_A) | d' \rangle \langle d' | \Gamma_j(\bar{b},\bar{s},\bar{s}_1,\ldots,\bar{s}_A) | dt \rangle \quad . \end{aligned} \tag{10.3}$$

Here the terms with $j = j'$ constitute the incoherent production cross-section. Due to our assumption of equal elastic scattering in the initial and final states, the matrix elements in (10.3) factorize in the nuclear variables, and all the terms in the sum become identical. Neglecting simultaneous production and scattering on the same target nucleon, the sum comes down to

$$\begin{aligned} \sum_t \int d^3r_t \, \rho(\bar{r}_t) \, \Gamma_{px}^{*}(\bar{b}'-\bar{s}_t) \, \Gamma_{px}(\bar{b}-\bar{s}_t) \simeq \frac{1}{k_p^2} \cdot T\left(\frac{\bar{b}+\bar{b}'}{2}\right) \int d^2\beta \, |f_{px}(\bar{\beta})|^2 \, e^{-i\bar{\beta}(\bar{b}-\bar{b}')} \\ = T\left(\frac{\bar{b}+\bar{b}'}{2}\right) \cdot \sigma_{px} \cdot \exp[-(\bar{b}-\bar{b}')^2/4a_{px}] \quad , \end{aligned} \tag{10.4}$$

where the last step is valid only for Gaussian amplitudes (2.21). The elastic scattering part of the nuclear matrix element in (10.3) was already evaluated in (5.6). When we neglect the difference between (A-1) and A the final formula reads

$$\begin{aligned} \frac{d\sigma^I}{d\Omega} = \frac{1}{\pi^2} \int d^2b \, d^2b' \, e^{i\bar{q}(\bar{b}-\bar{b}')} \int d^3r_t \, \varphi'^{*}(\bar{r}) \varphi(\bar{r}) \int d^3r' \, \varphi'^{*}(\bar{r}') \varphi(\bar{r}') \cdot T\left(\frac{\bar{b}+\bar{b}'}{2} + \frac{\bar{s}+\bar{s}'}{4}\right) \cdot \\ \cdot \exp[Z_1^{*}(\bar{b}',\bar{s}') + Z_1(\bar{b},\bar{s}) + Z_2(\bar{b},\bar{s};\bar{b}',\bar{s}')] \cdot \int d^2\beta \, |f_{px}(\bar{\beta})|^2 \, e^{-i\bar{\beta}(\bar{b}-\bar{b}'+\frac{\bar{s}-\bar{s}'}{2})} \end{aligned} \tag{10.5}$$

The natural way to continue is to perform an expansion in Z_1 and Z_2 but we shall not do this. Instead we want to discuss the total stripping cross-section. As the approach developed above only applies to individual stripping reactions it is not a convenient starting point and we prefer to use probability considerations.

The probability for the neutron to traverse the nucleus at impact parameter $\bar{b}_n = \bar{b} - \bar{s}/2$ without colliding in $\exp[-\sigma T(\bar{b}_n)]$. The probability for the proton to collide an impact parameter $\bar{b}_p = \bar{b} + \bar{s}/2$ is $(1-\exp[-\sigma T(\bar{b}_p)])$. The cross-section for stripping is then given by the product of these two probabilities properly weighted over the deuteron wave function. However, this argument does not take into account the intertwining of proton and neutron interactions caused by the eclipse term. When this is done we get

$$\begin{aligned} \sigma_{n,strip} &= \int d^2b \int d^3s \, |\varphi(\bar{s})|^2 e^{-\sigma T(\bar{b}_n)} \left\{1 - \exp\left[-\sigma T(\bar{b}_p) + \frac{\sigma \eta}{2} T(\bar{b}) e^{-\bar{s}^2/4a}\right]\right\} \\ &= \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \left\{ e^{-\sigma T(\bar{b})} - e^{-2Z_1(\bar{b},\bar{s})} \right\} \qquad (10.6) \\ &\simeq \sigma\left[2N_0(2\sigma) - N_0(\sigma) + 2M_1(2\sigma)\right] - \sigma\frac{\eta}{2} \cdot \frac{1}{1+R_d^2/4a} \cdot N_1(2\sigma) \; . \end{aligned}$$

Later on we shall show that this is the appropriate way to incorporate the eclipse correction. The result for proton stripping is of course the same.

When comparing with experimental data at 3.54 GeV/c we are still in a region where the difference between σ_{pp} and σ_{pn} is appreciable, and we would like to take it into account. As the resulting correction turns out to be very small we only consider the non-ecliptic term, the ecliptic term being itself small. We put

$$\sigma_{pp} = \sigma_{nn} = \sigma + \delta\sigma \qquad (10.7)$$

$$\sigma_{pn} = \sigma - \delta\sigma \; . \qquad (10.8)$$

For a nucleus with N_n neutrons and N_p protons the average proton-nucleus and neutron-nucleus cross-sections become

$$\sigma_p = \sigma - \frac{N_n - N_p}{N_n + N_p} \cdot \delta\sigma \qquad , \tag{10.9}$$

$$\sigma_n = \sigma + \frac{N_n - N_p}{N_n + N_p} \cdot \delta\sigma \qquad . \tag{10.10}$$

The non-ecliptic contribution to (10.6) is then modified to

$$\sigma'_{n,strip} = \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \, e^{-\sigma_n T(\bar{b}_n)} \left\{1 - e^{-\sigma_p T(\bar{b}_p)}\right\} \quad . \tag{10.11}$$

Now, as $\sigma_p + \sigma_n = 2\sigma$ we can write

$$\sigma'_{n,strip} \simeq \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \left\{1 - e^{-\sigma T_2(\bar{b},\bar{s})}\right\} - \sigma_n N_0(\sigma_n) \quad , \tag{10.12}$$

where the neglected contribution is extremely small. The correction to (10.6) therefore comes from the last term of (10.12). Expanding in the difference $\sigma_n - \sigma$ we obtain

$$\begin{aligned} \delta\sigma_{n,strip} &= - \frac{N_n - N_p}{N_n + N_p} \cdot \delta\sigma \cdot N_1(\sigma) \\ &= - \delta\sigma_{p,strip} \quad . \end{aligned} \tag{10.13}$$

For $\delta\sigma$ = 3.5 mb the corrections are 0.6, 1.9 and 5.7 mb for ^{27}Al, ^{64}Cu and ^{208}Pb respectively, They are included in the values of Table I, which were calculated with σ = 45 mb.

The cross-section for collisions where both proton and neutron collide can be obtained by similar arguments. The probability that both collide is obviously $(1-\exp[-\sigma T(\bar{b}_n)]) \cdot (1-\exp[-\sigma T(\bar{b}_p)])$. Correcting for eclipse contributions we get

$$\sigma_{n,p} = \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \left\{ 1 - e^{-\sigma T(\bar{b}_n)} - e^{-\sigma T(\bar{b}_p)} + e^{-\sigma T_2(\bar{b},\bar{s}) + \frac{\sigma \eta}{2} T(\bar{b}) e^{-\bar{s}^2/4a}} \right\}$$

$$= \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \left\{ 1 - 2e^{-\sigma T(\bar{b})} + e^{2Z_1(\bar{b},\bar{s})} \right\} \qquad (10.14)$$

$$\simeq 2\sigma \left[N_0(\sigma) - N_0(2\sigma) - M_1(2\sigma) \right] + \sigma \frac{\eta}{2} \frac{1}{1 + R_d^2/4a} \cdot N_1(2\sigma) \quad .$$

The sum of all inelastic reactions becomes

$$\sigma_{n,strip} + \sigma_{p,strip} + \sigma_{n,p} = \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \left\{ 1 - e^{2Z_1(\bar{b},\bar{s})} \right\}$$

$$\simeq 2\sigma \left[N_0(2\sigma) + M_1(2\sigma) \right] - \sigma \frac{\eta}{2} \frac{1}{1 + R_d^2/4a} N_1(2\sigma) \quad . \qquad (10.15)$$

We now remark that it is possible to test our probability arguments. If we add the elastic and dissociation cross-sections to the total reaction cross-section (10.15) we should recover the deuteron nucleus total cross-section. By adding (3.15) and (7.5) we get the coherent cross-section

$$\sigma_{c,el} + \sigma_{c,diss} = \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \left| 1 - e^{Z_1(\bar{b},\bar{s})} \right|^2$$

$$\simeq 2\sigma \left[N_0(\sigma) - N_0(2\sigma) \right] + 2\sigma \left[M_1(\sigma) - M_1(2\sigma) \right] \qquad (10.16)$$

$$- \sigma \frac{\eta}{2} \frac{1}{1 + R_d^2/4a} \cdot \left[N_1(\sigma) - N_1(2\sigma) \right] \quad .$$

The corresponding incoherent cross-section can be obtained directly from (8.5)

$$\sigma_{I,el} + \sigma_{I,diss} = \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \, e^{Z_1^*(\bar{b},\bar{s}) + Z_1(\bar{b},\bar{s})} \left\{ e^{Z_2(\bar{b},\bar{s};\bar{b},\bar{s})} - 1 \right\} , \qquad (10.17)$$

but is already included in (10.15) as we there used total nucleon-nucleon cross-sections and not pure collision cross-sections. Adding up all contributions we get

$$\sigma_{n,strip} + \sigma_{p,strip} + \sigma_{n,p} + \sigma_{c,el} + \sigma_{c,diss} =$$

$$= 2 \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot \left\{ 1 - e^{-Z_1(\bar{b},\bar{s})} \right\} \qquad (10.18)$$

$$\simeq 2\sigma \left[N_0(\sigma) + M_1(\sigma) \right] - \sigma \frac{\eta}{2} \frac{1}{1 + R_d^2/4a} N_1(\sigma) \quad .$$

Indeed the same result was found for the total cross-section in (4.2) and (4.7) as calculated from the optical theorem. We conclude that the optical theorem is valid not only for the over-all result but also for fixed impact parameter $\bar{b}$.

11. - ELASTIC PRODUCTION

With elastic production we mean reactions with an outgoing deuteron and one or more produced mesons. Since the deuteron is only loosely bound such reactions will be rare, but nevertheless may have a certain interest.

We first consider reactions which are caused by one of the nucleons in the deuteron. For such reactions the amplitude for coherent production is easily obtained from the considerations in Section 9. In (9.9) we just change the neutron-proton scattering state φ' into the deuteron wave function. In this formula we also neglect the term arising from simultaneous production and scattering on a target nucleon. This yields

$$F_c(\bar{q}) = 2i\,[f_{px}(0)+f_{nx}(0)]\cdot\int d^2b\; e^{i\bar{q}\bar{b}}\int d^2s\,|\varphi(\bar{s})|^2 \;\cdot$$
$$\cdot\int_{-\infty}^{\infty} dz'\; e^{iq_L(z'-\frac{z}{2})}\cdot A_\rho\left(\bar{b}+\frac{\bar{s}}{2},z'\right)\cdot \exp[Z_1(\bar{b},\bar{s})] \;. \qquad (11.1)$$

Here we have used the symmetry property $Z_1(\bar{b},s) = Z_1(\bar{b},-\bar{s})$ to transform the neutron collision amplitude into the proton collision amplitude. The result (11.1) is of course the impulse approximation prediction and, in particular, gives the selection rules predicted by this approximation.

A formula for the corresponding incoherent production is also easily established. In analogy with (10.4) we get after some calculations

$$\frac{d\sigma^I}{d\Omega} = \frac{1}{\pi^2}\int d^2b\int d^2b'\int d^2s\,|\varphi(\bar{s})|^2\int d^2s'\,|\varphi(\bar{s}')|^2\cdot T\left(\frac{\bar{b}_p+\bar{b}'_p}{2}\right) e^{i\bar{q}(\bar{b}-\bar{b}')}\;\cdot$$
$$\cdot \exp[Z_1^*(\bar{b}',\bar{s}')+Z_1(\bar{b},\bar{s})+Z_2(\bar{b},\bar{s};\bar{b}',\bar{s}')]\int d^2\beta\,|f_{px}(\bar{\beta})+f_{nx}(\bar{\beta})|^2 e^{i\bar{\beta}(\bar{b}_p-\bar{b}'_p)} \;. \qquad (11.2)$$

where we used the notation $\bar{b}_p = \bar{b} + \bar{s}/2$ and $\bar{b}_p' = \bar{b}' + \bar{s}'/2$. To arrive at this result we have several times used the symmetry of Z_1 and Z_2 under transformations $\bar{s} \to -\bar{s}$ and $\bar{s}' \to -\bar{s}'$. The canonical way to proceed is now to expand in the eclipse term of Z_1 and Z_2. As the calculations are straightforward we leave them to the interested reader and instead turn our attention to double particle production.

Double particle production reactions with outgoing deuterons are also possible. Of course, when the proton has produced two pions, say, the probability that it holds together with the neutron is quite small. On the other hand when the proton and the neutron produce one pion each they may very well hold together if the momentum transfers obtained are not too different. This process is analogous to the flat contribution to elastic scattering [cf. (5.27), (5.39) and (5.46)]. The actual derivation of the amplitude is similar to the derivation of (11.2). When we neglect terms of order $1/A$ we have

$$\begin{aligned}\frac{d\sigma^I}{d\Omega} = \frac{k_d^2}{4\pi^2} \int d^2b \int d^2b'\, e^{i\bar{q}(\bar{b}-\bar{b}')} \int d^2s\, |\varphi(\bar{s})|^2 \int d^2s'\, |\varphi(\bar{s}')|^2 \cdot \\ \cdot \exp[Z_1^*(\bar{b}',\bar{s}') + Z_1(\bar{b},\bar{s}) + Z_2(\bar{b},\bar{s};\bar{b}',\bar{s}')] \cdot T\left(\frac{\bar{b}_p+\bar{b}_p'}{2}\right) \cdot T\left(\frac{\bar{b}_n+\bar{b}_n'}{2}\right) \\ \cdot \Big[\int d^2q\, e^{i\bar{q}(\bar{b}_p'-\bar{b}_p)} \cdot |f_{px}(\bar{q})|^2 \int d^2q'\, e^{i\bar{q}'(\bar{b}_n'-\bar{b}_n)}\, |f_{nx}(\bar{q}')|^2 \\ + \int d^2q\, e^{i\bar{q}(\bar{b}_p'-\bar{b}_p)} f_{px}^*(\bar{q})\, f_{nx}(\bar{q}) \cdot \int d^2q'\, e^{i\bar{q}'(\bar{b}_n'-\bar{b}_n)} f_{nx}^*(\bar{q}') f_{px}(\bar{q}')\Big] ,\end{aligned} \qquad (11.3)$$

where the notations $\bar{b}_p = \bar{b} + \bar{s}/2$, $\bar{b}_n = \bar{b} - \bar{s}/2$, etc., are used.

In order to learn something from this complicated expression we consider production amplitudes of the form (2.21), i.e.,

$$f_{px}(\bar{q}) = f_{nx}(\bar{q}) = k_p \cdot a_{px} \cdot \eta_{px}\, e^{-a_{px}\cdot \bar{q}^2/2} , \qquad (11.4)$$

$$\sigma_{px} = \pi a_{px} |\eta_{px}|^2 . \qquad (11.5)$$

We also neglect eclipse terms. For this case the integrations over $\bar{b}'$ and $\bar{s}'$ are immediately performed in the usual approximation. We get

$$\frac{d\sigma^{I}}{d\Omega} = k_d^2 \, \frac{2\sigma_{px}^2}{\sigma} \, \frac{1}{1+R_d^2/4a_{px}} \, \underline{P} \cdot \frac{a_{px}}{2\pi} \, e^{-a_{px}\bar{q}^2/2} \quad , \tag{11.6}$$

$$\begin{aligned} \underline{P} &= \sigma \int d^2b \int d^2s \, |\varphi(\bar{s})|^2 \cdot T\left(\bar{b}+\frac{\bar{s}}{2\sqrt{2}}\right) T\left(\bar{b}-\frac{\bar{s}}{2\sqrt{2}}\right) e^{-\sigma T_2\left(\bar{b},\frac{\bar{s}}{\sqrt{2}}\right)} \\ &\simeq N_2(2\sigma) + \frac{1}{2}\left[M_2(2\sigma) - M_3(2\sigma)\right] \quad . \end{aligned} \tag{11.7}$$

This term gives an integrated cross-section

$$\sigma^{I} \simeq 2 \, \frac{\sigma_{px}^2}{\sigma} \cdot \frac{1}{1+R_d^2/4a_{px}} \, N_2(2\sigma) \quad . \tag{11.8}$$

The results can also be obtained from probability considerations.

Similar formulae can be worked out for the production of anti-deuterons, since deuteron and anti-deuteron have identical wave functions.

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- APPENDIX -

In this Appendix we collect definitions and numerical values for the effective nucleon numbers used in the text. The definitions are as follows :

$$N_0(\sigma) = \frac{1}{\sigma} \int d^2b \left\{ 1 - e^{-\sigma T(\bar{b})} \right\} \quad , \tag{A.1}$$

$$N_k(\sigma) = \frac{1}{k!} \int d^2b \cdot \left(\sigma T(\bar{b}) \right)^{k-1} T(\bar{b})\, e^{-\sigma T(\bar{b})} \quad , \tag{A.2}$$

$$M_k(\sigma) = \frac{1}{(k-1)!} \cdot \frac{R_d^2}{16} \int d^2b \left(\sigma T(\bar{b}) \right)^{k-1} \Delta T(\bar{b})\, e^{-\sigma T(\bar{b})} \quad , \tag{A.3}$$

$$\delta N(\sigma) = \frac{2\pi}{\sigma R_d^2} \int_0^\infty b_+ db_+ \int_0^\infty b_- db_-\, I_0\left(\frac{2b_+b_-}{R_d^2} \right) e^{-\frac{1}{R_d^2}(b_+^2 + b_-^2)} \cdot \left\{ 1 - e^{-\sigma T(b_+)} \right\} \left\{ 1 - e^{-\sigma T(b_-)} \right\} \quad , \tag{A.4}$$

$$\delta M_0(\sigma) = \sigma \left(\frac{R_d^2}{16} \right)^2 \int d^2b\, e^{-\sigma T(\bar{b})} \left[\left(\frac{d^2 T(b)}{db^2} \right)^2 + \left(\frac{1}{b} \cdot \frac{dT(b)}{db} \right)^2 \right] \quad . \tag{A.5}$$

The numerical values given in the Tables are calculated with a particle density of the Woods-Saxon form, with radius $R = 1.14 \cdot A^{1/3}$ fm and surface diffuseness parameter $d = 0.545$ fm. By interpolation it should be possible to get the effective nucleon numbers for any nucleus and any cross-section desired.

A	30	35	40	45	50	55
14.4	11.12	10.69	10.30	9.94	9.59	9.27
16.0	12.16	11.68	11.23	10.81	10.42	10.05
27.0	18.85	17.90	17.04	16.25	15.53	14.86
40.0	25.96	24.44	23.08	21.85	20.74	19.73
64.0	37.70	35.11	32.84	30.82	29.03	27.43
108.0	56.44	51.94	48.06	44.70	41.77	39.19
140.0	68.65	62.80	57.83	53.56	49.86	46.64
208.0	92.08	83.51	76.34	70.28	65.10	60.63

Table III : σ (mb)
Effective nucleon number $N_0(\frac{1}{2}\sigma)$

A	30	35	40	45	50	55
14.4	1.57	1.68	1.75	1.81	1.85	1.88
16.0	1.82	1.93	2.01	2.07	2.11	2.13
27.0	3.62	3.74	3.81	3.84	3.84	3.81
40.0	5.83	5.92	5.92	5.87	5.78	5.67
64.0	9.92	9.83	9.64	9.38	9.09	8.79
108.0	17.12	16.54	15.87	15.16	14.46	13.79
140.0	22.07	21.08	20.02	18.98	17.98	17.04
208.0	31.96	30.01	28.11	26.34	24.72	23.25

Table IV : σ (mb)
Effective nucleon number $\delta N(\frac{1}{2}\sigma)$

A	30	35	40	45	50	55
14.4	8.70	8.42	7.93	7.50	7.11	6.76
16.0	9.71	9.09	8.54	8.06	7.63	7.24
27.0	14.25	13.16	12.23	11.42	10.72	10.10
40.0	18.82	17.22	15.87	14.72	13.73	12.88
64.0	26.00	23.54	21.51	19.81	18.37	17.14
108.0	36.92	33.08	29.98	27.44	25.31	23.50
140.0	43.81	39.08	35.30	32.22	29.65	27.48
208.0	56.74	50.32	45.24	41.13	37.74	34.89

Table V : σ (mb)
Effective nucleon number $N_o(\sigma)$

A	30	35	40	45	50	55
14.4	6.45	5.90	5.44	5.06	4.73	4.44
16.0	6.89	6.29	5.80	5.38	5.02	4.71
27.0	9.55	8.62	7.87	7.25	6.73	6.28
40.0	12.13	10.87	9.87	9.05	8.37	7.79
64.0	16.07	14.31	12.92	11.80	10.87	10.08
108.0	21.95	19.42	17.45	15.86	14.56	13.47
140.0	25.62	22.61	20.27	18.39	16.86	15.57
208.0	32.46	28.54	25.51	23.10	21.12	19.47

Table VI : σ (mb)
Effective nucleon number $N_o(2\sigma)$

A	30	35	40	45	50	55
14.4	5.46	4.80	4.26	3.81	3.43	3.11
16.0	5.75	5.02	4.43	3.95	3.55	3.21
27.0	7.21	6.14	5.30	4.65	4.12	3.70
40.0	8.33	6.97	5.95	5.16	4.55	4.07
64.0	9.68	7.96	6.72	5.80	5.09	4.54
108.0	11.19	9.09	7.62	6.56	5.76	5.14
140.0	11.94	9.67	8.11	6.98	6.14	5.48
208.0	13.14	10.62	8.91	7.68	6.76	6.05

Table VII : σ (mb)

Effective nucleon number $N_1(\sigma)$

A	30	35	40	45	50	55
14.4	2.85	2.42	2.10	1.86	1.66	1.51
16.0	2.93	2.48	2.15	1.90	1.70	1.54
27.0	3.35	2.82	2.43	2.14	1.92	1.74
40.0	3.68	3.08	2.66	2.35	2.10	1.90
64.0	4.10	3.44	2.98	2.62	2.35	2.13
108.0	4.65	3.91	3.38	2.99	2.68	2.43
140.0	4.96	4.17	3.61	3.19	2.86	2.59
208.0	5.48	4.62	4.00	3.53	3.17	2.87

Table VIII : σ (mb)

Effective nucleon number $N_1(2\sigma)$

A	30	35	40	45	50	55
14.4	2.19	2.09	1.98	1.85	1.72	1.60
16.0	2.40	2.28	2.13	1.98	1.83	1.69
27.0	3.65	3.29	2.94	2.63	2.35	2.11
40.0	4.73	4.10	3.56	3.09	2.71	2.39
64.0	6.10	5.07	4.25	3.60	3.09	2.69
108.0	7.60	6.06	4.93	4.10	3.49	3.02
140.0	8.30	6.51	5.25	4.35	3.69	3.20
208.0	9.29	7.15	5.72	4.72	4.02	3.49

Table IX : σ (mb)
Effective nucleon number $N_2(\sigma)$

A	30	35	40	45	50	55
14.4	1.48	1.28	1.11	0.97	0.86	0.77
16.0	1.56	1.33	1.15	1.00	0.88	0.79
27.0	1.90	1.57	1.32	1.14	1.00	0.89
40.0	2.12	1.71	1.45	1.24	1.09	0.97
64.0	2.38	1.92	1.60	1.38	1.22	1.09
108.0	2.67	2.15	1.81	1.57	1.38	1.24
140.0	2.82	2.29	1.93	1.67	1.48	1.32
208.0	3.09	2.52	2.13	1.85	1.63	1.47

Table X : σ (mb)
Effective nucleon number $N_2(2\sigma)$

A	30	35	40	45	50	55
14.4	0.88	0.95	0.99	1.00	0.99	0.97
16.0	1.02	1.08	1.11	1.12	1.10	1.06
27.0	1.93	1.93	1.87	1.77	1.65	1.53
40.0	2.89	2.74	2.53	2.30	2.07	1.86
64.0	4.31	3.84	3.36	2.91	2.52	2.19
108.0	6.12	5.09	4.21	3.49	2.93	2.50
140.0	7.02	5.66	4.57	3.74	3.11	2.63
208.0	8.22	6.43	5.05	4.06	3.35	2.84

Table XI : σ (mb)
Effective nucleon number $N_3(\sigma)$

A	30	35	40	45	50	55
14.4	0.94	0.86	0.78	0.69	0.62	0.56
16.0	1.02	0.92	0.82	0.73	0.64	0.57
27.0	1.41	1.19	1.00	0.86	0.74	0.65
40.0	1.67	1.35	1.11	0.94	0.81	0.71
64.0	1.92	1.51	1.23	1.03	0.89	0.78
108.0	2.16	1.68	1.37	1.15	1.00	0.88
140.0	2.27	1.77	1.44	1.22	1.06	0.94
208.0	2.45	1.92	1.58	1.34	1.17	1.04

Table XII : σ (mb)
Effective nucleon number $N_3(2\sigma)$

A	30	35	40	45	50	55
14.4	0.79	0.83	0.85	0.86	0.87	0.87
16.0	0.87	0.90	0.92	0.93	0.94	0.93
27.0	1.30	1.32	1.33	1.32	1.31	1.28
40.0	1.71	1.71	1.70	1.67	1.63	1.59
64.0	2.29	2.26	2.21	2.15	2.08	2.01
108.0	3.09	3.00	2.89	2.77	2.66	2.55
140.0	3.55	3.42	3.27	3.13	2.98	2.85
208.0	4.34	4.13	3.93	3.73	3.54	3.36

Table XIII : σ (mb)
Effective nucleon number $M_1(\sigma)$

A	30	35	40	45	50	55
14.4	0.86	0.85	0.82	0.80	0.77	0.74
16.0	0.93	0.90	0.87	0.84	0.81	0.78
27.0	1.26	1.20	1.15	1.09	1.04	0.95
40.0	1.55	1.46	1.37	1.30	1.22	1.16
64.0	1.94	1.80	1.68	1.57	1.48	1.39
108.0	2.44	2.25	2.08	1.93	1.80	1.69
140.0	2.72	2.50	2.30	2.13	1.98	1.85
208.0	3.20	2.91	2.67	2.46	2.29	2.13

Table XIV : σ (mb)
Effective nucleon number $M_1(2\sigma)$

A	30	35	40	45	50	55
14.4	-0.25	-0.20	-0.14	-0.08	-0.02	0.03
16.0	-0.25	-0.19	-0.12	-0.06	0.00	0.06
27.0	-021	-0.09	0.02	0.11	0.19	0.26
40.0	-0.11	0.05	0.19	0.30	0.39	0.47
64.0	0.11	0.31	0.48	0.61	0.70	0.77
108.0	0.48	0.73	0.91	1.04	1.13	1.19
140.0	0.73	0.99	1.18	1.30	1.39	1.44
208.0	1.18	1.46	1.64	1.76	1.82	1.85

Table XV : σ (mb)
Effective nucleon number $M_2(\sigma)$

A	30	35	40	45	50	55
14.4	0.07	0.15	0.21	0.25	0.28	0.31
16.0	0.11	0.18	0.24	0.29	0.32	0.34
27.0	0.32	0.40	0.46	0.49	0.52	0.53
40.0	0.53	0.61	0.65	0.68	0.68	0.68
64.0	0.83	0.89	0.92	0.92	0.91	0.90
108.0	1.23	1.27	1.26	1.24	1.21	1.17
140.0	1.47	1.48	1.46	1.42	1.37	1.32
208.0	1.86	1.84	1.79	1.72	1.62	1.58

Table XVI : σ (mb)
Effective nucleon number $M_2(2\sigma)$

A	30	35	40	45	50	55
14.4	0.27	0.28	0.29	0.30	0.30	0.31
16.0	0.29	0.30	0.31	0.32	0.32	0.33
27.0	0.41	0.42	0.43	0.44	0.46	0.45
40.0	0.52	0.54	0.55	0.55	0.56	0.56
64.0	0.69	0.70	0.71	0.72	0.71	0.71
108.0	0.92	0.94	0.94	0.94	0.93	0.92
140.0	1.07	1.08	1.08	1.07	1.05	1.04
208.0	1.32	1.32	1.31	1.30	1.27	1.24

Table XVII : σ (mb)
Effective nucleon number $\delta M_0(2\sigma)$

A	30	35	40	45	50	55
14.4	-0.07	-2.68	-2.06	-2.24	-2.15	-2.31
16.0	-3.74	-3.14	-3.34	-3.54	-2.85	-3.52
27.0	-3.87	-3.84	3.78	3.45	3.82	3.73
40.0	-4.20	4.30	4.15	4.22	4.02	4.45
64.0	6.44	6.35	6.41	6.27	7.23	7.76
108.0	7.89	7.73	7.04	8.06	8.51	7.36
140.0	9.15	9.86	9.93	9.08	9.79	9.97
208.0	10.22	10.18	10.11	10.49	10.19	11.74

Table XVIII : σ (mb)
Effective nucleon number $\delta M_5(6\sigma)$

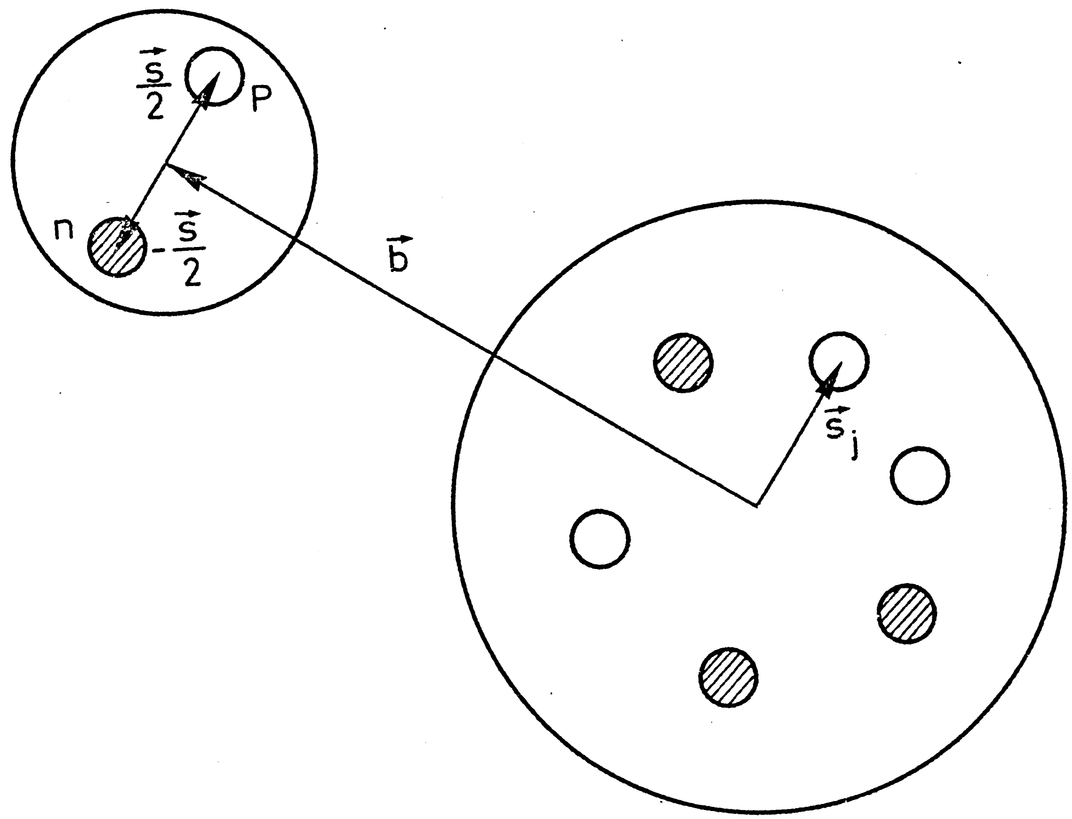
$\frac{\vec{s}}{2}$
P
n
$-\frac{\vec{s}}{2}$
$\vec{b}$
$\vec{s}_j$

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FIGURE CAPTION

Definitions of the transverse components $\bar{s}$ and $\bar{s}_j$ of the nucleon co-ordinate vectors in a deuteron-nucleus collision. The impact parameter between the deuteron c.m. and nuclear c.m. is denoted $\bar{b}$.